Spectral theory of a class of canonical systems with two singular endpoints

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This presentation is based on joint work with Matthias Langer:

- M. Langer and H. Woracek. "Direct and inverse spectral theorems for a class of canonical systems with two singular endpoints". manuscript in preparation.
- M. Langer and H. Woracek. "Indefinite Hamiltonian systems whose Titchmarsh–Weyl coefficients have no finite generalized poles of non-positive type". In: *Oper. Matrices* 7.3 (2013), pp. 477–555.
- M. Langer and H. Woracek. *Distributional representations of generalized Nevanlinna functions*. submitted. preprint: ASC Report 17. 2013.

This work is around for a long time. We presented parts of it

- at the conference IWOTA 2010 (Berlin),
- in a Seminar talk at the University of Vienna ($\sim$ 2011).

Since then it improved and expanded......

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These slides are available from my website

http://asc.tuwien.ac.at/index.php?id=woracek

The Canonical System

We consider 2×2-Hamiltonian systems without potential:

$$y'(t) = zJH(t)y(t), \quad t \in (a,b).$$

Here the *Hamiltonian* H shall be subject to

- $H(t) : (a,b) \to \mathbb{R}^{2\times 2}$,
- $H(t) \geq 0$, $t \in (a,b)$,
- $H \in L^1_{\mathrm{loc}}(a,b)$,
- H does not vanish identically on any set of positive measure,
- $z \in \mathbb{C}$ a parameter,
- $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

The Canonical System

This equation is the eigenvalue equation of a differential operator.

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Direct Problems: Given a Hamiltonian H, find information about spectral data of selfadjoint realisations.

Inverse Problems:

- Existence Theorems: Given some spectral data, does there exist a Hamiltonian H which leads to this data.
- Uniqueness Theorems: Which spectral data obtained from some Hamiltonian determine this Hamiltonian uniquely.

Outline

The Operator Model

(for a canonical system)

Not all what we say in this section is *strictly* correct.

We neglect the possible existence of indivisible intervals

and the need of forming equivalence classes.

The model space

Definition (The model space $L^2(H)$)

The *model space* $L^2(H)$ is the space of all $f : (a,b) \to \mathbb{C}^2$ with

$$\|f\|_H^2 := \int_a^b f(t)^* H(t) f(t)\, dt < \infty.$$

If endowed with the scalar product

$$(f,g)_H = \int_a^b g(t)^* H(t) f(t)\, dt, \quad f,g \in L^2(H),$$

the space $L^2(H)$ becomes a Hilbert space.

The model operator

Definition (The maximal operator $T_{\max}(H)$)

The (graph of the) *maximal operator* $T_{\max}(H)$ is

$$T_{\max}(H) = \Big\{ (f; g) \in L^2(H) \times L^2(H) : \\ f \text{ is locally absolutely continuous and } f' = JHg \Big\}$$

The operator $T_{\max}(H)$ is closed.

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Definition (The minimal operator $T_{\min}(H)$)

The *minimal operator* $T_{\min}(H)$ is $T_{\min}(H) = T_{\max}(H)^*$.

The operator $T_{\min}(H)$ is closed and symmetric. It is either selfadjoint, or completely nonselfadjoint.

Limit Circle vs. Limit Point Case

The spectral theory of $T_{\max}(H)$ depends on the growth of H towards the endpoints a and b.

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- H is in *limit circle case at* a, if $(x_0 \in (a,b))$

$$\int_a^{x_0} \operatorname{tr} H(t)\, dt < \infty \qquad \Big(\Leftrightarrow H \in L^1_{\mathrm{loc}}([a,b)) \Big).$$

- H is in *limit point case at* a, if $(x_0 \in (a,b))$

$$\int_a^{x_0} \operatorname{tr} H(t)\, dt = \infty.$$

Limit Circle vs. Limit Point Case

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- H is in *limit point case at* a, if $(x_0 \in (a,b))$

$$\int_a^{x_0} \operatorname{tr} H(t)\, dt = \infty.$$

Similar: *limit circle case at* b and *limit point case at* b.

Limit Circle vs. Limit Point Case

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- Case I, lc $\leftrightarrow$ lc: $(2, 2)$.
- Case II, lc $\leftrightarrow$ lp or lp $\leftrightarrow$ lc: $(1, 1)$.
- Case III, lp $\leftrightarrow$ lp: $(0, 0)$.

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- Case II, lc $\leftrightarrow$ lp or lp $\leftrightarrow$ lc: $(1,1)$.
- Case III, lp $\leftrightarrow$ lp: $(0,0)$.

In Case III, $T_{\min}(H) = T_{\max}(H)$. Hence, $T_{\min}(H)$ is selfadjoint and is the only selfadjoint realisation.

In the Cases I and II, there are many different selfadjoint realisations.

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The Weyl Construction in the Limit Circle Case

(lc $\leftrightarrow$ lp)

The Weyl construction

- $\forall\ \psi \in \operatorname{dom} T_{\max}(H): \quad \psi(a) := \lim_{x \searrow a} \psi(x)$ exists.

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- Denote by $\mathfrak{N}_z$ the set of all solutions of the canonical system. For each $z \in \mathbb{C}$ the map $\psi \mapsto \psi(a)$ is a bijection from $\mathfrak{N}_z$ onto $\mathbb{C}^2$.

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- Denote by $\mathfrak{N}_z$ the set of all solutions of the canonical system. For each $z \in \mathbb{C}$ the map $\psi \mapsto \psi(a)$ is a bijection from $\mathfrak{N}_z$ onto $\mathbb{C}^2$.
- Let $\theta(\cdot\,;z) = (\theta_i(\cdot\,;z))_{i=1}^2, \varphi(\cdot\,;z) = (\varphi_i(\cdot\,;z))_{i=1}^2 \in \mathcal{N}_z$ with $\theta(a;z) = (1,\,0)^T$, $\varphi(a;z) = (0,\,1)^T$. Then

$$q_H(z) := \lim_{x \nearrow b} \frac{\theta_1(x;z)\tau + \theta_2(x;z)}{\varphi_1(x;z)\tau + \varphi_2(x;z)}$$

exists locally uniformly on $\mathbb{C} \setminus \mathbb{R}$ as an analytic function in z, does not depend on $\tau \in \mathbb{R} \cup \{\infty\}$, and

$$\operatorname{Im} q_H(z) \geq 0, \quad z \in \mathbb{C}^+.$$

The Weyl construction

Direct Theorem: Let μ_H be the measure in the Herglotz-integral representation of q_H, so that μ_H is a positive Borel measure on $\mathbb{R}$ with $\int_{\mathbb{R}} (1+t^2)^{-1}\, d\mu_H(t) < \infty$. Set

$$A_D := \{(f;g) \in T_{\max}(H) : (1,0)f(a) = 0\}.$$

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$$A_D := \{(f;g) \in T_{\max}(H) : (1,0)f(a) = 0\}.$$

- $$(\Theta_H f)(t) := \int_a^b \varphi(x;t)^T H(x) f(x)\,dx, \quad t \in \mathbb{R},$$
$$f \in L^2(H),\ \ \sup(\operatorname{supp} f) < b,$$

  extends to an isomorphism of $L^2(H)$ onto $L^2(\mu_H)$ with

$$\Theta_H \circ A_D = M_{\mu_H} \circ \Theta_H.$$

- In particular, A_D has simple spectrum.

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$$A_D := \{(f;g)\in T_{\max}(H) : (1,0)f(a)=0\}.$$

- $$(\Theta_H^{-1}g)(x)=\int_{-\infty}^{\infty} g(t)\varphi(x;t)\,d\mu_H(t), \quad x\in(a,b),$$ $$g\in L^2(\mu_H),\ \operatorname{supp} g \text{ compact}.$$

The Weyl construction

Inverse Theorem:

- Let q be a function of Nevanlinna class (i.e., analytic with nonnegative imaginary part in $\mathbb{C}^+$).
  Then $\exists\ H :\ \text{lc} \leftrightarrow \text{lp}, q = q_H$.
  This Hamiltonian is (essentially) unique.

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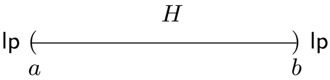
  Then $\exists\ H:\ \text{lc} \leftrightarrow \text{lp}, q = q_H$.
  This Hamiltonian is (essentially) unique.
- Let μ be a positive Borel measure on $\mathbb{R}$ with
  $\int_{\mathbb{R}} (1+t^2)^{-1}\, d\mu_H(t) < \infty$.
  Then $\exists\ H:\ \text{lc} \leftrightarrow \text{lp}, \mu = \mu_H$.
  This Hamiltonian is (a bit less, but still essentially) unique.

Simple Spectrum in the Limit Point Case

(lp $\leftrightarrow$ lp)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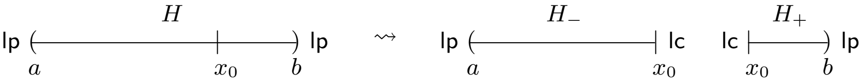
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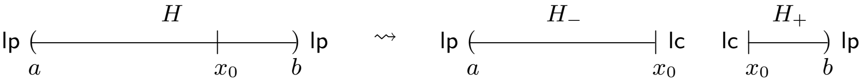
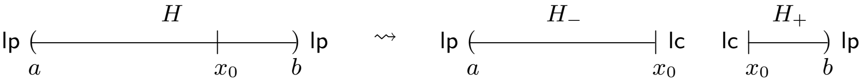
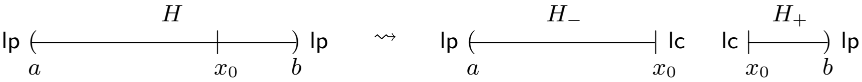


- Weyl-coefficients q_{H_-} and q_{H_+}.



- Weyl-coefficients q_{H_-} and q_{H_+}.
- $Q_H(z) := \dfrac{1}{q_{H_+}(z)+q_{H_-}(z)} \begin{pmatrix} q_{H_+}(z)q_{H_-}(z) & -q_{H_+}(z) \\ -q_{H_+}(z) & -1 \end{pmatrix}$

  is a 2×2-matrix valued Nevanlinna function.



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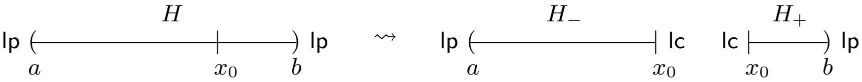
- Let Ω_H be the 2×2-matrix measure in the Herglotz-integral representation of Q_H. Then there exists a unitary operator $U : L^2(H) \to L^2(\Omega_H)$ with

$$U \circ T_{\min}(H) = M_x \circ U.$$



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Corollary

The spectral multiplicity of $T_{\min}(H)$ cannot exceed 2.

Kac's Theorem

Theorem

Let H be lp $\leftrightarrow$ lp.

- *The singular spectrum of $T_{\min}(H)$ is always simple.*
- *The absolutely continuous spectrum of $T_{\min}(H)$ is simple, if and only if the set*

$$\Big\{x \in \mathbb{R} : \lim_{\epsilon \downarrow 0} \operatorname{Im} q_{H_+}(x + i\epsilon) \text{ exists in } (0, \infty)\Big\} \cap \Big\{x \in \mathbb{R} : \lim_{\epsilon \downarrow 0} \operatorname{Im} q_{H_-}(-x + i\epsilon) \text{ exists in } (0, \infty)\Big\}$$

has Lebesgue measure zero.

De Branges class

Definition

Let H be lp $\leftrightarrow$ lp. We say that H is of de Branges class, if

- for one (and hence for all) $x_0 \in (a,b)$
  $\int_a^{x_0} \binom{0}{1}^* H(x) \binom{0}{1} \, dx < \infty, \quad \int_a^{x_0} m_{22}(x) \binom{1}{0}^* H(x) \binom{1}{0} \, dx < \infty$
  where $m_{22}(x) := \int_a^x \binom{0}{1}^T H(y) \binom{0}{1} \, dy$.

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  where $m_{22}(x) := \int_a^x \binom{0}{1}^T H(y) \binom{0}{1} \, dy$.

Theorem

Let H be lp $\leftrightarrow$ lp. If H is of de Branges class, then the selfadjoint extensions of $T_{\min}(H_-)$ have resolvents of Hilbert-Schmidt class. By Kac's Theorem, $T_{\min}(H)$ has simple spectrum.

Three Classes of Objects

(the main players)

The class $\mathbb{H}$ of Hamiltonians

For a Hamiltonian H and $x_0 \in (a,b)$ define $X_k : (a, x_0) \to \mathbb{C}^2$ by
$X_0(x) := \binom{1}{0}, \quad X_k(x) := \int_{x_0}^{x} JH(y)X_{k-1}(y)\,dy,\ k \in \mathbb{N}.$

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  where $m_{22}(x) := \int_a^{x} \binom{0}{1}^T H(y)\binom{0}{1}\,dy$.
  I.e., H is of de Branges class.

- $$\exists\, N \in \mathbb{N}_0 : \quad L^2(H|_{(a,x_0)}) \cap \operatorname{span}\{X_k : k \le N\} \neq \{0\} \quad (1)$$

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- $\exists\ N \in \mathbb{N}_0 : \quad L^2(H|_{(a,x_0)}) \cap \operatorname{span}\{X_k :\ k \leq N\} \neq \{0\}$ (1)

If $H \in \mathbb{H}$, we denote by $\Delta(H)$ the smallest number N with (1).

The class $\mathbb{H}$ of Hamiltonians

Example

Let $\alpha \in \mathbb{R}$ and set

$$H_\alpha(x) := \begin{pmatrix} x^{-\alpha} & 0 \\ 0 & 1 \end{pmatrix}, \qquad x \in (0, \infty).$$

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Then H_α is in lpc at ∞, and $\int_a^{x_0} \binom{0}{1}^* H(x) \binom{0}{1} \, dx < \infty$.

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$\alpha < 1$	lc $\leftrightarrow$ lp	
$1 \leq \alpha < 2$	lp $\leftrightarrow$ lp	class $\mathbb{H}$ with $\Delta(H_\alpha) = n$, $\alpha \in \left[2 - \frac{1}{n}, 2 - \frac{1}{n+1}\right), n \in \mathbb{N}$
$\alpha \geq 2$	lp $\leftrightarrow$ lp	not de Branges class (hence not $\mathbb{H}$)

The class $\mathbb{H}$ of Hamiltonians

Example

The Hamiltonian

$$H(x) := \begin{pmatrix} \frac{x^{-2}}{(\ln x)^2} & 0 \\ 0 & 1 \end{pmatrix}, \qquad x \in (0,1),$$

The class $\mathbb{H}$ of Hamiltonians

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- is lp $\leftrightarrow$ lp;
- is of de Branges class;
- is *not* of class $\mathbb{H}$.

The class $\mathbb{H}$ of Hamiltonians

Example (The Bessel Hamiltonian)

For $\alpha \geq 1$ set

$$H_\alpha(x) := \begin{pmatrix} x^{-\alpha} & 0 \\ 0 & x^{\alpha} \end{pmatrix}, \qquad x \in (0, \infty).$$

This Hamiltonian arises when rewriting the Bessel equation as a first-order system.

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We have $H_\alpha \in \mathbb{H}$ with $\Delta(H_\alpha) = \left\lfloor \frac{\alpha+1}{2} \right\rfloor$.

The class $\mathbb{H}$ of Hamiltonians

Example (The Bessel Hamiltonian)

Assume α not odd integer. Then

$$X_k(x) = \begin{cases} \begin{pmatrix} \displaystyle\sum_{\substack{l=0 \\ l\,\text{even}}}^{k} \mu_{l,k} x^l + \sum_{\substack{m=1 \\ m\,\text{odd}}}^{k-1} \lambda_{m,k} x^{\alpha+m} \\ 0 \end{pmatrix} & \text{if } k \text{ is even,} \\ \begin{pmatrix} 0 \\ \displaystyle\sum_{\substack{l=1 \\ l\,\text{odd}}}^{k} \mu_{l,k} x^{-\alpha+l} + \sum_{\substack{m=0 \\ m\,\text{even}}}^{k-1} \lambda_{m,k} x^{m} \end{pmatrix} & \text{if } k \text{ is odd.} \end{cases}$$

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If $\alpha \in 2\mathbb{N}_0 + 1$, additional logarithmic terms occur.

The class $\mathcal{N}_{<\infty}$ of generalised Nevanlinna functions

Definition

A function q is called a *generalised Nevanlinna function*, if

- q is meromorphic in $\mathbb{C} \setminus \mathbb{R}$;
- $q(\overline{z}) = \overline{q(z)}$, $z \in \rho(q)$ (where $\rho(q)$ denotes the domain of analyticity of q in $\mathbb{C} \setminus \mathbb{R}$);
- the reproducing kernel

$$K_q(w, z) := \frac{q(z) - \overline{q(w)}}{z - \overline{w}} \quad z, w \in \rho(q),$$

has a finite number negative squares.

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has a finite number negative squares.

We denote the set of all generalised Nevanlinna functions by $\mathcal{N}_{<\infty}$. If $q \in \mathcal{N}_{<\infty}$, we denote the actual number of negative squares of the kernel K_q by $\operatorname{ind}_- q$.

The subclass $\mathcal{N}_{<\infty}^{(\infty)}$

Definition

We denote by $\mathcal{N}_{<\infty}^{(\infty)}$ the set of all functions $q \in \mathcal{N}_{<\infty}$ such that

$$\lim_{z \xrightarrow{\measuredangle} i\infty} \frac{q(z)}{z^{2\kappa-1}} \in (-\infty, 0) \qquad \text{or} \qquad \lim_{z \xrightarrow{\measuredangle} i\infty} \left| \frac{q(z)}{z^{2\kappa-1}} \right| = \infty, \qquad (2)$$

where $\kappa := \operatorname{ind}_- q$ and "$\xrightarrow{\measuredangle}$" denotes the non-tangential limit.

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$$\lim_{z \xrightarrow{\measuredangle} i\infty} \frac{q(z)}{z^{2\kappa-1}} \in (-\infty, 0) \qquad \text{or} \qquad \lim_{z \xrightarrow{\measuredangle} i\infty} \left| \frac{q(z)}{z^{2\kappa-1}} \right| = \infty, \tag{2}$$

where $\kappa := \operatorname{ind}_- q$ and "$\xrightarrow{\measuredangle}$" denotes the non-tangential limit.

The significance of the condition in this definition is not that (2) holds *for some* κ, but that it holds *exactly for* $\kappa = \operatorname{ind}_- q$.

The class $\mathbb{M}$ of measures

Definition

Let μ be a positive Borel measure on $\mathbb{R}$. We say that μ belongs to the class $\mathbb{M}$ if

$$\exists\ N \in \mathbb{N}_0 : \quad \int_{\mathbb{R}} \frac{d\mu(t)}{(1+t^2)^{N+1}} < \infty. \tag{3}$$

If $\mu \in \mathbb{M}$, we denote by $\Delta(\mu)$ the smallest non-negative integer N such that (3) holds.

The Spectral Theory

(of Hamiltonians of class $\mathbb{H}$)

Regularized Boundary Values

Let $H \in \mathbb{H}$ be given, and fix $x_0 \in (a, b)$.

Regularized Boundary Values

Let $H \in \mathbb{H}$ be given, and fix $x_0 \in (a,b)$.

Theorem (Existence)

For each $z \in \mathbb{C}$ and each solution $\psi = (\psi_1, \psi_2)^T \in \mathfrak{N}_z$ the boundary value

$$\operatorname{rbv}_{z,1} \psi := \lim_{x \searrow a} \psi_1(x)$$

and the regularized boundary value

$$\operatorname{rbv}_{z,2} \psi := $$
$$-\lim_{x \searrow a} \Big[\sum_{l=0}^{\Delta(H)} z^l \big(\mathfrak{w}_l^{x_0}(x)\big)^* J \Big(\psi(x) - \lim_{t \searrow a} \psi_1(t) \sum_{k=\Delta(H)+1}^{2\Delta(H)-l} z^k \mathfrak{w}_k^{x_0}(x) \Big) \Big]$$

exist.

Regularized Boundary Values

Theorem (The Boundary Value Map)

For each $z \in \mathbb{C}$ *the map*

$$\mathrm{rbv}_z : \begin{cases} \mathfrak{N}_z & \rightarrow \quad \mathbb{C}^2 \\ \psi & \mapsto \quad (\mathrm{rbv}_{z,1}\,\psi,\ \mathrm{rbv}_{z,2}\,\psi)^T \end{cases}$$

is a bijection from $\mathfrak{N}_z$ *onto* $\mathbb{C}^2$*.*

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is a bijection from $\mathfrak{N}_z$ *onto* $\mathbb{C}^2$*.*

Theorem (Existence of a well-behaved Solution)

- *For each* $z \in \mathbb{C} \setminus \{0\}$ *there exists an (up to scalar multiples) unique solution* $\psi = (\psi_1, \psi_2) \in \mathfrak{N}_z \setminus \{0\}$ *such that* $\lim_{x \searrow a} \psi_2(x)$ *exists.*
- $$\lim_{x \searrow a} \psi_2(x) \text{ exists } \iff \psi|_{(a,x_0)} \in L^2(H|_{(a,x_0)}) \iff \operatorname{rbv}_{z,1}\psi = 0$$

Regularized Boundary Values

Theorem (Controlling x_0-dependence)

Let $x_0, x_0' \in (a,b)$, and let rbv_z and rbv'_z be the correspondingly defined regularized boundary value mappings.
There exists a polynomial $p \in \mathbb{R}[z]$ with $p(0) = 0$ and $\deg p \leq 2\Delta(H)$, such that

$$\mathrm{rbv}'_{z,2}\,\psi = \mathrm{rbv}_{z,2}\,\psi + \mathrm{rbv}_{z,1}\,\psi \cdot p(z), \qquad \psi \in \mathfrak{N}_z, \;\; z \in \mathbb{C}.$$

Singular Weyl coefficients

Let $H \in \mathbb{H}$ be given, and fix $x_0 \in (a, b)$.

Singular Weyl coefficients

Let $H \in \mathbb{H}$ be given, and fix $x_0 \in (a, b)$.

Theorem (Existence of limits)

For $z \in \mathbb{C}$ denote by $\theta(\cdot\,;z) = (\theta_1(\cdot\,;z), \theta_2(\cdot\,;z))^T$ and $\varphi(\cdot\,;z) = (\varphi_1(\cdot\,;z), \varphi_2(\cdot\,;z))^T$ the unique solutions in $\mathcal{N}_z$ with

$$\mathrm{rbv}_z\, \theta(\cdot\,;z) = (1,\, 0)^T, \qquad \mathrm{rbv}_z\, \varphi(\cdot\,;z) = (0,\, 1)^T.$$

Then, for each $\tau \in \mathbb{R} \cup \{\infty\}$, the limit

$$q_H(z) := \lim_{x \nearrow b} \frac{\theta_1(x;z)\tau + \theta_2(x;z)}{\varphi_1(x;z)\tau + \varphi_2(x;z)}$$

exists locally uniformly on $\mathbb{C} \setminus \mathbb{R}$ as an analytic function in z and does not depend on τ.

Singular Weyl coefficients

Definition

We call the function q_H constructed as such limit a(!) *singular Weyl coefficient* of H.

Singular Weyl coefficients

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Theorem (Some properties)

- *Each function q_H constructed as such limit belongs to the class $\mathcal{N}^{(\infty)}_{\Delta(H)}$.*
- *We have*

$$\theta(\cdot\,;z) - q_H(z)\varphi(\cdot\,;z) \in L^2\big(H|_{(x_0,b)}\big), \quad z \in \mathbb{C}\setminus\mathbb{R},$$

and this property characterizes the value $q_H(z)$ for each $z \in \mathbb{C}\setminus\mathbb{R}$.

Singular Weyl coefficients

Theorem (Controlling x_0-dependence)

Let $x_0, x_0' \in (a,b)$, and let q_H and q_H' be the correspondingly defined limits.
There exists a polynomial $p \in \mathbb{R}[z]$ with $p(0) = 0$ and $\deg p \leq 2\Delta(H)$, such that

$$q_H'(z) = q_H(z) + p(z).$$

Singular Weyl coefficients

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There exists a polynomial $p \in \mathbb{R}[z]$ *with* $p(0) = 0$ *and* $\deg p \leq 2\Delta(H)$*, such that*

$$q_H'(z) = q_H(z) + p(z).$$

Definition

Write $q_1 \sim q_2$ if $q_1 - q_2 \in \mathbb{R}[z] \wedge (q_1 - q_2)(0) = 0$.
We denote by $[q]_H$ the equivalence class modulo $\sim$ which contains some (and hence any) function q_H constructed as limit, and speak of $[q]_H$ as the(!) *singular Weyl coefficient* of H.

The Spectral Measure

Theorem

Let $H \in \mathbb{H}$ be given. There exists a unique positive Borel measure μ_H on $\mathbb{R}$ such that for any $q_H \in [q]_H$

$$\mu_H\big([s_1, s_2]\big) = \frac{1}{\pi} \lim_{\varepsilon \searrow 0} \lim_{\delta \searrow 0} \int_{s_1-\varepsilon}^{s_2+\varepsilon} \operatorname{Im} q_H(t + i\delta)\, dt,$$

$$-\infty < s_1 < s_2 < \infty.$$

We have $\mu_H \in \mathbb{M}$ and $\Delta(\mu_H) = \Delta(H)$.

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Definition

We call μ_H the *spectral measure* of H.

The Fourier Transform

Theorem

Let $H \in \mathbb{H}$ *be given, and let* $\varphi(\cdot\,;z) \in \mathfrak{N}_z$ *be the unique solution with* $\operatorname{rbv}_z \varphi(\cdot\,;z) = (0,\,1)^T$.

- $$(\Theta_H f)(t) := \int_a^b \varphi(x;t)^T H(x) f(x)\,dx, \quad t \in \mathbb{R}, \qquad f \in L^2(H),\ \ \sup(\operatorname{supp} f) < b,$$

  extends to an isomorphism of $L^2(H)$ *onto* $L^2(\mu_H)$ *with*

  $$\Theta_H \circ T_{\min}(H) = M_{\mu_H} \circ \Theta_H.$$

- $$(\Theta_H^{-1} g)(x) = \int_{-\infty}^{\infty} g(t)\varphi(x;t)\,d\mu_H(t), \quad x \in (a,b), \qquad g \in L^2(\mu_H),\ \ \operatorname{supp} g \text{ compact}.$$

Inverse Theorems

Theorem (Existence)

- *Let* $q \in \mathcal{N}_{<\infty}^{(\infty)}$ *with* $\operatorname{ind}_- q > 0$. *Then there exists a Hamiltonian* $H \in \mathbb{H}$ *with* $q \in [q]_H$.
- *Let* $\mu \in \mathbb{M}$ *with* $\Delta(\mu) > 0$. *Then there exists a Hamiltonian* $H \in \mathbb{H}$ *with* $\mu_H = \mu$.

Inverse Theorems

Hamiltonians H_1 and H_2 are *reparameterizations of each other*, if $\exists\ \phi : (a_2, b_2) \to (a_1, b_1)$ bijective, monotonically increasing, ϕ, ϕ^{-1} absolutely continuous, with $H_2(t) = H_1(\phi(t)) \cdot \phi'(t)$, $t \in (a_2, b_2)$.

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Theorem (Global Uniqueness)

Let $H_1, H_2 \in \mathbb{H}$ be given.

- *If there exist singular Weyl coefficients q_{H_1} and q_{H_2} such that $q_{H_1} - q_{H_2} \in \mathbb{R}[z]$, then ($\alpha := (q_{H_1} - q_{H_2})(0)$)*

$$H_1 \sim \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} H_2 \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}. \tag{4}$$

- *If $[q]_{H_1} = [q]_{H_2}$, then $H_1 \sim H_2$.*
- *If $\mu_{H_1} = \mu_{H_2}$, then there exists $\alpha \in \mathbb{R}$ such that (4) holds.*

Inverse Theorems

Theorem (Local Uniqueness)

Let $H_1, H_2 \in \mathbb{H}$ be given (both defined on (a,b)). Set

$$s_i(\tau) := \sup\Big\{x \in (a,b) : \int_a^x \sqrt{\det H_i(y)}\, dy < \tau\Big\}, \quad \tau > 0.$$

Then, for each $\tau > 0$, the following are equivalent.

- $H_1|_{(a,s_1(\tau))} \sim H_2|_{(a,s_2(\tau))}$.
- $\exists\; q_{H_1}, q_{H_2}, \beta \in (0,\pi) \quad \forall\, \varepsilon > 0$:

$$q_{H_1}(re^{i\beta}) - q_{H_2}(re^{i\beta}) = \mathrm{O}\big(e^{(-2\tau+\varepsilon)r\sin\beta}\big),\; r \to \infty.$$

- $\exists\; q_{H_1}, q_{H_2}, k \geq 0 \quad \forall\, \delta \in (0,\pi/2)$: $\quad \big($*in* $|\arg z - \pi/2| \leq \delta\big)$

$$q_{H_1}(z) - q_{H_2}(z) = \mathrm{O}\big((\operatorname{Im} z)^k e^{-2\tau \operatorname{Im} z}\big),\; |z| \to \infty.$$

Application to Schrödinger Operators

(and the perturbed Bessel operator)

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A class of potentials

Consider a 1-dimensional Schrödinger equation

$$-u''(x) + V(x)u(x) = \lambda u(x), \quad x \in (a, b).$$

We say $V \in \mathbb{K}$, if 1.−4. hold:

1. V real-valued and $V \in L^1_{\mathrm{loc}}(a, b)$;

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2. V can be represented as $V = \frac{\phi''}{\phi}$ with a function ϕ satisfying

$$\phi(x) > 0, \quad x \in (a,b),$$

$$\exists\; x_0 \in (a,b): \quad \phi|_{(a,x_0)} \in L^2(a,x_0), \phi^{-1}|_{(a,x_0)} \notin L^2(a,x_0)$$

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3. For one (and hence for all) $x_0 \in (a,b)$

$$\int_a^{x_0} \phi(x)^2 \int_x^{x_0} \frac{1}{\phi(t)^2}\, dt\, dx < \infty.$$

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Define function $\tilde{w}_k^{x_0} : (a,b) \to \mathbb{R}$ recursively by

$$\tilde{w}_0^{x_0}(x) := \frac{1}{\phi(x)}, \quad \tilde{w}_k^{x_0}(x) := \begin{cases} \phi(x) \int\limits_x^{x_0} \frac{1}{\phi(t)} \tilde{w}_{k-1}^{x_0}(t)\, dt\,, & k \text{ odd} \\ \frac{1}{\phi(x)} \int\limits_a^{x} \phi(t) \tilde{w}_{k-1}^{x_0}(t)\, dt\,, & k \text{ even} \end{cases}$$

4. $\exists\, N \in \mathbb{N}_0 : \quad \tilde{w}_N^{x_0}|_{(a,x_0)} \in L^2(a, x_0).$

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Denote be the minimal such number N by $\Delta(V)$.

A class of potentials

Example (The perturbed Bessel potential)

Consider a potential $V \in L^1_{\mathrm{loc}}(0,b)$ of the form

$$V(x) = \frac{l(l+1)}{x^2} + V_0(x)$$

where

- either $l > -1/2, \quad xV_0(x) \in L^1(0,x_0)$,
  or $\quad l = -1/2, \quad (\ln x)xV_0(x) \in L^1(0,x_0)$;
- lpc at b;
- the minimal operator is bounded from below (w.l.o.g. positive).

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- lpc at b;
- the minimal operator is bounded from below (w.l.o.g. positive).

Then $V \in \mathbb{K}$ with $\Delta(V) = \left\lfloor l + \frac{3}{2} \right\rfloor$.

A class of potentials

Example (Power asymptotics)

Assume $V = \frac{\phi''}{\phi}$ where $\phi > 0$ and

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Then $V \in \mathbb{K}$ with $\Delta(V) = \left\lfloor \frac{\alpha+1}{2} \right\rfloor$.

Each perturbed Bessel potential is of this kind, but there are also others. For example oscillatory potentials V with ϕ like

$$\phi(x) := x^{\alpha}\big(2 + \sin\frac{1}{x}\big), \quad x > 0.$$

Direct and Inverse Spectral Theorems

Let $V \in \mathbb{K}$ and keep $x_0 \in (a,b)$ fixed.

Theorem (Regularized Boundary Values)

For each $\lambda \in \mathbb{C}$ and each solution u at λ

- the boundary value exists:

$$\operatorname{rbv}^S_{\lambda,1} u := \lim_{x \searrow a} \big(\phi(x)u'(x) - \phi'(x)u(x)\big)$$

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Theorem (Regularized Boundary Values)

For each $\lambda \in \mathbb{C}$ and each solution u at λ

- the regularized boundary value exists:

$$\operatorname{rbv}^S_{\lambda,2} u := \lim_{x \searrow a} \Bigg[\sum_{k=0}^{\lfloor \frac{\Delta(V)-1}{2} \rfloor} \lambda^k \big(\tilde{w}^{x_0}_{2k+1}(x) u'(x) - (\tilde{w}^{x_0}_{2k+1})'(x) u(x) \big) + \\ + \begin{cases} \lambda^{\frac{\Delta(V)}{2}} \tilde{w}^{x_0}_{\Delta(V)}(x) u(x) & \text{if } \Delta(V) \text{ is even} \\ 0 & \text{if } \Delta(V) \text{ is odd} \end{cases} + \\ + \big(\operatorname{rbv}^S_{\lambda,1} u \big) \Bigg(\sum_{k=\lfloor \frac{\Delta(V)+1}{2} \rfloor}^{\Delta(V)-1} \sum_{l=0}^{2k-\Delta(V)} (-1)^l \lambda^k \tilde{w}^{x_0}_l(x) \tilde{w}^{x_0}_{2k-l+1}(x) \Bigg) \Bigg]$$

Direct and Inverse Spectral Theorems

Theorem (Singular Titchmarsh-Weyl Coefficient)

Let $\theta(\cdot;\lambda)$ *and* $\varphi(\cdot;\lambda)$ *be the solutions with*

$$\mathrm{rbv}_\lambda^S\,\theta(\cdot;\lambda) = \begin{pmatrix}1\\0\end{pmatrix}, \quad \mathrm{rbv}_\lambda^S\,\varphi(\cdot;\lambda) = \begin{pmatrix}0\\1\end{pmatrix}.$$

Then

- $m_V(\lambda) := \lim\limits_{x\nearrow b} \dfrac{\theta(x;\lambda)}{\varphi(x;\lambda)}, \quad \lambda \in \mathbb{C}\setminus[0,\infty),$ *exists locally uniformly;*
- $m_V \in \mathcal{N}_\kappa^{(\infty)}$ *with* $\kappa := \lfloor \frac{\Delta(V)}{2} \rfloor$*;*
- $\theta(\cdot;\lambda) - m_V(\lambda)\varphi(\cdot;\lambda) \in L^2(x_0,b)$.

We call m_V *a singular Titchmarsh-Weyl coefficient of* V*.*

Direct and Inverse Spectral Theorems

Theorem (Spectral Measure)

There exists a unique positive Borel measure μ_V with

$$\mu_V\left([s_1, s_2]\right) = \frac{1}{\pi} \lim_{\varepsilon \searrow 0} \lim_{\delta \searrow 0} \int_{s_1-\varepsilon}^{s_2+\varepsilon} \operatorname{Im} m_V(t + i\delta)\, dt, \qquad -\infty < s_1 < s_2 < \infty.$$

This measure satisfies

$$\mu_V((-\infty, 0)) = 0, \quad \int_{[0,\infty)} \frac{d\mu_V(t)}{(1+t)^{n+1}} < \infty$$

with $n := \Delta(V)$ and no smaller $n \in \mathbb{N}_0$.

Direct and Inverse Spectral Theorems

Theorem (Fourier Transform)

Let $V \in \mathbb{K}$ *be given, and let* $\varphi(\cdot\,;\lambda)$ *be the unique solution with* $\mathrm{rbv}_\lambda^S\,\varphi(\cdot\,;z) = (0,\,1)^T$. *Let* A_V *be the Friedrichs extension of the minimal operator.*

- $$(\Theta_V f)(t) := \int_a^b \varphi(x;t)^T f(x)\,dx, \quad t \in \mathbb{R},$$
  $$f \in L^2(a,b),\ \ \sup(\operatorname{supp} f) < b,$$

  extends to an isomorphism of $L^2(a,b)$ *onto* $L^2(\mu_V)$ *with*

  $$\Theta_V \circ A_V = M_{\mu_V} \circ \Theta_V.$$

- $$(\Theta_V^{-1} g)(x) = \int_0^\infty g(t)\varphi(x;t)\,d\mu_V(t), \quad x \in (a,b),$$
  $$g \in L^2(\mu_V),\ \ \operatorname{supp} g \subseteq (0,\infty)\ \textit{compact}.$$

Direct and Inverse Spectral Theorems

As for canonical systems x_0-dependence can be controlled.

Definition

For $V \in \mathbb{K}$ we denote by $[m]_V$ the equivalence class of a singular Titchmarsh-Weyl coefficient modulo "$m_1 - m_2 \in \mathbb{R}[z]$".

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The function ϕ in the representation of V is not unique; we can multiply ϕ with any positive constant c. Then m_V and μ_V are multiplied with c^{-2}.

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The function ϕ in the representation of V is not unique; we can multiply ϕ with any positive constant c. Then m_V and μ_V are multiplied with c^{-2}.

Theorem (Global Uniqueness)

Let $V_1, V_2 \in \mathbb{K}$ be given (defined on $(0, b_i)$, $i = 1, 2$). The following are equivalent:

- $b_1 = b_2$ *and* $V_1(x) = V_2(x)$, $x \in (0, b_i)$ *a.e.*
- $\exists\ c > 0 : \quad [m]_{V_1} = c[m]_{V_2}$.
- $\exists\ c > 0 : \quad \mu_{V_1} = c\mu_{V_2}$.

Direct and Inverse Spectral Theorems

Theorem (Local Uniqueness)

Let $V_1, V_2 \in \mathbb{H}$ be given (defined on $(0, b_i)$, $i = 1, 2$). Then, for each $\tau > 0$, the following are equivalent.

- $V_1(x) = V_2(x)$, $x \in (0, \tau)$ *a.e.*
- $\exists\, m_{V_1}, m_{V_2}, \beta \in (0, 2\pi), c > 0 \quad \forall\, \varepsilon > 0 :$

$$m_{V_1}(re^{i\beta}) - cm_{V_2}(re^{i\beta}) = \mathrm{O}\big(e^{(-2\tau+\varepsilon)\sqrt{r}\sin\frac{\beta}{2}}\big), \; r \to \infty.$$

- $\exists\, m_{V_1}, m_{V_2}, k \geq 0, c > 0 \quad \forall\, \delta \in (0, \pi) : \;\big($*in* $|\arg\lambda - \pi| \leq \delta\big)$

$$m_{V_1}(\lambda) - cm_{V_2}(\lambda) = \mathrm{O}\big(|\lambda|^k e^{-2\tau\,\mathrm{Im}\sqrt{\lambda}}\big), \; |\lambda| \to \infty,$$

where $\sqrt{\lambda}$ is the branch with $\mathrm{Im}\sqrt{\lambda} > 0$.

The End