Stability of the derivative of a canonical product

Harald Woracek

Vienna University of Technology

FWF (I 1536-N25) :: Joint Project :: RFBR (13-01-91002-ANF)

This presentation is based on:

- M. Langer and H. Woracek. "Stability of the derivative of a canonical product". In: *Complex Anal. Oper. Theory* (to appear), 44 pp. DOI: 10.1007/s11785-013-0315-5.
- M. Langer and H. Woracek. "Stability of N-extremal measures". 7pp. (submitted). Preprint in: ASC Report 05 (2013), Vienna University of Technology.
- H. Woracek. "Existence of zerofree functions N-associated to a de Branges Pontryagin space". In: *Monatsh. Math.* 162.4 (2011), pp. 453–506.

These slides are available from my website

http://asc.tuwien.ac.at/index.php?id=woracek

Outline

A perturbation problem

Given $\alpha = (\alpha_n)_{n\in\mathbb{N}}$, $\beta = (\beta_n)_{n\in\mathbb{N}}$ with

- $\alpha_n, \beta_n \in \mathbb{R}\backslash\{0\}$;

Given $\alpha = (\alpha_n)_{n\in\mathbb{N}}$, $\beta = (\beta_n)_{n\in\mathbb{N}}$ with

- $\alpha_n, \beta_n \in \mathbb{R}\backslash\{0\}$;
- α_n pairwise different, β_n pairwise different;

Given $\alpha = (\alpha_n)_{n\in\mathbb{N}}$, $\beta = (\beta_n)_{n\in\mathbb{N}}$ with

- $\alpha_n, \beta_n \in \mathbb{R}\backslash\{0\}$;
- α_n pairwise different, β_n pairwise different;
- the canonical products converge:

$$P_\alpha(z) := \lim_{r\to\infty} \prod_{|\alpha_n|\leqslant r} \Big(1-\frac{z}{\alpha_n}\Big), \quad P_\beta(z) := \lim_{r\to\infty} \prod_{|\beta_n|\leqslant r} \Big(1-\frac{z}{\beta_n}\Big).$$

Given $\alpha = (\alpha_n)_{n\in\mathbb{N}}$, $\beta = (\beta_n)_{n\in\mathbb{N}}$ with

- $\alpha_n, \beta_n \in \mathbb{R}\backslash\{0\}$;
- α_n pairwise different, β_n pairwise different;
- the canonical products converge:

$$P_\alpha(z) := \lim_{r\to\infty} \prod_{|\alpha_n|\leqslant r} \Big(1 - \frac{z}{\alpha_n}\Big), \quad P_\beta(z) := \lim_{r\to\infty} \prod_{|\beta_n|\leqslant r} \Big(1 - \frac{z}{\beta_n}\Big).$$

The question

How large may the perturbation $\gamma_n := \beta_n - \alpha_n$ be, such that still

$$\exists\, c, C > 0 : \quad c|P'_\alpha(\alpha_n)| \leqslant |P'_\beta(\beta_n)| \leqslant C|P'_\alpha(\alpha_n)|, \quad n \in \mathbb{N}.$$

- At places in the vicinity of which the sequence α is well separated and grows regularly, the sequence $\alpha' = (P'_\alpha(\alpha_n))_{n\in\mathbb{N}}$ behaves regularly and can be controlled.

- At places in the vicinity of which the sequence α is well separated and grows regularly, the sequence $\alpha' = (P'_\alpha(\alpha_n))_{n\in\mathbb{N}}$ behaves regularly and can be controlled.
- Points of α being close to each other give rise to peaks in α', and lumps of points being close to each other produce peaks which even may spread out over neighbouring points.

- At places in the vicinity of which the sequence α is well separated and grows regularly, the sequence $\alpha' = (P'_\alpha(\alpha_n))_{n\in\mathbb{N}}$ behaves regularly and can be controlled.
- Points of α being close to each other give rise to peaks in α', and lumps of points being close to each other produce peaks which even may spread out over neighbouring points.

The intuition

The perturbation γ must be smaller than the separation of α and must not allow that lumps of points appear/vanish in the vicinity of any point of α. Its maximal size

- *depends on the asymptotic growth of α,*
- *is limited relative to the separation of α,*
- *is limited relative to the local irregularity of α.*

Our aim is

. . . to make this intuition quantitatively precise.

Two toy examples

Example

Consider $\alpha_n := \left(n + \frac{1}{2}\right)^2$ and $\beta_n := \left(n + \frac{3}{2}\right)^2$, $n = 0, 1, 2, \ldots$

Two toy examples

Example

Consider $\alpha_n := \left(n+\frac{1}{2}\right)^2$ and $\beta_n := \left(n+\frac{3}{2}\right)^2$, $n = 0, 1, 2, \ldots$

Then

- $P_\alpha(z) = \cos\sqrt{z}, \quad P_\beta(z) = (1-2z)^{-1}\cos\sqrt{z}.$
- $P'_\alpha(\alpha_n) = \frac{(-1)^{n+1}\pi}{2n+3}, \quad P'_\beta(\beta_n) = \frac{(-1)^{n+2}\pi}{(2n+3)(2n+2)}.$

Two toy examples

Example

Consider $\alpha_n := \left(n + \frac{1}{2}\right)^2$ and $\beta_n := \left(n + \frac{3}{2}\right)^2$, $n = 0, 1, 2, \ldots$

Then

- $P_\alpha(z) = \cos\sqrt{z}, \quad P_\beta(z) = (1 - 2z)^{-1}\cos\sqrt{z}.$
- $P'_\alpha(\alpha_n) = \frac{(-1)^{n+1}\pi}{2n+3}, \quad P'_\beta(\beta_n) = \frac{(-1)^{n+2}\pi}{(2n+3)(2n+2)}.$

Conclusion

- $\gamma_n = \alpha_{n+1} - \alpha_n \asymp n \quad \not\Rightarrow \quad P'_\alpha(\alpha_n) \not\asymp P'_\beta(\beta_n).$

Two toy examples

Example

Consider $\alpha_n := \left(n + \frac{1}{2}\right)^2$ and $\beta_n := \left(n + \frac{3}{2}\right)^2$, $n = 0, 1, 2, \ldots$

Then

- $P_\alpha(z) = \cos\sqrt{z}, \quad P_\beta(z) = (1-2z)^{-1}\cos\sqrt{z}.$
- $P'_\alpha(\alpha_n) = \frac{(-1)^{n+1}\pi}{2n+3}, \quad P'_\beta(\beta_n) = \frac{(-1)^{n+2}\pi}{(2n+3)(2n+2)}.$

Conclusion

- $\gamma_n = \alpha_{n+1} - \alpha_n \asymp n \quad \not\Rightarrow \quad P'_\alpha(\alpha_n) \not\asymp P'_\beta(\beta_n).$

Notation

$$(\sigma_n)_{n\in\mathbb{N}} \asymp (\tau_n)_{n\in\mathbb{N}} \;:\Leftrightarrow\; \exists\, c, C > 0 : \quad c|\sigma_n| \leqslant |\tau_n| \leqslant C|\sigma_n|$$

Two toy examples

Example

Consider $\alpha_n := \left(n + \frac{1}{2}\right)^2$ and $\beta_n := \left(n + \frac{3}{2}\right)^2$, $n = 0, 1, 2, \dots$

Then

- $P_\alpha(z) = \cos\sqrt{z}, \quad P_\beta(z) = (1 - 2z)^{-1} \cos\sqrt{z}.$
- $P'_\alpha(\alpha_n) = \frac{(-1)^{n+1}\pi}{2n+3}, \quad P'_\beta(\beta_n) = \frac{(-1)^{n+2}\pi}{(2n+3)(2n+2)}.$

Conclusion

- $\gamma_n = \alpha_{n+1} - \alpha_n \asymp n \quad \not\Rightarrow \quad P'_\alpha(\alpha_n) \not\asymp P'_\beta(\beta_n).$

Our Theorems will show

- $\gamma_n = \mathrm{O}\left(\frac{n}{\log n \cdot (\log\log n)^2}\right) \quad \Rightarrow \quad P'_\alpha(\alpha_n) \asymp P'_\beta(\alpha_n + \gamma_n).$

Two toy examples

Example

Consider

$$\alpha_{2k-1} := k, \alpha_{2k} := -k, \ k = 1, 2, 3, \ldots, \qquad \alpha_0 := \frac{1}{10}.$$

Two toy examples

Example

Consider

$$\alpha_{2k-1} := k, \alpha_{2k} := -k, \ k = 1, 2, 3, \ldots, \qquad \alpha_0 := \frac{1}{10}.$$

- $P_\alpha(z) = \dfrac{1 - 10z}{\pi z} \sin(\pi z)$. Hence, $|P'_\alpha(\alpha_n)| \asymp 1$.

Two toy examples

Example

Consider

$$\alpha_{2k-1} := k, \alpha_{2k} := -k, \ k = 1, 2, 3, \ldots, \qquad \alpha_0 := \frac{1}{10}.$$

- $P_\alpha(z) = \dfrac{1 - 10z}{\pi z} \sin(\pi z)$. Hence, $|P'_\alpha(\alpha_n)| \asymp 1$.

Let $|\gamma_n| = \mathrm{O}\big(n^{-\varepsilon}\big)$ for some $\varepsilon > 0$ (with $\gamma_n \neq -\alpha_n$), and set $\beta_n := \alpha_n + \gamma_n$.

Two toy examples

Example

Consider

$$\alpha_{2k-1} := k, \alpha_{2k} := -k, \ k = 1, 2, 3, \dots, \qquad \alpha_0 := \frac{1}{10}.$$

- $P_\alpha(z) = \dfrac{1 - 10z}{\pi z} \sin(\pi z)$. Hence, $|P'_\alpha(\alpha_n)| \asymp 1$.

Let $|\gamma_n| = \mathrm{O}\big(n^{-\varepsilon}\big)$ for some $\varepsilon > 0$ (with $\gamma_n \neq -\alpha_n$), and set $\beta_n := \alpha_n + \gamma_n$.

- $P_\beta(z)$ is a sine type function. Hence, $|P'_\beta(\beta_n)| \asymp 1$.

Two toy examples

Example

Consider

$$\alpha_{2k-1} := k, \alpha_{2k} := -k,\ k = 1, 2, 3, \ldots, \qquad \alpha_0 := \frac{1}{10}.$$

- $P_\alpha(z) = \dfrac{1 - 10z}{\pi z} \sin(\pi z)$. Hence, $|P'_\alpha(\alpha_n)| \asymp 1$.

Let $|\gamma_n| = \mathrm{O}\big(n^{-\varepsilon}\big)$ for some $\varepsilon > 0$ (with $\gamma_n \neq -\alpha_n$), and set $\beta_n := \alpha_n + \gamma_n$.

- $P_\beta(z)$ is a sine type function. Hence, $|P'_\beta(\beta_n)| \asymp 1$.

Our Theorems will show

- $\gamma_n = \mathrm{O}\Big(\dfrac{1}{\log n \cdot (\log\log n)^2}\Big) \quad \Rightarrow \quad P'_\alpha(\alpha_n) \asymp P'_\beta(\alpha_n + \gamma_n).$

Well-separated sequences

For sequences α and γ denote

- α^+ the (finite or infinite) subsequence consisting of all positive elements of α arranged according to increasing modulus.
- $\gamma^{[+]}$ the correspondingly arranged subsequence of γ: $\gamma_n^{[+]} = \gamma_{k(n)} \Leftrightarrow \alpha_n^+ = \alpha_{k(n)}$.

$\alpha^-, \gamma^{[-]}$ ... analogous for the negative elements of α.

Well-separated sequences

For sequences α and γ denote

- α^+ the (finite or infinite) subsequence consisting of all positive elements of α arranged according to increasing modulus.
- $\gamma^{[+]}$ the correspondingly arranged subsequence of γ: $\gamma_n^{[+]} = \gamma_{k(n)} \Leftrightarrow \alpha_n^+ = \alpha_{k(n)}$.

$\alpha^-, \gamma^{[-]}$ ... analogous for the negative elements of α.

Theorem (communicated by A.Baranov)

Assume $\exists \rho \geqslant 0, c > 0 : \ |\alpha_{n+1}^{\pm} - \alpha_n^{\pm}| \geqslant c n^{\rho}$, *and*

Well-separated sequences

For sequences α and γ denote

- α^+ the (finite or infinite) subsequence consisting of all positive elements of α arranged according to increasing modulus.
- $\gamma^{[+]}$ the correspondingly arranged subsequence of γ: $\gamma_n^{[+]} = \gamma_{k(n)} \Leftrightarrow \alpha_n^+ = \alpha_{k(n)}$.

$\alpha^-, \gamma^{[-]}$ ... analogous for the negative elements of α.

Theorem (communicated by A.Baranov)

Assume $\exists \rho \geqslant 0, c > 0 : \ |\alpha_{n+1}^{\pm} - \alpha_n^{\pm}| \geqslant cn^{\rho}$, *and*

- $\sum_n \frac{|\gamma_n^{[\pm]}|}{|\alpha_n^{\pm}|} < \infty$,
- $\exists C > 0 : \ |\gamma_n^{[\pm]}| \leqslant C \frac{n^{\rho}}{\log n}$

Well-separated sequences

For sequences α and γ denote

- α^+ the (finite or infinite) subsequence consisting of all positive elements of α arranged according to increasing modulus.
- $\gamma^{[+]}$ the correspondingly arranged subsequence of γ: $\gamma_n^{[+]} = \gamma_{k(n)} \Leftrightarrow \alpha_n^+ = \alpha_{k(n)}$.

$\alpha^-, \gamma^{[-]}$ ... analogous for the negative elements of α.

Theorem (communicated by A.Baranov)

Assume $\exists \rho \geqslant 0, c > 0 : \ |\alpha_{n+1}^\pm - \alpha_n^\pm| \geqslant cn^\rho$, *and*

- $\sum_n \frac{|\gamma_n^{[\pm]}|}{|\alpha_n^\pm|} < \infty$,
- $\exists C > 0 : \ |\gamma_n^{[\pm]}| \leqslant C \frac{n^\rho}{\log n}$

Then $P'_\alpha(\alpha_n) \asymp P'_\beta(\alpha_n + \gamma_n)$.

Growth functions, separation and local irregularity

Growth functions

Definition

$\lambda : \mathbb{R}^+ \to \mathbb{R}^+$ is a *growth function* if

(gf1) λ is (for large r) differentiable, and $\lim_{r\to\infty} \lambda(r) = \infty$.

Growth functions

Definition

$\lambda : \mathbb{R}^+ \to \mathbb{R}^+$ is a *growth function* if

(gf1) λ is (for large r) differentiable, and $\lim_{r\to\infty} \lambda(r) = \infty$.

(gf2) $\rho_\lambda := \lim_{r\to\infty} \frac{\log \lambda(r)}{\log r} \in [0, \infty)$

Growth functions

Definition

$\lambda : \mathbb{R}^+ \to \mathbb{R}^+$ is a *growth function* if

(gf1) λ is (for large r) differentiable, and $\lim_{r\to\infty} \lambda(r) = \infty$.

(gf2) $\rho_\lambda := \lim_{r\to\infty} \frac{\log \lambda(r)}{\log r} \in [0, \infty)$

(gf3) $\lim_{r\to\infty} \left(r \frac{\lambda'(r)}{\lambda(r)} \Big/ \frac{\log \lambda(r)}{\log r} \right) = 1.$

Growth functions

Definition

$\lambda : \mathbb{R}^+ \to \mathbb{R}^+$ is a *growth function* if

(gf1) λ is (for large r) differentiable, and $\lim_{r\to\infty} \lambda(r) = \infty$.

(gf2) $\rho_\lambda := \lim_{r\to\infty} \frac{\log \lambda(r)}{\log r} \in [0, \infty)$

(gf3) $\lim_{r\to\infty} \left(r \frac{\lambda'(r)}{\lambda(r)} \Big/ \frac{\log \lambda(r)}{\log r} \right) = 1.$

Example

$\lambda(r) = r^a \cdot \left(\log_{(m_1)} r\right)^{b_1} \cdot \ldots \cdot \left(\log_{(m_n)} r\right)^{b_n}$, with

- $a \geqslant 0$, $m_i \in \mathbb{N}$, $m_1 < \ldots < m_n$,
- $b_1, \ldots, b_n \in \mathbb{R}$, with $b_1 > 0$ if $a = 0$,
- $\log_{(1)} r := \log r, \quad \log_{(k+1)} r := \log\left(\log_{(k)} r\right)$, $k \in \mathbb{N}$.

We have $\rho_\lambda = a$.

Properties of growth functions

- For large r the function λ is strictly increasing and bounded away from 0.
- $\lim_{r\to\infty} \frac{\lambda(Cr)}{\lambda(r)} = C^{\rho_\lambda}$ uniformly in C on compact subsets of $(0,\infty)$.
- Let $\sigma > 0$. Then (for large r)

$$\frac{\lambda(r)}{r^\sigma} \text{ is } \begin{cases} \text{increasing}, & \sigma < \rho_\lambda \\ \text{decreasing}, & \sigma > \rho_\lambda \end{cases}$$

Upper and lower densities

Definition

Let λ be a growth function and $(\xi_n)_{n\in\mathbb{N}}$, $\xi \in \mathbb{R}$, a sequence without finite accumulation point. Set

$$n_\xi(r) := \#\{n \in \mathbb{N} : \, |\xi_n| \leqslant r\}$$

- *Upper λ-density:* $\Delta_\lambda(\xi) := \limsup_{r\to\infty} \frac{n_\xi(r)}{\lambda(r)}$
- *Lower λ-density:* $\delta_\lambda(\xi) := \liminf_{r\to\infty} \frac{n_\xi(r)}{\lambda(r)}$

Upper and lower densities

Definition

Let λ be a growth function and $(\xi_n)_{n\in\mathbb{N}}$, $\xi \in \mathbb{R}$, a sequence without finite accumulation point. Set

$$n_\xi(r) := \#\{n \in \mathbb{N} : |\xi_n| \leqslant r\}$$

- *Upper λ-density:* $\Delta_\lambda(\xi) := \limsup_{r\to\infty} \frac{n_\xi(r)}{\lambda(r)}$
- *Lower λ-density:* $\delta_\lambda(\xi) := \liminf_{r\to\infty} \frac{n_\xi(r)}{\lambda(r)}$

Lemma

Let λ be a growth function and $(\xi_n)_{n\in\mathbb{N}}$, $0 < \xi_1 \leqslant \xi_2 \leqslant \xi_3 \leqslant \dots$, a sequence without finite accumulation point. Then

$$\delta_\lambda(\xi) = \liminf_{n\to\infty} \frac{n}{\lambda(\xi_n^+)}, \qquad \Delta_\lambda(\xi) = \limsup_{n\to\infty} \frac{n}{\lambda(\xi_n^+)}$$

Upper and lower densities

Example

Let λ be a growth function, and set $\xi_n := \lambda^{-1}(n)$. Then

$$\delta_\lambda(\xi) = \Delta_\lambda(\xi) = 1.$$

Upper and lower densities

Example

Let λ be a growth function, and set $\xi_n := \lambda^{-1}(n)$. Then

$$\delta_\lambda(\xi) = \Delta_\lambda(\xi) = 1.$$

Remark

- For each sequence ξ without finite accumulation point, there exists a growth function λ with $0 < \Delta_\lambda(\xi) < \infty$.

Upper and lower densities

Example

Let λ be a growth function, and set $\xi_n := \lambda^{-1}(n)$. Then

$$\delta_\lambda(\xi) = \Delta_\lambda(\xi) = 1.$$

Remark

- For each sequence ξ without finite accumulation point, there exists a growth function λ with $0 < \Delta_\lambda(\xi) < \infty$.
- It is not always possible to choose λ such that

$$\Delta_\lambda(\xi) < \infty \text{ and } \delta_\lambda(\xi) > 0.$$

This is due to possible existence of large clusters of points in ξ.

Separation and local irregularity

Definition

Let $\xi = (\xi_n)_{n\in\mathbb{N}}$, $\xi \in \mathbb{R}\backslash\{0\}$, and let $\rho > 1$.

Separation and local irregularity

Definition

Let $\xi = (\xi_n)_{n\in\mathbb{N}}$, $\xi \in \mathbb{R}\backslash\{0\}$, and let $\rho > 1$.

Define

- $s_\xi(n) := \inf\left\{|\xi_k - \xi_n| : \ k \in \mathbb{N}, \xi_k \neq \xi_n\right\}$

Separation and local irregularity

Definition

Let $\xi = (\xi_n)_{n\in\mathbb{N}}$, $\xi \in \mathbb{R}\backslash\{0\}$, and let $\rho > 1$.

Define

- $s_\xi(n) := \inf \left\{|\xi_k - \xi_n| : \; k \in \mathbb{N}, \, \xi_k \neq \xi_n\right\}$
- $r_\xi(\rho, n) := \#\left\{k \in \mathbb{N} : \; \dfrac{\xi_k}{\xi_n} \in \left(\dfrac{1}{\rho}, \rho\right)\right\}$

Separation and local irregularity

Definition

Let $\xi = (\xi_n)_{n\in\mathbb{N}}$, $\xi \in \mathbb{R}\backslash\{0\}$, and let $\rho > 1$.

Define

- $s_\xi(n) := \inf\left\{|\xi_k - \xi_n| : \ k \in \mathbb{N},\, \xi_k \neq \xi_n\right\}$
- $r_\xi(\rho, n) := \#\left\{k \in \mathbb{N} : \ \dfrac{\xi_k}{\xi_n} \in \left(\dfrac{1}{\rho}, \rho\right)\right\}$

Example

Consider $\xi_n := n^\sigma$ with $\sigma \geqslant 1$. Then

$$s_\xi(n) \asymp n^{\sigma-1}, \qquad r_\xi(\rho, n) = \left\lfloor \left(\rho^{\frac{1}{\sigma}} - \rho^{-\frac{1}{\sigma}}\right) n \right\rfloor, \ \rho > 1.$$

The role of r_ξ

Lemma

Let λ *be a growth function,* $\rho > 1$, *and* $(\xi_n)_{n\in\mathbb{N}}$, $0 < \xi_1 \leqslant \xi_2 \leqslant \xi_3 \leqslant \ldots$, *a sequence without finite accumulation point.*

The role of r_ξ

Lemma

*Let λ be a growth function, $\rho > 1$, and $(\xi_n)_{n\in\mathbb{N}}$,
$0 < \xi_1 \leqslant \xi_2 \leqslant \xi_3 \leqslant \ldots$, a sequence without finite accumulation point.*

- *Assume $\Delta_\lambda(\xi) < \infty,\ \delta_\lambda(\xi) > 0.$ Then*

$$r_\xi(\rho, n) = \mathrm{O}(n), \qquad \sum_{\substack{k\in\mathbb{N} \\ \frac{\xi_k}{\xi_n}\in(\frac{1}{\rho},\rho)}} \frac{1}{k} = \mathrm{O}(1), \quad n \to \infty.$$

The role of r_ξ

Lemma

Let λ *be a growth function,* $\rho > 1$*, and* $(\xi_n)_{n\in\mathbb{N}}$*,*
$0 < \xi_1 \leqslant \xi_2 \leqslant \xi_3 \leqslant \ldots$*, a sequence without finite accumulation point.*

- *Assume* $\Delta_\lambda(\xi) < \infty,\ \delta_\lambda(\xi) > 0$*. Then*

$$r_\xi(\rho, n) = \mathrm{O}(n), \qquad \sum_{\substack{k\in\mathbb{N} \\ \frac{\xi_k}{\xi_n}\in(\frac{1}{\rho},\rho)}} \frac{1}{k} = \mathrm{O}(1), \quad n \to \infty.$$

- *Assume* $\xi_n = \lambda^{-1}(n)$*. Then*

$$r_\xi(\rho, n) \asymp n \ (\textit{if } \rho_\lambda > 0), \qquad \sum_{\substack{k\in\mathbb{N}\colon k\neq n \\ \frac{\xi_k}{\xi_n}\in(\frac{1}{\rho},\rho)}} \frac{1}{|\xi_k - \xi_n|} = \mathrm{O}\left(\frac{n\log n}{\xi_n}\right)$$

The stability theorems

Sequences under consideration

Remember: For a sequence α denote by

- α^+ the (finite or infinite) subsequence consisting of all positive elements of α arranged according to increasing modulus.
- α^- ... analogous for the negative elements of α.

Sequences under consideration

Remember: For a sequence α denote by

- α^+ the (finite or infinite) subsequence consisting of all positive elements of α arranged according to increasing modulus.
- α^- ... analogous for the negative elements of α.

Definition

Let $\mathbb{S}$ the set of all sequences $\alpha = (\alpha_n)_{n=1}^{\infty}$ with

(S1) $\alpha_n \in \mathbb{R}\backslash\{0\}$, pairwise distinct, without finite accumulation point.

(S2) $\lim\limits_{n\to\infty} \dfrac{n}{\alpha_n^+} = \lim\limits_{n\to\infty} \dfrac{n}{|\alpha_n^-|} \in [0,\infty)$

(S3) $\lim\limits_{r\to\infty} \sum\limits_{|\alpha_n|\leqslant r} \dfrac{1}{\alpha_n} \in \mathbb{R}$

The general situation

Theorem

Let $\alpha \in \mathbb{S}$ *and let* Λ *be a growth function, such that*

- $\frac{\Lambda(r)}{r}$ *is (for large* r*) non-increasing or non-decreasing.*
- $\sum_{n \in \mathbb{N}} \frac{1}{\Lambda(|\alpha_n|)} < \infty$.

The general situation

Theorem

Let $\alpha \in \mathbb{S}$ and let Λ be a growth function, such that

- $\frac{\Lambda(r)}{r}$ *is (for large r) non-increasing or non-decreasing.*
- $\sum_{n\in\mathbb{N}} \frac{1}{\Lambda(|\alpha_n|)} < \infty$.

Let $\beta \in \mathbb{S}$, set $\gamma := \beta - \alpha$, and assume

(A) $|\gamma_n| = \mathrm{O}\left(\frac{|\alpha_n|}{\Lambda(|\alpha_n|)}\right)$.

(B) $\left(\frac{\gamma_n}{s_\alpha(n)}\right)_{n\in\mathbb{N}} \in \ell^1$.

(C) $\exists\, \rho > 1 : \ |\gamma_n| = \mathrm{O}\left(\frac{s_\alpha(n)}{r_\alpha(\rho,n)}\right)$.

The general situation

Theorem

Let $\alpha \in \mathbb{S}$ and let Λ be a growth function, such that

- $\frac{\Lambda(r)}{r}$ *is (for large r) non-increasing or non-decreasing.*
- $\sum_{n\in\mathbb{N}} \frac{1}{\Lambda(|\alpha_n|)} < \infty$.

Let $\beta \in \mathbb{S}$, set $\gamma := \beta - \alpha$, and assume

(A) $|\gamma_n| = \mathrm{O}\left(\frac{|\alpha_n|}{\Lambda(|\alpha_n|)}\right)$.

(B) $\left(\frac{\gamma_n}{s_\alpha(n)}\right)_{n\in\mathbb{N}} \in \ell^1$.

(C) $\exists\, \rho > 1 : \ |\gamma_n| = \mathrm{O}\left(\frac{s_\alpha(n)}{r_\alpha(\rho,n)}\right)$.

Then $|P'_\alpha(\alpha_n)| \asymp |P'_\beta(\beta_n)|$.

Mild regularity

Theorem

Let $\alpha \in \mathbb{S}$ *and let* Λ *be a growth function, such that*

- $\frac{\Lambda(r)}{r}$ *is (for large* r*) non-increasing or non-decreasing.*
- $\sum_{n\in\mathbb{N}} \frac{1}{\Lambda(|\alpha_n|)} < \infty$.

Mild regularity

Theorem

Let $\alpha \in \mathbb{S}$ and let Λ be a growth function, such that

- $\frac{\Lambda(r)}{r}$ *is (for large r) non-increasing or non-decreasing.*
- $\sum_{n\in\mathbb{N}} \frac{1}{\Lambda(|\alpha_n|)} < \infty$.

- α^+ infinite $\Rightarrow$ $\exists\lambda^+ :$ $\Delta_{\lambda^+}(\alpha^+) < \infty,$ $\delta_{\lambda^+}(\alpha^+) > 0$
  α^- infinite $\Rightarrow$ $\exists\lambda^- :$ $\Delta_{\lambda^-}(\alpha^-) < \infty,$ $\delta_{\lambda^-}(\alpha^-) > 0$

Mild regularity

Theorem

Let $\alpha \in \mathbb{S}$ and let Λ be a growth function, such that

- $\frac{\Lambda(r)}{r}$ *is (for large r) non-increasing or non-decreasing.*
- $\sum_{n\in\mathbb{N}} \frac{1}{\Lambda(|\alpha_n|)} < \infty$.

- α^+ infinite $\Rightarrow$ $\exists\lambda^+ :\ \Delta_{\lambda^+}(\alpha^+) < \infty,\ \delta_{\lambda^+}(\alpha^+) > 0$
  α^- infinite $\Rightarrow$ $\exists\lambda^- :\ \Delta_{\lambda^-}(\alpha^-) < \infty,\ \delta_{\lambda^-}(\alpha^-) > 0$

Let $\beta \in \mathbb{S}$, set $\gamma := \beta - \alpha$, and assume

(A) $\quad |\gamma_n| = \mathrm{O}\left(\frac{|\alpha_n|}{\Lambda(|\alpha_n|)}\right)$.

Mild regularity

Theorem

Let $\alpha \in \mathbb{S}$ and let Λ be a growth function, such that

- $\frac{\Lambda(r)}{r}$ *is (for large r) non-increasing or non-decreasing.*
- $\sum_{n\in\mathbb{N}} \frac{1}{\Lambda(|\alpha_n|)} < \infty$.

- α^+ infinite $\Rightarrow$ $\exists\lambda^+ :\ \Delta_{\lambda^+}(\alpha^+) < \infty,\ \delta_{\lambda^+}(\alpha^+) > 0$
  α^- infinite $\Rightarrow$ $\exists\lambda^- :\ \Delta_{\lambda^-}(\alpha^-) < \infty,\ \delta_{\lambda^-}(\alpha^-) > 0$

Let $\beta \in \mathbb{S}$, set $\gamma := \beta - \alpha$, and assume

(A) $|\gamma_n| = \mathrm{O}\left(\frac{|\alpha_n|}{\Lambda(|\alpha_n|)}\right)$.

(B) $\left(\frac{\gamma_n}{s_\alpha(n)}\right)_{n\in\mathbb{N}} \in \ell^1$.

(C) $\exists\, \rho > 1 :\ |\gamma_n| = \mathrm{O}\left(\frac{s_\alpha(n)}{r_\alpha(\rho,n)}\right)$.

Mild regularity

Theorem

Let $\alpha \in \mathbb{S}$ and let Λ be a growth function, such that

- $\frac{\Lambda(r)}{r}$ *is (for large r) non-increasing or non-decreasing.*
- $\sum_{n\in\mathbb{N}} \frac{1}{\Lambda(|\alpha_n|)} < \infty$.

- α^+ infinite $\Rightarrow$ $\exists \lambda^+ :\ \Delta_{\lambda^+}(\alpha^+) < \infty,\ \delta_{\lambda^+}(\alpha^+) > 0$
  α^- infinite $\Rightarrow$ $\exists \lambda^- :\ \Delta_{\lambda^-}(\alpha^-) < \infty,\ \delta_{\lambda^-}(\alpha^-) > 0$

Let $\beta \in \mathbb{S}$, set $\gamma := \beta - \alpha$, and assume

(A) $\quad |\gamma_n| = \mathrm{O}\left(\frac{|\alpha_n|}{\Lambda(|\alpha_n|)}\right)$.

(B1) $\quad \frac{|\gamma_n^{[+]}|}{s_{\alpha^+}(n)} = \mathrm{O}\left(\frac{1}{n}\right), \qquad \frac{|\gamma_n^{[-]}|}{s_{\alpha^-}(n)} = \mathrm{O}\left(\frac{1}{n}\right)$.

(C) $\quad \exists\, \rho > 1 :\ |\gamma_n| = \mathrm{O}\left(\frac{s_\alpha(n)}{r_\alpha(\rho,n)}\right)$.

Mild regularity

Theorem

Let $\alpha \in \mathbb{S}$ and let Λ be a growth function, such that

- $\frac{\Lambda(r)}{r}$ *is (for large r) non-increasing or non-decreasing.*
- $\sum_{n\in\mathbb{N}} \frac{1}{\Lambda(|\alpha_n|)} < \infty$.
- α^+ infinite $\Rightarrow \exists\lambda^+ : \Delta_{\lambda^+}(\alpha^+) < \infty, \; \delta_{\lambda^+}(\alpha^+) > 0$
  α^- infinite $\Rightarrow \exists\lambda^- : \Delta_{\lambda^-}(\alpha^-) < \infty, \; \delta_{\lambda^-}(\alpha^-) > 0$

Let $\beta \in \mathbb{S}$, set $\gamma := \beta - \alpha$, and assume

(A) $\quad |\gamma_n| = \mathrm{O}\left(\frac{|\alpha_n|}{\Lambda(|\alpha_n|)}\right)$.

(B1) $\quad \frac{|\gamma_n^{[+]}|}{s_{\alpha^+}(n)} = \mathrm{O}\left(\frac{1}{n}\right), \qquad \frac{|\gamma_n^{[-]}|}{s_{\alpha^-}(n)} = \mathrm{O}\left(\frac{1}{n}\right)$.

Mild regularity

Theorem

Let $\alpha \in \mathbb{S}$ and let Λ be a growth function, such that

- $\frac{\Lambda(r)}{r}$ *is (for large r) non-increasing or non-decreasing.*
- $\sum_{n\in\mathbb{N}} \frac{1}{\Lambda(|\alpha_n|)} < \infty$.
- α^+ infinite $\Rightarrow$ $\exists\lambda^+ : \; \Delta_{\lambda^+}(\alpha^+) < \infty, \; \delta_{\lambda^+}(\alpha^+) > 0$
  α^- infinite $\Rightarrow$ $\exists\lambda^- : \; \Delta_{\lambda^-}(\alpha^-) < \infty, \; \delta_{\lambda^-}(\alpha^-) > 0$

Let $\beta \in \mathbb{S}$, set $\gamma := \beta - \alpha$, and assume

(A) $\quad |\gamma_n| = \mathrm{O}\Big(\frac{|\alpha_n|}{\Lambda(|\alpha_n|)}\Big)$.

(B1) $\quad \frac{|\gamma_n^{[+]}|}{s_{\alpha^+}(n)} = \mathrm{O}\big(\frac{1}{n}\big), \qquad \frac{|\gamma_n^{[-]}|}{s_{\alpha^-}(n)} = \mathrm{O}\big(\frac{1}{n}\big)$.

Then $|P'_\alpha(\alpha_n)| \asymp |P'_\beta(\beta_n)|$.

Strong regularity. I

Theorem

Let $\alpha \in \mathbb{S}$ and let Λ be a growth function, such that

- $\frac{\Lambda(r)}{r}$ *is (for large r) non-increasing or non-decreasing.*
- $\sum_{n\in\mathbb{N}} \frac{1}{\Lambda(|\alpha_n|)} < \infty$.

Strong regularity. I

Theorem

Let $\alpha \in \mathbb{S}$ and let Λ be a growth function, such that

- $\frac{\Lambda(r)}{r}$ *is (for large r) non-increasing or non-decreasing.*
- $\sum_{n\in\mathbb{N}} \frac{1}{\Lambda(|\alpha_n|)} < \infty$.

- α^+ infinite $\Rightarrow$ $\exists\lambda^+ : \Delta_{\lambda^+}(\alpha^+) < \infty, \ \delta_{\lambda^+}(\alpha^+) > 0$
  α^- infinite $\Rightarrow$ $\exists\lambda^- : \Delta_{\lambda^-}(\alpha^-) < \infty, \ \delta_{\lambda^-}(\alpha^-) > 0$

Strong regularity. I

Theorem

Let $\alpha \in \mathbb{S}$ and let Λ be a growth function, such that

- $\frac{\Lambda(r)}{r}$ *is (for large r) non-increasing or non-decreasing.*
- $\sum_{n\in\mathbb{N}} \frac{1}{\Lambda(|\alpha_n|)} < \infty$.

- α^+ infinite $\Rightarrow$ $\exists\lambda^+ :$ $\alpha^+(n) = (\lambda^+)^{-1}(n)$
  α^- infinite $\Rightarrow$ $\exists\lambda^- :$ $\alpha^-(n) = -(\lambda^-)^{-1}(n)$

Strong regularity. I

Theorem

Let $\alpha \in \mathbb{S}$ and let Λ be a growth function, such that

- $\frac{\Lambda(r)}{r}$ *is (for large r) non-increasing or non-decreasing.*
- $\sum_{n\in\mathbb{N}} \frac{1}{\Lambda(|\alpha_n|)} < \infty$.

- α^+ infinite $\Rightarrow$ $\exists \lambda^+ :$ $\alpha^+(n) = (\lambda^+)^{-1}(n)$
  α^- infinite $\Rightarrow$ $\exists \lambda^- :$ $\alpha^-(n) = -(\lambda^-)^{-1}(n)$

Let $\beta \in \mathbb{S}$, set $\gamma := \beta - \alpha$, and assume

(A) $\quad |\gamma_n| = \mathrm{O}\left(\frac{|\alpha_n|}{\Lambda(|\alpha_n|)}\right).$

Strong regularity. I

Theorem

Let $\alpha \in \mathbb{S}$ and let Λ be a growth function, such that

- $\frac{\Lambda(r)}{r}$ *is (for large r) non-increasing or non-decreasing.*
- $\sum_{n\in\mathbb{N}} \frac{1}{\Lambda(|\alpha_n|)} < \infty$.

- α^+ infinite $\Rightarrow$ $\exists\lambda^+ :\ \alpha^+(n) = (\lambda^+)^{-1}(n)$
  α^- infinite $\Rightarrow$ $\exists\lambda^- :\ \alpha^-(n) = -(\lambda^-)^{-1}(n)$

Let $\beta \in \mathbb{S}$, set $\gamma := \beta - \alpha$, and assume

(A) $\quad |\gamma_n| = \mathrm{O}\Big(\frac{|\alpha_n|}{\Lambda(|\alpha_n|)}\Big)$.

(B1) $\quad \frac{|\gamma_n^{[+]}|}{s_{\alpha^+}(n)} = \mathrm{O}\big(\frac{1}{n}\big), \qquad \frac{|\gamma_n^{[-]}|}{s_{\alpha^-}(n)} = \mathrm{O}\big(\frac{1}{n}\big)$.

Strong regularity. I

Theorem

Let $\alpha \in \mathbb{S}$ and let Λ be a growth function, such that

- $\frac{\Lambda(r)}{r}$ *is (for large r) non-increasing or non-decreasing.*
- $\sum_{n\in\mathbb{N}} \frac{1}{\Lambda(|\alpha_n|)} < \infty$.

- α^+ infinite $\Rightarrow \exists\lambda^+ : \alpha^+(n) = (\lambda^+)^{-1}(n)$
  α^- infinite $\Rightarrow \exists\lambda^- : \alpha^-(n) = -(\lambda^-)^{-1}(n)$

Let $\beta \in \mathbb{S}$, set $\gamma := \beta - \alpha$, and assume

(A) $\quad |\gamma_n| = \mathrm{O}\left(\frac{|\alpha_n|}{\Lambda(|\alpha_n|)}\right)$.

(B2) $\quad \frac{|\gamma_n^{[+]}|}{s_{\alpha^+}(n)} = \mathrm{O}\left(\frac{\alpha_n^+}{n s_{\alpha^+}(n)} \frac{1}{\log n}\right), \quad \frac{|\gamma_n^{[-]}|}{s_{\alpha^-}(n)} = \mathrm{O}\left(\frac{\alpha_n^+}{n s_{\alpha^+}(n)} \frac{1}{\log n}\right)$.

Strong regularity. I

Theorem

Let $\alpha \in \mathbb{S}$ and let Λ be a growth function, such that

- $\frac{\Lambda(r)}{r}$ *is (for large r) non-increasing or non-decreasing.*
- $\sum_{n\in\mathbb{N}} \frac{1}{\Lambda(|\alpha_n|)} < \infty$.

- α^+ infinite $\Rightarrow \exists\lambda^+ : \alpha^+(n) = (\lambda^+)^{-1}(n)$
  α^- infinite $\Rightarrow \exists\lambda^- : \alpha^-(n) = -(\lambda^-)^{-1}(n)$

Let $\beta \in \mathbb{S}$, set $\gamma := \beta - \alpha$, and assume

(A) $|\gamma_n| = \mathrm{O}\Big(\frac{|\alpha_n|}{\Lambda(|\alpha_n|)}\Big)$.

(B2) $\frac{|\gamma_n^{[+]}|}{s_{\alpha^+}(n)} = \mathrm{O}\Big(\frac{\alpha_n^+}{ns_{\alpha^+}(n)}\frac{1}{\log n}\Big)$, $\frac{|\gamma_n^{[-]}|}{s_{\alpha^-}(n)} = \mathrm{O}\Big(\frac{\alpha_n^+}{ns_{\alpha^+}(n)}\frac{1}{\log n}\Big)$.

Then $|P'_\alpha(\alpha_n)| \asymp |P'_\beta(\beta_n)|$.

Strong regularity. II

Theorem

Let $\alpha \in \mathbb{S}$ *and let* Λ *be a growth function, such that*

- $\frac{\Lambda(r)}{r}$ *is (for large* r*) non-increasing or non-decreasing.*
- $\sum_{n\in\mathbb{N}} \frac{1}{\Lambda(|\alpha_n|)} < \infty$.

Strong regularity. II

Theorem

Let $\alpha \in \mathbb{S}$ and let Λ be a growth function, such that

- $\frac{\Lambda(r)}{r}$ *is (for large r) non-increasing or non-decreasing.*
- $\sum_{n\in\mathbb{N}} \frac{1}{\Lambda(|\alpha_n|)} < \infty$.
- α^+ infinite $\Rightarrow\ \exists\lambda^+ :\ \alpha^+(n) = (\lambda^+)^{-1}(n)$
  α^- infinite $\Rightarrow\ \exists\lambda^- :\ \alpha^-(n) = -(\lambda^-)^{-1}(n)$

Strong regularity. II

Theorem

Let $\alpha \in \mathbb{S}$ and let Λ be a growth function, such that

- $\frac{\Lambda(r)}{r}$ *is (for large r) non-increasing or non-decreasing.*
- $\sum_{n\in\mathbb{N}} \frac{1}{\Lambda(|\alpha_n|)} < \infty$.
- α^+ infinite $\Rightarrow$ $\exists\lambda^+ : \ \alpha^+(n) = (\lambda^+)^{-1}(n)$
  α^- infinite $\Rightarrow$ $\exists\lambda^- : \ \alpha^-(n) = -(\lambda^-)^{-1}(n)$

- $\frac{\lambda^+(r)}{\Lambda(r)}$ non-increasing (same for λ^-)
- either $\rho_{\lambda^+} > 0$ or $\frac{\log\lambda^+(r)}{\log r}$ non-increasing (same for λ^-)

Strong regularity. II

Theorem

Let $\alpha \in \mathbb{S}$ and let Λ be a growth function, such that

- $\frac{\Lambda(r)}{r}$ *is (for large r) non-increasing or non-decreasing.*
- $\sum_{n\in\mathbb{N}} \frac{1}{\Lambda(|\alpha_n|)} < \infty$.
- α^+ infinite $\Rightarrow$ $\exists\lambda^+ : \ \alpha^+(n) = (\lambda^+)^{-1}(n)$
  α^- infinite $\Rightarrow$ $\exists\lambda^- : \ \alpha^-(n) = -(\lambda^-)^{-1}(n)$

- $\frac{\lambda^+(r)}{\Lambda(r)}$ non-increasing (same for λ^-)
- either $\rho_{\lambda^+} > 0$ or $\frac{\log\lambda^+(r)}{\log r}$ non-increasing (same for λ^-)

Let $\beta \in \mathbb{S}$, set $\gamma := \beta - \alpha$, and assume

(A) $\quad |\gamma_n| = \mathrm{O}\Big(\frac{|\alpha_n|}{\Lambda(|\alpha_n|)}\Big)$.

Strong regularity. II

Theorem

Let $\alpha \in \mathbb{S}$ and let Λ be a growth function, such that

- $\frac{\Lambda(r)}{r}$ *is (for large r) non-increasing or non-decreasing.*
- $\sum_{n\in\mathbb{N}} \frac{1}{\Lambda(|\alpha_n|)} < \infty.$
- α^+ infinite $\Rightarrow$ $\exists\lambda^+ :\ \alpha^+(n) = (\lambda^+)^{-1}(n)$
  α^- infinite $\Rightarrow$ $\exists\lambda^- :\ \alpha^-(n) = -(\lambda^-)^{-1}(n)$

- $\frac{\lambda^+(r)}{\Lambda(r)}$ non-increasing (same for λ^-)
- either $\rho_{\lambda^+} > 0$ or $\frac{\log\lambda^+(r)}{\log r}$ non-increasing (same for λ^-)

Let $\beta \in \mathbb{S}$, set $\gamma := \beta - \alpha$, and assume

(A) $\quad |\gamma_n| = \mathrm{O}\Big(\frac{|\alpha_n|}{\Lambda(|\alpha_n|)}\Big).$

(B2) $\quad \frac{|\gamma_n^{[+]}|}{s_{\alpha^+}(n)} = \mathrm{O}\Big(\frac{\alpha_n^+}{n s_{\alpha^+}(n)} \frac{1}{\log n}\Big), \quad \frac{|\gamma_n^{[-]}|}{s_{\alpha^-}(n)} = \mathrm{O}\Big(\frac{\alpha_n^+}{n s_{\alpha^+}(n)} \frac{1}{\log n}\Big).$

Strong regularity. II

Theorem

Let $\alpha \in \mathbb{S}$ and let Λ be a growth function, such that

- $\frac{\Lambda(r)}{r}$ *is (for large r) non-increasing or non-decreasing.*
- $\sum_{n\in\mathbb{N}} \frac{1}{\Lambda(|\alpha_n|)} < \infty$.
- α^+ infinite $\Rightarrow \exists\lambda^+ : \alpha^+(n) = (\lambda^+)^{-1}(n)$
  α^- infinite $\Rightarrow \exists\lambda^- : \alpha^-(n) = -(\lambda^-)^{-1}(n)$

- $\frac{\lambda^+(r)}{\Lambda(r)}$ non-increasing (same for λ^-)
- either $\rho_{\lambda^+} > 0$ or $\frac{\log \lambda^+(r)}{\log r}$ non-increasing (same for λ^-)

Let $\beta \in \mathbb{S}$, set $\gamma := \beta - \alpha$, and assume

(A) $\quad |\gamma_n| = \mathrm{O}\left(\frac{|\alpha_n|}{\Lambda(|\alpha_n|)}\right).$

Strong regularity. II

Theorem

Let $\alpha \in \mathbb{S}$ and let Λ be a growth function, such that

- $\frac{\Lambda(r)}{r}$ *is (for large r) non-increasing or non-decreasing.*
- $\sum_{n\in\mathbb{N}} \frac{1}{\Lambda(|\alpha_n|)} < \infty$.
- α^+ infinite $\Rightarrow$ $\exists\lambda^+ :\ \alpha^+(n) = (\lambda^+)^{-1}(n)$
  α^- infinite $\Rightarrow$ $\exists\lambda^- :\ \alpha^-(n) = -(\lambda^-)^{-1}(n)$
- $\frac{\lambda^+(r)}{\Lambda(r)}$ non-increasing (same for λ^-)
- either $\rho_{\lambda^+} > 0$ or $\frac{\log\lambda^+(r)}{\log r}$ non-increasing (same for λ^-)

Let $\beta \in \mathbb{S}$, set $\gamma := \beta - \alpha$, and assume

(A) $\qquad |\gamma_n| = \mathrm{O}\Big(\frac{|\alpha_n|}{\Lambda(|\alpha_n|)}\Big)$.

Then $|P'_\alpha(\alpha_n)| \asymp |P'_\beta(\beta_n)|$.

Returning to the example

Example

Consider $\alpha_n := n^\sigma$, $n \in \mathbb{N}$, with $\sigma > 1$.

Returning to the example

Example

Consider $\alpha_n := n^\sigma$, $n \in \mathbb{N}$, with $\sigma > 1$.

- $\alpha_n = (\lambda^+)^{-1}(n)$ for $\lambda^+(r) := r^{\frac{1}{\sigma}}$.

Returning to the example

Example

Consider $\alpha_n := n^\sigma$, $n \in \mathbb{N}$, with $\sigma > 1$.

- $\alpha_n = (\lambda^+)^{-1}(n)$ for $\lambda^+(r) := r^{\frac{1}{\sigma}}$.
- We may use, e.g., $\Lambda(r) := r^{\frac{1}{\sigma}} \log r (\log\log r)^2$:

Returning to the example

Example

Consider $\alpha_n := n^\sigma$, $n \in \mathbb{N}$, with $\sigma > 1$.

- $\alpha_n = (\lambda^+)^{-1}(n)$ for $\lambda^+(r) := r^{\frac{1}{\sigma}}$.
- We may use, e.g., $\Lambda(r) := r^{\frac{1}{\sigma}} \log r (\log\log r)^2$:

$$\frac{\Lambda(r)}{r} = r^{\frac{1}{\sigma}-1} \log r (\log\log r)^2 \text{ non-increasing (for large } r\text{)}$$

$$\sum_{n\in\mathbb{N}} \frac{1}{\Lambda(|\alpha_n|)} = \sum_{n\in\mathbb{N}} \frac{1}{n \cdot \sigma \log n \cdot (\log \sigma + \log\log n)^2} < \infty$$

$$\frac{\lambda^+(r)}{\Lambda(r)} = \frac{1}{\log r (\log\log r)^2} \text{ non-increasing}$$

$$\rho_{\lambda^+} = \frac{1}{\sigma} > 0$$

Returning to the example

Example

Consider $\alpha_n := n^\sigma$, $n \in \mathbb{N}$, with $\sigma > 1$.

General Theorem:

$$\sum_{n\in\mathbb{N}} \frac{|\gamma_n|}{n^{\sigma-1}} < \infty.$$

Returning to the example

Example

Consider $\alpha_n := n^\sigma$, $n \in \mathbb{N}$, with $\sigma > 1$.

General Theorem: $$\sum_{n\in\mathbb{N}} \frac{|\gamma_n|}{n^{\sigma-1}} < \infty.$$

Mild regularity: $$|\gamma_n| = \mathrm{O}\big(n^{\sigma-2}\big).$$

Returning to the example

Example

Consider $\alpha_n := n^\sigma$, $n \in \mathbb{N}$, with $\sigma > 1$.

General Theorem: $$\sum_{n\in\mathbb{N}} \frac{|\gamma_n|}{n^{\sigma-1}} < \infty.$$

Mild regularity: $$|\gamma_n| = \mathrm{O}\big(n^{\sigma-2}\big).$$

Strong regularity (I or II): $$|\gamma_n| = \mathrm{O}\left(\frac{n^{\sigma-1}}{\log n \cdot (\log\log n)^2}\right).$$

Returning to the example

Example

Consider $\alpha_n := n^\sigma$, $n \in \mathbb{N}$, with $\sigma > 1$.

General Theorem: $$\sum_{n\in\mathbb{N}} \frac{|\gamma_n|}{n^{\sigma-1}} < \infty.$$

Mild regularity: $$|\gamma_n| = \mathrm{O}\big(n^{\sigma-2}\big).$$

Strong regularity (I or II): $$|\gamma_n| = \mathrm{O}\Big(\frac{n^{\sigma-1}}{\log n \cdot (\log\log n)^2}\Big).$$

Remember: $|\gamma_n| \asymp n^{\sigma-1}$ is not allowed!

Some applications

(In-)determinate measures

Definition

Let μ be a positive Borel measure on $\mathbb{R}$ which has all power moments, and set

$$s_n := \int_{\mathbb{R}} t^n \, d\mu(t), \quad n = 0, 1, 2, \dots$$

- μ is *determinate*, if there is no other measure ν with $\int_{\mathbb{R}} t^n \, d\nu(t) = s_n$, $n = 0, 1, 2, \dots$
- μ is *indeterminate* otherwise.

(In-)determinate measures

Definition

Let μ be a positive Borel measure on $\mathbb{R}$ which has all power moments, and set

$$s_n := \int_{\mathbb{R}} t^n \, d\mu(t), \quad n = 0, 1, 2, \ldots$$

- μ is *determinate*, if there is no other measure ν with $\int_{\mathbb{R}} t^n \, d\nu(t) = s_n$, $n = 0, 1, 2, \ldots$
- μ is *indeterminate* otherwise.

Convention: We always assume that $\operatorname{supp} \mu$ is not finite.

(In-)determinate measures

Definition

Let μ be a positive Borel measure on $\mathbb{R}$ which has all power moments, and set

$$s_n := \int_{\mathbb{R}} t^n \, d\mu(t), \quad n = 0, 1, 2, \ldots$$

- μ is *determinate*, if there is no other measure ν with $\int_{\mathbb{R}} t^n \, d\nu(t) = s_n$, $n = 0, 1, 2, \ldots$
- μ is *indeterminate* otherwise.

Convention: We always assume that $\operatorname{supp} \mu$ is not finite.

Proposition

Assume μ is indeterminate. Then μ is discrete.

Nevanlinna parameterisation

Theorem

Assume μ is indeterminate.

Nevanlinna parameterisation

Theorem

Assume μ is indeterminate.

Then there exist four entire functions A, B, C, D, such that

$$\int_{\mathbb{R}} \frac{d\nu(t)}{t-z} = \frac{A(z)\tau(z)+B(z)}{C(z)\tau(z)+D(z)}$$

establishes a bijection between

$$\{\nu : \int_{\mathbb{R}} t^n \, d\nu(t) = s_n, n = 0, 1, 2, \dots\}$$

and $$\mathcal{N}_0 := \{\tau : \textit{analytic in } \mathbb{C}^+, \operatorname{Im} \tau(z) \geqslant 0\}.$$

Nevanlinna parameterisation

Theorem

Assume μ is indeterminate.

Then there exist four entire functions A, B, C, D, such that

$$\int_{\mathbb{R}} \frac{d\nu(t)}{t - z} = \frac{A(z)\tau(z) + B(z)}{C(z)\tau(z) + D(z)}$$

establishes a bijection between

$$\{\nu : \int_{\mathbb{R}} t^n \, d\nu(t) = s_n, n = 0, 1, 2, \dots\}$$

and $$\mathcal{N}_0 := \{\tau : \text{analytic in } \mathbb{C}^+, \operatorname{Im} \tau(z) \geqslant 0\}.$$

Note: μ itself appears in the first of these sets, and hence corresponds to some parameter τ_0.

N-extremal measures

Definition

Let μ be a positive Borel measure on $\mathbb{R}$ which has all power moments.

μ is *N-extremal*, if the space $\mathbb{C}[z]$ of polynomials is dense in $L^2(\mu)$.

N-extremal measures

Definition

Let μ be a positive Borel measure on $\mathbb{R}$ which has all power moments.

μ is *N-extremal*, if the space $\mathbb{C}[z]$ of polynomials is dense in $L^2(\mu)$.

Theorem

μ is N-extremal if and only if

N-extremal measures

Definition

Let μ be a positive Borel measure on $\mathbb{R}$ which has all power moments.

μ is *N-extremal*, if the space $\mathbb{C}[z]$ of polynomials is dense in $L^2(\mu)$.

Theorem

μ is N-extremal if and only if

either

- *μ is determinate,*

or

- *μ is indeterminate and corresponds to a constant parameter in the Nevanlinna parameterisation.*

The Hamburger class

Definition

The *Hamburger class* $\mathcal{H}$ is the set of all entire function F with

The Hamburger class

Definition

The *Hamburger class* $\mathcal{H}$ is the set of all entire function F with

- F is a transcendental entire function,
- F has minimal exponential type,
- F has only real and simple zeros, say $(y_n)_{n=1}^{\infty}$, and

$$\lim_{n\to\infty} \frac{|y_n|^l}{|F'(y_n)|} = 0, \qquad l = 0, 1, 2, \ldots .$$

The Hamburger class

Definition

The *Hamburger class* $\mathcal{H}$ is the set of all entire function F with

- F is a transcendental entire function,
- F has minimal exponential type,
- F has only real and simple zeros, say $(y_n)_{n=1}^{\infty}$, and

$$\lim_{n\to\infty} \frac{|y_n|^l}{|F'(y_n)|} = 0, \qquad l = 0, 1, 2, \dots .$$

Remark

If $F \in \mathcal{H}$ and $F(0) = 1$, then $F(z) = \lim_{r\to\infty} \prod_{|y_n|\leqslant r} \left(1 - \frac{z}{y_n}\right)$.

Indeterminate and N-extremal measures

Theorem (A.Borichev, M.Sodin)

Consider $\mu = \sum_{n=1}^{\infty} \mu_n \delta_{x_n}$. *Then* μ *is indeterminate and N-extremal if and only if*

Indeterminate and N-extremal measures

Theorem (A.Borichev, M.Sodin)

Consider $\mu = \sum_{n=1}^{\infty} \mu_n \delta_{x_n}$. *Then* μ *is indeterminate and N-extremal if and only if*

- $F(z) := \lim_{r\to\infty} \prod_{|x_n|\leqslant r} \left(1 - \frac{z}{x_n}\right) \in \mathcal{H}$;
- $\sum_{n=1}^{\infty} |x_n|^l \mu_n < \infty$ *for all* $l = 0, 1, \ldots$;
- $\sum_{n=1}^{\infty} \frac{1}{\mu_n |F'(x_n)|^2 (1+x_n^2)} < \infty$;
- *for every function* $G \in \mathcal{H}$ *with* $\frac{F}{G}$ *entire,*

$$\sum_{k=1}^{\infty} \frac{1}{\mu_{n(k)} |G'(x_{n(k)})|^2} = \infty,$$

where $(x_{n(k)})_{k=1}^{\infty}$ *is the sequence of zeros of* G.

Applying the stability theorem

Theorem

Let $\mu = \sum_{n=1}^{\infty} \mu_n \delta_{\alpha_n}$ *be indeterminate and N-extremal. Let* $(\beta_n)_{n=1}^{\infty}$, $\beta_n \in \mathbb{R}$ *pairwise distinct, and let* $\nu_n > 0$.

Applying the stability theorem

Theorem

Let $\mu = \sum_{n=1}^{\infty} \mu_n \delta_{\alpha_n}$ *be indeterminate and N-extremal. Let* $(\beta_n)_{n=1}^{\infty}$, $\beta_n \in \mathbb{R}$ *pairwise distinct, and let* $\nu_n > 0$.

Choose a growth function Λ*, such that*

- $\frac{\Lambda(r)}{r}$ *is (for large* r*) non-increasing or non-decreasing.*
- $\sum_{n\in\mathbb{N}} \frac{1}{\Lambda(|\alpha_n|)} < \infty$.
- $|\beta_n - \alpha_n| = \mathrm{O}\left(\frac{|\alpha_n|}{\Lambda(|\alpha_n|)}\right)$.
- $\left(\frac{\beta_n - \alpha_n}{s_\alpha(n)}\right)_{n\in\mathbb{N}} \in \ell^1$.
- $\exists\, \rho > 1 : \; |\beta_n - \alpha_n| = \mathrm{O}\left(\frac{s_\alpha(n)}{r_\alpha(\rho,n)}\right)$.

Applying the stability theorem

Theorem

Let $\mu = \sum_{n=1}^{\infty} \mu_n \delta_{\alpha_n}$ *be indeterminate and N-extremal. Let* $(\beta_n)_{n=1}^{\infty}$, $\beta_n \in \mathbb{R}$ *pairwise distinct, and let* $\nu_n > 0$.

Choose a growth function Λ, *such that*

- $\frac{\Lambda(r)}{r}$ *is (for large* r*) non-increasing or non-decreasing.*
- $\sum_{n\in\mathbb{N}} \frac{1}{\Lambda(|\alpha_n|)} < \infty$.
- $|\beta_n - \alpha_n| = \mathrm{O}\left(\frac{|\alpha_n|}{\Lambda(|\alpha_n|)}\right)$.
- $\left(\frac{\beta_n - \alpha_n}{s_\alpha(n)}\right)_{n\in\mathbb{N}} \in \ell^1$.
- $\exists\, \rho > 1 : \ |\beta_n - \alpha_n| = \mathrm{O}\left(\frac{s_\alpha(n)}{r_\alpha(\rho,n)}\right)$.

Then $\nu := \sum_{n\in\mathbb{N}} \nu_n \delta_{\beta_n}$ *is indeterminate and N-extremal.*

Using regularity ?

The versions of our stability theorem for regularly distributed sequences *do not* apply *immediately* to this problem.

Using regularity ?

The versions of our stability theorem for regularly distributed sequences *do not* apply *immediately* to this problem.

The reason is that a subsequence of a regularly behaving sequence is not necessarily itself regularly behaving.

Canonical systems

Let $H : [0, L) \to \mathbb{R}^{2\times 2}$ be such that

- $H \in L^1_{loc}([0, L))$,
- $H(t) \geqslant 0$, $t \in [0, L)$,
- H does not vanish identically on any set of positive measure.

Canonical systems

Let $H : [0, L) \to \mathbb{R}^{2\times 2}$ be such that

- $H \in L^1_{loc}([0, L))$,
- $H(t) \geqslant 0$, $t \in [0, L)$,
- H does not vanish identically on any set of positive measure.

The *canonical system* with *Hamiltonian* H is the equation

$$y'(t) = zJH(t)y(t), \quad t \in [0, L).$$

Here z is a complex parameter and $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Canonical systems

Let $H : [0, L) \to \mathbb{R}^{2\times 2}$ be such that

- $H \in L^1_{loc}([0, L))$,
- $H(t) \geqslant 0$, $t \in [0, L)$,
- H does not vanish identically on any set of positive measure.

The *canonical system* with *Hamiltonian* H is the equation

$$y'(t) = zJH(t)y(t), \quad t \in [0, L).$$

Here z is a complex parameter and $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Assume that *Weyl's limit point case* takes place at L, i.e.,

$$\int_0^L \operatorname{tr} H(t)\, dt = \infty.$$

The Weyl coefficient

Let $W(x,z) = (w_{ij}(x,z))_{i,j=1,2}$ be the solution of the initial value problem

$$\frac{d}{dt}W(t,z)J = zW(t,z)H(t), \quad t \in [0,L), \qquad W(0,z) = I,$$

The Weyl coefficient

Let $W(x,z) = (w_{ij}(x,z))_{i,j=1,2}$ be the solution of the initial value problem

$$\frac{d}{dt}W(t,z)J = zW(t,z)H(t), \quad t \in [0,L), \qquad W(0,z) = I,$$

Then, for each $\tau \in \mathbb{R} \cup \{\infty\}$, the limit

$$\lim_{t \nearrow L} \frac{w_{11}(t,z)\tau + w_{12}(t,z)}{w_{21}(t,z)\tau + w_{22}(t,z)} =: q_H(z)$$

exists locally uniformly on $\mathbb{C}\backslash\mathbb{R}$ and does not depend on τ.

The Weyl coefficient

Let $W(x,z) = (w_{ij}(x,z))_{i,j=1,2}$ be the solution of the initial value problem

$$\frac{d}{dt}W(t,z)J = zW(t,z)H(t), \quad t \in [0,L), \qquad W(0,z) = I,$$

Then, for each $\tau \in \mathbb{R} \cup \{\infty\}$, the limit

$$\lim_{t \nearrow L} \frac{w_{11}(t,z)\tau + w_{12}(t,z)}{w_{21}(t,z)\tau + w_{22}(t,z)} =: q_H(z)$$

exists locally uniformly on $\mathbb{C}\backslash\mathbb{R}$ and does not depend on τ.

q_H is the *Titchmarsh–Weyl coefficient* (or *Weyl m-function*) of H.

The Weyl coefficient

Let $W(x,z)=(w_{ij}(x,z))_{i,j=1,2}$ be the solution of the initial value problem

$$\frac{d}{dt}W(t,z)J=zW(t,z)H(t),\quad t\in[0,L),\qquad W(0,z)=I,$$

Then, for each $\tau\in\mathbb{R}\cup\{\infty\}$, the limit

$$\lim_{t\nearrow L}\frac{w_{11}(t,z)\tau+w_{12}(t,z)}{w_{21}(t,z)\tau+w_{22}(t,z)}=:q_H(z)$$

exists locally uniformly on $\mathbb{C}\backslash\mathbb{R}$ and does not depend on τ.

q_H is the *Titchmarsh–Weyl coefficient* (or *Weyl m-function*) of H.

It is characterised by

$$y(t)=\begin{pmatrix}w_{11}(t,z)\\w_{12}(t,z)\end{pmatrix}-q_H(z)\begin{pmatrix}w_{21}(t,z)\\w_{22}(t,z)\end{pmatrix}\in L^2(H)$$

Spectral theorems

Theorem (Direct Theorem)

The Titchmarsh–Weyl coefficient of H belongs to the Nevanlinna class $\mathcal{N}_0$, i.e.,

- q_H *is analytic on* $\mathbb{C}\backslash\mathbb{R}$;
- $q_H(\overline{z}) = \overline{q_H(z)}$, $z \in \mathbb{C}\backslash\mathbb{R}$;
- $\operatorname{Im} q_H(z) \geqslant 0$ *for* $\operatorname{Im} z > 0$.

Spectral theorems

Theorem (Direct Theorem)

The Titchmarsh–Weyl coefficient of H belongs to the Nevanlinna class $\mathcal{N}_0$, i.e.,

- q_H *is analytic on* $\mathbb{C}\backslash\mathbb{R}$;
- $q_H(\overline{z}) = \overline{q_H(z)}$, $z \in \mathbb{C}\backslash\mathbb{R}$;
- $\operatorname{Im} q_H(z) \geqslant 0$ *for* $\operatorname{Im} z > 0$.

Theorem (Inverse Theorem)

Let $q \in \mathcal{N}_0$. Then there exists H, such that $q = q_H$.

Spectral theorems

Definition

Two Hamiltonians H_1, H_2 are *reparameterisations* of each other, if there exists $\varphi : (0, L_2) \to (0, L_1)$ with

- φ is absolutely continuous, increasing, and bijective;
- φ^{-1} is absolutely continuous;
- $H_2(t) = H_1(\varphi(t)) \cdot \varphi'(t)$, $t \in (0, L_2)$ a.e.

Spectral theorems

Definition

Two Hamiltonians H_1, H_2 are *reparameterisations* of each other, if there exists $\varphi : (0, L_2) \to (0, L_1)$ with

- φ is absolutely continuous, increasing, and bijective;
- φ^{-1} is absolutely continuous;
- $H_2(t) = H_1(\varphi(t)) \cdot \varphi'(t)$, $t \in (0, L_2)$ a.e.

Theorem (Uniqueness Theorem)

Let H_1, H_2 be given. If $q_{H_1} = q_{H_2}$, then H_1 and H_2 are reparameterisations of each other.

Indivisible intervals

Definition

Let H be a Hamiltonian on $[0, L)$, and let $(x_1, x_2) \subseteq (0, L)$ be nonempty. Then (x_1, x_2) is *indivisible* (for H), if

$$H(t) = h(t) \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} \cdot \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}^T, \quad t \in (x_1, x_2) \text{ a.e.}$$

Indivisible intervals

Definition

Let H be a Hamiltonian on $[0, L)$, and let $(x_1, x_2) \subseteq (0, L)$ be nonempty. Then (x_1, x_2) is *indivisible* (for H), if

$$H(t) = h(t) \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} \cdot \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}^T, \quad t \in (x_1, x_2) \text{ a.e.}$$

- $h(t) = \operatorname{tr} H(t)$, $t \in (x_1, x_2)$ a.e.

Indivisible intervals

Definition

Let H be a Hamiltonian on $[0, L)$, and let $(x_1, x_2) \subseteq (0, L)$ be nonempty. Then (x_1, x_2) is *indivisible* (for H), if

$$H(t) = h(t) \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} \cdot \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}^T, \quad t \in (x_1, x_2) \text{ a.e.}$$

- $h(t) = \operatorname{tr} H(t)$, $t \in (x_1, x_2)$ a.e.
- (x_1, x_2) is indivisible, if and only if $\det H(t) = 0$ for $t \in (x_1, x_2)$ and $\ker H(t)$ is constant on (x_1, x_2).

Indivisible intervals

Definition

Let H be a Hamiltonian on $[0, L)$, and let $(x_1, x_2) \subseteq (0, L)$ be nonempty. Then (x_1, x_2) is *indivisible* (for H), if

$$H(t) = h(t) \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} \cdot \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}^T, \quad t \in (x_1, x_2) \text{ a.e.}$$

- $h(t) = \operatorname{tr} H(t)$, $t \in (x_1, x_2)$ a.e.
- (x_1, x_2) is indivisible, if and only if $\det H(t) = 0$ for $t \in (x_1, x_2)$ and $\ker H(t)$ is constant on (x_1, x_2).
- The angle ϕ is unique $\mod \pi$ and is the *type* of (x_1, x_2).

Indivisible intervals

Definition

Let H be a Hamiltonian on $[0, L)$, and let $(x_1, x_2) \subseteq (0, L)$ be nonempty. Then (x_1, x_2) is *indivisible* (for H), if

$$H(t) = h(t) \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} \cdot \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}^T, \quad t \in (x_1, x_2) \text{ a.e.}$$

- $h(t) = \operatorname{tr} H(t)$, $t \in (x_1, x_2)$ a.e.
- (x_1, x_2) is indivisible, if and only if $\det H(t) = 0$ for $t \in (x_1, x_2)$ and $\ker H(t)$ is constant on (x_1, x_2).
- The angle ϕ is unique $\mod \pi$ and is the *type* of (x_1, x_2).
- The *length* of (x_1, x_2) is $\int_{x_1}^{x_2} \operatorname{tr} H(t)\, dt$.

Indivisible intervals

Definition

Let H be a Hamiltonian on $[0, L)$, and let $(x_1, x_2) \subseteq (0, L)$ be nonempty. Then (x_1, x_2) is *indivisible* (for H), if

$$H(t) = h(t) \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} \cdot \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}^T, \quad t \in (x_1, x_2) \text{ a.e.}$$

- $h(t) = \operatorname{tr} H(t)$, $t \in (x_1, x_2)$ a.e.
- (x_1, x_2) is indivisible, if and only if $\det H(t) = 0$ for $t \in (x_1, x_2)$ and $\ker H(t)$ is constant on (x_1, x_2).
- The angle ϕ is unique $\mod \pi$ and is the *type* of (x_1, x_2).
- The *length* of (x_1, x_2) is $\int_{x_1}^{x_2} \operatorname{tr} H(t)\, dt$.
- Each indivisible interval is contained in a maximal indivisible interval.

Hamiltonians ending with indivisible intervals

Definition

Let H be a Hamiltonian on $[0, L)$. We say that H *ends with at least N indivisible intervals*, if

$$\exists\ 0 \leqslant x_0 < x_1 < \cdots < x_N = L : \quad \text{each interval } (x_{i-1}, x_i) \text{ is maximal indivisible.}$$

Hamiltonians ending with indivisible intervals

Definition

Let H be a Hamiltonian on $[0, L)$. We say that H *ends with at least N indivisible intervals*, if

$$\exists\ 0 \leqslant x_0 < x_1 < \cdots < x_N = L : \quad \text{each interval } (x_{i-1}, x_i) \text{ is maximal indivisible.}$$

Remark

The role of the indivisible interval ending at L is different than the role of the others. Namely, its length is infinite whereas the length of the others is finite.

Hamiltonians ending with indivisible intervals

Definition

Let H be a Hamiltonian on $[0, L)$. We say that H *ends with at least* N *indivisible intervals*, if

$$\exists\ 0 \leqslant x_0 < x_1 < \cdots < x_N = L : \qquad \text{each interval } (x_{i-1}, x_i) \text{ is maximal indivisible.}$$

Remark

The role of the indivisible interval ending at L is different than the role of the others. Namely, its length is infinite whereas the length of the others is finite.

Note: H ends with (at least) 1 indivisible interval, means – in essence – that H is in limit circle case at L.

Hamiltonians ending with indivisible intervals

Theorem

Let H be a Hamiltonian, and let $N \in \mathbb{N}$.

Hamiltonians ending with indivisible intervals

Theorem

Let H be a Hamiltonian, and let $N \in \mathbb{N}$.

Then H ends with at least N indivisible intervals, if and only if

Hamiltonians ending with indivisible intervals

Theorem

Let H be a Hamiltonian, and let $N \in \mathbb{N}$.

Then H ends with at least N indivisible intervals, if and only if

- *q_H is meromorphic in $\mathbb{C}$*

Hamiltonians ending with indivisible intervals

Theorem

Let H be a Hamiltonian, and let $N \in \mathbb{N}$.

Then H ends with at least N indivisible intervals, if and only if

- *q_H is meromorphic in $\mathbb{C}$*
- *let α_n be the poles of q_H, then* $\lim_{r\to\infty} \sum_{|\alpha_n| \leqslant r} \frac{1}{\alpha_n} \in \mathbb{R}$

Hamiltonians ending with indivisible intervals

Theorem

Let H be a Hamiltonian, and let $N \in \mathbb{N}$.

Then H ends with at least N indivisible intervals, if and only if

- *q_H is meromorphic in $\mathbb{C}$*
- *let α_n be the poles of q_H, then* $\lim_{r\to\infty} \sum_{|\alpha_n|\leqslant r} \frac{1}{\alpha_n} \in \mathbb{R}$
- $\lim_{n\to\infty} \frac{n}{\alpha_n^+} = \lim_{n\to\infty} \frac{n}{|\alpha_n^-|} \in [0,\infty)$

Hamiltonians ending with indivisible intervals

Theorem

Let H be a Hamiltonian, and let $N \in \mathbb{N}$.

Then H ends with at least N indivisible intervals, if and only if

- *q_H is meromorphic in $\mathbb{C}$*
- *let α_n be the poles of q_H, then* $\lim_{r\to\infty} \sum_{|\alpha_n|\leqslant r} \frac{1}{\alpha_n} \in \mathbb{R}$
- $\lim_{n\to\infty} \frac{n}{\alpha_n^+} = \lim_{n\to\infty} \frac{n}{|\alpha_n^-|} \in [0,\infty)$
- *let σ_n be the negative residuum of q_H at α_n, then*

$$\sum_{n\in\mathbb{N}} \frac{\alpha_n^{2N-4}}{P'_\alpha(\alpha_n)\sigma_n} < \infty.$$

Applying stability theorems

Theorem

Let H be a Hamiltonian which ends with at least N indivisible intervals, let α_n be the poles of q_H and σ_n its negative residues.

Let $\beta_n \in \mathbb{S}$ be such that the hypothesis of any of our stability theorems are fullfilled, and let $\tau_n \asymp \sigma_n$.

Then the Hamiltonian $\tilde{H}$ whose Titchmarsh-Weyl coefficient has poles β_n with negative residues τ_n ends with at least N indivisible intervals.

The Krein class

Definition

The *Krein class* $\mathbb{K}$ is the set of all entire functions F with

The Krein class

Definition

The *Krein class* $\mathbb{K}$ is the set of all entire functions F with

- The zeroes α_n of F are simple.
- $\sum_{n\in\mathbb{N}} \left| \operatorname{Im} \frac{1}{\alpha_n} \right| < \infty.$
- $\exists l \in \mathbb{N} : \sum_{n\in\mathbb{N}} \frac{1}{|\alpha_n|^l |F'(\alpha_n)|} < \infty.$
- $\exists p$ polynomial, s.t. for $z \in \mathbb{C} \backslash \{\alpha_n : n \in \mathbb{N}\}$

$$\frac{1}{F(z)} = p(z) + \sum_{n\in\mathbb{N}} \frac{1}{F'(\alpha_n)} \left(\frac{1}{z - \alpha_n} + \frac{1}{\alpha_n} + \ldots + \frac{z^{l-2}}{\alpha_n^{l-1}} \right)$$

The Krein class: real zeroes

Definition

Let $l \in \mathbb{N}$. Denote by $\mathbb{K}_l$ the set of all entire functions F with

The Krein class: real zeroes

Definition

Let $l \in \mathbb{N}$. Denote by $\mathbb{K}_l$ the set of all entire functions F with

- $F(\mathbb{R}) \subseteq \mathbb{R}$, $F(0) = 1$, all zeroes of F are real.
- $\exists l \in \mathbb{N} : \ \sum_{n\in\mathbb{N}} \frac{1}{|\alpha_n|^l |F'(\alpha_n)|} < \infty$.
- $\exists p$ polynomial with $\deg p \leqslant l-2$, s.t. for $z \in \mathbb{C}\backslash\{\alpha_n : n \in \mathbb{N}\}$

$$\frac{1}{F(z)} = p(z) + \sum_{n\in\mathbb{N}} \frac{1}{F'(\alpha_n)} \left(\frac{1}{z-\alpha_n} + \frac{1}{\alpha_n} + \ldots + \frac{z^{l-2}}{\alpha_n^{l-1}} \right)$$

The Krein class: real zeroes

Definition

Let $l \in \mathbb{N}$. Denote by $\mathbb{K}_l$ the set of all entire functions F with

- $F(\mathbb{R}) \subseteq \mathbb{R}$, $F(0) = 1$, all zeroes of F are real.
- $\exists l \in \mathbb{N}: \displaystyle\sum_{n \in \mathbb{N}} \frac{1}{|\alpha_n|^l |F'(\alpha_n)|} < \infty$.
- $\exists p$ polynomial with $\deg p \leqslant l-2$, s.t. for $z \in \mathbb{C} \backslash \{\alpha_n : n \in \mathbb{N}\}$

$$\frac{1}{F(z)} = p(z) + \sum_{n \in \mathbb{N}} \frac{1}{F'(\alpha_n)} \left(\frac{1}{z - \alpha_n} + \frac{1}{\alpha_n} + \ldots + \frac{z^{l-2}}{\alpha_n^{l-1}} \right)$$

Remark

We have $\mathbb{K}_l \subseteq \mathbb{K}_{l+1} \subseteq \mathbb{K}$, $l \in \mathbb{N}$. Moreover, $\bigcup_{l \in \mathbb{N}} \mathbb{K}_l$ is the set of all real $F \in \mathbb{K}$ with only real zeros.

Applying stability theorems

Theorem

Let $\alpha, \beta \in \mathbb{S}$ and $l \in \mathbb{N}$. Assume that the hypothesis of any of our stability theorems are fullfilled. Then

$$P_\alpha \in \mathbb{K}_l \quad \Leftrightarrow \quad P_\beta \in \mathbb{K}_l$$