Directing functionals and de Branges space completions in almost Pontryagin spaces

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It is very powerful !

... and yet – seemingly – neither widely known nor widely used.

Outline

1. Three Representation Theorems

2. Directing functionals
   - Motivation
   - Aspect 1: Representations in spaces $L^2(\mu)$
   - Aspect 2: Representations in reproducing kernel spaces
   - Directing functionals in Pontryagin- or Krein spaces

3. Representations in almost Pontryagin spaces of analytic functions
   - Almost Pontryagin spaces
   - Representations in Ω-spaces

Three Representation Theorems

It is a basic problem to...

... find normal forms (representations) for operators.

In other words:

... find unitary equivalences to particular operators in particular spaces.

Recall some notation:

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- T is completely non-selfadjoint, if $\bigcap_{\zeta\in\mathbb{C}\setminus\mathbb{R}} \operatorname{ran}(T-\zeta) = \{0\}$.
- T is regular, if its set of points of regular type equals $\mathbb{C}$.

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Definition

If $\mathcal{H}$ is a Hilbert space of functions, $S(\mathcal{H})$ is the multiplication operator defined on its natural maximal domain.

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Assume

- *both deficiency indices of T are $\leqslant 1$,*
- *T is completely non-selfadjoint.*

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Assume

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Then

- *$\exists\,\mu$ positive Borel measure on $\mathbb{R}$,*
- *$\exists\,\mathcal{F}:\mathcal{H}\to L^2(\mu)$ isometric,*

such that

$$\mathcal{F}\circ T\subseteq S(L^2(\mu))\circ\mathcal{F}.$$

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Definition

A Herglotz space $\mathcal{A}$ is the reproducing kernel space of functions analytic on $\mathbb{C}\backslash\mathbb{R}$, generated by the kernel of a Herglotz function:

$$N_q(w,z) = \frac{q(z) - \overline{q(w)}}{z - \overline{w}}$$

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Then

- *$\exists\,\mathcal{A}$ Herglotz space,*
- *$\exists\,\mathcal{F}:\mathcal{H}\to\mathcal{A}$ unitary,*

such that

$$\mathcal{F}\circ T = S(\mathcal{A})\circ\mathcal{F}.$$

Theorem 3 gives a representation in a reproducing kernel space of entire functions.

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- For a function f denote $f^{\#}(z) := \overline{f(\overline{z})}$.

Definition

A de Branges space $\mathcal{B}$ is a reproducing kernel space of entire functions satisfying

- $\forall f \in \mathcal{B}\ \forall w \in \mathbb{C}\setminus\mathbb{R}: \quad f(w) = 0 \quad \Rightarrow \quad \dfrac{f(z)}{z-w} \in \mathcal{B}$
- $\forall f \in \mathcal{B}\ \forall w \in \mathbb{C}\setminus\mathbb{R}: \quad f(w) = 0 \quad \Rightarrow \quad \left\| \dfrac{z-\overline{w}}{z-w} f(z) \right\|_{\mathcal{B}} = \|f\|_{\mathcal{B}}$
- $\forall f \in \mathcal{B}: \quad f^{\#} \in \mathcal{B}$ and $\|f^{\#}\|_{\mathcal{B}} = \|f\|_{\mathcal{B}}$

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Then

- *$\exists\, \mathcal{B}$ de Branges space,*
- *$\exists\, \mathcal{F} : \mathcal{H} \to \mathcal{B}$ unitary,*

such that

$$\mathcal{F} \circ T = S(\mathcal{B}) \circ \mathcal{F}.$$

Directing functionals

In many concrete situations one is given a non-complete positive semidefinite space $\mathcal{L}$ and a symmetric operator S in $\mathcal{L}$.

Applying representation theorems requires to pass to the completion $\mathcal{H}$ of $\mathcal{L}$ and the closure T of S in this completion.

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This brings up a difficulty:

- $\mathcal{L}$ and S are given directly by the data of the problem under consideration.
- $\mathcal{H}$ and T are often not easily accessible
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- $\mathcal{H}$ and T are often not easily accessible
  (simply by the inconstructive nature of the completion process).

At this point directing functionals come into play!

- They allow to produce explicit representations of S.
- They allow to draw conclusions about the closure T.

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$\Phi : \mathcal{L} \times \mathbb{R} \to \mathbb{C}$ is a directing functional for S, if:

- $\forall \zeta \in \mathbb{R}$: $x \mapsto \Phi(x, \zeta)$ is linear.
- $\forall x \in \mathcal{L}\ \exists\, \Omega_x$ open neighbourhood of $\mathbb{R}$:
  $\zeta \mapsto \Phi(x, \zeta)$ has analytic continuation to Ω_x.
- $\forall x \in \mathcal{L}, \zeta \in \mathbb{R}$: $x \in \operatorname{ran}(S - \zeta) \;\Leftrightarrow\; \Phi(x, \zeta) = 0$.
- $\exists\, x \in \mathcal{L}$: $\zeta \mapsto \Phi(x, \zeta)$ does not vanish identically.

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Assume

- *we have a directing functional for S, say Φ.*

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- *$\exists\, \mu$ positive Borel measure on $\mathbb{R}$,*

such that

- *$\mathcal{F} : x \mapsto \Phi(x, \cdot)$ is isometry of $\mathcal{L}$ into $L^2(\mu)$,*
- *$\mathcal{F} \circ S \subseteq S(L^2(\mu)) \circ \mathcal{F}$.*

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$\Phi : \mathcal{L} \times \mathbb{C} \to \mathbb{C}$ is a universal directing functional for S, if:

- $\forall \zeta \in \mathbb{C}$: $x \mapsto \Phi(x, \zeta)$ is linear.
- $\forall x \in \mathcal{L}$: $\zeta \mapsto \Phi(x, \zeta)$ is analytic on $\mathbb{C}$.
- $\forall x \in \mathcal{L}, \zeta \in \mathbb{C}$: $x \in \operatorname{ran}(S - \zeta) \;\Leftrightarrow\; \Phi(x, \zeta) = 0$.
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- $\exists\, \mathcal{B}$ *de Branges space,*

such that

- $\mathcal{F} : x \mapsto \Phi(x,\cdot)$ *is isometry of $\mathcal{L}$ onto dense subspaces of $\mathcal{B}$,*
- $\mathcal{F} \circ S \subseteq S(\mathcal{B}) \circ \mathcal{F}$, *and even* $S(\mathcal{B}) = \mathrm{Clos}_{\mathcal{B}\times\mathcal{B}}[(\mathcal{F}\times\mathcal{F})(S)]$.

In particular, T is regular.

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Theorem

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Assume

- *both deficiency indices of T are 1,* ⇜ *assumption on $\mathbb{C}\backslash\mathbb{R}$*
- *we have a universal directing functional for S, say Φ,*
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In particular, T is regular. ⇜ *conclusion on $\mathbb{R}$*

In the previous representation theorem:

T closed symmetric operator in a Hilbert space $\mathcal{H}$. Assume

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Then

- $\exists\, \mathcal{B}$ de Branges space,
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- both deficiency indices of T are 1,
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- we have a universal directing functional for S, say Φ, ⇜ T regular
- $\exists\, x \in \mathcal{L}$: $\Phi(x, \cdot) = 1$. ⇜ explicit representation

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Example (Power moment problem)

Let

- $(c_j)_{j=0}^{\infty}$ be a sequence of real numbers,
- $\mathbb{C}[z]$ be the linear space of polynomials,
- S be the multiplication operator $p(z) \mapsto zp(z)$, $p \in \mathbb{C}[z]$.

Define an inner product on $\mathbb{C}[z]$ by sesquilinearity and

$$\left[z^k, z^l\right] := c_{k+l}, \quad k, l = 0, 1, 2, \ldots,$$

and assume that $[.,.]$ is positive semidefinite.

Define $\Phi : \mathbb{C}[z] \times \mathbb{C} \to \mathbb{C}$ by

$$\Phi(p, z) := p(z).$$

Then S is symmetric, and Φ is a universal directing functional for S.

Directing functionals in Pontryagin- or Krein spaces

- Versions of directing functionals for Pontryagin- or Krein spaces focus mainly on Aspect 1, representations in spaces – analogue to – $L^2(\mu)$.
- Aspect 2, representations in reproducing kernel spaces, are only implicitly contained in the literature.

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Remark

The notions of directing functionals and universal directing functionals become indistinct.

Representations in almost Pontryagin spaces of analytic functions

Definition

$\mathcal{Q}$ is an almost Pontryagin space (aPs), if $\mathcal{Q} = \mathcal{H}[\dot{+}]\mathcal{K}$ with

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- An aPs is endowed with the natural inner product and topology.
- $\mathrm{ind}_0\,\mathcal{Q}$ denotes the dimension of the isotropic part of $\mathcal{Q}$.
- An aPs $\mathcal{Q}$ is a Pontryagin space (Ps), if and only if $\mathrm{ind}_0\,\mathcal{Q} = 0$.

Definition

Let $\mathcal{L}$ be an inner product space.

$\langle \iota, \mathcal{Q} \rangle$ is an aPs-completion of $\mathcal{L}$, if

- $\mathcal{Q}$ is an aPs,
- $\iota : \mathcal{L} \to \mathcal{Q}$ is linear, isometric, and $\iota(\mathcal{L})$ is dense in $\mathcal{Q}$.

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Remark

- An inner product space $\mathcal{L}$ has an aPs-completion, if and only if $\mathrm{ind}_- \mathcal{L} < \infty$.

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Remark

- An inner product space $\mathcal{L}$ has an aPs-completion, if and only if $\operatorname{ind}_- \mathcal{L} < \infty$.
- If $\operatorname{ind}_- \mathcal{L} < \infty$, then there exist many aPs-completion.

  In fact:
  - There exists a unique Ps-completion.
  - For each $\Delta \geqslant 1$ there exist many aPs-completions with $\operatorname{ind}_0 \mathcal{Q} = \Delta$.

Definition

Let Ω be a nonempty set. $\mathcal{Q}$ is a reproducing kernel aPs on Ω, if

- $\mathcal{Q}$ is an aPs,
- the elements of $\mathcal{A}$ are complex-valued functions on Ω, and the linear operations on $\mathcal{A}$ are given pointwise.
- $\forall \eta \in \Omega$: point evaluation $\chi_\eta : \mathcal{A} \to \mathbb{C}$ is continuous.

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$\Phi : \mathcal{L} \times \Omega \to \mathbb{C}$ is a strong Ω-directing functional for S, if:

- $\forall \zeta \in \mathbb{C}$: $x \mapsto \Phi(x, \zeta)$ is linear.
- $\forall x \in \mathcal{L}$: $\zeta \mapsto \Phi(x, \zeta)$ is analytic on Ω.
- $\forall x \in \mathcal{L}, \zeta \in \Omega \backslash \mathbb{R}$: $x \in \operatorname{ran}(S - \zeta) \Leftrightarrow \Phi(x, \zeta) = 0$.
- There is no nonempty open subset O of Ω, such that $\Phi|_{\mathcal{L} \times O} = 0$.

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- There is no nonempty open subset O of Ω, such that $\Phi|_{\mathcal{L} \times O} = 0$.

- One important difference to (universal) directing functionals: The condition

$$x \in \operatorname{ran}(S - \zeta) \Leftrightarrow \Phi(x, \zeta) = 0$$

is required only for nonreal points.

This is not magic: An assumption along $\mathbb{R}$ will appear in the theorem.

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- For f analytic, denote by $\mathfrak{d}_f(w)$ the multiplicity of w as zero of f.
- For a set $\mathcal{B}$ of analytic functions, denote $\mathfrak{d}_{\mathcal{B}}(w) = \inf_{f\in\mathcal{B}} \mathfrak{d}_f(w)$.

Definition

Let $\Omega \subseteq \mathbb{C}$ be open and nonempty.

An Ω-space $\mathcal{B}$ is a reproducing kernel aPs of functions analytic in Ω satisfying

- $\forall w \in \Omega\ \forall f \in \mathcal{B}$: $\quad f^{(\mathfrak{d}_{\mathcal{B}}(w))}(w) = 0 \quad \Longrightarrow \quad \dfrac{f(z)}{z-w} \in \mathcal{B}$
- $\forall w \in \Omega\ \forall f, g \in \mathcal{B}$: $\quad f^{(\mathfrak{d}_{\mathcal{B}}(w))}(w) = g^{(\mathfrak{d}_{\mathcal{B}}(w))}(w) = 0 \quad \Longrightarrow$
$$\Big[\frac{z-\overline{w}}{z-w} f(z), \frac{z-\overline{w}}{z-w} g(z)\Big]_{\mathcal{B}} = \big[f(z), g(z)\big]_{\mathcal{B}}$$
- There is no connected component of Ω where all functions of $\mathcal{B}$ vanish identically.

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Main Theorem (weak – but readable – variant)

Let $\mathcal{L}$ be an inner product space with $\operatorname{ind}_- \mathcal{L} < \infty$, S a symmetric operator in $\mathcal{L}$, and $\Omega \subseteq \mathbb{C}$ open and nonempty.

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- *there exists an aPs-completion $\langle \iota, \mathcal{A} \rangle$ of $\mathcal{L}$, such that*
    - *both deficiency indices of $T := \operatorname{Clos}_{\mathcal{A}\times\mathcal{A}} \big[(\iota \times \iota)(S) \big]$ are 1,*
    - $\bigcap_{\eta \in \mathbb{C}\setminus\mathbb{R}} \operatorname{ran}(T - \eta) \subseteq \mathcal{A}^\circ$,
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$\Omega \cup (\mathbb{C}\setminus\mathbb{R}) \subseteq r(\tilde{T})$ for the closure of S in a suitable aPs-completion.

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This talk was based on my paper

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The paper and these slides are available from my website

`http://www.asc.tuwien.ac.at/~woracek`