Indefinite de Branges’ theory and Differential operators with singular coefficients

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- Review of the positive definite theory
- Two examples of indefinite notions
- The indefinite theory
- Application: Hamiltonian systems with two singular endpoints
- Application: Spectral functions of Krein strings

Many exceptional cases will be omitted and many technical details will be ignored.

If read literally, many statements are incorrect. Statements being affected will be marked in the text.

Review of the Positive Definite Theory

Hamiltonian systems

A Hamiltonian is a function

- $H : (\sigma_0, \sigma_1) \to \mathbb{R}^{2\times 2}$ measurable;
- $H(t) \geq 0$, $H \in L^1_{\text{loc}}((\sigma_0, \sigma_1))$;

The Hamiltonian system with Hamiltonian H is the differential equation

$$y'(x) = zJH(x)y(x),\ x \in (\sigma_0, \sigma_1)\,,$$

where

$$J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\,.$$

Hamiltonian systems

Classical spectral theory deals with the case (lcc at σ_0)

$$\int_{\sigma_0}^{\sigma_0+\epsilon} \operatorname{tr} H(t)\, dt < \infty$$

$\rightsquigarrow$ initial value problem.

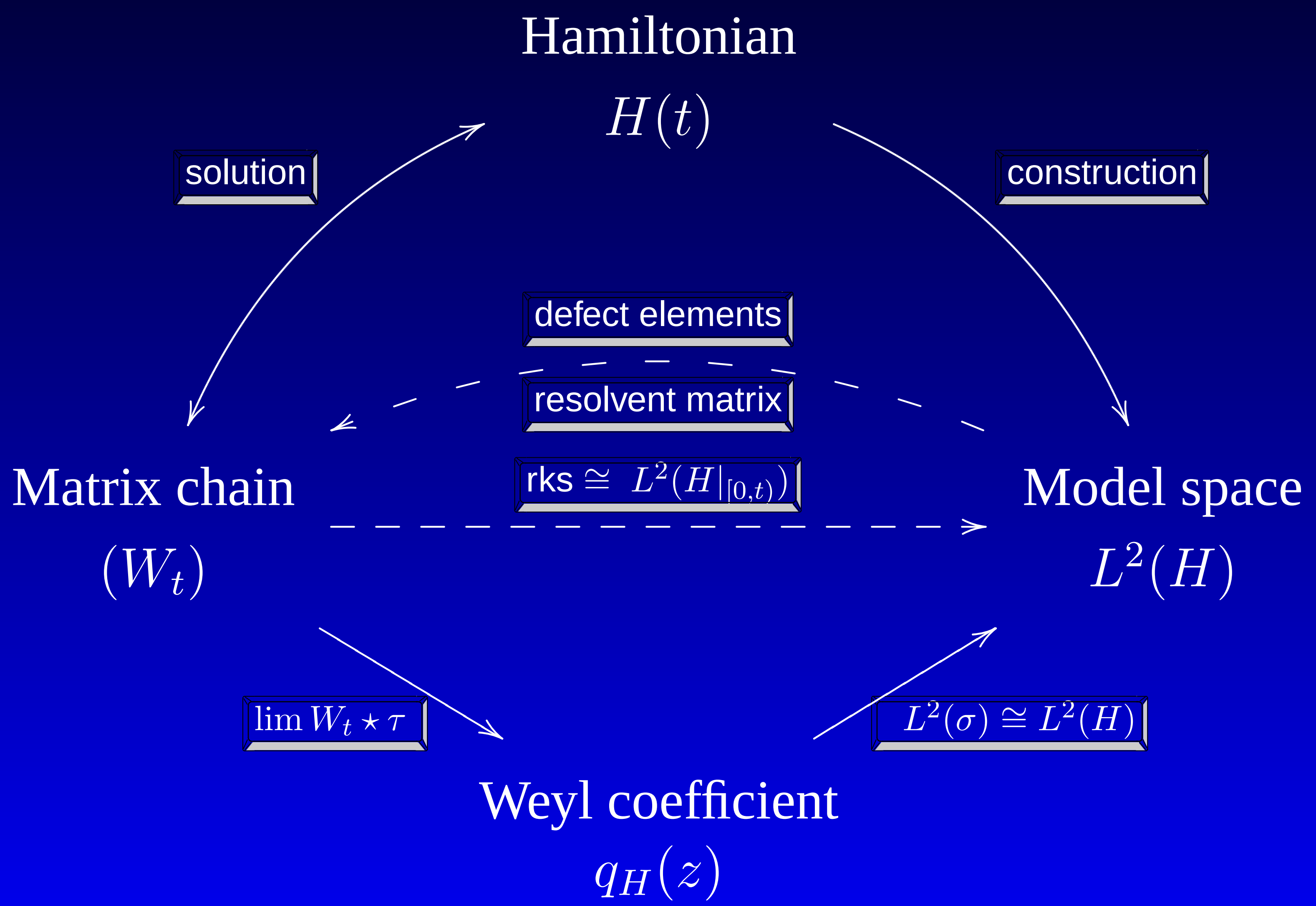
Depending whether (lcc or lpc at σ_1)

$$\int_{\sigma_0}^{\sigma_1} \operatorname{tr} H(t)\, dt < \infty \quad \text{or} \quad = \infty$$

the associated differential operator behaves differently.

We will always consider the case that limit point case takes place at the right endpoint.

Hamiltonian systems: LCC$\leftrightarrow$LPC



Hamiltonian systems: LCC$\leftrightarrow$LPC

Theorem (Direct Spectral Theorem).

The selfadjoint differential operator associated with H is unitarily equivalent to the multiplication operator in the space $L^2(\sigma)$, where σ is the positive Borel measure in the Herglotz-integral representation of the Weyl-coefficient q_H.

Method of proof: Operator theory.

'boundary triples in Hilbert spaces'

'Krein's theory of entire operators'

Hamiltonian systems: LCC$\leftrightarrow$LPC

Theorem (Inverse Spectral Theorem)**.**

Let σ be a positive Borel measure in $\mathbb{R}$ with $\int_{-\infty}^{\infty} \frac{d\sigma(t)}{1+t^2} < \infty$. Then there exists an essentially unique Hamiltonian H, such that σ is the measure in the Herglotz-integral representation of the Weyl-coefficient q_H.

Method of proof: Operator theory / complex analysis.
'de Branges theory of Hilbert spaces of entire functions'

De Branges spaces

Definition (De Branges Hilbert space)**.** A Hilbert space $\mathcal{H}$ ($\neq \{0\}$) is called a dB-space, if

(dB1) $\mathcal{H}$ is a reproducing kernel Hilbert space of entire functions.

(dB2) If $F \in \mathcal{H}$, then $F^{\#}(z) := \overline{F(\overline{z})} \in \mathcal{H}$, and

$$[F^{\#}, G^{\#}] = [G, F] \,.$$

(dB3) If $F \in \mathcal{H}$ and $z_0 \in \mathbb{C} \setminus \mathbb{R}$ with $F(z_0) = 0$, then $\frac{z-\overline{z_0}}{z-z_0} F(z) \in \mathcal{H}$, and

$$\Big[\frac{z-\overline{z_0}}{z-z_0} F(z), \frac{z-\overline{z_0}}{z-z_0} G(z)\Big] = [F, G] \,.$$

De Branges spaces

A dB-spaces is characterized by one single entire function.

Theorem. Let $\mathcal{H}$ be a dB-space. Then there exists an entire function E with $|E(\overline{z})| < |E(z)|$, $z \in \mathbb{C}^+$, such that the reproducing kernel of $\mathcal{H}$ is equal to

$$K(w,z) := i\frac{E(z)\overline{E(w)} - \overline{E(\overline{z})}E(\overline{w})}{2(z-\overline{w})}\,.$$

Conversely, if E is any entire function with $|E(\overline{z})| < |E(z)|$, $z \in \mathbb{C}^+$, then this kernel generates a dB-space.

If $\mathcal{H}$ is generated by E, we write $\mathcal{H} = \mathcal{H}(E)$.

More concretely

De Branges spaces

An important subclass of dB-spaces is formed by those spaces which are closed with respect to forming difference quotients, i.e. satisfy

(DQ) If $F \in \mathcal{H}$ and $z_0 \in \mathbb{C}$, then $\frac{F(z)-F(z_0)}{z-z_0} \in \mathcal{H}$.

De Branges spaces

Theorem (Existence- and Ordering-Theorem)**.** Let μ be a positive Borel measure on $\mathbb{R}$ with $\int_{-\infty}^{\infty} \frac{d\mu(t)}{1+t^2} < \infty$, and consider the set $\mathrm{Sub}(\mu)$ of all dB-spaces with (DQ) which are isometrically contained in $L^2(\mu)$. Then

- $\mathrm{Sub}(\mu)$ is totally ordered with respect to inclusion
- $\mathrm{Clos}_{L^2(\mu)} \bigcup \{\mathcal{H} : \mathcal{H} \in \mathrm{Sub}(\mu)\} = L^2(\mu)$
  $\dim \bigcap \{\mathcal{H} : \mathcal{H} \in \mathrm{Sub}(\mu)\} \leq 1$
- For each $\mathcal{H} \in \mathrm{Sub}(\mu)$,
  $\dim \left(\mathcal{H} / \mathrm{Clos} \bigcup \{\mathcal{L} \in \mathrm{Sub}(\mu) : \mathcal{L} \subsetneq \mathcal{H}\}\right) \leq 1$
  $\dim \left(\bigcap \{\mathcal{L} \in \mathrm{Sub}(\mu) : \mathcal{L} \supsetneq \mathcal{H}\} / \mathcal{H}\right) \leq 1$

Hamiltonians $\leftrightsquigarrow$ dB-spaces

The significance of de Branges' theory for Hamiltonian systems originates in the following fact.

Theorem ( ). Let H be a Hamiltonian, let $(W_t)_{t\in[\sigma_0,\sigma_1)}$ be the fundamental solution, and let μ be the measure in the Herglotz-integral representation of its Weyl-coefficient. Then

$$\mathrm{Sub}(\mu) = \Big\{ \mathcal{H}\big(W_{t,11}(z) - iW_{t,12}(z)\big) : t \in (\sigma_0,\sigma_1) \Big\}.$$

TWO EXAMPLES FOR INDEFINITE NOTIONS

Nevanlinna-Pick Interpolation

The Nevanlinna-Pick interpolation problem:

Let $z_1, \ldots, z_n \in \mathbb{C}^+$ and $w_1, \ldots, w_n \in \mathbb{C}$ be given. Does there exist a function q which

- is analytic in $\mathbb{C}^+$;
- satisfies $\operatorname{Im} q(z) \geq 0$, $z \in \mathbb{C}^+$;
- satisfies $q(z_i) = w_i$, $i = 1, \ldots, n$.

Nevanlinna-Pick Interpolation

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- satisfies $\operatorname{Im} q(z) \geq 0$, $z \in \mathbb{C}^+$;
- satisfies $q(z_i) = w_i$, $i = 1, \dots, n$.

Theorem. The answer is ‘yes’ if and only if

$$\mathbb{P} := \left(\frac{w_i - \overline{w_j}}{z_i - \overline{z_j}} \right)_{i,j=1}^{n} \geq 0 \,.$$

Nevanlinna-Pick Interpolation

Method of proof:

Step 1: An analytic function q in $\mathbb{C}^+$ satisfies $\operatorname{Im} q(z) \geq 0$, $z \in \mathbb{C}^+$, if and only if each matrix

$$\Big(\frac{q(\zeta_i) - \overline{q(\zeta_j)}}{\zeta_i - \overline{\zeta_j}}\Big)_{i,j=1}^{N}, \quad N \in \mathbb{N},\ \zeta_1, \dots, \zeta_N \in \mathbb{C}^+,$$

is positive semidefinite.

Step 2: Use extension theory and spectral theory of symmetric and selfadjoint operators in a Hilbert space.

Nevanlinna-Pick Interpolation

What if the matrix $\mathbb{P}$ is not positive semidefinite.

Can we at least find solutions of the interpolation problem which are ‘not too far away’ from being analytic with nonnegative imaginary part ?

Nevanlinna-Pick Interpolation

Definition (generalized Nevanlinna function)**.** We say $q \in \mathcal{N}_{<\infty}$ if

- q is meromorphic in $\mathbb{C}^+$;
- the supremum $\operatorname{ind}_- q$ of the numbers of negative squares of the following matrices is finite:

$$\left(\frac{q(\zeta_i) - \overline{q(\zeta_j)}}{\zeta_i - \overline{\zeta_j}}\right)_{i,j=1}^{N}, \quad N \in \mathbb{N},\ \zeta_1, \dots, \zeta_N \in \rho(q)$$

Functions in $\mathcal{N}_{<\infty}$ are indeed not too far away from being analytic with nonnegative imaginary part.

Nevanlinna-Pick Interpolation

Theorem (). Let $z_1, \dots, z_n \in \mathbb{C}^+$ and $w_1, \dots, w_n \in \mathbb{C}$ be given. Then there exists a function $q \in \mathcal{N}_{<\infty}$ such that

$$q(z_i) = w_i, \; i = 1, \dots, n\,.$$

This function can be chosen such that $\operatorname{ind}_- q$ equals the number of negative squares of $\mathbb{P}$.

<u>Method of proof:</u> Use extension and spectral theory of symmetric and selfadjoint operators in almost Pontryagin spaces .

The Bessel equation

The Bessel equation is the eigenvalue problem with singular endpoint 0

$$-u''(x) + \frac{\nu^2 - \frac{1}{4}}{x^2} u(x) = \lambda u(x),\ x > 0$$

Here ν is a parameter $\nu > \frac{1}{2}$ and λ is the eigenvalue parameter.

The Bessel equation

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Here ν is a parameter $\nu > \frac{1}{2}$ and λ is the eigenvalue parameter.

Rewriting this equation as a first-order-system, making a substitution in the independent variable, and setting $\alpha := 2\nu - 1$, $\lambda = z^2$, yields a Hamiltonian system with

$$H_\alpha(x) = \begin{pmatrix} x^{-\alpha} & 0 \\ 0 & x^\alpha \end{pmatrix}$$

The Bessel equation

If $\alpha < 1$ limit circle case takes place at 0, and we can apply the classical theory. All ingredients can be computed explicitly; e.g., the Weyl-coefficient of H_α turns out to be

$$q_\alpha(z) := c_\alpha z^\alpha \,,$$

where c_α is an appropriate constant, and an appropriate branch of the power z^α is used.

In particular, Weyl theory gives us a Fourier transform into a space $L^2(\sigma)$ with a scalar measure σ.

The Bessel equation

What if $\alpha \geq 1$.

Does there still exist a Fourier transform into a space $L^2(\sigma)$ with a scalar measure σ ?
Does the function $q_\alpha(z) := c_\alpha z^\alpha$ have any meaning for the spectral theory of the equation ?

The Bessel equation

What if $\alpha \geq 1$.

Does there still exist a Fourier transform into a space $L^2(\sigma)$ with a scalar measure σ ?
Does the function $q_\alpha(z) := c_\alpha z^\alpha$ have any meaning for the spectral theory of the equation ?

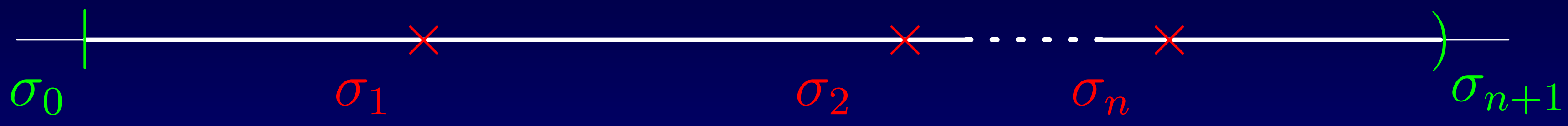
We will return later and answer these questions. For now:

Observation. Assume that α is not an odd integer. Then we have $q_\alpha \in \mathcal{N}_{<\infty}$. In fact,

$$\operatorname{ind}_- q_\alpha = \left\lfloor \frac{\alpha+1}{2} \right\rfloor .$$

THEORY OF INDEFINITE DE BRANGES SPACES AND HAMILTONIAN SYSTEMS

General Hamiltonians



- $\sigma_0 =$ initial point, $\sigma_{n+1} =$ endpoint.
- $\sigma_1, \ldots, \sigma_n =$ singularities.

$\mathfrak{H}$ consists of data :

- $-\infty < \sigma_0 < \sigma_1 < \ldots < \sigma_{n+1} \leq \infty$
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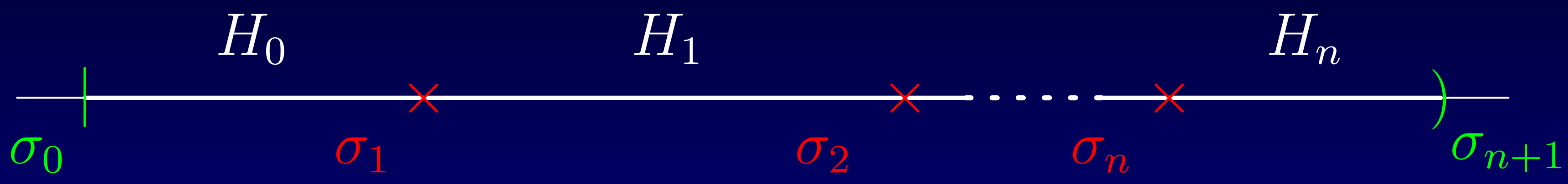
General Hamiltonians

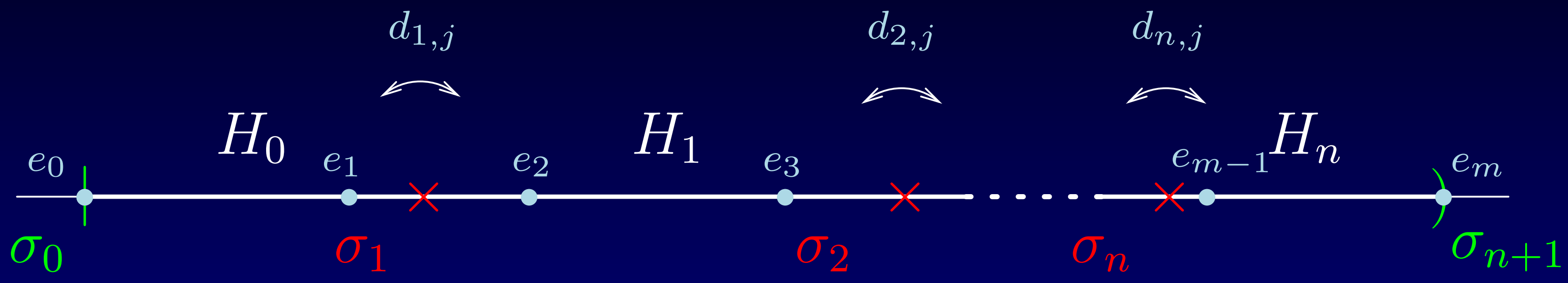


- $H_0, \ldots, H_n =$ Hamiltonians; lpc at $\sigma_1, \ldots, \sigma_n$
- H_0 lcc at σ_0; we consider H_n lpc at σ_{n+1}
- growth of H_i towards singularity is restricted

$\mathfrak{h}$ consists of data :

- $-\infty < \sigma_0 < \sigma_1 < \ldots < \sigma_{n+1} \leq \infty$
- $H_i : (\sigma_i, \sigma_{i+1}) \to \mathbb{R}^{2\times 2}$
-
-

General Hamiltonians

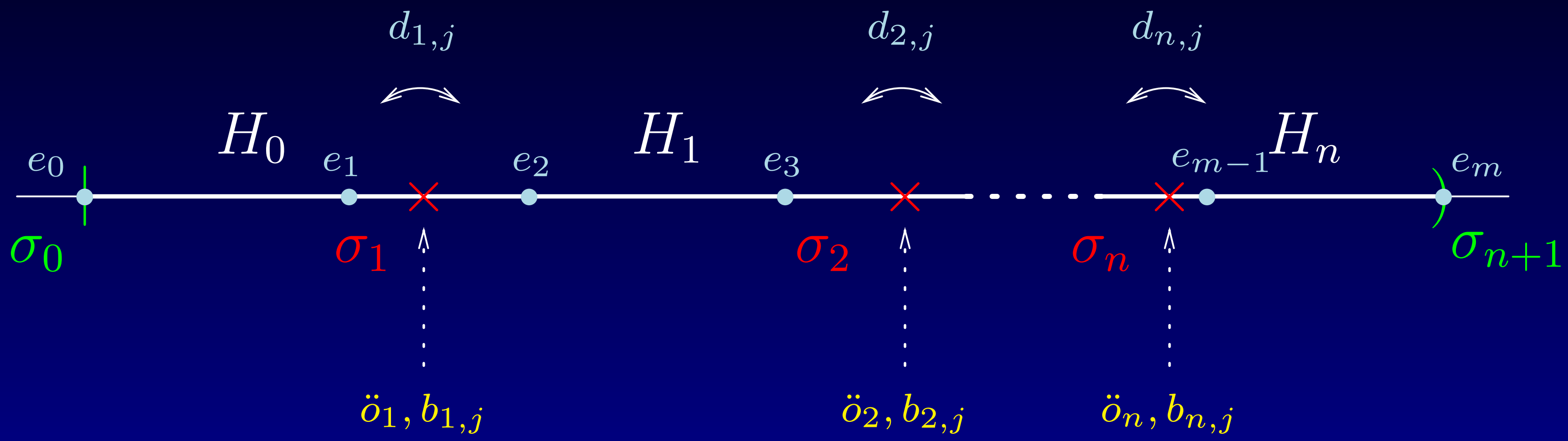


- $d_{i,j}$ = interface conditions for regularized boundary values
- 'interface conditions' are relative to $\{e_0, \ldots, e_m\}$

$\mathfrak{h}$ consists of data :

- $-\infty < \sigma_0 < \sigma_1 < \ldots < \sigma_{n+1} \leq \infty$
- $H_i : (\sigma_i, \sigma_{i+1}) \to \mathbb{R}^{2\times 2}$
- $\sigma_0 = e_0 < e_1 < \ldots < e_m = \sigma_{n+1}, \quad d_{i,0}, \ldots, d_{i,2\Delta_i - 1} \in \mathbb{R}$
-

General Hamiltonians



- $\ddot{o}_i, b_{i,j}$ = contribution concentrated in the singularity

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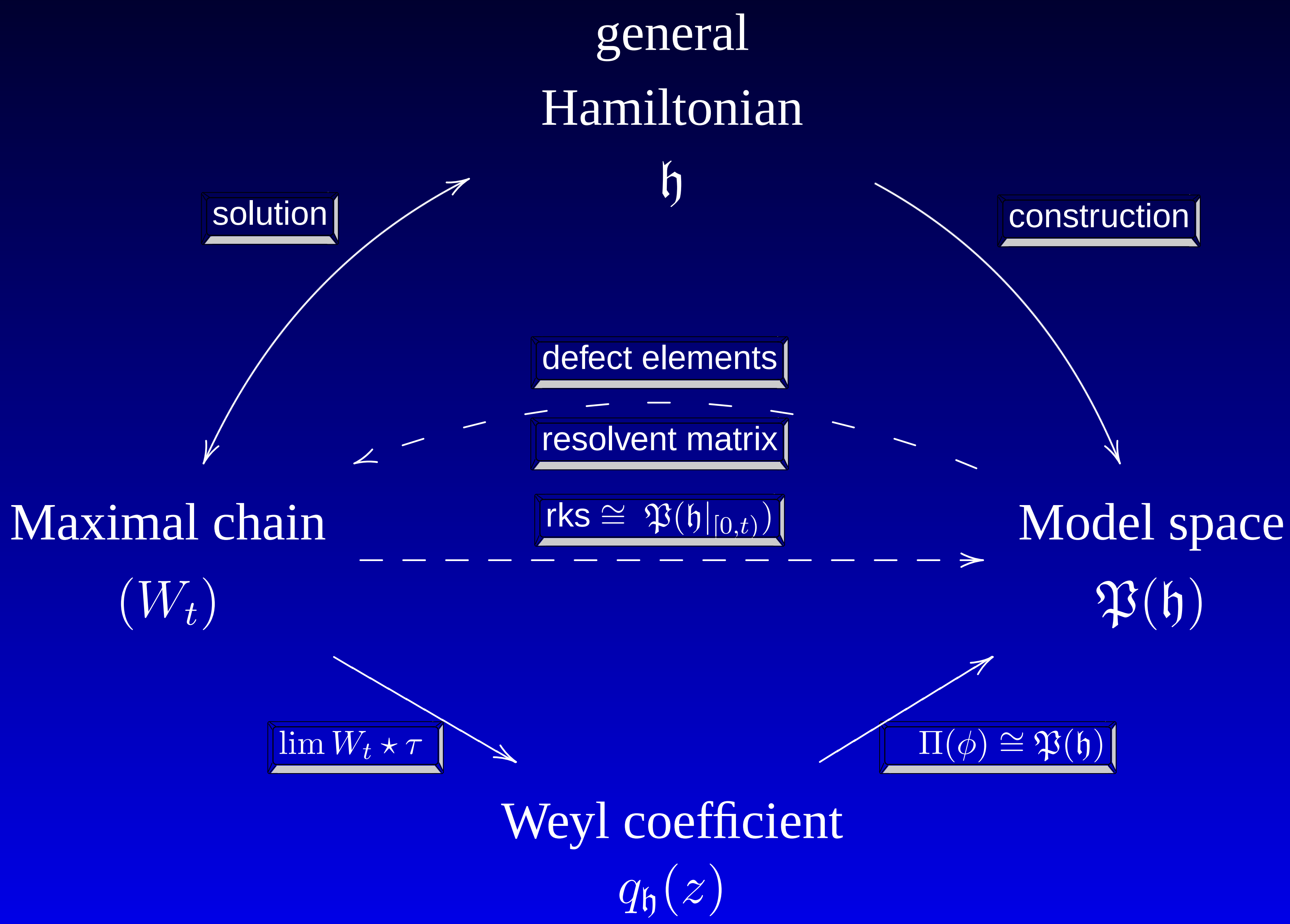
$\mathfrak{h}$ consists of data :

- $-\infty < \sigma_0 < \sigma_1 < \ldots < \sigma_{n+1} \leq \infty$
- $H_i : (\sigma_i, \sigma_{i+1}) \to \mathbb{R}^{2\times 2}$
- $\sigma_0 = e_0 < e_1 < \ldots < e_m = \sigma_{n+1}, \quad d_{i,0}, \ldots, d_{i,2\Delta_i - 1} \in \mathbb{R}$
- $\ddot{o}_i \in \mathbb{N}_0, b_{i,1}, \ldots, b_{i,\ddot{o}_i+1} \in \mathbb{R}$

Theory of indefinite systems



Theory of indefinite systems

Theorem (Direct Spectral Theorem)**.**

The selfadjoint differential operator associated with $\mathfrak{h}$ is unitarily equivalent to the multiplication operator in the space $\Pi(\phi)$, where ϕ is the distribution in the representation of the Weyl-coefficient $q_{\mathfrak{h}}$.

Method of proof: Operator theory.

'boundary triples in Pontryagin spaces'

'Entire operators in Pontryagin spaces'

Theory of indefinite systems

Theorem (Inverse Spectral Theorem)**.**

Let ϕ be a distribution of the class $\mathcal{F}$. Then there exists an essentially unique general Hamiltonian $\mathfrak{h}$, such that ϕ is the distribution in the representation of the Weyl-coefficient $q_{\mathfrak{h}}$.

Method of proof: Operator theory / complex analysis.
'de Branges almost Pontryagin spaces'

De Branges aPs

Definition (De Branges aPs)**.** An aPs $\mathcal{A}$ ($\neq \{0\}$) is called a dB-space, if

(dB1) The elements of $\mathcal{A}$ are entire functions and for each $w \in \mathbb{C}$ the point evaluation functional $\chi_w : F \mapsto F(w)$ is continuous.

(dB2) If $F \in \mathcal{A}$, then $F^\#(z) := \overline{F(\overline{z})} \in \mathcal{A}$, and

$$[F^\#, G^\#] = [G, F] \,.$$

(dB3) If $F \in \mathcal{A}$ and $z_0 \in \mathbb{C} \setminus \mathbb{R}$ with $F(z_0) = 0$, then $\frac{z-\overline{z_0}}{z-z_0} F(z) \in \mathcal{A}$, and

$$[\frac{z-\overline{z_0}}{z-z_0} F(z), \frac{z-\overline{z_0}}{z-z_0} G(z)] = [F, G] \,.$$

De Branges aPs

If the dB-space $\mathcal{A}$ is nondegenerated (i.e. a Pontryagin space) then there exists a reproducing kernel $K(w,z)$ for $\mathcal{A}$. In this case, the space is characterized by one single entire function.

Theorem. Let $\mathcal{P}$ be a nondegenerated dB-space. Then there exists an entire function E of the class $\boxed{\mathcal{H}B_{<\infty}}$, such that the reproducing kernel of $\mathcal{P}$ is equal to

$$K(w,z) := i\,\frac{E(z)\overline{E(w)} - \overline{E(\overline{z})}E(\overline{w})}{2(z-\overline{w})}\,.$$

Conversely, if $E \in \mathcal{H}B_{<\infty}$, then this kernel generates a nondegenerated dB-space.

If $\mathcal{P}$ is generated by E, we write $\mathcal{P} = \mathcal{P}(E)$.

De Branges aPs

Again, dB-spaces closed with respect to forming difference quotients, i.e. satisfying

(DQ) If $F \in \mathcal{H}$ and $z_0 \in \mathbb{C}$, then $\frac{F(z)-F(z_0)}{z-z_0} \in \mathcal{H}$;

play an important role.

De Branges aPs

Theorem (Existence- and Ordering-Theorem)**.** Let $\phi \in \mathcal{F}$, and consider the set $\mathrm{Sub}(\phi)$ of all dB-spaces with (DQ) which are 'isometrically contained in $\Pi(\phi)$'. Then

- $\mathrm{Sub}(\mathcal{A})$ is totally ordered with respect to inclusion
- $\mathrm{Clos}_{\Pi(\phi)} \bigcup \{\mathcal{A} : \ \mathcal{A} \in \mathrm{Sub}(\phi)\} = \Pi(\phi)$
  $\dim \bigcap \{\mathcal{A} : \ \mathcal{A} \in \mathrm{Sub}(\phi)\} \leq 1$
- For each $\mathcal{A} \in \mathrm{Sub}(\phi)$,
  $\dim \left(\mathcal{H} / \mathrm{Clos} \bigcup \{\mathcal{B} \in \mathrm{Sub}(\phi) : \mathcal{B} \subsetneq \mathcal{A}\}\right) \leq 1$
  $\dim \left(\bigcap \{\mathcal{B} \in \mathrm{Sub}(\phi) : \mathcal{B} \supsetneq \mathcal{A}\} / \mathcal{A}\right) \leq 1$

De Branges aPs

Let $\phi \in \mathcal{F}$, denote by $q_\phi \in \mathcal{N}_{<\infty}$ the function represented by ϕ, and consider the chain $\operatorname{Sub}(\phi)$. Then we can ask for the behaviour of the values $\operatorname{ind}_- \mathcal{A}$ and $\operatorname{ind}_0 \mathcal{A}$ when $\mathcal{A}$ varies through $\operatorname{Sub}(\phi)$.

De Branges aPs

Let $\phi \in \mathcal{F}$, denote by $q_\phi \in \mathcal{N}_{<\infty}$ the function represented by ϕ, and consider the chain $\mathrm{Sub}(\phi)$. Then we can ask for the behaviour of the values $\mathrm{ind}_- \mathcal{A}$ and $\mathrm{ind}_0 \mathcal{A}$ when $\mathcal{A}$ varies through $\mathrm{Sub}(\phi)$.

Observation. The function $\mathcal{A} \mapsto \mathrm{ind}_- \mathcal{A}$ is nondecreasing, and

$$\max_{\mathcal{A}\in\mathrm{Sub}(\phi)} \mathrm{ind}_- \mathcal{A} = \mathrm{ind}_- q_\phi \,.$$

The chain $\mathrm{Sub}(\phi)$ is the union of disjoint intervals where $\mathrm{ind}_- \mathcal{A}$ is constant.

De Branges aPs

Theorem. Let $\phi \in \mathcal{F}$.

- There are only finitely many $\mathcal{A} \in \mathrm{Sub}(\phi)$ with $\mathrm{ind}_0 \mathcal{A} \neq 0$.
- If $\mathcal{A}_1 \subsetneq \mathcal{A}_2 \subsetneq \mathcal{A}_3$ are three consequtive members of $\mathrm{Sub}(\phi)$, then

$$\mathrm{ind}_0 \mathcal{A}_1 \geq \mathrm{ind}_0 \mathcal{A}_2 > 0 \;\Rightarrow\; \mathrm{ind}_0 \mathcal{A}_2 > \mathrm{ind}_0 \mathcal{A}_3$$

- If $\mathcal{A}_1 \subsetneq \mathcal{A}_2$ are consequtive members of $\mathrm{Sub}(\phi)$, then $|\,\mathrm{ind}_0 \mathcal{A}_1 - \mathrm{ind}_0 \mathcal{A}_2| \leq 1$.
- If $\mathcal{P}_1, \mathcal{A}_2, \mathcal{P}_3 \in \mathrm{Sub}(\phi)$ with $\mathcal{P}_1 \subsetneq \mathcal{A}_2 \subsetneq \mathcal{P}_3$ and $\mathrm{ind}_0 \mathcal{P}_1 = \mathrm{ind}_0 \mathcal{P}_3 = 0$, then

$$\mathrm{ind}_0 \mathcal{A}_2 > 0 \;\Rightarrow\; \mathrm{ind}_- \mathcal{A}_1 < \mathrm{ind}_- \mathcal{A}_3$$

Hamiltonians $\leftrightsquigarrow$ dB-spaces

The relation of general Hamiltonians with dB-spaces is:

Theorem (). Let $\mathfrak{h}$ be a general Hamiltonian, let $(W_t)_{t\in I}$ be the associated maximal chain, and let ϕ be the distribution in the representation of its Weyl-coefficient. Then

$$\Big\{\mathcal{A}\in \mathrm{Sub}(\phi):\ \mathrm{ind}_0\,\mathcal{A}=0\Big\}=$$
$$=\Big\{\mathcal{P}\Big(W_{t,11}(z)-iW_{t,12}(z)\Big):\ t\in I\Big\}.$$

Set $\mathcal{P}_t := \mathcal{P}(W_{t,11}(z)-iW_{t,12}(z))$. The number $\mathrm{ind}_-\,\mathcal{P}_t$ is constant on the intervals (σ_{i-1},σ_i) and takes different values on different intervals.

Hamiltonians $\leftrightsquigarrow$ dB-spaces

Singularities are related to degenerated members of $\mathrm{Sub}(\phi)$:

Theorem (). Let $t \in (\sigma_{i-1}, \sigma_i)$ and $s \in (\sigma_i, \sigma_{i+1})$. Then either

- There exists $\mathcal{A} \in \mathrm{Sub}(\phi)$ with

$$\mathcal{P}_t \subseteq \mathcal{A} \subseteq \mathcal{P}_s, \quad \mathrm{ind}_0 \mathcal{A} > 0\,.$$

- There exist $\mathcal{P}_-, \mathcal{P}_+ \in \mathrm{Sub}(\phi)$ with

$$\mathrm{ind}_0 \mathcal{P}_- = \mathrm{ind}_0 \mathcal{P}_+ = 0, \quad \dim \mathcal{P}_+/\mathcal{P}_1 = 1\,,$$
$$\mathrm{ind}_- \mathcal{P}_- = \mathrm{ind}_- \mathcal{P}_t,\ \mathrm{ind}_- \mathcal{P}_+ = \mathrm{ind}_- \mathcal{P}_s\,.$$

APPLICATION: HAMILTONIAN SYSTEMS WITH TWO SINGULAR ENDPOINTS

Hamiltonian systems: LPC$\leftrightarrow$LPC

If a Hamiltonian H is in lcc at the left and lpc at the right endpoint, we know that we have a Fourier transform onto a space $L^2(\sigma)$ with some scalar positive Borel measure σ. A measure σ appears in this way if and only if $\int_{-\infty}^{\infty} \frac{d\sigma(t)}{1+t^2} < \infty$.

What if H is in lpc at both endpoints ?

Hamiltonian systems: LPC$\leftrightarrow$LPC

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Observation. In general there cannot exist a Fourier transform of the above kind: There exist Hamiltonians for which the associated selfadjoint differential operator has spectrum with spectral multiplicity 2.

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What if H is in lpc at both endpoints ?

Observation. In general there cannot exist a Fourier transform of the above kind: There exist Hamiltonians for which the associated selfadjoint differential operator has spectrum with spectral multiplicity 2.

We can use indefinite theory to find a class of Hamiltonians where it still works !

Hamiltonian systems: LPC$\leftrightarrow$LPC

Definition. We say that $H \in \mathbb{H}$, if ($x_0 \in (\sigma_0, \sigma_1)$ fixed)

- $\int_{\sigma_0}^{x_0} H(t)_{22}\, dt < \infty$;
- $\int_{\sigma_0}^{x_0} \big(\int_t^{x_0} H(s)_{11}\, ds \big) \cdot H(t)_{22}\, dt < \infty$;
- Define functions $h_k : (\sigma_0, x_0) \to \mathbb{C}^2$ recursively by

$$h_0(x) := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad h_k(x) := \int_{x_0}^{x} JH(y) h_{k-1}(y)\, dy,\ k \in \mathbb{N}\,.$$

  Then there exists $f \in \operatorname{span}\big\{h_k :\ k \leq N\big\} \setminus \{0\}$ with $\int_{\sigma_0}^{x_0} f(t)^* H(t) f(t)\, dt < \infty$.

We denote the minimal number N such that this is possible by $\Delta(H)$. We always have $\Delta(H) > 0$.

Hamiltonian systems: LPC↔LPC

Definition. We say $\mu \in \mathbb{M}$ if

- μ is a scalar valued positive Borel measure on $\mathbb{R}$;
- there exists a number $n \in \mathbb{N}_0$, such that

$$\int_{-\infty}^{\infty} \frac{d\mu(t)}{(1+t^2)^{n+1}} < \infty \,.$$

We denote the minimal number n such that this integral is finite by $\Delta(\mu)$.

Hamiltonian systems: LPC$\leftrightarrow$LPC

Theorem (Direct & Inverse Spectral Theorem)**.**
If $H \in \mathbb{H}$, there exists a Fourier transform onto a space $L^2(\mu)$ with some $\mu \in \mathbb{M}$, $\Delta(\mu) > 0$. Conversely, if $\mu \in \mathbb{M}$, $\Delta(\mu) > 0$, there exists an essentially unique Hamiltonian such that μ arises in this way. Thereby we have $\Delta(\mu) = \Delta(H)$.

Hamiltonian systems: LPC$\leftrightarrow$LPC

Theorem (Direct & Inverse Spectral Theorem)**.**
If $H \in \mathbb{H}$, there exists a Fourier transform onto a space $L^2(\mu)$ with some $\mu \in \mathbb{M}$, $\Delta(\mu) > 0$. Conversely, if $\mu \in \mathbb{M}$, $\Delta(\mu) > 0$, there exists an essentially unique Hamiltonian such that μ arises in this way. Thereby we have $\Delta(\mu) = \Delta(H)$.

Method of proof:

Step 1; Preparation

Step 2; Constructing the Fourier transform

Step 3; The inverse construction

Hamiltonian systems: LPC$\leftrightarrow$LPC

- The measure μ and the Fourier transform can be constructed via regularized boundary values .
- If H is of diagonal form, the conditions for $H \in \mathbb{H}$ can be rewritten to a much simpler form .
- The above results apply immediately to Sturm-Liouville operators without a potential term.
- Sturm-Liouville operators in Schrödinger form can theoretically also be embedded in the theory. However, it is necessary to apply a Liouville transformation, and this is complicated.
- The Bessel equation revisited .

APPLICATION: SPECTRAL FUNCTIONS OF KREIN STRINGS

Krein strings

A string is a pair $\mathrm{S}[L, \mathfrak{m}]$ where $L \in [0, \infty]$ and $\mathfrak{m}$ is a positive, possibly unbounded, Borel measure supported on $[0, L]$. We think of L as its length and of $\mathfrak{m}$ measuring its mass.

To each string a boundary value problem is associated, namely

$$f'(x) + z \int_0^\infty f(y)\, d\mathfrak{m}(y),\ x \in \mathbb{R}, \qquad f'(0-) = 0\,.$$

Krein strings

A string is a pair $\mathrm{S}[L, \mathfrak{m}]$ where $L \in [0, \infty]$ and $\mathfrak{m}$ is a positive, possibly unbounded, Borel measure supported on $[0, L]$. We think of L as its length and of $\mathfrak{m}$ measuring its mass.

To each string a boundary value problem is associated, namely

$$f'(x) + z \int_0^\infty f(y)\, d\mathfrak{m}(y), \; x \in \mathbb{R}, \qquad f'(0-) = 0\,.$$

Definition. A positive Borel measure τ on $\mathbb{R}$ is called a (canonical) spectral measure of the string $\mathrm{S}[L, \mathfrak{m}]$, if there exists an (appropriately normalized) Fourier transform of $L^2(\mathfrak{m})$ onto $L^2(\tau)$.

Krein strings

Theorem (Direct & Inverse Spectral Theorem). In order that a given positive Borel measure τ is a spectral measure of some string, it is necessary that

- $\int_{\mathbb{R}} \frac{d\tau(\lambda)}{1+|\lambda|} < \infty$;
- either $\operatorname{supp}\tau \subseteq [0,\infty)$, or τ is discrete and has exactly one point mass in $(-\infty, 0)$.

Conversely, if $\int_{\mathbb{R}} \frac{d\tau(\lambda)}{1+|\lambda|} < \infty$ and $\operatorname{supp}\tau \subseteq [0,\infty)$, then τ is a spectral measure of some string, and this string is uniquely determined by τ.

Krein strings

What if $\operatorname{supp}\tau$ intersects $(-\infty, 0)$?

Krein strings

What if $\operatorname{supp}\tau$ intersects $(-\infty, 0)$?

We can use indefinite theory to answer this question !

Krein strings

What if $\operatorname{supp}\tau$ intersects $(-\infty, 0)$?

Theorem (). Let τ be a positive Borel measure on $\mathbb{R}$ with $\operatorname{supp}\tau \not\subseteq [0, \infty)$. If τ is a spectral measure of some string, then the following conditions (SM_1)–(SM_7) hold.

(SM_1)
(SM_2)
(SM_3)

(SM_4)
(SM_5)

(SM_6)
(SM_7)

Conversely, if τ satisfies (SM_1)–(SM_7), then there exists a unique string such that τ is a spectral measure of this string.

Krein strings

Method of proof:

Step 1: Let τ be a discrete measure with $\int_{\mathbb{R}} \frac{d\tau(\lambda)}{1+|\lambda|} < \infty$ and $|\operatorname{supp}\tau \cap (-\infty, 0)| = 1$. Then

$$Q(z) := z \int_{\mathbb{R}} \frac{d\tau(\lambda)}{\lambda - z^2} \in \mathcal{N}_{<\infty}\,.$$

Consider the general Hamiltonian $\mathfrak{h}$ whose Weyl-coefficient equals Q. Then, in order that τ is a spectral function of some string, it is necessary and sufficient that $\mathfrak{h}$ has a certain form.

Step 2: A general Hamiltonian is of the mentioned form, if and only if its Weyl-coefficient is meromorphic in $\mathbb{C}$, and the locations and residues of its poles have correct asymptotics.

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THE END

The matrix chain (W_t)

Let H be a Hamiltonian defined on (σ_0, σ_1). Then W_t, $t \in [\sigma_0, \sigma_1)$, denotes the unique solution of the initial value problem

$$\frac{d}{dt} W_t(z) J = z W_t(z) H(t), \; x \in [\sigma_0, \sigma_1) \,,$$

$$W_0(z) = I \,.$$

The model space $L^2(H)$

Supressing some technicalities which arise from ‘indivisible intervals’, we have

$$L^2(H) := \left\{ f : (\sigma_0, \sigma_1) \to \mathbb{C}^2 : \int_{\sigma_0}^{\sigma_1} f(t)^T H(t) f(t)\, dt < \infty \right\}$$

$$T_{max}(H) := \Big\{ (f; g) \in L^2(H)^2 : f \text{ absolutely continuous}, \\ f(t) = JH(t)g(t), \text{ a.e.} \Big\}$$

$$\Gamma(H)(f; g) := f(\sigma_0),\ (f; g) \in T_{max}(H)$$

The reproducing kernel space

The kernel

$$K_{W_t}(w,z) := \frac{W_t(z)JW_t(w)^* - J}{z - \overline{w}}$$

is positive definite, thus generates a reproducing kernel Hilbert space $\mathfrak{K}(W_t)$. The elements of $\mathfrak{K}(W_t)$ are entire 2-vector-functions.

The operator $\mathcal{S}(W_t)$ of multiplication by z is a symmetry with defect 2. The map $\Gamma(W_t) : f \mapsto f(0)$ is a boundary map for $\mathcal{S}(W_t)$.

The reproducing kernel space

The boundary triple $\langle L^2(H|_{(\sigma_0,t)}), T_{min}(H|_{(\sigma_0,t)}), \Gamma(H|_{(\sigma_0,t)})\rangle$ is isomorphic to $\langle \mathfrak{K}(W_t), \mathcal{S}(W_t), \Gamma(W_t)\rangle$. The isomorphism of $L^2(H|_{(\sigma_0,t)})$ to $\mathfrak{K}(W_t)$ is given by

$$f(x) \mapsto \int_{\sigma_0}^{t} W_x(z)H(x)f(x)\,dx\,.$$

W_t from defect elements

Let $y_1(z,x) = (y_1(z,x)_2, y_1(z,x)_2)^T$ and $y_2(z,x) = (y_2(z,x)_2, y_2(z,x)_2)^T$ be the elements of $\ker(T_{max}(H|_{(\sigma_0,t)} - z)$, such that $y_1(z,\sigma_0) = (1,0)^T$ and $y_2(z,\sigma_0) = (0,1)^T$. Then

$$W_t(z) = \begin{pmatrix} y_1(z,t)_1 & y_1(z,t)_2 \\ y_2(z,t)_1 & y_2(z,t)_2 \end{pmatrix}$$

W_t as resolvent matrix

Consider

$$S_1 := \Big\{(x;y) \in T_{max}(H|_{(\sigma_0,t)}) : \\ \pi_{l,1}\Gamma(H|_{(\sigma_0,t)})(x;y) = 0, \pi_r\Gamma(H|_{(\sigma_0,t)})(x;y) = 0\Big\}$$

$$u : (x;y) \mapsto \pi_{l,2}\Gamma(H|_{(\sigma_0,t)})(x;y),\ (x;y) \in T_{max}(H|_{(\sigma_0,t)})$$

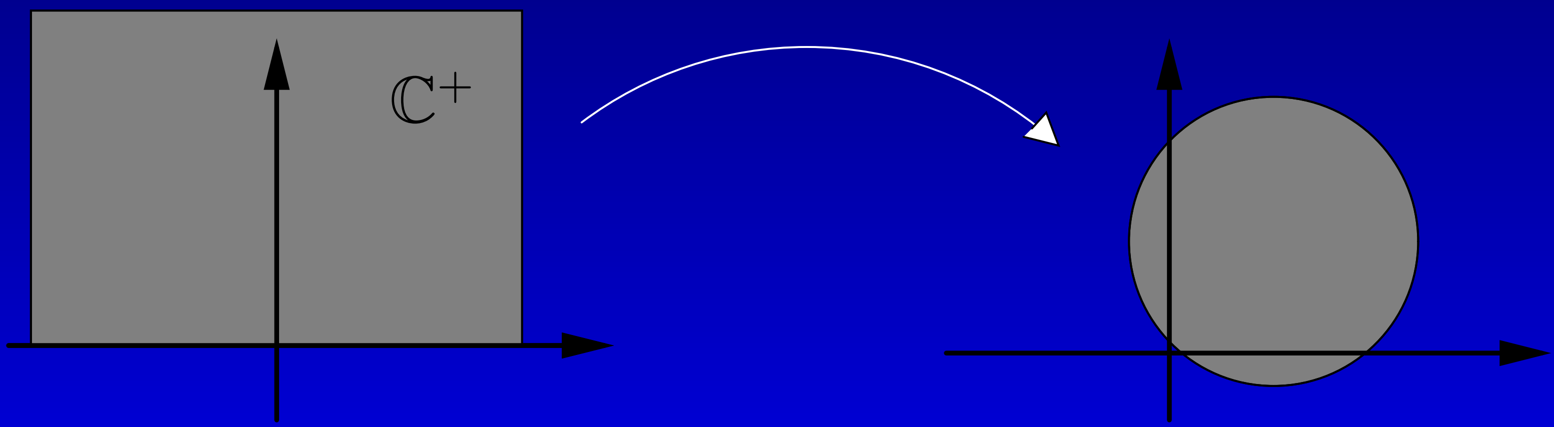
Then S_1 is symmetric with defect 1 and $u|_{S_1^*}$ is continuous. The matrix function W_t is a u-resolvent matrix of S_1.

The Weyl coefficient $q_H(z)$

For $W = (w_{ij})_{i,j=1}^2 \in \mathbb{C}^{2\times 2}$ and $\tau \in \mathbb{C}$ denote

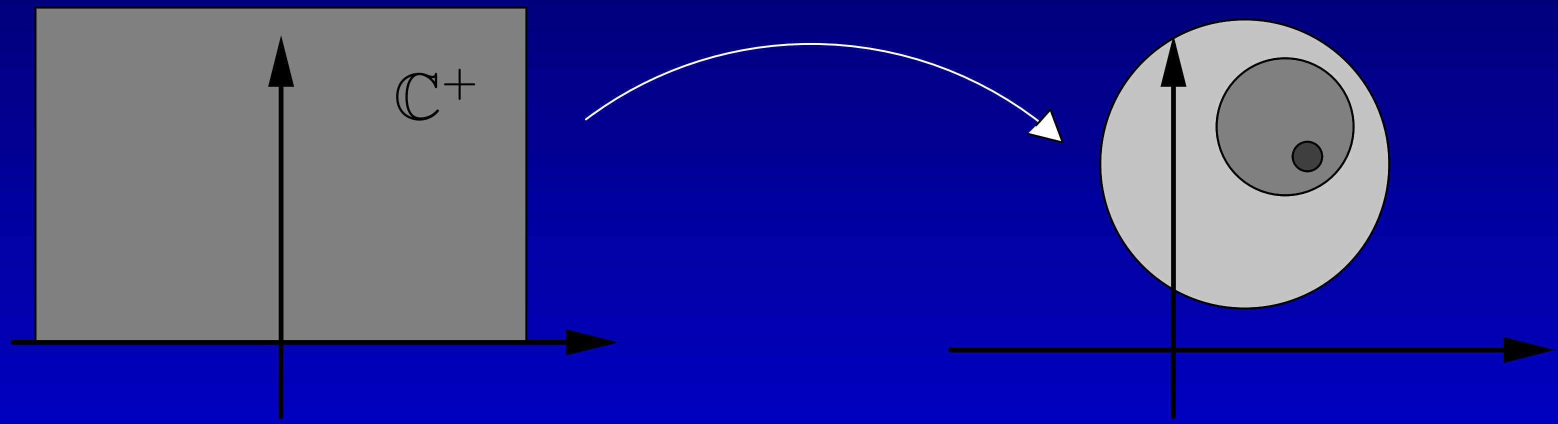
$$W \star \tau := \frac{w_{11}\tau + w_{12}}{w_{21}\tau + w_{22}}$$

The assignment $\tau \mapsto W \star \tau$ maps the upper half plane to some (general) disk:



The Weyl coefficient $q_H(z)$

Let $(W_t)_{t\in[\sigma_0,\sigma_1)}$ be the matrix chain associated with the Hamiltonian H. The assignments $\tau \mapsto W_t \star \tau$ map $\mathbb{C}^+$ to a nested sequence of disks contained in $\mathbb{C}^+$. The disk $W_t \star \mathbb{C}^+$ is contained in the upper half plane and its radius is $[\int_{\sigma_0}^{t} \operatorname{tr} H(x)\, dx]^{-1}$.



The Weyl coefficient $q_H(z)$

For each $z \in \mathbb{C}^+$ the limit

$$q_H(z) := \lim_{t \nearrow \sigma_1} W_t(z) \star \tau$$

exists, and does not depend on $\tau \in \mathbb{C}^+$.

- The function q_H is analytic in $\mathbb{C}^+$;
- $\operatorname{Im} q_H(z) \geq 0$, $z \in \mathbb{C}^+$.

The Fourier transform

Consider the Herglotz integral representation

$$q_H(z) = a + bz + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) d\sigma(t)$$

of the Weyl coefficient q_H. The map

$$f(x) \mapsto \int_{\sigma_0}^{\sigma_1} (0,1) W_x(z) H(x) f(x)\, dx$$

is an isomorphism of $L^2(H)$ onto $L^2(\sigma)$ (in fact, an isomorphism of boundary triples).

Concrete definition

Theorem. Let E be entire with $|E(\overline{z})| < |E(z)|$, $z \in \mathbb{C}^+$. Then the dB-space whose reproducing kernel is given by means of E equals the set of all entire functions F with

- $\frac{F}{E}$ and $\frac{F^\#}{E}$ are of bounded type and nonpositive mean type in $\mathbb{C}^+$;
- $\int_{-\infty}^{\infty} \left|\frac{F(t)}{E(t)}\right|^2 dt < \infty$;
- The (square of the) norm in $\mathcal{H}(E)$ is given by the above integral.

Almost Pontryagin spaces

Definition. An aPs is an inner product space of the form

$$\mathcal{A} = \mathcal{A}_+ [\dot{+}] \mathcal{A}_- [\dot{+}] \mathcal{A}_0$$

where

- $\mathcal{A}_+$ is a Hilbert space;
- $\mathcal{A}_-$ is a finite-dimensional negative definite space;
- $\mathcal{A}_0$ is a finite-dimensional neutral space.

We endow $\mathcal{A}$ with the product topology, then $\mathcal{A}$ becomes a Banach space, and set $\operatorname{ind}_- \mathcal{A} := \dim \mathcal{A}_-$, $\operatorname{ind}_0 \mathcal{A} := \dim \mathcal{A}_0$. We speak of a Pontryagin space, if $\mathcal{A}_0 = \{0\}$.

The class $\mathcal{N}_{<\infty}$

Theorem. We have $q \in \mathcal{N}_{<\infty}$ if and only if there exist

- $n \in \mathbb{N}$, points $a_j \in \mathbb{C}^+ \cup \mathbb{R}$ and multiplicities α_j, $j = 1, \ldots, n$,
- $m \in \mathbb{N}$, points $b_j \in \mathbb{C}^+ \cup \mathbb{R}$ and multiplicities β_j, $j = 1, \ldots, m$,
- $\tilde{q}$ analytic in $\mathbb{C}^+$ with $\operatorname{Im} \tilde{q}(z) \geq 0$, $z \in \mathbb{C}^+$,

such that

$$q(z) = \frac{\prod_{j=1}^{n} [(z - a_j)(z - \overline{a_j})]^{\alpha_j}}{\prod_{j=1}^{m} [(z - b_j)(z - \overline{b_j})]^{\beta_j}} \, \tilde{q}(z) \, .$$

The class $\mathcal{N}_{<\infty}$

Theorem. We have $q \in \mathcal{N}_{<\infty}$ if and only if there exists

- a distribution ϕ on $\mathbb{R} \cup \{\infty\}$ which, off some finite set of points, coincides with a positive (possibly unbounded) Borel measure,
- a rational function r analytic and real valued along $\mathbb{R}$,

such that

$$q(z) = r(z) + \phi\Big(\big[\frac{1}{t-z} - \frac{t}{1+t^2}\big](1+t^2)\Big)$$

The Weyl coefficient $q_{\mathfrak{h}}(z)$

The limit

$$q_{\mathfrak{h}}(z) := \lim_{t \nearrow \sigma_{n+1}} W_t(z) \star \tau$$

exists as a meromorphic function locally uniformly on $\mathbb{C} \setminus \mathbb{R}$ and does not depend on $\tau \in \mathbb{C}^+$.

The function $q_{\mathfrak{h}}$ belongs $\mathcal{N}_{<\infty}$, and

$$\operatorname{ind}_- q_{\mathfrak{h}} := \sum_{i=1}^{n} \Big(\Delta_i + [\frac{\ddot{o}_i}{2}]\Big) + \#\Big\{1 \leq i \leq n : \ \ddot{o}_i \text{ odd}, c_{i,1} < 0\Big\}\,.$$

The Fourier transform

Let ϕ be the distribution on $\overline{\mathbb{R}}$ which represents $q_{\mathfrak{h}}$ as

$$q(z) = r(z) + \phi\Big([\frac{1}{t-z} - \frac{t}{1+t^2}](1+t^2) \Big) .$$

This distribution, including the poles of r, generates an 'L^2'-like Pontryagin space $\Pi(\phi)$ (in fact, a 'multiplication operator'-like Pontryagin space boundary triple).

There exists an isomorphism of $\mathfrak{P}(\mathfrak{h})$ onto $\Pi(\phi)$ (in fact, an isomorphism of boundary triples).

The model space $\mathfrak{P}(\mathfrak{h})$

Given a general Hamiltonian $\mathfrak{h}$ we construct an operator model, which is a Pontryagin space boundary triple

$$\langle \mathfrak{P}(\mathfrak{h}), T(\mathfrak{h}), \Gamma(\mathfrak{h}) \rangle$$

The actual construction is quite involved and too complicated to be elaborated here.

The model space $\mathfrak{P}(\mathfrak{h})$

If $J = [s_-, s_+] \subseteq (\sigma_i, \sigma_{i+1})$, there exists an isometric and homeomorphic embedding

$$\iota_J : L^2(H_i|_J) \to \mathfrak{P}(\mathfrak{h})$$

If $J \subseteq J'$, then

$$\begin{tikzcd} L^2(H_i|_J) \arrow[r, "\iota_J"] \arrow[d, "\subseteq"'] & \mathfrak{P}(\mathfrak{h}) \\ L^2(H_i|_{J'}) \arrow[ur, "\iota_{J'}"'] & \end{tikzcd}$$

Indefinite Hermite-Biehler class

Definition. Denote by $\mathcal{H}B_{<\infty}$ the set of all entire functions E, such that E and $E^{\#}$ have no common nonreal zeros, $E^{-1}E^{\#}$ is not constant, and the reproducing kernel

$$K_E(w,z) := \frac{i}{2}\frac{E(z)\overline{E(w)} - E^{\#}(z)E(\overline{w})}{z-\overline{w}}$$

has a finite number negative squares.

Indefinite Hermite-Biehler class

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$$K_E(w,z) := \frac{i}{2}\frac{E(z)\overline{E(w)} - E^\#(z)E(\overline{w})}{z-\overline{w}}$$

has a finite number negative squares.

Observation. An entire function E satisfies $|E(\overline{z})| < |E(z)|$, $z \in \mathbb{C}^+$, if and only if $E \in \mathcal{H}B_{<\infty}$ and the above kernel is positive semidefinite.

Reproducing kernel space of W_t

The kernel

$$K_{W_t}(w,z) := \frac{W_t(z)JW_t(w)^* - J}{z-\overline{w}}$$

has a finite number of negative squares, thus generates a reproducing kernel Pontryagin space $\mathfrak{K}(W_t)$. The elements of $\mathfrak{K}(W_t)$ are entire 2-vector-functions.
The operator $\mathcal{S}(W_t)$ of multiplication by z is a symmetry with defect 2. The map $\Gamma(W_t) : f \mapsto f(0)$ is a boundary map for $\mathcal{S}(W_t)$.

Reproducing kernel space of W_t

There exists an isomorphism Φ_t of the boundary triples $\langle \mathfrak{K}(W_t), \mathcal{S}(W_t), \Gamma(W_t)\rangle$, $\langle \mathfrak{P}^2(\mathfrak{h}|_{(\sigma_0,t)}), S(\mathfrak{h}|_{(\sigma_0,t)}), \Gamma(\mathfrak{h}|_{(\sigma_0,t)})\rangle$.

If $J := [s_-, s_+] \subseteq (\sigma_{i-1}, \sigma_i)$, then the map

$$\lambda_J : f(x) \mapsto \int_{s_-}^{s_+} W_x(z)H(x)f(x)\,dx$$

is an isomorphism of $L^2(H_i|_{[s_-,s_+]})$ onto $\mathfrak{K}(W_{s_+})[-]\mathfrak{K}(W_{s_-})$. We have

$$\begin{tikzcd} L^2(H_i|_J) \arrow[r, "\iota_J"] \arrow[d, "\lambda_J"'] & \mathfrak{P}(\mathfrak{h}|_{(\sigma_0,s_+)}) \\ \mathfrak{K}(W_{s_+})[-]\mathfrak{K}(W_{s_-}) \arrow[ur, "\Phi_{s_+}"'] & \end{tikzcd}$$

W_t as resolvent matrix

For $t \in I$ consider

$$S_1 := \Big\{(x;y) \in T(\mathfrak{h}|_{(\sigma_0,t)}) :$$
$$\pi_{l,1}\Gamma(\mathfrak{h}|_{(\sigma_0,t)})(x;y) = 0, \pi_r\Gamma(\mathfrak{h}|_{(\sigma_0,t)})(x;y) = 0\Big\}$$

$$u : (x;y) \mapsto \pi_{l,2}\Gamma(\mathfrak{h}|_{(\sigma_0,t)})(x;y),\ (x;y) \in T(\mathfrak{h}|_{(\sigma_0,t)})$$

Then S_1 is symmetric with defect 1 and $u|_{S_1^*}$ is continuous. The matrix function W_t is a u-resolvent matrix of S_1.

W_t from defect elements

Let $\phi_z, \psi_z \in \ker(T(\mathfrak{h}|_{(\sigma_0,t)}))$ be such that

$$\pi_l\Gamma(\mathfrak{h}|_{(\sigma_0,t)})(\phi_z; z\phi_z) = \begin{pmatrix}1\\0\end{pmatrix},\ \pi_l\Gamma(\mathfrak{h}|_{(\sigma_0,t)})(\psi_z; z\psi_z) = \begin{pmatrix}0\\1\end{pmatrix}$$

Then

$$W_t(z) = \begin{pmatrix}\pi_r\Gamma(\mathfrak{h}|_{(\sigma_0,t)})(\phi_z; z\phi_z)^T\\ \pi_r\Gamma(\mathfrak{h}|_{(\sigma_0,t)})(\psi_z; z\psi_z)^T\end{pmatrix}$$

The maximal chain (W_t)

Constructing (W_t) from $\mathfrak{h}$: On each interval (σ_{i-1}, σ_i) the matrix function W_t shall be a solution of the differential equation

$$\frac{d}{dt} W_t(z) J = z W_t(z) H_i(t), \; x \in (\sigma_{i-1}, \sigma_i)$$

- On $[\sigma_0, \sigma_1)$, (W_t) is the unique solution with $W_{\sigma_0} = I$.

The maximal chain (W_t)

Constructing (W_t) from $\mathfrak{h}$: On each interval (σ_{i-1}, σ_i) the matrix function W_t shall be a solution of the differential equation

$$\frac{d}{dt} W_t(z) J = z W_t(z) H_i(t), \; x \in (\sigma_{i-1}, \sigma_i)$$

- On $[\sigma_0, \sigma_1)$, (W_t) is the unique solution with $W_{\sigma_0} = I$.

Theorem. Each solution on (σ_{i-1}, σ_i) has regularized boundary values at the singularities.

- Once $W_t|_{(\sigma_{i-1}, \sigma_i)}$ is known, choose for $W_t|_{(\sigma_i, \sigma_{i+1})}$ the unique solution whose regularized boundary values at σ_i+ fit those of $W_t|_{(\sigma_{i-1}, \sigma_i)}$ at σ_i-, the interface parameters $d_{i,j}$, and the jump parameters $\ddot{o}_i, c_{i,j}$.

The maximal chain (W_t)

Constructing $\mathfrak{h}$ from (W_t):

- Since $\det W_t(z) = 1$, on each interval (σ_{i-1}, σ_i), the Hamiltonian function H_i can be computed immediately from the differential equation $\frac{d}{dt} W_t(z) J = z W_t(z) H_i(t)$.
- The parameters $d_{i,j}, \ddot{o}_i, c_{i,j}$ associated with a singularity can be computed via a set a recursive formulas from the Taylor coefficients of entries of W_t and the solutions of the Hamiltonian differential equation assuming prescribed values at the points e_i.

The class $\mathcal{F}$ of distributions

Starting point is the indefinite analogue of the Herglotz-integral representation:

Theorem. We have $q \in \mathcal{N}_{<\infty}$ if and only if there exists

- a distribution ϕ on $\mathbb{R} \cup \{\infty\}$ which, off some finite set of points, coincides with a positive (possibly unbounded) Borel measure,
- a rational function r analytic and real valued along $\mathbb{R}$,

such that

$$q(z) = r(z) + \phi\Big([\frac{1}{t-z} - \frac{t}{1+t^2}] (1+t^2) \Big)$$

The class $\mathcal{F}$ of distributions

Strictly speaking, the class $\mathcal{F}$ consists of all pairs (r, ϕ) where r and ϕ are as above.

To shorten notation, we just write $\phi \in \mathcal{F}$, and intuitively understand by ϕ a distribution on $\mathbb{R} \cup \{\infty\}$ plus a finite sum of Dirac distributions (and their derivatives) at nonreal points.

Preparation

A Hamiltonian belongs to the class $\mathbb{H}$, if and only if it can be considered as a part of a general Hamiltonian $\mathfrak{h}$ with $\operatorname{ind}_- \mathfrak{h} > 0$. This general Hamiltonian can be chosen to have a certain simple form.

In turn, a general Hamiltonian $\mathfrak{h}$ is of the mentioned form, if and only if its Weyl-coefficient satisfies $\operatorname{ind}_- q_{\mathfrak{h}} > 0$ and

$$\lim_{z \hat{\to} i\infty} \frac{q_{\mathfrak{h}}(z)}{z^{2\kappa-1}} \in (-\infty, 0) \text{ or } \lim_{z \hat{\to} i\infty} \left| \frac{q_{\mathfrak{h}}(z)}{z^{2\kappa-1}} \right| = \infty ,$$

where $\kappa := \operatorname{ind}_- q_{\mathfrak{h}}$

Constructing FT

Given $H \in \mathbb{H}$, we build the general Hamiltonian according to Step 1. The Fourier transform of the Pontryagin space $\mathcal{P}(\mathfrak{h})$ onto $\Pi(\phi)$, where ϕ is the distribution in the integral representation of the Weyl-coefficient $q_{\mathfrak{h}}$, can be restricted to obtain a Fourier transform onto a space $L^2(\mu)$.

Inverse construction

Given $\mu \in \mathbb{M}$ with $\Delta(\mu) > 0$, consider the function

$$Q(z) := \int_{-\infty}^{\infty} \Big(\frac{1}{t-z} - (t+z) \sum_{k=1}^{\Delta(\mu)+1} \frac{(1+z^2)^{k-1}}{(1+t^2)^k} \Big) \, d\mu(t) \, .$$

Then $Q \in \mathcal{N}_{<\infty}$ and has the asymptotics mentioned in Step 1.

Hence, the general Hamiltonian whose Weyl-coefficient equals Q has a part being a Hamiltonian $H \in \mathbb{H}$. This Hamiltonian does the job.

Regularized boundary values

Let $H \in \mathbb{H}$ and fix $x_0 \in (\sigma_0, \sigma_1)$. Then there exists a unique sequence $(\mathfrak{w}_k)_{k \in \mathbb{N}_0}$ of absolutely continuous functions on (σ_0, σ_1), such that

$$\mathfrak{w}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (\mathfrak{w}_{l+1})' = JH\mathfrak{w}_l,\ l \geq 0$$

$$\mathfrak{w}_l \in L^2\big(H|_{(\sigma_0, x_0)}\big),\ l \geq \Delta(H), \quad \mathfrak{w}_l(x_0) \in \operatorname{span}\Big\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\Big\},\ l \geq 0\,.$$

For each $z \in \mathbb{C}$, the space $\mathfrak{N}_z$ of all solutions $\psi(x; z)$ of the Hamiltonian system is a linear space of dimension 2. For $z = 0$, this space is trivial; it contains all constant functions.

Regularized boundary values

Theorem. For each $\psi(.;z) \in \mathfrak{N}_z$ the following limits exist (and do not depend on the choice of x_0).

$$\begin{aligned} \operatorname{rbv}_1 \psi(.;z) := \lim_{x \searrow \sigma_0} \Bigg[\sum_{l=0}^{\Delta} z^l \mathfrak{w}_l(x)^* J \bigg(\psi(x;z) + \\ + (0,1)\psi(.;z) \sum_{k=\Delta+1}^{2\Delta-l} z^k \mathfrak{w}_k(x) \bigg) \Bigg] \end{aligned}$$

$$\operatorname{rbv}_2 \psi(.;z) := \lim_{x \searrow \sigma_0} (0,1)\psi(x;z)$$

The map $\psi(.;z) \mapsto \operatorname{rbv} \psi(.;z) := (\operatorname{rbv}_1 \psi(.;z), \operatorname{rbv}_2 \psi(.;z))$ is a bijection of $\mathfrak{N}_z$ onto $\mathbb{C}^2$.

Regularized boundary values

Theorem. Let $\phi = (\phi_1, \phi_2)^T$ and $\theta = (\theta_1, \theta_2)^T$ be the unique elements of $\mathfrak{N}_z$ such that $\operatorname{rbv} \phi(.; z) = (1, 0)$ and $\operatorname{rbv} \theta(.; z) = (0, 1)$. Then the limit

$$q(z) = \lim_{x \nearrow \sigma_1} \frac{\phi_1(x; z)\tau + \phi_2(x; z)}{\theta_1(x; z)\tau + \theta_2(x; z)}$$

exists locally uniformly on $\mathbb{C} \setminus \mathbb{R}$ and does not depend on $\tau \in \mathbb{R}$. We have

$$\phi(.; z) - q(z)\theta(.; z) \in L^2\left(H|_{(x_0,\sigma_1)}\right) .$$

Regularized boundary values

Theorem. The measure μ and the Fourier transform Θ are given by the formulas

$$\mu\big((s_1,s_2)\big) = \lim_{\delta\searrow 0}\lim_{\varepsilon\searrow 0}\int_{s_1+\delta}^{s_2-\delta} \operatorname{Im} q(t+i\varepsilon)\,dt\,,$$

$$-\infty < s_1 < s_2 < \infty$$

$$(\Theta f)(t) = \int_{\sigma_0}^{\sigma_1} \theta(x;t)^* H(x) f(x)\,dx, \quad t \in \mathbb{R},$$

$$f \in L^2(H),\ \sup\operatorname{supp} f < \sigma_1$$

$$(\Theta^{-1} g)(x) = \int_{-\infty}^{\infty} g(t)\theta(x;t)\,d\mu(t), \quad x \in (a,b),$$

$g \in L^2(\mu),\ \operatorname{supp} g$ compact

Diagonal Hamiltonians

The conditions (I) and (HS) need no further simplification; they are obvious growth conditions on H. The condition (Δ) is more involved.

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Let H be a diagonal Hamiltonian. We denote by Λ the operator which assigns to a scalar function $f:(\sigma_0,x_0)\to\mathbb{C}$ the function

$$(\Lambda f)(x) := \int_{\sigma_0}^{x}\Big(\int_{x_0}^{t} f(s)\,H(s)_{11}(s)ds\Big)H(s)_{22}(t)\,dt$$

and is defined whenever all integrals exist.

Diagonal Hamiltonians

Theorem. Let H be a diagonal Hamiltonian which satisfies (I) and (HS), and define

$$N := \sup\left\{n \in \mathbb{N}_0 :\ \Lambda^n 1 \notin L^2(H_{11})\right\} \in \mathbb{N}_0 \cup \{\infty\}\,.$$

Then (Δ) holds if and only if $N < \infty$. Moreover,

$$\Delta(H) = \begin{cases} 2N+1 & , \ \int_{x_0}^{x} (\Lambda^N 1)(t)\, H(t)_{11}\, dt \in L^2(H_{22}) \\ 2N+2 & , \ \int_{x_0}^{x} (\Lambda^N 1)(t)\, H(t)_{11}\, dt \notin L^2(H_{22}) \end{cases}$$

$\hookleftarrow$

Conditions (I) and (HS)

(I) $\int_{\sigma_0}^{x_0} H(t)_{22}\,dt < \infty;$

(HS) $\int_{\sigma_0}^{x_0} \left(\int_t^{x_0} H(s)_{11}\,ds \right) \cdot H(t)_{22}\,dt < \infty.$

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Condition (Δ)

(Δ) Define functions $h_k : (\sigma_0, x_0) \to \mathbb{C}^2$ recursively by

$$h_0(x) := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad h_k(x) := \int_{x_0}^{x} JH(y)h_{k-1}(y)\,dy,\ k \in \mathbb{N}\,.$$

Then there exists $f \in \operatorname{span}\left\{h_k :\ k \leq N\right\} \setminus \{0\}$ with $\int_{\sigma_0}^{x_0} f(t)^* H(t) f(t)\,dt < \infty$.

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Sturm-Liouville operators

Consider an equation of the form

$$-(p(x)y'(x))' + q(x)y(x) = \lambda w(x)y(x)\,.$$

If the potential term q vanishes identically, this equation can be rewritten immediately to the Hamiltonian system with

$$H(x) := \begin{pmatrix} w(x) & 0 \\ 0 & \frac{1}{p(x)} \end{pmatrix}$$

Hence, it is simple to derive corresponding spectral results for equations of this kind.

The Bessel equation revisited

As we have already said, for the Bessel equation the transformation to a Hamiltonian system can be carried out explicitly; the arising Hamiltonian being

$$H_\alpha(x) = \begin{pmatrix} x^{-\alpha} & 0 \\ 0 & x^\alpha \end{pmatrix}$$

This Hamiltonian belongs to $\mathbb{H}$ and $\Delta(H_\alpha) = \left\lfloor \frac{\alpha+1}{2} \right\rfloor$.

If α is not an odd integer, the Weyl-coefficient of the general Hamiltonian used in the proofs is nothing but the function $q_\alpha(z) := c_\alpha z^\alpha$. For $\alpha \in 2\mathbb{N} - 1$ it includes a logarithmic term.

Conditions (SM_1)–(SM_3)

($\mathbf{SM_1}$) The set $\operatorname{supp}\tau \cap (-\infty, 0) \neq \emptyset$ contains exactly one point.

($\mathbf{SM_2}$) The measure τ is discrete, and has no point mass at 0.

Write

$$\operatorname{supp}\tau = \{\xi\} \cup \{\xi_1, \xi_2, \xi_3, \dots\}$$

with $\xi < 0 < \xi_1 < \xi_2 < \xi_3 < \dots$, and denote by σ and $\sigma_1, \sigma_2, \sigma_3, \dots$ the weights of the point masses of τ at the points ξ and $\xi_1, \xi_2, \xi_3, \dots$, respectively.

($\mathbf{SM_3}$)

$$\sum_k \frac{\sigma_k}{\xi_k} < \infty\,.$$

Conditions (SM$_4$), (SM$_5$)

(SM$_4$) The limit $\lim_{k\to\infty}\frac{k^2}{\xi_k}$ exists in $[0,\infty)$.

Set $\Gamma(z) := \prod_k \left(1-\frac{z}{\xi_k}\right)$.

(SM$_5$)
$$\sum_k \xi_k^{-3}\frac{1}{\Gamma'(\xi_k)^2\sigma_k} < \infty\,.$$

Conditions (SM_6), (SM_7)

Set $\Xi(x) := \left[\sum_k \frac{(-x)\Gamma(x)^2}{(1-\frac{x}{\xi_k})^2} \cdot \xi_k^{-3} \frac{1}{\Gamma'(\xi_k)^2\sigma_k}\right]^{-1}$.

($\mathbf{SM_6}$) $0 < \sigma \leq \Xi(\xi)$.

($\mathbf{SM_7}$) If $\sigma = \Xi(\xi)$, then

$$\sum_k \xi_k^{-2} \frac{1}{\Gamma'(\xi_k)^2\sigma_k} = \infty\,.$$