Stability of order and type of a measure

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These slides are available from my website

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Outline

The type of a measure

Definition

Let μ be a finite positive Borel measure on $\mathbb{R}$. The type of μ is the supremum $T[\mu]$ of all $a \in [0, \infty)$ such that

$$\operatorname{span}\left\{e^{itz} : |t| \leqslant a\right\}$$

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is not dense in $L^2(\mu)$.

Finiteness of μ is a superfluous assumption. The natural class to consider is measures with at most polynomial growth:

$$\mu \quad \text{such that} \quad \exists\, N \in \mathbb{N} : \int_{\mathbb{R}} \frac{d\mu(x)}{1 + x^{2N}} < \infty.$$

This (and more) follows from equivalent ways to define $T[\mu]$.

Fact

Let $\mathcal{E}(a)$ be the Fourier image of all C^∞-functions with support in $(-a,a)$.
Then $T[\mu]=\sup\{a\geqslant 0:\mathcal{E}(a)$ not dense in $L^2(\mu)\}$.

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Fact

Given μ let $\mathcal{C}[\mu]$ be the set of all de Branges Hilbert spaces of entire functions $\mathcal{H}(E)$ with

- E is of Cartwright class and has no real zeroes,
- $\mathcal{H}(E)$ is contained isometrically in $L^2(\mu)$.

Then $T[\mu]$ is the supremum of the exponential types of functions in

$$\mathcal{L} := \bigcup_{\mathcal{H}(E)\in\mathcal{C}[\mu]} \mathcal{H}(E),$$

equivalently, the supremum of exponential types of functions E, $\mathcal{H}(E) \in \mathcal{C}[\mu]$.

A perturbation result

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Definition (Majorisation up to an exponentially small error)

For two measures μ and $\tilde{\mu}$ write $\mu \preccurlyeq \tilde{\mu}$, if

$$\exists\ \delta > 0,\ c_0, c_1, c_2 \geqslant 0 \qquad \forall\ x \in \mathbb{R}:$$
$$\mu\big((x - e^{-\delta|x|}, x + e^{-\delta|x|})\big) \leqslant$$
$$c_0\tilde{\mu}\big((x - c_1 e^{-\delta|x|}, x + c_1 e^{-\delta|x|})\big) + c_2 e^{-2\delta|x|}.$$

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Theorem (A.Borichev, M.Sodin 2011)

If $\mu\preccurlyeq\tilde\mu$, then $T[\mu]\leqslant T[\tilde\mu]$.

Type and spectral density

Let m be a nondecreasing bounded function on a finite interval $[0, L]$, and consider the Krein-Feller operator $-D_m D_x$. Let

- $(x_n)_{n\in\mathbb{N}}$ be its spectrum arranged as increasing sequence,
- μ its spectral measure,
- $\hat{\mu}$ the symmetrised measure $\hat{\mu} := \sum_{n\in\mathbb{N}} \mu(\{x_n\})\big(\delta_{\{\sqrt{x_n}\}} + \delta_{\{-\sqrt{x_n}\}}\big)$.

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Theorem (M.G.Krein 1951)

$$\lim_{n\to\infty} \frac{n}{\sqrt{x_n}} = \frac{1}{2} T[\hat{\mu}] = \int_0^L \sqrt{m'(x)}\, dx.$$

Notice

A corresponding theorem holds for canonical systems (M.G.Krein 1951 / L.de Branges 1961).

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Fact

For each $p \in (0,1)$ there exists a unique probability measure ν which is self-similar with weight p: for all $M \subseteq [0,1]$ Borel set

$$\nu(M) = p \cdot \nu\big(S_1^{-1}(M)\big) + (1-p) \cdot \nu\big(S_2^{-1}(M)\big).$$

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Theorem (U.Freiberg 2000)

Assume $\log(\frac{p}{3}) / \log(\frac{1-p}{3}) \notin \mathbb{Q}$. *Let*

- $\rho \in (0,1)$ *the solution of* $(\frac{p}{3})^\rho + (\frac{1-p}{3})^\rho = 1$,
- m *the distribution function of* ν, $(x_n)_{n\in\mathbb{N}}$ *the spectrum of* $-D_m D_x$,
- $N(r) := \#\{n : \sqrt{x_n} \leqslant r\}$ *the counting function.*

Then $\lim_{r\to\infty} N(r)/r^\rho \in (0,\infty)$.

Observe that...

...the type of the spectral measure cannot distinguish between different p. It is always 0.

Still, on a finer scale, the spectra for different p **are** different.

Let $\rho_n > 0$, $q_n \in \mathbb{R}$, and consider the Jacobi operator

$$J = \begin{pmatrix} q_0 & \rho_0 & & \\ \rho_0 & q_1 & \rho_1 & \\ & \rho_1 & q_2 & \ddots \\ & & \ddots & \ddots \end{pmatrix}$$

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Theorem (M.Riesz 1923)

Assume that J is limit circle case (type C). Let

- μ *a spectral measure,*
- $(x_n^+)_{n\in\mathbb{N}}$ *the positive spectral points arranged increasingly,*
- $(x_n^-)_{n\in\mathbb{N}}$ *the negative spectral points arranged decreasingly.*

Then

$$\lim_{n\to\infty} \frac{n}{x_n^+} = \lim_{n\to\infty} \frac{n}{|x_n^-|} = T[\mu] = 0.$$

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Assume the Jacobi parameters have power-like asymptotics

$$\rho_n = n^{\beta_1}\left(x_0 + \frac{x_1}{n} + \mathrm{O}(n^{-2})\right), \quad q_n = n^{\beta_2}\left(y_0 + \frac{y_1}{n} + \mathrm{O}(n^{-2})\right),$$

with $x_0 > 0$, $y_0 \neq 0$, and $x_1, y_1 \in \mathbb{R}$.

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with $x_0 > 0$, $y_0 \neq 0$, and $x_1, y_1 \in \mathbb{R}$.

Fact

Necessary conditions that J is in limit circle case are

- $\beta_1 > 1$ (Carleman 1926),
- $\beta_1 - \beta_2 \geqslant 0$ with $|y_0| \leqslant 2x_0$ if $\beta_1 = \beta_2$ (Wouk 1953).

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Theorem (Yu.M.Berezanskii 1956)

Assume $\beta_1 - \beta_2 > 1$. Then J is in limit circle case and for all $\varepsilon > 0$

$$\limsup_{r\to\infty} \frac{N(r)}{r^{\frac{1}{\beta_1}-\varepsilon}} = \infty, \qquad \limsup_{r\to\infty} \frac{N(r)}{r^{\frac{1}{\beta_1}+\varepsilon}} = 0.$$

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Theorem (R.Pruckner 2017)

Assume $\beta_1 - \beta_2 \in [0,1]$ *with* $|y_0| < 2x_0$ *if* $\beta_1 = \beta_2$. *Then* J *is in limit circle case and*

$$\limsup_{r\to\infty} \frac{N(r)}{r^{\frac{1}{\beta_1}}} > 0, \qquad \limsup_{r\to\infty} \frac{N(r)}{r^{\frac{1}{\beta_1}}(\ln^{[m]} r)^{1-\frac{1}{\beta_1}}} < \infty, m \in \mathbb{N},$$

where $\ln^{[0]} x := x$ *and* $\ln^{[m]} x := \ln(\ln^{[m-1]} x)$.

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Still, on a finer scale, the spectra of different J **are** different.

Order and λ-type of a measure

The basic problem (informally)

Does there exist an underlying concept of

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...... and what can one say about it.

Let μ be a positive Borel measure on $\mathbb{R}$ with at most power growth, let $\mathcal{C}[\mu]$ be the chain of all de Branges Hilbert spaces of entire functions $\mathcal{H}(E)$ with

- E is of Cartwright class and has no real zeroes,
- $\mathcal{H}(E)$ is contained isometrically in $L^2(\mu)$,

and set

$$\mathcal{L} := \bigcup_{\mathcal{H}(E)\in\mathcal{C}[\mu]} \mathcal{H}(E).$$

Recall that...

... $T[\mu]$ is the supremum of the exponential types of functions in $\mathcal{L}$.

Definition

The order of μ is the supremum $\rho[\mu]$ of the orders of functions in $\mathcal{L} = \bigcup_{\mathcal{H}(E)\in\mathcal{C}[\mu]} \mathcal{H}(E)$.

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Notice

- The order of μ is always in $[0,1]$.
- For each given $\rho \in [0,1]$ there exists μ such that $\rho[\mu] = \rho$.

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Example

Let μ be a discrete measure $\mu = \sum m_n \cdot \delta_{a_n}$.
Let γ be the convergence exponent of (a_n).

- Then $\rho[\mu] \leqslant \gamma$.
- If μ is limit circle (i.e., $\mathcal{C}[\mu]$ has maximal element), then $\rho[\mu] = \gamma$.

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- Typical examples are

$$\lambda(r) = r^a \cdot \big(\log r\big)^{b_1} \cdot \big(\log\log r\big)^{b_2} \cdot \ldots \cdot \big(\log^{[m]} r\big)^{b_m} \quad (r \text{ large}),$$

where $a \geqslant 0$ and $b_1, \ldots, b_m \in \mathbb{R}$ such that $\log r = \mathrm{o}(\lambda(r))$.

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Definition

The λ-type of an entire function f is the infimum of all $\alpha > 0$ with

$$\exists\, \beta > 0 : \quad |f(z)| \leqslant \beta e^{\alpha \cdot \lambda(|z|)},\ z \in \mathbb{C}.$$

Equivalently: λ-type of $f = \limsup_{|z| \to \infty} \dfrac{\log|f(z)|}{\lambda(|z|)}$.

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Notice

- The λ-type of μ is understood in $[0,\infty]$.
- For each given λ and $\tau \in [0,\infty]$ there exists μ such that $\tau_\lambda[\mu] = \tau$.

Fact (limit circle case is easy...)

Assume μ is limit circle, i.e., $\mathcal{C}[\mu]$ has a maximal element $\mathcal{H}(E_0)$.

- Then $\mathcal{L} = \mathcal{H}(E_0)$ and the order (λ-type, resp.) of μ equals the order (λ-type, resp.) of E_0.
- μ is discrete, and the order (λ-type, resp.) can be read off the support of μ.

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Notice

Can be used for constructing limit point example:

1. assume E_0 has a certain growth (order or λ-type),
2. consider the de Branges–chain in $\mathcal{H}(E_0)$,
3. append an infinite polynomial chain.

Then

- the resulting limit circle measure has the same order and λ-type as E_0.

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A complete solution seems out of reach ......

...... any results welcome !

Growth dependent Borichev-Sodin type perturbations

Recall that...

Majorisation up an exponentially small error: $\mu \preccurlyeq \tilde{\mu}$, if

$$\exists\ \delta > 0,\ c_0, c_1, c_2 \geqslant 0 \qquad \forall\ x \in \mathbb{R}:$$
$$\mu\big((x - e^{-\delta|x|}, x + e^{-\delta|x|})\big) \leqslant$$
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Definition (Majorisation up a λ-small error)

Let λ be a growth function.

For two measures μ and $\tilde{\mu}$ write $\mu \preceq_\lambda \tilde{\mu}$, if

$$\exists\ c_0, c_1, c_2 \geqslant 0 \qquad \forall\ x \in \mathbb{R}:$$

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Example

Measures which arise from discrete ones when point masses spread over small intervals.

μ is a bump measure if there exist $\delta > 0$ and disjoint intervals I_n with

- $\mu(I_n) \neq 0$ and $\operatorname{supp} \mu \subseteq \bigcup I_n$,
- length of I_n is $\leqslant \exp(-\delta \cdot |\max I_n|)$.

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Pick $a_n \in I_n$ and set $\tilde{\mu} := \sum \mu(I_n) \cdot \delta_{a_n}$. Then $\mu \preceq_{\delta r} \tilde{\mu}$ and $\tilde{\mu} \preceq_{\delta r} \mu$.

Fast Growth Theorem

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- Assume μ is majorised by $\tilde{\mu}$.
- We think of $\tilde{\mu}$ as the *unperturbed* measure and of μ as the *perturbed*.
- $\tilde{\mathcal{L}} = \bigcup_{\mathcal{H}(E)\in\mathcal{C}[\tilde{\mu}]} \mathcal{H}(E)$.

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Fast Growth Theorem (informally)

Assume suitable a priori knowledge connecting the maximal growth of functions in $\tilde{\mathcal{L}}$ with the size of the perturbation. Then $\tilde{\mathcal{L}} \subseteq L^2(\mu)$ and:

1. **either** $\tilde{\mathcal{L}}$ *dense in* $L^2(\mu)$,
2. **or** $\operatorname{Clos}_{L^2(\mu)} \tilde{\mathcal{L}}$ *contains functions growing faster than those in* $\tilde{\mathcal{L}}$.

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This means that...

...density may be lost, but this loss must be balanced with occurrance of fast growing functions in the closure.

Parameters occur which should be fitted to each other.

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  Quantifies the a priori knowledge and strength of the conclusion.
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Fast Growth Theorem (quantitatively precise)

Assume: $\mu, \tilde{\mu}$ *have at most power growth,* $\tilde{\mu}$ *infinite index of determinacy*

- $\mu \preceq_{\lambda_1} \tilde{\mu}$
- $\forall\, f \in \tilde{\mathcal{L}}: \ \limsup_{|z|\to\infty} \frac{\log|f(z)|}{\lambda(|z|)} < \infty$ *and* $\limsup_{x\to\pm\infty} \frac{\log|f(x)|}{\lambda(|x|)} \leqslant c$
- $\exists\, c^+ > c: \ \lambda_1 \geqslant 2c^+\lambda$

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Then: $\tilde{\mathcal{L}} \subseteq L^2(\mu)$, *and*

1. **either** $\operatorname{Clos}_{L^2(\mu)} \tilde{\mathcal{L}} = L^2(\mu)$
2. **or** $\exists\, f \in \operatorname{Clos}_{L^2(\mu)} \tilde{\mathcal{L}}:$

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Assume $r = \mathrm{O}(\lambda_1(r))$ and use $\lambda(r) := r, c := 0$.

Assume: $\mu, \tilde{\mu}$ have at most power growth, $\tilde{\mu}$ infinite index of determinacy

- $\mu \preceq_{\lambda_1} \tilde{\mu}$
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- $\exists\, c^+ > c :\ \lambda_1 \geqslant 2c^+\lambda$ ...automatic ✓

Then: $\tilde{\mathcal{L}} \subseteq L^2(\mu)$, and

1. **either** $\mathrm{Clos}_{L^2(\mu)}\, \tilde{\mathcal{L}} = L^2(\mu)$
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  Conclusion: $\exists\, f \in \mathrm{Clos}_{L^2(\mu)} \mathbb{C}[z] : \limsup_{x\to\pm\infty} \dfrac{\log|f(x)|}{|x|^\alpha} \geqslant \dfrac{1}{2} - \varepsilon.$

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In particular: $\rho[\mu] = \rho[\tilde\mu]$ *and* $\tau_\lambda[\mu] = \tau_\lambda[\tilde\mu]$.

Corollary

Assume: $\mu, \tilde{\mu}$ *have at most power growth,* $\mu, \tilde{\mu}$ *infinite index of determinacy*

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This means that...

...an a priori estimate gives rise to equality of orders.

Example

Let μ be a bump measure with at most power growth.

- Let I_n be supporting intervals, and pick $a_n \in I_n$ $(a_n \neq 0)$.
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Then

$$\rho[\mu] \leqslant \gamma$$

with equality if

$$\exists\, N \in \mathbb{N} : \quad \sum \frac{1}{\mu(I_n)|P'(a_n)| \cdot |a_n|^N} < \infty.$$

Definition

For a growth function λ and a measure μ set

$$h_\lambda[\mu](\phi) := \sup\Big\{ \limsup_{r\to\infty} \frac{\log|f(re^{i\phi})|}{\lambda(r)} : f \in \bigcup_{\mathcal{H}\in\mathcal{C}[\mu]} \mathcal{H} \Big\},$$

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Notice

For $\lambda(r) = r^a \cdot \big(\log r\big)^{b_1} \cdot \big(\log\log r\big)^{b_2} \cdot \ldots \cdot \big(\log^{[m]} r\big)^{b_m}$, we have $\rho_\lambda = a$.

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Set $\theta := \frac{\pi}{2}$ *if* $\rho_\lambda \leqslant \frac{1}{2}$ *and* $\theta := \frac{\pi}{2}(\frac{1}{\rho_\lambda} - 1)$ *if* $\frac{1}{2} < \rho_\lambda \leqslant 1$.

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Observe that...

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  ...automatic ✓
- $\exists\, c^+ > c : \ \lambda_1 \geqslant 2c^+\lambda$ ...automatic ✓

Then:

$$\underline{h}_\lambda[\mu](\phi) \leqslant h_\lambda[\tilde{\mu}](\phi), \quad |\phi - \frac{\pi}{2}| < \theta.$$

If $\rho_\lambda \in (\frac{1}{2}, 1), \lambda(r) = \mathrm{o}(r^{\rho_\lambda})$, or $\rho_\lambda = 1$, also for "$= \theta$".

Quasi-Monotonicity Theorem (large λ_1)

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Then:

$$\tau_\lambda[\mu] \leqslant \tau_\lambda[\tilde{\mu}].$$