Reproducing kernel almost Pontryagin spaces

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FWF (I 1536-N25) :: Joint Project :: RFBR (13-01-91002-ANF)

This presentation is based on:

- M. Kaltenbäck, H. Winkler, and H. Woracek. "Almost Pontryagin spaces". In: *Oper. Theory Adv. Appl.* 160 (2005), pp. 253–271.
- H. de Snoo and H. Woracek. "Sums, couplings, and completions of almost Pontryagin spaces". In: *Linear Algebra Appl.* 437.2 (2012), pp. 559–580.
- H. Woracek. "Reproducing kernel almost Pontryagin spaces". 40pp. (submitted). Preprint in: ASC Report 14 (2014), Vienna University of Technology.
- H. Woracek. "Directing functionals and de Branges space completions in the almost Pontryagin space setting". manuscript in preparation.
- M. Langer and H. Woracek. "A Pontryagin space approach to the index of determinacy of a measure". manuscript in preparation.

These slides are available from my website

http://asc.tuwien.ac.at/index.php?id=woracek

Outline

Almost Pontryagin Spaces

Definition of aPs

A triple $\langle \mathcal{A}, [\cdot,\cdot]_{\mathcal{A}}, \mathcal{O} \rangle$ is an *almost Pontryagin space* (*aPs* for short), if

- $\mathcal{A}$ is a linear space,
- $[\cdot,\cdot]_{\mathcal{A}}$ is an inner product on $\mathcal{A}$,
- $\mathcal{O}$ is a topology on $\mathcal{A}$,

such that the following axioms hold:

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A triple $\langle \mathcal{A}, [\cdot,\cdot]_{\mathcal{A}}, \mathcal{O}\rangle$ is an *almost Pontryagin space* (*aPs* for short), if

(aPs1) The topology $\mathcal{O}$ is a Hilbert space topology on $\mathcal{A}$ (i.e., it is induced by some inner product which turns $\mathcal{A}$ into a Hilbert space).

(aPs2) The inner product $[\cdot,\cdot]_{\mathcal{A}}$ is $\mathcal{O}$-continuous (i.e., it is continuous as a map of $\mathcal{A} \times \mathcal{A}$ into $\mathbb{C}$ where $\mathcal{A} \times \mathcal{A}$ carries the product topology $\mathcal{O} \times \mathcal{O}$ and $\mathbb{C}$ the euclidean topology).

(aPs3) There exists an $\mathcal{O}$-closed linear subspace $\mathcal{M}$ of $\mathcal{A}$ with finite codimension in $\mathcal{A}$, such that $\langle \mathcal{M}, [\cdot,\cdot]_{\mathcal{A}}|_{\mathcal{M}\times\mathcal{M}}\rangle$ is a Hilbert space.

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If $\langle \mathcal{A}, [\cdot,\cdot]_{\mathcal{A}}, \mathcal{O} \rangle$ and $\langle \mathcal{B}, [\cdot,\cdot]_{\mathcal{B}}, \mathcal{T} \rangle$ are almost Pontryagin spaces, a map $\psi : \mathcal{A} \to \mathcal{B}$ is an *isomorphism*, if it is linear, isometric, and homeomorphic.

The role of the topology

Let $\langle \mathcal{A}, [.,.]_{\mathcal{A}} \rangle$ be an inner product space. Denote $\mathcal{A}^{\circ} := \{x \in \mathcal{A} : [x,y]_{\mathcal{A}} = 0, y \in \mathcal{A}\}$.

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- Assume that $[.,.]_\mathcal{A}$ is nondegenerated (i.e., $\mathcal{A}^\circ = \{0\}$). Then being an aPs is a property of the inner product alone: there exists at most one topology $\mathcal{O}$ s.t. $\langle \mathcal{A}, [.,.]_\mathcal{A}, \mathcal{O} \rangle$ is an aPs.

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- If $[.,.]_{\mathcal{A}}$ is degenerated (i.e., $\mathcal{A}^\circ \neq \{0\}$), $\dim \mathcal{A} = \infty$, and $\mathcal{O}$ is a topology s.t. $\langle \mathcal{A}, [.,.]_{\mathcal{A}}, \mathcal{O} \rangle$ is an aPs, then there exists a topology $\mathcal{T}$, $\mathcal{T} \neq \mathcal{O}$, s.t. $\langle \mathcal{A}, [.,.]_{\mathcal{A}}, \mathcal{T} \rangle$ is an aPs.

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- $\langle \mathcal{A}, [.,.]_{\mathcal{A}}, \mathcal{O} \rangle$ is a nondegenerated aPs if and only if $\langle \mathcal{A}, [.,.]_{\mathcal{A}} \rangle$ is a Pontryagin space and $\mathcal{O}$ is its Pontryagin space topology.

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- If $[.,.]_{\mathcal{A}}$ is degenerated (i.e., $\mathcal{A}^{\circ} \neq \{0\}$), $\dim \mathcal{A} = \infty$, and $\mathcal{O}$ is a topology s.t. $\langle \mathcal{A}, [.,.]_{\mathcal{A}}, \mathcal{O} \rangle$ is an aPs, then there exists a topology $\mathcal{T}$, $\mathcal{T} \neq \mathcal{O}$, s.t. $\langle \mathcal{A}, [.,.]_{\mathcal{A}}, \mathcal{T} \rangle$ is an aPs.
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- $\langle \mathcal{A}, [.,.]_{\mathcal{A}}, \mathcal{O} \rangle$ is a nondegenerated and positive definite aPs if and only if $\langle \mathcal{A}, [.,.]_{\mathcal{A}} \rangle$ is a Hilbert space and $\mathcal{O}$ is its Hilbert space topology.

Example

For $a > 0$, the *Paley-Wiener space* is

$$\begin{aligned}\mathcal{P}W_a :=& \big\{ F \text{ entire} : F \text{ exponential type } \leq a, F|_{\mathbb{R}} \in L^2(\mathbb{R}) \big\} \\ =& \Big\{ F : \exists f \in L^2([-a,a]) \text{ s.t. } F(z) = \int_{[-a,a]} f(t) e^{-itz}\, dt \Big\}.\end{aligned}$$

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Set

$$[F,G] := \int_{\mathbb{R}} F(t)\overline{G(t)}\, dt - \pi F(0)\overline{G(0)}, \quad F, G \in \mathcal{P}W_a,$$

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and let $\mathcal{P}W_a$ be endowed with the subspace topology of $L^2(\mathbb{R})$. Then

$$\mathcal{P}W_a \text{ is } \begin{cases} \text{Hilbert space} & , \quad a < \pi \\ \text{aPs } (\dim \mathcal{A}^\circ = 1) & , \quad a = \pi \\ \text{Pontryagin space} & , \quad a > \pi \end{cases}$$

Equivalent definitions of aPs

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- There exists a Pontryagin space which (isometrically) contains $\mathcal{A}$ as a closed subspace.
- There exists a Hilbert space inner product $(.,.)$ on $\mathcal{A}$, and G bounded selfadjoint in $\langle \mathcal{A}, (.,.) \rangle$ s.t. (E spectral measure of G)

$$[x, y]_{\mathcal{A}} = (Gx, y), \quad x, y \in \mathcal{A},$$

$$\exists \varepsilon > 0 : \ \dim \operatorname{ran} E((-\infty, \varepsilon]) < \infty.$$

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Let $\mathcal{F} \subseteq \mathcal{A}'$ be *point separating on* $\mathcal{A}^\circ$, i.e. assume

$$\mathcal{A}^\circ \cap \bigcap_{\varphi \in \mathcal{F}} \ker \varphi = \{0\},$$

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- $\mathcal{A}' = \{[\cdot, y]_{\mathcal{A}} : y \in \mathcal{A}\} + \operatorname{span} \mathcal{F}$.

The notion of a completion

Definition

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Two aPs-completions $\langle \iota_i, \mathcal{A}_i \rangle$, $i = 1, 2$, are *isomorphic*, if there exists an isomorphism $\varphi : \mathcal{A}_1 \to \mathcal{A}_2$ with $\varphi \circ \iota_1 = \iota_2$.

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We speak of a *Hilbert-space completion* or a *Pontryagin-space completion*, if

$$\operatorname{ind}_- \mathcal{A} = 0, \dim \mathcal{A}^\circ = 0 \text{ or } \dim \mathcal{A}^\circ = 0, \text{ resp.}$$

Example

Consider $\mathcal{L} := \bigcup_{0<a<\pi} \mathcal{P}W_a$ and set

$$[F, G] := \int_{\mathbb{R}} F(t)\overline{G(t)}\, dt - \pi F(0)\overline{G(0)}, \quad F, G \in \mathcal{L}.$$

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- The norm $F \mapsto [F, F]^{\frac{1}{2}}$ is not equivalent to the $L^2(\mathbb{R})$-norm on $\mathcal{L}$.

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Then

- $\langle \mathcal{L}, [.,.] \rangle$ is positive definite.
- The norm $F \mapsto [F, F]^{\frac{1}{2}}$ is not equivalent to the $L^2(\mathbb{R})$-norm on $\mathcal{L}$.
- $\langle \mathcal{P}W_\pi, [.,.] \rangle$ with $\iota : F \mapsto F$ is an aPs completion of $\mathcal{L}$.

Completions: Existence

Let $\langle \mathcal{L}, [.,.]_{\mathcal{L}} \rangle$ be an inner product space. Set

$$\operatorname{ind}_- \mathcal{L} := \sup \{ \dim \mathcal{N} : \mathcal{N} \text{ negative definite subspace of } \mathcal{L} \}.$$

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Proposition

Let $\mathcal{L}$ be an inner product space. The following are equivalent:

- $\mathrm{ind}_- \mathcal{L} < \infty$.
- *$\mathcal{L}$ has an aPs-completion.*
- *$\mathcal{L}$ has a Pontryagin-space completion.*

Completions: Description ?

Task: describe the totality of completions of $\mathcal{L}$ (up to isomorphism).

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Proposition

Let $\mathcal{L}$ be an inner product space with $\operatorname{ind}_- \mathcal{L} < \infty$. Then each two Pontryagin-space completions of $\mathcal{L}$ are isomorphic.

Completions: Description ?

Task: describe the totality of completions of $\mathcal{L}$ (up to isomorphism).

Example

Let $\langle \mathcal{L}, (.,.)_{\mathcal{L}} \rangle$ be a Hilbert space, $f_1, \ldots, f_n : \mathcal{L} \to \mathbb{C}$ be linear with

$$\mathcal{L}' \cap \operatorname{span}\{f_1, \ldots, f_n\} = \{0\}.$$

Set

$$\mathcal{A} := \mathcal{L} \times \mathbb{C}^n, \quad \iota(x) := \big(x; (f_i(x))_{i=1}^n\big),$$
$$\big[(x; (\xi_i)_{i=1}^n), (y; (\eta_i)_{i=1}^n)\big]_{\mathcal{A}} := (x, y)_{\mathcal{L}},$$
$$\big((x; (\xi_i)_{i=1}^n), (y; (\eta_i)_{i=1}^n)\big)_{\mathcal{A}} := (x, y)_{\mathcal{L}} + \sum_{i=1}^n \xi_i \overline{\eta_i}.$$

Then $\langle \iota, \mathcal{A} \rangle$ is an aPs-completion of $\mathcal{L}$ with $\dim \mathcal{A}^\circ = n$.

The intrinsic dual

Let $\mathcal{L}$ be an inner product space with $\operatorname{ind}_- \mathcal{L} < \infty$.

Definition

Let $\varphi : \mathcal{L} \to \mathbb{C}$ be linear. We write $\varphi \in \mathcal{L}^{\curlywedge}$, if

$$\forall (x_n)_{n\in\mathbb{N}}, x_n \in \mathcal{L} :$$
$$\Big([x_n, x_n]_{\mathcal{L}} \to 0,\ [x_n, x]_{\mathcal{L}} \to 0, x \in \mathcal{L} \Big) \Rightarrow \varphi(x_n) \to 0.$$

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- $\mathcal{L}^{\curlywedge}$ can be interpreted as the topological dual w.r.t. a certain seminorm on $\mathcal{L}$.

Completions: Description

For an aPs-completion $\langle \iota, \mathcal{A} \rangle$ of $\mathcal{L}$, set

$$\iota^*(\mathcal{A}') := \{ f \circ \iota : f \in \mathcal{A}' \}.$$

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Theorem

The map $\langle \iota, \mathcal{A} \rangle \mapsto \iota^*(\mathcal{A}')$ *induces a bijection between*

- *the set of isomorphy classes of aPs-completions of* $\mathcal{L}$,

and

- *the set of those linear subspaces of the algebraic dual* $\mathcal{L}^*$ *of* $\mathcal{L}$ *which contain* $\mathcal{L}^{\curlywedge}$ *with finite codimension.*

For each aPs-completion it holds that

$$\dim \left(\iota^*(\mathcal{A}') \big/ \mathcal{L}^{\curlywedge} \right) = \dim \mathcal{A}^\circ.$$

Reproducing Kernel Spaces

Continuity of point evaluations

For a set Ω and $\eta \in \Omega$ denote by $\chi_\eta : \mathbb{C}^\Omega \to \mathbb{C}$ the *point-evaluation functional* $\chi_\eta : f \mapsto f(\eta)$.

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Definition

Let Ω be a set. An aPs $\mathcal{A}$ is a *reproducing kernel aPs on* Ω, if

(rk1) $\mathcal{A} \subseteq \mathbb{C}^\Omega$ (linear operations defined pointwise);

(rk2) $\forall \eta \in \Omega : \chi_\eta|_\mathcal{A} \in \mathcal{A}'$.

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Being a reproducing kernel aPs is a property of the inner product alone (regardless whether it is nondegenerated or degenerated):

Proposition

If $\langle \mathcal{A}, [.,.]_\mathcal{A} \rangle$ *is an inner product space with* $(\mathrm{rk1})$, *then there exists at most one topology* $\mathcal{O}$ *on* $\mathcal{A}$ *such that* $\langle \mathcal{A}, [.,.]_\mathcal{A}, \mathcal{O} \rangle$ *is a reproducing kernel aPs.*

Continuity of point evaluations

Example

For each $a > 0$, the Paley-Wiener space $\mathcal{P}W_a$ endowed with the inner product

$$[F, G] := \int_{\mathbb{R}} F(t)\overline{G(t)}\, dt - \pi F(0)\overline{G(0)}, \quad F, G \in \mathcal{P}W_a,$$

and the subspace topology of $L^2(\mathbb{R})$ is a reproducing kernel aPs of entire functions.

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Remember

$$\mathcal{P}W_a \text{ is } \begin{cases} \text{Hilbert space}, & a < \pi \\ \text{aPs } (\dim \mathcal{A}^\circ = 1), & a = \pi \\ \text{Pontryagin space}, & a > \pi \end{cases}$$

Kernel functions ?

- Let $\mathcal{A}$ be a reproducing kernel Pontryagin space (i.e., a nondegerated reproducing kernel aPs). Then

$$\exists!\, K : \Omega \times \Omega \to \mathbb{C} : \quad \begin{aligned} & K(w,.) \in \mathcal{A}, w \in \Omega, \\ & f(w) = [f, K(w,.)]_{\mathcal{A}}, f \in \mathcal{A}, w \in \Omega. \end{aligned}$$

This function is called the *reproducing kernel* of $\mathcal{A}$.

Kernel functions ?

- Let $\mathcal{A}$ be a reproducing kernel Pontryagin space (i.e., a nondegerated reproducing kernel aPs). Then

$$\exists!\, K : \Omega \times \Omega \to \mathbb{C} : \quad K(w,.) \in \mathcal{A}, w \in \Omega,$$
$$f(w) = [f, K(w,.)]_{\mathcal{A}}, f \in \mathcal{A}, w \in \Omega.$$

  This function is called the *reproducing kernel* of $\mathcal{A}$.

- Let $\mathcal{A}$ be a degenerated reproducing kernel aPs. Then there cannot exist a function K with these properties:

$$f(w) = [f, K(w,.)]_{\mathcal{A}} = 0, \quad f \in \mathcal{A}^{\circ}, w \in \Omega.$$

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Example

Let $a > 0$, $a \neq \pi$. The reproducing kernel of $\langle \mathcal{P}W_a, [.,.] \rangle$ is

$$K(w,z) := \frac{\sin[a(z-\overline{w})]}{\pi(z-\overline{w})} + \frac{1}{\pi - a} \cdot \frac{\sin[a\overline{w}]}{\overline{w}} \frac{\sin[az]}{z}.$$

Almost reproducing kernels

Definition

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(aRK1) K is a hermitian kernel on Ω, i.e.,

$$K(z,w) = \overline{K(w,z)}, \quad z,w \in \Omega,$$

(aRK2) $K(w,.) \in \mathcal{A}, \quad w \in \Omega,$

(aRK3) There exists data $\delta = ((w_i)_{i=1}^n ; (\gamma_i)_{i=1}^n) \in \Omega^n \times \mathbb{R}^n$ where $n := \dim \mathcal{A}^\circ$, such that

$$\forall f \in \mathcal{A}, w \in \Omega :$$

$$f(w) = [f, K(w,.)]_{\mathcal{A}} + \sum_{i=1}^n \gamma_i \cdot \chi_{w_i}(f)\overline{\chi_{w_i}(K(w,.))}.$$

Almost reproducing kernels: Existence

Theorem

Let $\mathcal{A}$ be a reproducing kernel aPs, set $n := \dim \mathcal{A}^\circ$, and let $(w_i)_{i=1}^n \in \Omega^n$ be such that

$$\mathcal{A}^\circ \cap \bigcap_{i=1}^{n} \ker \chi_{w_i} = \{0\}.$$

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Then there exists a closed and nowhere dense exceptional set $E \subseteq \mathbb{R}^n$, such that for each $(\gamma_i)_{i=1}^n \in \mathbb{R}^n \setminus E$ there exists an almost reproducing kernel of $\mathcal{A}$ with data $\delta := ((w_i)_{i=1}^n; (\gamma_i)_{i=1}^n)$.

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- Such choices of $(w_i)_{i=1}^n \in \Omega^n$ certainly exist since $\{\chi_w : w \in \Omega\}$ is point separating.

Almost reproducing kernels: Properties

For a hermitian kernel K we denote by $\operatorname{ind}_- K \in \mathbb{N}_0 \cup \{\infty\}$ the supremum of the numbers of negative squares of quadratic forms

$$\sum_{i,j=1}^{n} K(w_j, w_i)\xi_i\overline{\xi_j} \quad \text{where} \quad n \in \mathbb{N},\ w_1, \ldots, w_n \in \Omega.$$

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Assume K is an almost reproducing kernel of $\mathcal{A}$ with data δ. Then

- $\mathcal{A}^\circ \cap \bigcap_{i=1}^n \ker \chi_{w_i} = \{0\}$,
- $\operatorname{ind}_- K < \infty$,
- $\gamma_i \neq 0, \quad i, i = 1, \dots, n$,
- $K(w_i, w_j) = \delta_{ij} \frac{1}{\gamma_i}, \quad i, i = 1, \dots, n.$

Almost reproducing kernels: Uniqueness

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Let $\mathcal{A}$ be a reproducing kernel aPs, set $n := \dim \mathcal{A}^\circ$, and let K_1 and K_2 be almost reproducing kernels for $\mathcal{A}$ with corresponding data δ_1 and δ_2, respectively.

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- Due to the Existence Theorem, $\mathcal{A}$ has many different almost reproducing kernels.

Almost reproducing kernels: Description

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Then there exists a unique reproducing kernel aPs, such that K is the almost reproducing kernel of $\mathcal{A}$ with data $\delta = ((w_i)_{i=1}^n; (\gamma_i)_{i=1}^n)$.

Reproducing kernel space completions ?

Let $\mathcal{L}$ be an inner product space whose elements are functions.

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Consider the space $\mathcal{L} := \bigcup_{0<a<\pi} \mathcal{P}W_a$ endowed with

$$[F, G] := \int_{\mathbb{R}} F(t)\overline{G(t)}\, dt - \pi F(0)\overline{G(0)}, \quad F, G \in \mathcal{L}.$$

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Then $\mathcal{L}$ is positive definite, and

- $\mathcal{L}$ is isometrically and densely contained in the (degenerated) reproducing kernel aPs $\langle \mathcal{P}W_\pi, [.,.]\rangle$.
- There does not exist a reproducing kernel Pontryagin space which contains $\mathcal{L}$ isometrically and densely.
- There does not exist a reproducing kernel Hilbert space which contains $\mathcal{L}$ isometrically.

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Example

Let μ be a positive Borel measure on the real line which is compactly supported and not discrete, and consider the space $\mathcal{L}$ of all polynomials endowed with

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Then there does not exist a reproducing kernel aPs which contains $\mathcal{L}$ isometrically.

Topologising the intrinsic dual

Proposition

Let $\mathcal{L}$ be an inner product space with $\operatorname{ind}_- \mathcal{L} < \infty$. Then, for each aPs-completion $\langle \iota, \mathcal{A} \rangle$ of $\mathcal{L}$, it holds that

$$\mathcal{L}^{\curlywedge} = \iota^*\big(\{[.,y]_{\mathcal{A}} : y \in \mathcal{A}\}\big) = \big\{x \mapsto [\iota x, y]_{\mathcal{A}} : y \in \mathcal{A}\big\}.$$

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- The map $\iota^*|_{\mathcal{A}'}$ is injective since $\iota(\mathcal{L})$ is dense in $\mathcal{A}$.

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Definition

Let $\mathcal{T}^{\curlywedge}$ be the topology induced by the norm

$$\|\phi\|_{\curlywedge} := \|(\iota^*|_{\mathcal{A}'})^{-1}\phi\|_{\mathcal{A}'}, \quad \phi \in \mathcal{L}^{\curlywedge}.$$

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(B) $\dim \left(\left[\mathcal{L}^{\curlywedge} + \operatorname{span}\{\chi_w|_{\mathcal{L}} : w \in \Omega\} \right] \Big/ \mathcal{L}^{\curlywedge} \right) < \infty$,

(C) $\mathcal{L}^{\curlywedge} \cap \operatorname{span}\{\chi_w|_{\mathcal{L}} : w \in \Omega\}$ *is* $\mathcal{T}^{\curlywedge}$*-dense in* $\mathcal{L}^{\curlywedge}$*.*

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These conditions can be reformulated in a concrete way. It holds that

$$(\mathrm{B}) \Leftrightarrow (\mathrm{B}') \qquad (\mathrm{C}) \Rightarrow (\mathrm{C}') \qquad (\mathrm{B}) \wedge (\mathrm{C}') \Rightarrow (\mathrm{C})$$

(B') There exist $N \in \mathbb{N}$ and $(w_i)_{i=1}^N \in M^N$, such that the following implication holds. If $(f_n)_{n\in\mathbb{N}}$ is a sequence of elements of $\mathcal{L}$ with

$$\lim_{n\to\infty} [f_n, f_n]_{\mathcal{L}} = 0, \quad \lim_{n\to\infty} [f_n, g]_{\mathcal{L}} = 0, \; g \in \mathcal{L},$$
$$\lim_{n\to\infty} \chi_{w_i}(f_n) = 0, \; i = 1, \dots, N,$$

then $\lim_{n\to\infty} \chi_w(f_n) = 0$, $w \in \Omega$.

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(C') If $(f_n)_{n\in\mathbb{N}}$ is a sequence of elements of $\mathcal{L}$ with

$$\lim_{n,m\to\infty}[f_n - f_m, f_n - f_m]_{\mathcal{L}} = 0, \quad \lim_{n\to\infty}[f_n - f_m, g]_{\mathcal{L}} = 0, \; g \in \mathcal{L},$$
$$\lim_{n\to\infty} \chi_w(f_n) = 0, \; w \in \Omega,$$

then $\lim_{n\to\infty}[f_n, g]_{\mathcal{L}} = 0$, $g \in \mathcal{L}$.

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Theorem

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- We call this unique space the *reproducing kernel completion of $\mathcal{L}$*.
- The number $\Delta(\mathcal{L}) := \dim \mathcal{A}^\circ$ where $\mathcal{A}$ is the reproducing kernel completion of $\mathcal{L}$ is an important geometric invariant of $\mathcal{L}$.

A motivating example: The Hamburger power moment problem

The Hamburger power moment problem

Given $(s_n)_{n=0}^{\infty}$, $s_n \in \mathbb{R}$, *does there exist a positive Borel measure on* $\mathbb{R}$ *with* $s_n = \int_{\mathbb{R}} t^n \, d\mu(t)$, $n = 0, 1, 2, \ldots$ *?*

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Consider the inner product

$$\left[\sum_i \alpha_i t^i, \sum_j \beta_j t^j \right] := \sum_{i,j} s_{i+j} \cdot \alpha_i \overline{\beta_j}$$

on the space $\mathbb{C}[z]$ of all polynomials. Then $\langle \mathbb{C}[z], [.,.] \rangle$ is positive semidefinite.

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- The solution μ is unique (*determinate* case).
- There exist infinitely many solutions (*indeterminate* case).

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- The solution μ is unique (*determinate* case).
- There exist infinitely many solutions (*indeterminate* case).

Let S be the multiplication operator $Sp(z) := zp(z)$ on $\mathbb{C}[z]$. Let $\mathcal{H}$ be the Hilbert space completion of $\langle \mathbb{C}[z], [.,.] \rangle$, and let T be the closure of S in $\mathcal{H}$. Then one of the following holds.

- T is selfadjoint (determinate case).
- T is symmetric with defect index $(1,1)$ (indeterminate case).

The Nevanlinna parameterisation

Theorem

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There exist four entire functions A, B, C, D, such that

$$\int_{\mathbb{R}} \frac{d\mu(t)}{t-z} = \frac{A(z)\tau(z) + B(z)}{C(z)\tau(z) + D(z)}$$

establishes a bijection between $\{\mu : \text{solution}\}$ and $\mathcal{N}_0 := \{\tau : \text{analytic in } \mathbb{C}^+, \operatorname{Im}\tau(z) \geq 0\}$.

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The operator T is entire with respect to the gauge $u := 1$. The matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is the u-resolvent matrix of T.

The three term recurrence

Given μ with all power moments, let p_n, $n \in \mathbb{N}_0$, be the polynomials with degree n and positive leading coefficient, such that $\{p_n : n \in \mathbb{N}_0\}$ is orthonormal w.r.t. $[p, q] := \int_{\mathbb{R}} p\overline{q}\, d\mu$.

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Theorem

There exist unique $a_n > 0$ and $b_n \in \mathbb{R}$, s.t. $(p_{-1} := 0)$

$$zp_n(z) = a_{n+1}p_{n+1}(z) + b_n p_n(z) + a_n p_{n-1}(z), \quad n \in \mathbb{N}_0$$

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The operator T is unitarily equivalent to the operator in ℓ^2 defined by the Jacobi matrix

$$J := \begin{pmatrix} b_0 & a_1 & 0 & 0 & 0 & \cdots \\ a_1 & b_1 & a_2 & 0 & 0 & \cdots \\ 0 & a_2 & b_2 & a_3 & 0 & \cdots \\ \vdots & & \ddots & \ddots & \ddots & \end{pmatrix}$$

(In-)determinate measures

Definition

Let μ be a positive measure with all power moments. Then μ is called *determinate* if it is uniquely determined by the sequence of its power moments, and *indeterminate* otherwise.

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μ is determinate if and only the polynomials are dense in $L^2(\mu)$.

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Being (in-)determinate means that the moment problem for

$$s_n := \int_{\mathbb{R}} t^n \, d\mu(t), \quad n = 0, 1, 2, \ldots,$$

which is by definition solvable, is actually (in-)determinate.

The index of determinacy

Definition

For μ determinate and $w \in \mathbb{C}$ set

$$\operatorname{ind}_w(\mu) := \sup \left\{ k \in \mathbb{N}_0 : \ |t-w|^{2k} d\mu(t) \text{ determinate} \right\} \in \mathbb{N}_0 \cup \{\infty\}.$$

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Theorem

Let μ be determinate.

- *If $\operatorname{ind}_w(\mu) = \infty$ for some $w \in \mathbb{C}$, then $\operatorname{ind}_w(\mu) = \infty$ for all $w \in \mathbb{C}$.*
- *Assume $\operatorname{ind}_w(\mu) < \infty$ for some $w \in \mathbb{C}$. Then μ is discrete and $\operatorname{ind}_w(\mu)$ is constant on $\mathbb{C} \setminus \operatorname{supp} \mu$; denote this constant by $\operatorname{ind}(\mu)$.*
- *Assume $\operatorname{ind}_w(\mu) < \infty$ for some $w \in \mathbb{C}$. Then $\operatorname{ind}_w(\mu) = \operatorname{ind}(\mu) + 1$, $w \in \operatorname{supp} \mu$.*

The index of determinacy

For each $k \in \mathbb{N}$ the (infinite, still well-defined) matrix J^k defines a linear operator V_k on ℓ^2 by taking the closure of the operator defined by the action of J^k on the subspace of finite sequences.

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Theorem

Let μ be determinate. Then the following are equivalent.

- μ has finite index of determinacy.
- There exists $N \in \mathbb{N}$ such that $V, \ldots, V_N$ are selfadjoint, but V_{N+1} is not.

The index of determinacy

For each $k \in \mathbb{N}$ the (infinite, still well-defined) matrix J^k defines a linear operator V_k on ℓ^2 by taking the closure of the operator defined by the action of J^k on the subspace of finite sequences.

Theorem

Let μ be determinate. Then the following are equivalent.

- μ has finite index of determinacy.
- There exists $N \in \mathbb{N}$ such that $V, \ldots, V_N$ are selfadjoint, but V_{N+1} is not.

If μ has finite index of determinacy, then $N = \operatorname{ind}(\mu) + 1$.

A class of distributions

Definition

(1) Let μ be a distribution on $\mathbb{R}$. We write $\mu \in \mathcal{D}_{<\infty}$, if

$\exists N \in \mathbb{N}_0, c_1, \ldots, c_N \in \mathbb{R}, \mu$ positive measure on $\mathbb{R} \setminus \{c_1, \ldots, c_N\}$:

$$\mu(f) = \int_{\mathbb{R}\setminus\{c_1,\ldots,c_N\}} f \, d\mu, \quad f \in C_{00}^{\infty}(\mathbb{R}), \operatorname{supp} f \subseteq \mathbb{R} \setminus \{c_1, \ldots, c_N\}$$

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(2) We say $\mu \in \mathcal{D}_{<\infty}$ has all power moments, if $\int_{|t|\geq t_0} |t|^n \, d\mu(t) < \infty$, $n \in \mathbb{N}$, provided $t_0 > \max\{|c_1|, \ldots, |c_N|\}$.

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(3) Let $\mathcal{R}_{<\infty}$ be the set of formal expressions $\rho := \sum_{i=1}^{m} \sum_{l=0}^{k_i} a_{il} \delta_{w_i}^{(l)}$ where $w_i \in \mathbb{C}^+$ pairwise different, $k_i \in \mathbb{N}_0$, $a_{il} \in \mathbb{C}$ with $a_{ik_i} \neq 0$.

A class of distributions

Definition

Let $(\mu, \rho) \in \mathcal{D}_{<\infty} \times \mathcal{R}_{<\infty}$ and assume that μ has all power moments. For f which is $C^\infty(\mathbb{R})$ with $f(t) = O(|t|^n)$, $t \to \infty$, and locally holomorphic at w_i, define

$$(\mu, \rho)(f) := \mu(f) + \sum_{i=1}^{m} \sum_{l=0}^{k_i} \Big(a_{il} \cdot [f]^{(l)}(w_i) + \overline{a_{il}} \cdot [f]^{(l)}(\overline{w_i}) \Big), \quad n \in \mathbb{N}_0$$

The indefinite moment problem

Given $(s_n)_{n=0}^{\infty}$, $s_n \in \mathbb{R}$, *does there exist* $(\mu, \rho) \in \mathcal{D}_{<\infty} \times \mathcal{R}_{<\infty}$ *with* $s_n = (\mu, \rho)(t^n)$, $n \in \mathbb{N}_0$ *?*

Existence of solutions

For a sequence $(s_n)_{n=0}^{\infty}$ of real numbers, set $\mathcal{L} := \mathbb{C}[z]$ and

$$\Big[\sum_i \alpha_i t^i, \sum_j \beta_j t^j\Big] := \sum_{i,j} s_{i+j} \cdot \alpha_i \overline{\beta_j}.$$

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The inner product space $\langle \mathbb{C}[z], [.,.] \rangle$ has finite negative index.

Existence of solutions

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Assume the indefinite moment problem is solvable.

Existence of solutions

Theorem

Assume the indefinite moment problem is solvable.

Then there exists a number $\Delta \in \mathbb{N}_0 \cup \{\infty\}$, such that ($\kappa_0 := \operatorname{ind}_- \mathcal{L}$)

n	0	$\cdots$	κ_0	$\kappa_0 + 1$	$\cdots$	$\kappa_0 + \Delta$	$\kappa_0 + \Delta$	$\cdots$
# *solutions* $\operatorname{ind}_- = n$	0	$\cdots$	1	0	$\cdots$	∞	∞	$\cdots$

This includes the extremal case as follows:

- *If $\Delta = 0$, the number of solutions is ∞ for all $n \geq \kappa_0$;*
- *If $\Delta = \infty$, the number of solutions is 0 for all $n > \kappa_0$.*

Parameterization of solutions

Let $\mathcal{K}_\kappa^\Delta$ be the set of all function τ meromorphic in $\mathbb{C}^+$, such that the maximal number of quadratic forms

$$Q(\xi_1, \dots, \xi_m; \eta_0, \dots, \eta_{\Delta-1}) := \sum_{i,j=1}^{m} \frac{\tau(w_i) - \overline{\tau(w_j)}}{w_i - \overline{w_j}} \xi_i \overline{\xi_j} + \sum_{k=0}^{\Delta-1} \sum_{i=1}^{m} \mathrm{Re}\left(z_i^k \xi_i \overline{\eta_k} \right)$$

where $m \in \mathbb{N}_0$, $w_1, \dots, w_m \in \mathbb{C}^+$, equals κ.

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Theorem

Assume the indefinite moment problem has $\Delta < \infty$.

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where $m\in\mathbb{N}_0$, $w_1,\dots,w_m\in\mathbb{C}^+$, equals κ.

Theorem

Assume the indefinite moment problem has $\Delta<\infty$.

There exist four entire functions A,B,C,D, such that

$$(\mu,\rho)\left(\frac{1}{t-z}\right) = \frac{A(z)\tau(z)+B(z)}{C(z)\tau(z)+D(z)}$$

establishes a bijection between $\{(\mu,\rho):\mathrm{ind}_-(\mu,\rho)=\kappa, \text{solution}\}$ and $\mathcal{K}_{\kappa-\kappa_0}^\Delta$.

The significance of completions

The positive definite case:

- The moment problem is solvable and indeterminate if and only if $\mathcal{L}$ has a reproducing kernel Hilbert space completion.
- Assume the moment problem is solvable and determinate, and let μ be its unique solution. Then $\operatorname{ind}(\mu) < \infty$ if and only if $\mathcal{L}$ has a reproducing kernel aPs-completion. If $\operatorname{ind}(\mu) < \infty$, then $\operatorname{ind}(\mu) = \Delta(\mathcal{L}) - 1$.

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The indefinite case:

- Assume the indefinite moment problem is solvable. Then $\Delta < \infty$ if and only if $\mathcal{L}$ has a reproducing kernel aPs-completion. If $\Delta < \infty$, then $\Delta = \Delta(\mathcal{L})$.
- Assume the indefinite moment problem is solvable with $\Delta < \infty$. The functions A, B, C, D occur from (an aPs-version) of Krein's resolvent matrix.

Directing Functionals

Let $\mathcal{L}$ be an inner product space whose elements are analytic functions.

- Can one improve the general conditions for existence of a reproducing kernel aPs-completion of $\mathcal{L}$ due to analyticity ?
- If there exists a reproducing kernel aPs-completion, are its elements again analytic ?

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An answer is obtained from an aPs-version of Krein's method of directing functionals.

Sets of semi-Φ-regularity

Definition

Let $\mathcal{L}$ be an inner product space, let S be a linear relation in $\mathcal{L}$, let $\Omega \subseteq \mathbb{C}$ and $M \subseteq \Omega$, and $\Phi : \mathcal{L} \times \Omega \to \mathbb{C}$.

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$$r_{\subseteq}(S, \Phi) := \{\eta \in \Omega : \operatorname{ran}(S - \eta) \subseteq \ker \Phi(\cdot, \eta)\}$$
$$r_{\supseteq}(S, \Phi) := \{\eta \in \Omega : \operatorname{ran}(S - \eta) \supseteq \ker \Phi(\cdot, \eta)\}$$
$$r(S, \Phi) := r_{\subseteq}(S, \Phi) \cap r_{\supseteq}(S, \Phi)$$

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$$\begin{aligned} r_{\supseteq}^{\text{app}}(S, \Phi; M) := \Big\{ &\eta \in \Omega : \ \forall x \in \ker \Phi(\cdot, \eta)\, \exists (x_n)_{n \in \mathbb{N}} \text{ s.t.} \\ & x_n \in \operatorname{ran}(S - \eta), \\ & \lim_{n\to\infty} [x_n, x_n]_{\mathcal{X}} = [x, x]_{\mathcal{X}}, \quad \lim_{n\to\infty} [x_n, y]_{\mathcal{X}} = [x, y]_{\mathcal{X}}, \ y \in \mathcal{L}, \\ & \lim_{n\to\infty} \Phi(x_n, w) = \Phi(x, w), \ w \in M \Big\}. \end{aligned}$$

Directing functionals in aPs

Definition

Let $\mathcal{L}$ be an inner product space, let S be a symmetric linear relation in $\mathcal{L}$, let $\Omega \subseteq \mathbb{C}$, and let $\Phi : \mathcal{L} \times \Omega \to \mathbb{C}$.

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We call Φ a *directing functional for* S, if it satisfies the following axioms.

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(DF1) For each $w \in \Omega$ the function $\Phi(\cdot, w) : \mathcal{L} \to \mathbb{C}$ is linear.

(DF2) The set Ω is open. For each $x \in \mathcal{L}$ the function $\Phi(x, \cdot) : \Omega \to \mathbb{C}$ is analytic.

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- (DF3) There is no nonempty open subset O of Ω, such that $\Phi|_{\mathcal{L}\times O} = 0$.
- (DF4) The set $r_{\subseteq}(S, \Phi)$ has accumulation points in each connected component of $\Omega \setminus \mathbb{R}$.
- (DF5) The set $r_{\supseteq}^{\mathrm{app}}(S, \Phi; \Omega \setminus \mathbb{R})$ has nonempty intersection with both half-planes $\mathbb{C}^+$ and $\mathbb{C}^-$.

Directing functionals in aPs

Example

Let $(s_n)_{n=0}^{\infty}$, $s_n \in \mathbb{R}$, be given and consider:

- $\mathcal{L} := \mathbb{C}[z]$ with $[.,.]$;
- $S := \{(p(z); zp(z)) : p \in \mathbb{C}[z]\}$;
- $\Omega := \mathbb{C}$;
- $\Phi(p, w) := p(w)$.

Then

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- $\Phi(1, w) = 1$, hence $\Phi(1, \cdot)$ vanishes nowhere;

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- $\Phi(p, \cdot) = p$ is entire;
- $\Phi(1, w) = 1$, hence $\Phi(1, \cdot)$ vanishes nowhere;
- $\forall w \in \mathbb{C} : \operatorname{ran}(S - w) = \big\{p \in \mathbb{C}[z] : p(w) = 0\big\} = \ker \Phi(\cdot, w)$, hence $r(S, \Phi) = \mathbb{C}$.

aPs-completion of spaces of analytic functions

Theorem

Let $\mathcal{L}$ be an inner product space with $\operatorname{ind}_- \mathcal{L} < \infty$, let S be a symmetric linear relation in $\mathcal{L}$, let $\Omega \subseteq \mathbb{C}$, and let $\Phi : \mathcal{L} \times \Omega \to \mathbb{C}$ be a directing functional for S.

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Assume that

- $\exists M \subseteq r_{\subseteq}(S, \Phi)$ s.t. M has accumulation points in each connected component of $\Omega \setminus \mathbb{R}$, and

$$\dim \left(\left[\mathcal{L}^{\curlywedge} + \operatorname{span}\{\Phi(\cdot, w) : w \in M\}\right] \Big/ \mathcal{L}^{\curlywedge} \right) < \infty;$$

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$$\dim \Big(\big[\mathcal{L}^{\curlywedge} + \operatorname{span}\{\Phi(\cdot, w) : w \in M\}\big] \Big/ \mathcal{L}^{\curlywedge} \Big) < \infty;$$
- Either $\mathcal{L}^{\curlywedge} \cap \operatorname{span}\big\{\Phi(\cdot, w) : w \in r_{\subseteq}(S, \Phi),$
$$\Phi(\cdot, w) \in \mathcal{L}^{\curlywedge} + \operatorname{span}\{\Phi(\cdot, w) : w \in M\}\big\}$$
or $\mathcal{L}^{\curlywedge} \cap \operatorname{span}\big\{\Phi(\cdot, w) : w \in r^{\mathrm{app}}_{\supseteq}(S, \Phi; \Omega \setminus \mathbb{R}) \setminus \mathbb{R}\big\}$
is dense in $\mathcal{L}^{\curlywedge}$ w.r.t. $\mathcal{T}^{\curlywedge}$.

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Under these assumptions:

- There exists a unique reproducing kernel aPs $\mathcal{B}$, such that $\Phi_{\mathcal{L}} : x \mapsto \Phi(x, \cdot)$ maps $\mathcal{L}$ isometrically onto a dense subspace of $\mathcal{B}$.
- The elements of $\mathcal{B}$ are analytic on Ω.
- $\mathrm{Clos}_{\mathcal{B}} \left[(\Phi_{\mathcal{L}} \times \Phi_{\mathcal{L}})(S) \right] = S(\mathcal{B})$.
  Here $S(\mathcal{B})$ is the multiplication operator in $\mathcal{B}$.

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Under these assumptions:

Concerning the geometry of $\mathcal{B}$, we have

- $$\begin{aligned}\Phi^*_{\mathcal{L}}(\mathcal{B}') &= \mathcal{L}^{\curlywedge} + \operatorname{span}\big\{\Phi(\cdot, w) : w \in M\big\} \\ &= \mathcal{L}^{\curlywedge} + \operatorname{span}\big\{\Phi(\cdot, w) : w \in \Omega\big\}\end{aligned}$$
- $\operatorname{ind}_0 \mathcal{B} = \dim \Big(\big[\mathcal{L}^{\curlywedge} + \operatorname{span}\{\Phi(\cdot, w) : w \in \Omega\}\big] \Big/ \mathcal{L}^{\curlywedge} \Big)$
- The set $\{w \in \Omega : \mathfrak{d}_{\mathcal{B}}(w) > 0\}$ is discrete.
  Here $\mathfrak{d}_{\mathcal{B}}(w)$ is the minimal multiplicity of w as a zero of some element of $\mathcal{B} \setminus \{0\}$.

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Let $\mathcal{L}$ be an inner product space with $\operatorname{ind}_- \mathcal{L} < \infty$, let S be a symmetric linear relation in $\mathcal{L}$, let $\Omega \subseteq \mathbb{C}$, and let $\Phi : \mathcal{L} \times \Omega \to \mathbb{C}$ be a directing functional for S.

Under these assumptions:

Concerning the operator theory of $S(\mathcal{B})$, we have

- $S(\mathcal{B})$ is of defect $(1, 1)$;
- $\Omega \subseteq r(S(\mathcal{B}))$
- $\operatorname{ran}(S(\mathcal{B}) - w) = \ker \chi_w^{(\mathfrak{d}_{\mathcal{B}}(w))}|_{\mathcal{B}},\ w \in \Omega$

De Branges space completions

Definition

An inner product space $\mathcal{L}$ whose elements are entire functions is called *algebraic de Branges space*, if

- If $f \in \mathcal{L}, w \in \mathbb{C} \setminus \mathbb{R}$ with $f(w) = 0$, then $\frac{f(z)}{z-w} \in \mathcal{L}$. We have

$$\left[\frac{z-\overline{w}}{z-w} f(z), \frac{z-\overline{w}}{z-w} g(z)\right]_{\mathcal{L}} = \left[f(z), g(z)\right]_{\mathcal{L}},$$
$$f, g \in \mathcal{B}, f(w) = g(w) = 0.$$

- If $f \in \mathcal{L}$ then $f^{\#}(z) := \overline{f(\overline{z})} \in \mathcal{L}$. We have

$$\left[f^{\#}, g^{\#}\right]_{\mathcal{L}} = [g, f]_{\mathcal{L}}, \quad f, g \in \mathcal{L}.$$

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If in addition $\mathcal{L}$ is a reproducing kernel aPs, then $\mathcal{L}$ is called a *de Branges aPs*.

De Branges space completions

Theorem

Let $\mathcal{L}$ be an algebraic de Branges space. If $\mathcal{L}$ has a reproducing kernel aPs-completion, then this completion is a de Branges aPs.

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