Direct and Inverse Spectral Problems for 2-dimensional Hamiltonian Systems

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We are going to survey the spectral theory of 2-dimensional Hamiltonian Systems without a potential term.

Many results are classical, dating back to the 1950's or 60's (for particular cases, even much earlier). Others are more recent, and taken from work of various authors.

We will also see a few results shown by myself (with different coauthors).

These slides are available from my website

http://asc.tuwien.ac.at/index.php?id=woracek

Not all what is written on these slides is *strictly* correct.

We will occasionally neglect some
technical difficulties and/or exeptional cases.

Each such instance will be clearly marked.

We consider 2×2-Hamiltonian systems without potential:

$$y'(t) = zJH(t)y(t), \quad t \in (s_-, s_+).$$

Here the *Hamiltonian* H shall be subject to

- $H(t) : (s_-, s_+) \to \mathbb{R}^{2\times2}$,
- $H(t) \geq 0$, $t \in (s_-, s_+)$,
- $H \in L^1_{\mathrm{loc}}(s_-, s_+)$,
- H does not vanish identically on any set of positive measure,
- $z \in \mathbb{C}$ a parameter,
- $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

This equation is the eigenvalue equation of a differential operator.

We investigate the spectral theory of its selfadjoint realizations.

Direct Problems: Given a Hamiltonian H, find information about spectral data of selfadjoint realizations.

Inverse Problems:

- Existence Theorems: Given some spectral data, does there exist a Hamiltonian H which leads to this data.
- Uniqueness Theorems: Which spectral data obtained from some Hamiltonian determine this Hamiltonian uniquely.

Outline

The Operator Model

Let H be a Hamiltonian on (s_-, s_+), let $(a,b) \subseteq (s_-, s_+)$ and $\phi \in \mathbb{R}$. Then (a,b) is *H-indivisible of type ϕ*, if

$$H(t) = h(t) \cdot \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} \big(\cos\phi, \sin\phi\big), \quad t \in (a,b),$$

with some scalar function $h \in L^1_{\mathrm{loc}}(a,b)$.

Definition (The model space $L^2(H)$)

The *model space* $L^2(H)$ is the space of all $f : (s_-, s_+) \to \mathbb{C}^2$ with

- $\|f\|_H^2 := \int_{s_-}^{s_+} f(t)^* H(t) f(t)\, dt < \infty.$
- If $(a,b) \subseteq (s_-, s_+)$ is indivisible of type ϕ, then

$$\big(\cos\phi, \sin\phi\big) f(t) = \text{ constant on } (a,b).$$

The Operator Model

In the definition of $L^2(H)$, we tacitly understand that two functions f, g with

$$H(t)f(t) = H(t)g(t), \quad t \in (s_-, s_+) \text{ a.e.},$$

are identified.

If endowed with the scalar product

$$(f, g)_H = \int_{s_-}^{s_+} g(t)^* H(t) f(t)\, dt, \quad f, g \in L^2(H),$$

the space $L^2(H)$ becomes a Hilbert space.

The Operator Model

Here we suppress some technical terms.

Definition (The maximal operator $T_{\max}(H)$)

The (graph of the) *maximal operator* $T_{\max}(H)$ is

$$T_{\max}(H) = \Big\{(f;g) \in L^2(H) \times L^2(H) : \\ f \text{ is locally absolutely continuous and } f' = JHg\Big\}$$

The operator $T_{\max}(H)$ is closed.

The Operator Model

Definition (The minimal operator $T_{\min}(H)$)

The *minimal operator* $T_{\min}(H)$ is $T_{\min}(H) = T_{\max}(H)^*$.

The operator $T_{\min}(H)$ is closed and symmetric. It is either selfadjoint, or completely nonselfadjoint.

Limit Circle vs. Limit Point Case

The spectral theory of $T_{\max}(H)$ depends on the growth of H towards the endpoints s_- and s_+.

- H is in *limit circle case at* s_-, if $(x_0 \in (s_-, s_+))$

$$\int_{s_-}^{x_0} \operatorname{tr} H(t)\, dt < \infty \qquad \Big(\Leftrightarrow H \in L^1_{\mathrm{loc}}([s_-, s_+)) \Big).$$

- H is in *limit point case at* s_-, if $(x_0 \in (s_-, s_+))$

$$\int_{s_-}^{x_0} \operatorname{tr} H(t)\, dt = \infty.$$

Similar: *limit circle case at* s_+ and *limit point case at* s_+.

The Three Fundamental Cases

We know the operator $T_{\min}(H)$ is closed and symmetric.

Its deficiency indices are always finite and equal.

- Case I, lc ↔ lc: $(2,2)$.
- Case II, lc ↔ lp or lp ↔ lc: $(1,1)$.
- Case III, lp ↔ lp: $(0,0)$.

In Case III, $T_{\min}(H) = T_{\max}(H)$. Hence, $T_{\min}(H)$ is selfadjoint and is the only selfadjoint realization.

In the Cases I and II, there are many different selfadjoint realizations.

Boundary Values

If s_- is in limit circle case, each $f = (f_1, f_2)^T \in \operatorname{dom} T_{\max}(H)$ has a continuous extension to s_-. Similar for s_+.

Selfadjoint realizations can be described with boundary conditions.

- Assume Case lc $\leftrightarrow$ lp. Then (for example)

$$A_D := \big\{(f; g) \in T_{\max}(H) : f_1(s_-) = 0\big\}$$

  is a selfadjoint restriction of $T_{\max}(H)$.

- Assume Case lc $\leftrightarrow$ lc. Then (for example) for each $\tau \in \mathbb{R} \cup \{\infty\}$

$$A_{D,\tau} = \big\{(f; g) \in T_{\max}(H) : \; f_1(s_-) = 0, \tau f_1(s_+) + f_2(s_+) = 0\big\}$$

  is a selfadjoint restrictions of $T_{\max}(H)$.

Reparameterizations

Two Hamiltonians H_1 on H_2 defined on (s^1_-, s^1_+) and (s^2_-, s^2_+), respectively, are *reparameterizations of each other*, if there exists

$$\phi : (s^2_-, s^2_+) \to (s^1_-, s^1_+)$$

such that

- ϕ is bijective and monotonically increasing,
- ϕ and ϕ^{-1} are both absolutely continuous,
- $H_2(t) = H_1(\phi(t)) \cdot \phi'(t)$ for $t \in (s^2_-, s^2_+)$.

If H_1 and H_2 are reparameterizations of each other, their operator models are unitarily equivalent.

Examples: 1. The Schrödinger Equation

Consider the equation $(0 < T < \infty)$

$$-y''(t) + V(t)y(t) = zy(t), \quad t \in [0, T],$$

where the *potential* $V(t)$ belongs to $L^1([0, T])$.

Let y_1 and y_2 be the solutions of $-y''(t) + V(t)y(t) = 0$ with

$$y_1(0) = 0, y_1'(0) = 1, \quad y_2(0) = 1, y_2'(0) = 0,$$

and define

$$H(t) := \begin{pmatrix} y_1(t)^2 & y_1(t)y_2(t) \\ y_1(t)y_2(t) & y_2(t)^2 \end{pmatrix}, \quad t \in [0, T].$$

Then H is a Hamiltonian which is lc $\leftrightarrow$ lc.

Examples: 1. The Schrödinger Equation

A function $y(t)$ solves the Schrödinger equation with potential $V(t)$ and parameter z, if and only if the function

$$u(t) = \begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix} \cdot \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}$$

solves the Hamiltonian system with Hamiltonian $H(t)$ and parameter z.

The operator models of the Schrödinger equation and of the Hamiltonian system are unitarily equivalent.

Examples: 2. The String Equation

Here we ignore some technicalities.

Let $L > 0$, and μ be a positive Borel measure on $\mathbb{R}$ with $\operatorname{supp}\mu \subseteq [0, L]$ and $\mu(\{L\}) = 0$. Consider the integral equation boundary value problem with complex parameter z:

$$y'(t) + \int_{[0,t]} zy(u)d\mu(u) = 0, \quad y'(0-) = 0.$$

Set $m(t) := \mu((-\infty, t))$, and

$$\hat{m}(x) = \begin{cases} \inf\left\{t \geq 0 : \ x \leq m(t)\right\}, & x \in [0, m(L)] \\ L &, \quad x > m(L) \end{cases}$$

Examples: 2. The String Equation

Here we ignore some technicalities.

Define

$$H(x) := \begin{pmatrix} \hat{m}(x)^2 & \hat{m}(x) \\ \hat{m}(x) & 1 \end{pmatrix}, \quad x \in (0, \infty).$$

Then H is a Hamiltonian which is lc ↔ lp.

The operator models of the string equation and of the Hamiltonian system with this Hamiltonian are unitarily equivalent.

Examples: 3. The Hamburger Moment Problem

Let $(s_n)_{n\geq 0}$ be a sequence of real numbers. Is this sequence the sequence of power moments of some positive Borel measure on the real line ? That is, does there exist a positive Borel measure μ on $\mathbb{R}$ with

$$s_n = \int_{\mathbb{R}} t^n \, d\mu(t), \quad n \geq 0 \quad ?$$

The answer is *yes*, if and only if

$$\det \left[(s_{i+j})_{i,j=0}^{N} \right] \geq 0, \quad N \geq 0.$$

Examples: 3. The Hamburger Moment Problem

Assume that $D_N = \det[(s_{i+j})_{i,j=0}^N] \geq 0$, $N \geq 0$.

Either

- the solution of the Hamburger moment problem, i.e., the measure having $(s_n)_{n\geq 0}$ as its moment sequence, is unique,

or

- there exist infinitely many such measures.

If $D_N = 0$ for some $N \geq 0$, then the solution is unique and is a discrete measure with finitely many pointmasses.

Examples: 3. The Hamburger Moment Problem

Assume that $D_N = \det[(s_{i+j})_{i,j=0}^N] > 0$, $N \geq 0$.

Set

$$E_N = \det\left[(s_{i+j+1})_{i,j=0}^N\right], \quad C_N = \det\left[(s_{i+j-1})_{i,j=0}^N\right] \ (s_{-1} = 0),$$

$$l_0 = 1, \quad l_N = (E_N^2 + C_N^2)\big(D_{N-1} D_N\big)^{-1}, \ N \geq 1,$$

$$t_0 = 0, \quad t_N = \sum_{n=0}^{N-1} l_n, \ n \geq 1, \qquad T = \lim_{N\to\infty} t_n,$$

$$\theta_0 = \frac{\pi}{2}, \quad \theta_N = \begin{cases} \arctan\left(-\frac{E_N}{C_N}\right), & C_N \neq 0 \\ \frac{\pi}{2}, & C_N = 0 \end{cases}$$

Examples: 3. The Hamburger Moment Problem

Define

$$H(t) = \begin{pmatrix} \cos\theta_N \\ \sin\theta_N \end{pmatrix} \big(\cos\theta_N, \sin\theta_N\big),\ t \in [t_N, t_{N-1}), \quad N \geq 0.$$

Then H is a Hamiltonian which is

- lc $\leftrightarrow$ lp, if the solution is unique,
- lc $\leftrightarrow$ lc, if the solution is not unique.

The set of solutions of the Hamburger moment problem coincides with the set of all *spectral measures* of the Hamiltonian H.

Case lc ↔ lc. Spectral Measures

Denote by $W(t,z) = (w_{ij}(t,z))_{i,j=1}^2$ the solution of

$$\frac{d}{dt} W(t,z) J = z W(t,z) H(t), \quad t \in [s_-, s_+], \qquad W(s_-, z) = I.$$

Then, for each $\tau \in \mathbb{R} \cup \{\infty\}$, the function

$$q_{H,\tau}(z) = \frac{w_{11}(s_+, z)\tau + w_{12}(s_+, z)}{w_{21}(s_+, z)\tau + w_{22}(s_+, z)}$$

belongs to the *Nevanlinna class*, that is,

- q_H is analytic in $\mathbb{C} \setminus \mathbb{R}$ and $q_H(\overline{z}) = \overline{q_H(z)}$,
- $\operatorname{Im} q_H(z) \geq 0$ for $\operatorname{Im} z > 0$.

Case lc ↔ lc. Spectral Measures

We can represent $q_{H,\tau}$ as (*Herglotz integral representation*)

$$q_{H,\tau}(z) = a_{H,\tau} + b_{H,\tau} z + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) d\mu_{H,\tau}(t), \quad z \in \mathbb{C} \setminus \mathbb{R},$$

with

- $a_{H,\tau} \in \mathbb{R}$, $b_{H,\tau} \geq 0$,
- μ_H positive Borel measure with $\int_{\mathbb{R}} \frac{d\mu_{H,\tau}(t)}{1+t^2} < \infty$.

The measure $\mu_{H,\tau}$ can be computed from $q_{H,\tau}$ by means of the Stieltjes Inversion Formula, and the constant $b_{H,\tau}$ from the behaviour of $q_{H,\tau}$ towards $i\infty$.

Each measure $\mu_{H,\tau}$ obtained in this way is called a *spectral measure of H*.

Case lc ↔ lc. Fourier Transforms

Here we ignore the exceptional case $b_{H,\tau} > 0$ and technicalities.

Theorem (Integral transforms, lc ↔ lp)

Let $\tau \in \mathbb{R} \cup \{\infty\}$. A unitary map $U_\tau : L^2(H) \to L^2(\mu_{H,\tau})$ is defined by

$$(U_\tau f)(x) = \int_{s_-}^{s_+} \big(w_{21}(t,x), w_{22}(t,x)\big) H(t) f(t) dt.$$

Its inverse $U_\tau^{-1} : L^2(\mu_{H,\tau}) \to L^2(H)$ is given as

$$(U_\tau^{-1} F)(t) = \int_{-\infty}^{\infty} \begin{pmatrix} w_{21}(t,x) \\ w_{22}(t,x) \end{pmatrix} F(x) d\mu_{H,\tau}(x).$$

Case lc ↔ lc. Fourier Transforms

Here we ignore the exceptional case $b_{H,\tau} > 0$ and technicalities.

Theorem (Unitary equivalence)

Let $\tau \in \mathbb{R} \cup \{\infty\}$. The selfadjoint realization

$$A_{D,\tau} = \big\{(f;g) \in T_{\max}(H) : \; f_1(s_-) = 0, \tau f_1(s_+) + f_2(s_+) = 0\big\}$$

is unitarily equivalent to the the multiplication operator M_x in $L^2(\mu_{H,\tau})$ via U_τ, that is,

$$U_\tau \circ A_{D,\tau} = M_x \circ U_\tau.$$

Case lc ↔ lc. A Direct Theorem

Theorem (Direct Spectral Theorem)

- *Each selfadjoint realization has compact resolvents.*
- *The spectrum of each selfadjoint realization defined by separated boundary conditions is simple.*
- *Let* (λ_n^+) *and* (λ_n^-) *denote the sequences of positive and negative, respectively, eigenvalues of a selfadjoint realization arranged according to increasing modulus. Then*

$$\lim \frac{n}{\lambda_n^+} = \lim \frac{n}{\lambda_n^-} = \frac{1}{\pi} \int_{s_-}^{s_+} \sqrt{\det H(t)}\, dt.$$

Case lc $\leftrightarrow$ lc. A Uniqueness Theorem

A Hamiltonian H is not uniquely determined by the spectrum of one of its selfadjoint realizations. It may happen that H_1 and H_2 are different (not reparameterizations of each other), and still

$$\sigma(A^1_{D,0}) = \sigma(A^2_{D,0}).$$

Theorem (Inverse Theorem / Uniqueness)

Assume that two Hamiltonians H_1 and H_2 satisfy

$$\sigma(A^1_{D,0}) = \sigma(A^2_{D,0}) \quad \textit{and} \quad \sigma(A^1_{D,\infty}) = \sigma(A^2_{D,\infty}),$$

then H_1 and H_2 are equal (up to a reparameterization).

Case lc ↔ lc. Existence Theorems

Theorem (Characterization of spectra)

Let (λ_n) be a sequence of pairwise different real numbers. Then there exists a Hamiltonian H in lc ↔ lc *with $\{\lambda_n\} = \sigma(A_{D,0})$, if and only if all λ_n are nonzero, and*

- *the limits $\lim \frac{n}{\lambda_n^+}$ and $\lim \frac{n}{\lambda_n^-}$ exist in $[0, \infty)$ and are equal, where (λ_n^+) and (λ_n^-) denote the sequences of positive and negative elements of (λ_n),*
- $\lim_{R\to\infty} \sum_{|\lambda_n|\le R} \frac{1}{\lambda_n}$ *exists in $\mathbb{R}$,*
- $\sum_n \frac{1}{|\lambda_n|^2 |A'(\lambda_n)|} < \infty$ *where* $A(z) = \lim_{R\to\infty} \prod_{|\lambda_n|\le R} \Big(1 - \frac{z}{\lambda_n}\Big)$.

Case lc ↔ lc. Existence Theorems

Theorem (Characterization of pairs of spectra)

Let (λ_n) and (μ_n) be two sequences of pairwise different real numbers. Then there exists a Hamiltonian H in lc ↔ lc *with $\{\lambda_n\} = \sigma(A_{D,0})$ and $\{\mu_n\} = \sigma(A_{D,\infty})$, if and only if all λ_n are nonzero, the point zero is among the μ_n's, and*

- *the sequences (λ_n) and (μ_n) interlace,*
- $\lim \frac{n}{\lambda_n^+} = \lim \frac{n}{\lambda_n^-} \in [0, \infty)$ *and* $\lim_{R\to\infty} \sum_{|\lambda_n|\leq R} \frac{1}{\lambda_n}$ *exists in* $\mathbb{R}$*,*
- $\sum_n \frac{1}{|\lambda_n|^2 |A'(\lambda_n) B(\lambda_n)|} < \infty$ *where*

$$A(z) = \lim_{R\to\infty} \prod_{|\lambda_n|\leq R} \left(1 - \frac{z}{\lambda_n}\right), \quad B(z) = z \lim_{R\to\infty} \prod_{0<|\mu_n|\leq R} \left(1 - \frac{z}{\mu_n}\right).$$

Case lc ↔ lp. The Weyl Construction

The cases lc ↔ lp and lp ↔ lc are fully analogous. We confine attention to lc ↔ lp.

Theorem (Weyl coefficient. Existence)

Denote by $W(t,z) = (w_{ij}(t,z))_{i,j=1}^2$ *the solution of*

$$\frac{d}{dt}W(t,z)J = zW(t,z)H(t), \quad t \in [s_-, s_+), \qquad W(s_-,z) = I.$$

Then, for each $\tau \in \mathbb{R} \cup \{\infty\}$ *the limit*

$$q_H(z) = \lim_{t \nearrow s_+} \frac{w_{11}(t,z)\tau + w_{12}(t,z)}{w_{21}(t,z)\tau + w_{22}(t,z)}$$

exists locally uniformly on $\mathbb{C} \setminus \mathbb{R}$. *It does not depend on* τ.

Case lc ↔ lp. The Weyl Construction

Theorem (Weyl coefficient. Properties)

The function q_H belongs to the Nevanlinna class, that is,

- q_H *is analytic in* $\mathbb{C} \setminus \mathbb{R}$ *and* $q_H(\overline{z}) = \overline{q_H(z)}$,
- $\operatorname{Im} q_H(z) \geq 0$ *for* $\operatorname{Im} z > 0$.

We can therefore represent q_H as

$$q_H(z) = a_H + b_H z + \int_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{t}{1+t^2}\Big) d\mu_H(t), \quad z \in \mathbb{C} \setminus \mathbb{R},$$

with

- $a_H \in \mathbb{R}$, $b_H \geq 0$,
- μ_H positive Borel measure with $\int_{\mathbb{R}} \frac{d\mu_H(t)}{1+t^2} < \infty$.

The measure μ_H obtained in this way is called the *spectral measure of* H.

Case lc ↔ lp. A Direct Theorem

Here we ignore the exceptional case $b_H > 0$ and technicalities.

Theorem (Direct Spectral Theorem)

A unitary map $U : L^2(H) \to L^2(\mu_H)$ is defined by

$$(Uf)(x) = \int_{s_-}^{s_+} \big(w_{21}(t,x), w_{22}(t,x)\big) H(t) f(t) dt.$$

It intertwines A_D and the multiplication operator M_x in $L^2(\mu_H)$:

$$U \circ A_D = M_x \circ U.$$

Its inverse $U^{-1} : L^2(\mu_H) \to L^2(H)$ is given as

$$(U^{-1}F)(t) = \int_{-\infty}^{\infty} \begin{pmatrix} w_{21}(t,x) \\ w_{22}(t,x) \end{pmatrix} F(x) d\mu_H(x).$$

Case lc $\leftrightarrow$ lp. The Inverse Theorem

The following Existence and Uniqueness Theorem is *the* major result in the spectral theory of Hamiltonian systems.

Theorem (Inverse Spectral Theorem)

Let a function q in the Nevanlinna class be given. Equivalently, let $a \in \mathbb{R}$, $b \geq 0$, and a positive Borel measure μ with $\int_{\mathbb{R}} \frac{d\mu(t)}{1+t^2}$ be given.

Then there exists a (up to reparameterization) unique Hamiltonian in lc $\leftrightarrow$ lp *whose Weyl coefficient equals q.*

Case lc ↔ lp. A Local Uniqueness Theorem

Theorem (Local uniqueness)

Let H_1 and H_2 be Hamiltonians defined on (s_-^1, s_+^1) and (s_-^2, s_+^2), respectively. For $a > 0$ set

$$s_a^j = \sup\Big\{t \in [s_-^j, s_+^j) : \int_{s_-^j}^t \sqrt{\det H_j(x)}\,dx < a\Big\}, \quad j = 1, 2\,.$$

Then the following are equivalent.

- $H_1\big|_{(s_-^1, s_a^1)}$ *and* $H_2\big|_{(s_-^2, s_a^2)}$ *are reparameterizations of each other.*
- $q_{H_1}(z) - q_{H_2}(z) = O\big((\operatorname{Im} z)^3 e^{-2a \operatorname{Im} z}\big), \quad z \hat{\to} i\infty.$

Case lc ↔ lp. Semibounded Spectrum

Theorem (Consequence of semibounded spectrum)

Let H be given with $\inf \operatorname{supp} \mu_H > -\infty$. Then there exist unique $L \in (0, \infty]$ and $\nu : [0, L) \to [0, +\infty)$, such that

- *ν is nondecreasing, right-continuous, and normalized by $\nu(0) \in [0, \pi)$ and $\nu(t) - \nu(t-) < \pi$,*
- *H is (a reparameterization of)*

$$H_\nu(x) = \begin{cases} \begin{pmatrix} [\cot \nu(t)]^2 & -\cot \nu(t) \\ -\cot \nu(t) & 1 \end{pmatrix} & \text{if } \nu(t) \notin \pi\mathbb{Z} \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \text{if } \nu(t) \in \pi\mathbb{Z} \end{cases}$$

Case lc ↔ lp. Semibounded Spectrum

The question converse to the above theorem is: Given ν with the properties stated in the theorem. Is the spectral measure of H_ν semibounded from below ?

The answer is *unknown*.

Case lc ↔ lp. Semibounded Spectrum

Theorem (The case of finite negativity)

The following are equivalent:

- H *is (up to a reparameterization) equal to* H_ν *with* ν *being (in addition) bounded.*
- $(-\infty, 0) \cap \operatorname{supp} \mu_H$ *is a finite set.*

Case lc $\leftrightarrow$ lp. Discrete Spectrum

It is a question of interest for which Hamiltonians their selfadjoint realizations have discrete spectrum (equivalently, have compact resolvents).

The answer is *unknown*.

What is easy to see is that for each discrete subset M of $\mathbb{R}$, there exist (infinitely many) Hamiltonians H which are lc $\leftrightarrow$ lp and such that

$$\sigma(A_D) = M.$$

Case lc ↔ lp. Discrete Spectrum

Write $H(t) = (h_{ij}(t))_{i,j=1}^2$, and set $B(t) = \int_{s_-}^t h_{12}(x)dx$.

For $r > 0$ set

$$M_1(r) = \Bigg\{ \lambda \in \mathbb{R} \setminus \{0\} : \limsup_{t \nearrow s_+} \left(\int_{s_-}^{t} h_{22}(x) e^{-2\lambda B(x)} dx \int_{t}^{s_+} h_{11}(x) e^{2\lambda B(x)} dx \right) \le \frac{r}{\lambda^2} \Bigg\},$$

$$M_2(r) = \Bigg\{ \lambda \in \mathbb{R} \setminus \{0\} : \limsup_{t \nearrow s_+} \left(\int_{s_-}^{t} h_{11}(x) e^{2\lambda B(x)} dx \int_{t}^{s_+} h_{22}(x) e^{-2\lambda B(x)} dx \right) \le \frac{r}{\lambda^2} \Bigg\}.$$

Case lc ↔ lp. Discrete Spectrum

In the literature this theorem is only stated. We have not seen a proof.

Theorem (Discreteness of spectrum)

If μ_H is discrete, then

$$\mathbb{R} \setminus \{0\} \subseteq M_1(1) \cup M_2(1).$$

If there exist sequences $\lambda_i \to +\infty$ and $\mu_i \to -\infty$ with

$$\{\lambda_i : i \in \mathbb{N}\} \cup \{\mu_i : i \in \mathbb{N}\} \subseteq \bigcup_{r<\frac{1}{4}} \big(M_1(r) \cup M_2(r)\big),$$

then μ_H is discrete.

Case lc ↔ lp. Discrete Spectrum

Theorem (The diagonal case)

Assume that H is diagonal, that is,

$$H(x) = \begin{pmatrix} h_1(x) & 0 \\ 0 & h_2(x) \end{pmatrix}, \quad x \in (s_-, s_+) \text{ a.e.}$$

Then μ_H is discrete, if and only if either

$$\int_{s_-}^{s_+} h_1(x)\,dx < \infty \text{ and } \lim_{x \nearrow s_+} \Big(\int_x^{s_+} h_1(t)\,dt \cdot \int_{s_-}^{x} h_2(t)\,dt \Big) = 0,$$

or

$$\int_{s_-}^{s_+} h_2(x)\,dx < \infty \text{ and } \lim_{x \nearrow s_+} \Big(\int_x^{s_+} h_2(t)\,dt \cdot \int_{s_-}^{x} h_1(t)\,dt \Big) = 0.$$

Case lc ↔ lp. Hilbert-Schmidt Property

Contrasting discreteness of the spectrum, the property that the spectrum is discrete with square summable eigenvalues can be characterized in general.

Theorem (Characterization of Hilbert-Schmidt class)

The resolvents of selfadjoint realizations belong to the Hilbert-Schmidt class, if and only if there exists an angle $\phi \in [0, \pi)$ such that (here $\xi_\alpha = \binom{\cos\alpha}{\sin\alpha}$)

- $\displaystyle\int_{s_-}^{s_+} \xi_\phi^T H(t)\xi_\phi \, dt < \infty$,
- $\displaystyle\int_{s_-}^{s_+} \Big(\int_{s_-}^{t} \xi_{\phi+\frac{\pi}{2}}^T H(u)\xi_{\phi+\frac{\pi}{2}} du \Big) \xi_\phi^T H(t)\xi_\phi \, dt < \infty$.

Case lp ↔ lp. Vector-Valued L^2-Spaces

Let $\Omega = (\Omega_{ij})_{i,j=1}^n$ be a positive $n \times n$-matrix valued Borel measure on $\mathbb{R}$, that is, a map of Borel sets to positive semidefinite $n \times n$-matrices which is σ-additive and satisfies $\Omega(\emptyset) = 0$.

Set $\rho(\Delta) = \operatorname{tr} \Omega(\Delta)$, then ρ is a finite positive Borel measure on $\mathbb{R}$ and each entry Ω_{ij} is absolutely continuous w.r.t. ρ. The *symmetric derivative* of Ω w.r.t. ρ is

$$\frac{d\Omega}{d\rho}(x) = \lim_{\varepsilon \downarrow 0} \frac{\Omega\big((x-\varepsilon, x+\varepsilon)\big)}{\rho\big((x-\varepsilon, x+\varepsilon)\big)}, \quad x \in \mathbb{R} \ \rho\text{-a.e.}$$

$L^2(\Omega)$ is the space of all $f : \mathbb{R} \to \mathbb{C}^n$ which are ρ-measurable and $\big(f(x), \frac{d\Omega}{d\rho}(x) f(x)\big)_{\mathbb{C}^n} \in L^1(\rho)$. It is endowed with

$$(f,g)_{L^2(\Omega)} = \int_{\mathbb{R}} \big(f(x), \frac{d\Omega}{d\rho}(x) g(x)\big)_{\mathbb{C}^n} \, d\rho(x).$$

Case lp ↔ lp. Matrix Weyl Function

Choose a point $s_0 \in (s_-, s_+)$. Then

- $H_+ = H|_{(s_0,s_+)}$ is a Hamiltonian lc ↔ lp.
- $H_- = H|_{(s_-,s_0)}$ is a Hamiltonian in lp ↔ lc,

The 2×2-matrix valued function

$$Q_H(z) = \frac{1}{q_{H_+}(z) + q_{H_-}(z)} \begin{pmatrix} q_{H_+}(z)q_{H_-}(z) & -q_{H_+}(z) \\ -q_{H_+}(z) & -1 \end{pmatrix}$$

belongs to the 2×2-*Nevanlinna class*, that is,

- Q_H is analytic in $\mathbb{C} \setminus \mathbb{R}$ and $Q_H(\overline{z}) = Q_H(z)^*$,
- $\operatorname{Im} q_H(z) = \frac{1}{2i}(Q_H(z) - Q_H(z)^*)$ is positive semidefinite for each z with $\operatorname{Im} z > 0$.

Case lp ↔ lp. Matrix Weyl Function

We can represent Q_H as $(z \in \mathbb{C} \setminus \mathbb{R})$

$$Q_H(z) = a_H + b_H z + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) \cdot (1+t^2) d\Omega_H(t),$$

where

- $a_H, b_H \in \mathbb{C}^{2\times 2}$, with $a_H = a_H^*$ and b_H positive semidefinite,
- Ω_H positive 2×2-matrix valued Borel measure.

Case lp ↔ lp. The Titchmarsh-Kodaira formula

Here we ignore the exceptional case $b_H \neq 0$.

Theorem (Unitary equivalence)

The selfadjoint operator $T_{min}(H)$ is unitarily equivalent to the multiplication operator in the space $L^2(\Omega_H)$. That is, there exists a unitary operator $U : L^2(H) \to L^2(\Omega_H)$ which intertwines $T_{\min}(H)$ and M_x:

$$U \circ T_{\min}(H) = M_x \circ U.$$

The action of U can again be described as an integral transform.

Corollary

The spectral multiplicity of $T_{\min}(H)$ cannot exceed 2.

Case lp ↔ lp. Local Spectral Multiplicity

Let E be the projection valued spectral measure of $T_{\min}(H)$, and let σ_1, σ_2 be scalar positive Borel measures with $\sigma_2 \ll \sigma_1 \sim E$ such that $T_{\min}(H)$ is unitarily equivalent to the multiplication operator in $L^2(\sigma_1) \oplus L^2(\sigma_2)$.

Set (*layers of spectrum*)

$$Y_l = \Big\{x \in \mathbb{R} : \frac{d\sigma_l}{d\sigma_1}(x) \in (0, \infty]\Big\}, \quad l = 1, 2,$$

and (*spectral multiplicity function*)

$$N_H(x) := \#\big\{l \in \{1, 2\} : x \in Y_l\big\}, \quad x \in \mathbb{R},\ \sigma_1\text{-a.e.}$$

Case lp $\leftrightarrow$ lp. Local Spectral Multiplicity

Theorem

We have $N_H(x) = \operatorname{rank} \frac{d\Omega_H}{d\rho}(x)$ *for* $x \in \mathbb{R}$, *ρ-a.e. (notice* $\rho \sim E$*).*

Denote by λ the Lebesgue measure, and set $\mu := \mu_{H_+} + \mu_{H_-}$.

Decompose

$$E = E_{ac} + E_s \quad \text{with} \quad E_{ac} \ll \lambda,\ E_s \perp \lambda,$$

and further

$$E_s = E_{s,ac} + E_{s,s} \quad \text{with} \quad E_{s,ac} \ll \mu,\ E_{s,s} \perp \mu.$$

Moreover, decompose

$$\mu = \mu_{ac} + \mu_s \quad \text{with} \quad \mu \ll \lambda,\ \mu \perp \lambda.$$

Case lp ↔ lp. Local Spectral Multiplicity

Set ($x \in \mathbb{R}$, μ-a.e.)

$$r(x) = \begin{cases} 2\,, & \frac{d\mu_{H_+}}{d\mu}(x), \frac{d\mu_{H_-}}{d\mu}(x) \in (0,\infty] \\ 1\,, & \text{exactly one of } \frac{d\mu_{H_+}}{d\mu}(x), \frac{d\mu_{H_-}}{d\mu}(x) \text{ is nonzero} \\ 0\,, & \frac{d\mu_{H_+}}{d\mu}(x), \frac{d\mu_{H_-}}{d\mu}(x) = 0 \end{cases}$$

Theorem (Computation of the multiplicity function)

We have

- $E_{ac} \sim \mu_{ac}$ *and* $N_H(x) = r(x)$, E_{ac}*-a.e.*
- $E_{s,ac} \sim \mathbb{1}_{\{r(x)=2\}} d\mu_s$ *and* $N_H(x) = 1$, $E_{s,ac}$*-a.e.*
- $N_H(x) = 1$, $E_{s,s}$*-a.e.*

Case lp ↔ lp. Simple Spectrum

Corollary

The singular spectrum of $T_{\min}(H)$ is always simple.

Theorem (Characterization of simplicity)

The operator $T_{\min}(H)$ has simple spectrum if and only if the set

$$\left\{x \in \mathbb{R} : \lim_{\epsilon \downarrow 0} \operatorname{Im} q_{H_+}(x + i\epsilon) \text{ exists in } (0, \infty)\right\} \cap \left\{x \in \mathbb{R} : \lim_{\epsilon \downarrow 0} \operatorname{Im} q_{H_-}(-x + i\epsilon) \text{ exists in } (0, \infty)\right\}$$

has Lebesgue measure zero.

Case lp $\leftrightarrow$ lp. Simple Spectrum

An explicit sufficient condition for simplicity is:

Theorem

Assume that H_+ has the Hilbert-Schmidt property, i.e., that there exists $\phi \in [0, \pi)$ with

- $\displaystyle\int_{s_0}^{s_+} \xi_\phi^T H(t) \xi_\phi \, dt < \infty,$
- $\displaystyle\int_{s_0}^{s_+} \Big(\int_{s_0}^{t} \xi_{\phi+\frac{\pi}{2}}^T H(u) \xi_{\phi+\frac{\pi}{2}} du \Big) \xi_\phi^T H(t) \xi_\phi \, dt < \infty.$

Then the spectrum of $T_{\min}(H)$ is simple.