Order and type of canonical systems. A survey.

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These slides are available from my website

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Outline

Introduction to the problem

We consider a 2-dimensional canonical system

$$f'(x) = zJH(x)f(x), \quad x \in (0, L),$$

where the Hamiltonian H satisfies

- $H : (0, L) \to \mathbb{R}^{2\times 2}$,
- $H(x) \geqslant 0$, $x \in (0, L)$ a.e.,
- $H \in L^1_{\mathrm{loc}}(0, L)$,
- H does not vanish identically on any set of positive measure,
- $z \in \mathbb{C}$ a parameter,
- $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

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- $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

We *always* assume $\int_0^x \operatorname{tr} H(y)\, dy < \infty, \quad x < L.$

Fundamental solution

The fundamental solution of H is the unique 2×2-matrix valued solution $W(x,z) = (w_{ij}(x,z))_{i,j=1}^2$ of the initial value problem

$$\begin{cases} \frac{d}{dx}W(x,z)J = zW(x,z)H(x), \quad x\in[0,L), \\ W(0,z) = I. \end{cases}$$

For technical reasons one passes to the transpose in the original equation.

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For technical reasons one passes to the transpose in the original equation.

For each $x\in(0,L)$ the entries of $W(x,z)$ are entire functions in z.

The de Branges chain

For each $x \in (0, L)$ the kernel

$$K_x(w, z) := \frac{w_{12}(x, z)w_{11}(x, \overline{w}) - w_{11}(x, z)w_{12}(x, \overline{w})}{z - \overline{w}}$$

is positive semidefinite and generates a reproducing kernel Hilbert space $\mathcal{H}_x$ of entire functions. This space is a de Branges space.

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Let $0 < x < y < L$. Then $\mathcal{H}_x \subseteq \mathcal{H}_y$ and the inclusion map is contractive.

If x is not inner point of an indivisible interval, the inclusion $\mathcal{H}_x \subseteq \mathcal{H}_y$ is isometric.

An interval $(a, b) \subseteq (0, L)$ is indivisible, if $H(x) = h(x)\xi_\phi\xi_\phi^T$ for $x \in (a, b)$ a.e., where $\xi_\phi = (\cos\phi,\ \sin\phi)^T$ and $h(x) > 0$ is a scalar-valued function.

Order and type

Definition

For $x \in [0, L)$ let $\rho(x)$ be the order of the function $w_{11}(x, \cdot)$. That is, the infimum of all $\rho > 0$ such that there exist $\alpha, \beta > 0$ with

$$|w_{11}(x, z)| \leqslant \alpha \exp\big(\beta \cdot |z|^{\rho}\big), \quad z \in \mathbb{C}. \tag{1}$$

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Definition

Let $x \in [0, L)$ and assume that $\rho(x) < \infty$. Let $\tau(x)$ be the type of the function $w_{11}(x, \cdot)$ w.r.t. its order. That is, the infimum of all $\beta > 0$ such that (1) holds for some $\alpha > 0$ and $\rho = \rho(x)$.

Order and type

Type can also be considered w.r.t. a growth function $\lambda(r)$ (instead of r^ρ). For example:

$$\lambda(r) = r^\rho \cdot (\log r)^{\alpha_1} \cdot (\log\log r)^{\alpha_2} \cdots$$

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Definition

The type $\tau_\lambda(x)$ w.r.t. $\lambda(r)$ is the infimum of all $\beta > 0$ such that

$$|w_{11}(x,z)| \leqslant \alpha \exp\big(\beta \cdot \lambda(|z|)\big), \quad z \in \mathbb{C}.$$

Order and type

THE PRINCIPLE PROBLEM:

Given H, compute $\rho(x)$ and $\tau(x)$, or $\tau_\lambda(x)$ for prescribed λ.

Krein–de Branges formula

Theorem (M.G.Krein 1951 / L.de Branges 1961)

Exponential type (i.e., type w.r.t. $\lambda(r) = r$) is given by the formula

$$\tau_r(x) = \int_0^x \sqrt{\det H(y)}\, dy.$$

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$$\tau_r(x) = \int_0^x \sqrt{\det H(y)}\, dy.$$

It *cannot be expected* that for other speeds of growth $\tau_\lambda(x)$ can be described by a nearly as neat formula, and the same for $\rho(x)$.

Properties of order and type

Theorem (explicit: A.D.Baranov and H.Woracek 2006)

- *The functions* $w_{ij}(x, \cdot)$, $i, j = 1, 2$, *all have the same order and* λ*-type.*
- $\rho(x)$ *and* $\tau_\lambda(x)$ *are the maximum of orders or types, respectively, of elements of the space* $\mathcal{H}_x$.

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Corollary

The functions $\rho : x \mapsto \rho(x)$ *and* $\tau_\lambda : x \mapsto \tau_\lambda(x)$ *are nondecreasing.*

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Corollary

The functions $\rho : x \mapsto \rho(x)$ and $\tau_\lambda : x \mapsto \tau_\lambda(x)$ are nondecreasing.

- The functions ρ and τ_λ are neither necessarily left-continuous nor right-continuous.
- For each finite sequence $\lambda_1,\dots,\lambda_n$ of growth functions with $\lambda_i \leqslant \lambda_{i+1}$, and $0<\alpha_1\leqslant\ldots\leqslant\alpha_n$, there exists a Hamiltonian H such that for some points $0<x_1<\cdots<x_n<L$

$$\tau_{\lambda_i}(x_i)=\alpha_i, \quad i=1,\dots,n.$$

Properties of order and type

It is an open question...

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For $\rho(x)$ the answer is expected to be "yes".
The same for $\tau_\lambda(x)$ provided $\lambda(r) = o(r)$.

This restriction is natural:

- If $\lambda(r) = r$ a function is of the form $\tau_\lambda(x)$ if and only if it is nondecreasing and absolutely continuous.
- If $r = o(\lambda(r))$, $\tau_\lambda(x)$ is identically 0.

Theorems I. Limit circle case

In this chapter assume that

$$\int_0^L \operatorname{tr} H(x)\, dx < \infty.$$

Then the monodromy matrix $W(L,z)$ of the system exists and we may investigate $\rho(L)$.

Romanov's Theorem

Theorem (R.V.Romanov 2016)

Let $\alpha \in (0,1]$. *Assume:* $\exists C > 0\ \forall R > 1\ \exists H^\star$ *composed of* $N^\star < \infty$ *indivisible intervals* $(x^\star_{j-1}, x^\star_j)$ *with angles* $\phi^\star_j$ $\exists a^\star_j \in (0,1]$:

- $\sum_{j=1}^{N^\star} \frac{1}{(a^\star_j)^2} \int_{x^\star_{j-1}}^{x^\star_j} \|H(x) - H^\star(x)\|\, dx \leqslant CR^{\alpha-1}$
- $\sum_{j=1}^{N^\star} (a^\star_j)^2 \big[x^\star_j - x^\star_{j-1}\big] \leqslant CR^{\alpha-1}$
- $\sum_{j=1}^{N^\star-1} \ln\big[1 + |\sin(\phi^\star_{j+1} - \phi^\star_j)| \cdot (a^\star_{j+1} a^\star_j)^{-1}\big] \leqslant CR^\alpha$
- $|\ln a^\star_1| + |\ln a^\star_{N^\star}| + \sum_{j=1}^{N^\star-1} \big|\ln \frac{a^\star_{j+1}}{a^\star_j}\big| \leqslant CR^\alpha$

Then $\rho(L) \leqslant \alpha$.

In *very simplified words*: If H can be approximated well by finite dimensional Hamiltonians, the order is small.

Romanov's Theorem

This result is sharp:

For each $\rho \in (0,1)$ there exists a Hamiltonian H such that $\rho(L) = \rho$ and that the conditions of the theorem are satisfied for all $\alpha \in (\rho, 1]$.

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We don't know what to expect.

Hamburger moment problems

A sequence $(s_n)_{n=0}^{\infty}$ is a Hamburger moment sequence if $s_n = \int_{\mathbb{R}} x^n \, d\mu(x)$ with some positive Borel measure μ.

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Associated to a Hamburger moment sequence is the Jacobi operator J. Its eigenvalue equation can be written as a three-term recurrance:

$$zP_n(z) = \rho_n P_{n+1}(z) + q_n P_n(z) + \rho_{n-1} P_{n-1}(z), \quad n = 0, 1, 2, \ldots$$

with ρ_n, q_n being the Jacobi parameters of $(s_n)_{n=0}^{\infty}$.

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with ρ_n, q_n being the Jacobi parameters of $(s_n)_{n=0}^\infty$.

The three-term recurrance of a Hamburger moment sequence can be rewritten as a canonical system: $H(x) := \xi_{\phi_n} \xi_{\phi_n}^T$, $x \in [x_{n-1}, x_n)$, where

$$q_n = -\frac{1}{l_n}\Big[\cot(\phi_{n+1} - \phi_n) + \cot(\phi_n - \phi_{n-1})\Big]$$

$$\frac{1}{\rho_n} = |\sin(\phi_{n+1} - \phi_n)| \sqrt{l_n l_{n+1}}, \quad x_n := \sum_{k=1}^n l_k.$$

Livšic's Theorem

Theorem (M.S.Livšic 1939)

Let H be the Hamiltonian arising from an indeterminate moment sequence $(s_n)_{n=1}^\infty$. Then

$$\rho(L) \geqslant \limsup_{n\to\infty} \frac{2n \ln n}{\ln s_{2n}}.$$

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In Livšic's estimate equality does not always hold. The gap between the left and right hand sides can be arbitrarily close to 1.

The Berg-Szwarc formula

Theorem (C.Berg and R.Szwarc 2014)

Let H be the Hamiltonian arising from an indeterminate moment sequence $(s_n)_{n=1}^{\infty}$, and let $P_n(z) = \sum_{k=0}^{n} b_{k,n} z^k$, $n \in \mathbb{N}_0$, be the orthogonal polynomials associated with this sequence. Then

$$\rho(L) = \limsup_{k \to \infty} \frac{-2k \ln k}{\ln \sum\limits_{n=k}^{\infty} b_{k,n}^2}.$$

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$$\rho(L)=\limsup_{k\to\infty}\frac{-2k\ln k}{\ln\sum\limits_{n=k}^{\infty}b_{k,n}^2}.$$

This formula is unlikely to be of much *practical* use, since it requires knowledge of *all* coefficients of orthogonal polynomials.

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Definition

Let $\vec{l}$, $\vec{\phi}$ be sequences of real numbers with $l_n > 0$ and $\phi_{n+1} \not\equiv \phi_n \mod \pi$. The Hamburger Hamiltonian $H_{\vec{l},\vec{\phi}}$ is

$$H_{\vec{l},\vec{\phi}}(x) := \xi_{\phi_n}\xi_{\phi_n}^T,\ x \in [x_{n-1}, x_n), \qquad x_n := \sum_{k=1}^{n} l_k.$$

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Corollary

Let $H_{\vec{l},\vec{\phi}}$ be a Hamburger Hamiltonian in the limit circle case. Then

$$\rho(L) \geqslant \limsup_{k\to\infty} \frac{-k\ln k}{\ln \prod_{n=1}^{k-1} |\sin(\phi_{k+1} - \phi_k)|\sqrt{l_k l_{k+1}}}.$$

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We introduce measures for decay and smoothness of a Hamburger Hamiltonian $H_{\vec{l},\vec{\phi}}$.

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- Smoothness of angles:

$$\Delta_\phi := \sup\left\{\tau \geqslant 0 : \frac{1}{n}\sum_{k=n}^{2n-1} |\sin(\phi_{n+1} - \phi_n)| = \mathrm{O}(n^{-\tau})\right\}$$

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- Speed of possible convergence of angles:

$$\Lambda := \sup_{\phi\in[0,\pi)} \sup\Big\{\tau \geqslant 0 : \sum_{j=n}^{\infty} l_j |\sin(\phi_j - \phi)| = \mathrm{O}\big(n^{1-\Delta_l^+ - \tau}\big)\Big\}$$

An upper estimate for a Hamburger Hamiltonian

Theorem (R.Pruckner and R.V.Romanov and H.Woracek ⟦to appear⟧)

Let $H_{\vec{l},\vec{\phi}}$ be a Hamburger Hamiltonian in limit circle case (i.e. $\sum l_n < \infty$).

- *Case $\Delta_l + \Delta_\phi \geqslant 2$, $(\Delta_l, \Delta_\phi, \Lambda) \neq (1,1,0)$:*

$$\rho(L) \leqslant \frac{1}{\Delta_l + \Delta_\phi}. \qquad (2)$$

- *Case $\Delta_l + \Delta_\phi < 2$, $\Lambda \geqslant 2\Delta_\phi$: Also (2).*
- *Case $\Delta_l + \Delta_\phi < 2$, $\Lambda < 2\Delta_\phi$:*

$$\rho(L) \leqslant \frac{1 - \Delta_\phi + \frac{1}{2}\Lambda}{\Delta_l - \Delta_\phi + \Lambda}.$$

In *simplified words*: If $\vec{l}$ decays fast and ϕ_n behave smoothly, the order is small.

An upper estimate for a Hamburger Hamiltonian

Combining the upper bound (1st case) with the lower bound obtained from the Berg-Szwarc formula allows to compute $\rho(L)$ for Hamburger Hamiltonians whose lengths l_n and angle differences $|\phi_{n+1} - \phi_n|$ behave not too wildly and decay sufficently fast.

Corollary

Assume

- $\tau > 1$, $\sigma > 0$, *and* $\tau + \sigma > 2$
- $l_n = n^{-\tau} \cdot r_n$ *with* $\frac{\ln r_n}{\ln n} \to 0$
- $|\phi_{n+1} - \phi_n|$ *is nonincreasing and* $\mathrm{O}(n^{-\sigma})$

Then

$$\rho(L) = \frac{1}{\tau + \sigma}.$$

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It is an open question...

whether this bound is sharp at every point of the critical region (3^{rd} case) "$\Delta_l + \Delta_\phi < 2$, $\Lambda_\phi < 2\Delta_\phi$".

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whether this bound is sharp at every point of the critical region (3rd case) "$\Delta_l + \Delta_\phi < 2$, $\Lambda_\phi < 2\Delta_\phi$".

We tend to believe the answer is "yes".

Theorems II. Limit point case

In this chapter assume that

$$\int_0^L \operatorname{tr} H(x)\,dx = \infty.$$

Then the Weyl coefficient

$$Q_H(z) := \lim_{x\nearrow L} \frac{w_{11}(x,z)\tau + w_{12}(x,z)}{w_{21}(x,z)\tau + w_{22}(x,z)}, \quad z \in \mathbb{C}\backslash\mathbb{R}$$

of the system exists independently of $\tau \in \mathbb{R} \cup \{\infty\}$, is an analytic function in $\mathbb{C}^+$ with nonnegative imaginary part in this half-plane, and satisfies $Q_H(\overline{z}) = \overline{Q_H(z)}$.

Uniform order

Definition

Assume Q_H is meromorphic in the whole plane. The uniform order $\bar{\rho}(H)$ of H is the convergence exponent of the sequence $(\frac{1}{\omega_n})$, where ω_n are the nonzero poles of Q_H.

If Q_H is not meromorphic throughout $\mathbb{C}$, set $\bar{\rho}(H) := \infty$.

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If H ends with an indivisible interval (L', L), then $\bar{\rho}(H) = \rho(L')$.

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This is a straight generalisation of the limit circle situation:

If H ends with an indivisible interval (L', L), then $\bar{\rho}(H) = \rho(L')$.

Uniform order is maybe best understood from an operator theoretic viewpoint: $\bar{\rho}(H)$ is the infimum of all $\alpha > 0$ such that selfadjoint realisations of the canonical system have resolvents belonging to the Schatten-class $\mathfrak{S}_\alpha$.

The Hilbert-Schmidt condition

Theorem (explicit: M.Kaltenbäck and H.Woracek 2007)

A Hamiltonian H has the property $\mathfrak{S}_2$, if and only if there exists an angle $\phi \in [0, \pi)$ with

- $\displaystyle\int_0^L \xi_\phi^T H(x) \xi_\phi \, dx < \infty$,
- $\displaystyle\int_0^L \xi_{\phi+\frac{\pi}{2}}^T G(x) \xi_{\phi+\frac{\pi}{2}} \cdot \xi_\phi^T H(x) \xi_\phi \, dx < \infty$ *where* $G(x) := \int_0^x H(y) \, dy$.

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For $\phi = 0$ these conditions take the simple form $(H(x) = (h_{ij}(x))_{i,j=1}^2)$

$$\int_0^L h_{11}(x) \, dx < \infty, \qquad \int_0^L \Big(\int_0^x h_{22}(y) \, dy \Big) h_{11}(x) \, dx.$$

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$$\int_0^L h_{11}(x) \, dx < \infty, \qquad \int_0^L \Big(\int_0^x h_{22}(y) \, dy \Big) h_{11}(x) \, dx.$$

Hence, the property $\mathfrak{S}_2$ relates to a theorem of de Branges about Hamiltonians which are in the limit point case at their left endpoint.

Krein strings

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Associated to a string is the Krein-Feller differential operator $-D_m D_x$. Its eigenvalue equation can be written as an integral boundary value problem:

$$\begin{cases} f(x) - f(0) + z \int_{[0,x]} (x - y) f(y)\, dm(y) = 0, \quad x \in [0, L]\,, \\ f'(0-) = 0 \end{cases}$$

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The equation of a string can be rewritten as a canonical system:

$$H(x) := \begin{pmatrix} 1 & -m(x) \\ -m(x) & m(x)^2 \end{pmatrix}, \quad x \in [0, L).$$

dating back to Krein...

Let $m : [0, \infty) \to [0, \infty)$ be nondecreasing and bounded, and consider

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dating back to Krein...

Let $m : [0,\infty) \to [0,\infty)$ be nondecreasing and bounded, and consider

$$H(x) := \begin{pmatrix} 1 & -m(x) \\ -m(x) & m(x)^2 \end{pmatrix}, \quad x \in [0,\infty).$$

Theorem (I.S.Kac and M.G.Krein 1958)

Q_H is meromorphic in the whole plane if and only if $\lim_{x\to\infty} x(m(\infty) - m(x)) = 0.$

Theorem (M.G.Krein ≈1952)

H has the property $\mathfrak{S}_1$ if and only if $\int_0^\infty x\,dm(x) < \infty.$

Kac's Theorems

Theorem (I.S.Kac 1962)

Assume $\lim_{x\to\infty} x(m(\infty)-m(x))=0$ *and let* $\alpha\in\{2,3,\ldots\}$. *Then* H *has the property* $\mathfrak{S}_\alpha$ *if and only if*

$$\int_Q U(x_1,x_2)U(x_2,x_3)\cdot\dots\cdot U(x_{\alpha-1},x_\alpha)U(x_\alpha,x_1)\,dx_1\cdots dx_\alpha<\infty,$$

where

$$U(x,s):=\begin{cases} m(\infty)-m(s)\,, & x\leqslant s\\ m(\infty)-m(x)\,, & x>s\end{cases}$$

and

$$Q:=\big\{(x_1,\ldots,x_\alpha)\in\mathbb{R}^\alpha:\,0\leqslant x_1\leqslant\cdots\leqslant x_\alpha\big\}.$$

Kac's Theorems

Theorem (I.S.Kac 1986)

Assume $\int_0^\infty x\,dm(x) < \infty$ *and let* $\alpha \in (0,1)$. *Then* H *has the property* $\mathfrak{S}_\alpha$ *if and only if*

$$\int_0^L \int_0^{s_x(l)} \big[u_x(t)\big]^{\alpha-1}\,dt\,dm(x) < \infty,$$

where

$$u_x(s) := s\big(m(x+s) - m(x-s)\big), \quad s \in [0, \min\{x, L-x\}),$$

$$s_x(t) := \sup\big\{s \in [0, \min\{x, L-x\}) :\ u_x(s) \leqslant t\big\}.$$

Kac's Theorem

It is an open question...

how to characterise the property $\mathfrak{S}_\alpha$ when $\alpha > 1$ but not an integer.

Kac's Theorem

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how to characterise the property $\mathfrak{S}_\alpha$ when $\alpha > 1$ but not an integer.

Maybe interpolating between α's might help ?

Non-uniform order

- Can one define a notion of *order of a measure* generalising that of the type of a measure ?

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Most of the common definitions of the type of a measure (density of exponentials, density of Fourier transforms of fast decaying functions) are intrinsically related to exponential type and cannot be adapted for small orders.

Non-uniform order

- Can one define a notion of *order of a measure* generalising that of the type of a measure ?

Most of the common definitions of the type of a measure (density of exponentials, density of Fourier transforms of fast decaying functions) are intrinsically related to exponential type and cannot be adapted for small orders.

For a meaningful definition of the order of a measure (say, a finite measure), one needs an appropriate *testing space*.

This space is found by considering the canonical system whose Weyl coefficient is the Cauchy-transform of the measure.

Non-uniform order

Definition

Let H be a Hamiltonian. We define the non-uniform order $\rho(L)$ as

$$\rho(L) := \sup\left\{\rho(x) : x \in [0, L)\right\} \in [0, 1].$$

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The order of measure is now defined as follows: Let μ be given, let H be the Hamiltonian whose Weyl coefficient is $\int_{\mathbb{R}} \frac{d\mu(x)}{x-z}$, and let the order $\rho(\mu)$ of μ be $\rho(L)$.

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Let $\mathcal{H}_x$ be the spaces of the de Branges chain of H. Then $\rho(L)$ is the supremum of orders of elements of the linear space

$$\mathcal{L} := \bigcup_{x \in [0,L)} \mathcal{H}_x.$$

Non-uniform order

- $\rho(L) \leqslant \bar{\rho}(L)$.
- It may happen that $\rho(L) = 0$ whereas $\bar{\rho}(L) = \infty$.
- For each $\rho \in [0, 1]$ there exist measures μ with nontrivial absolutely continuous part, such that the order of μ is ρ.

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- In order to determine or estimate non-uniform order, one can of course use the very definition, and try to estimate $\rho(x)$ by means of limit circle theorems for every x. Such investigations, however, have not been undertaken yet.
- The non-uniform order of a semibounded (finite) measure cannot exceed $\frac{1}{2}$. This, however, has rather trivial reasons.
- Otherwise, we do not know *any* theorem which determines or estimates the non-uniform order for a significant class of Hamiltonians.

Borichev-Sodin type perturbation

A notion of majorisation of measures was introduced by A.Borichev and M.Sodin (2011) for the case $\rho = 1$ to investigate the type of a measure.

We are intereted in orders < 1.

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Definition

Let $\rho \in (0,1)$. Write $\mu_1 \preceq \mu_2$, if

$$\exists\, c_0, c_1, c_2 \text{ with } c_1 \geqslant 1, c_0, c_2 \geqslant 0 \quad \forall\, x \in \mathbb{R}:$$
$$\mu_1\big((x - e^{-|x|^\rho}, x + e^{-|x|^\rho})\big) \leqslant c_0 \mu_2\big((x - c_1 e^{-|x|^\rho}, x + c_1 e^{-|x|^\rho})\big) + c_2 e^{-|x|^\rho}.$$

Borichev-Sodin type perturbation

Theorem (A.D.Baranov and H.Woracek ⟦recent⟧)

Let $\rho \in (0,1)$ *and assume that* $\rho(\mu_1), \rho(\mu_2) < \rho$.

If $\mu_1 \preceq \mu_2$ *and* $\mu_2 \preceq \mu_1$, *then* $\rho(\mu_1) = \rho(\mu_2)$.

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