Homogeneous de Branges spaces

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joint work with B.Eichinger

De Branges' Hilbert spaces of entire functions

In the late 1950's Louis de Branges initiated a theory of certain reproducing kernel Hilbert spaces.

Definition

A *de Branges space* is a Hilbert space $\mathcal{H}$ $(\neq \{0\})$ which satisfies:

- the elements of $\mathcal{H}$ are entire functions
- $\forall w \in \mathbb{C}$: $F \mapsto F(w)$ continuous on $\mathcal{H}$
- $\forall w \in \mathbb{C}, F \in \mathcal{H}$:

$$F(w) = 0 \;\Rightarrow\; \frac{F(z)}{z-w} \in \mathcal{H} \wedge \left\| \frac{z-\overline{w}}{z-w} F(z) \right\|_{\mathcal{H}} = \|F\|_{\mathcal{H}}$$

- $\forall F \in \mathcal{H}$: $F^{\#}(z) := \overline{F(\overline{z})} \in \mathcal{H} \wedge \|F^{\#}\|_{\mathcal{H}} = \|F\|_{\mathcal{H}}$

These spaces are very closely related to the spectral theory of certain symmetric operators with deficiency index $(1,1)$. Namely, *entire operators* in the sense of Mark G. Krein.

Also a close relation with complex analysis exists. Namely, de Branges spaces can be considered as *shift-coinvariant subspaces* of the Hardy space $H^2(\mathbb{C}^+)$.

We do not discuss these aspects further.

Example (Polynomials)

Let $s_0, \ldots, s_{2N} \in \mathbb{R}$ such that

$$\begin{pmatrix} s_0 & s_1 & \cdots & s_N \\ s_1 & & \iddots & \vdots \\ \vdots & \iddots & & \vdots \\ s_N & \cdots & \cdots & s_{2N} \end{pmatrix}$$

is positive definite. Define

$$\mathcal{H} := \Big\{ \sum_{n=0}^{N} \alpha_n z^n \Big| \alpha_n \in \mathbb{C} \Big\}$$

$$\Big(\sum_{n=0}^{N} \alpha_n z^n, \sum_{m=0}^{N} \beta_m z^m \Big)_{\mathcal{H}} := \sum_{n,m=0}^{N} s_{n+m} \alpha_n \overline{\beta_m}$$

Then $\langle \mathcal{H}, (.,.)_{\mathcal{H}} \rangle$ is a de Branges space.

Example (Fourier transform and Paley-Wiener spaces)

Let $(\mathcal{F}f)(z) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t)e^{-itz}\,\mathrm{d}t$ for $f \in L^2(\mathbb{R})$.

The Paley-Wiener Theorem says: Let $a > 0$, then

$$F \text{ entire with exponential type } \leq a \;\wedge\; F|_{\mathbb{R}} \in L^2(\mathbb{R})$$
$$\Longleftrightarrow$$
$$\exists f \in L^2(-a,a)\colon\; F(z) = (\mathcal{F}f)(z)$$

Define, for each $a > 0$, the *Paley-Wiener space* of type a as

$$\mathcal{P}W_a := \mathcal{F}\big(L^2(-a,a)\big), \quad (F,G)_{\mathcal{P}W_a} := \int_{\mathbb{R}} F(x)\overline{G(x)}\,\mathrm{d}x$$

Then $\langle \mathcal{P}W_a, (.,.)_{\mathcal{P}W_a} \rangle$ is a de Branges space.

De Branges spaces can be characterised via the form of their *reproducing kernel*: Let $K(w,z)$ be the function with

$$\forall w \in \mathbb{C}, F \in \mathcal{H}\colon\ \big(F, K(w, \_)\big)_{\mathcal{H}} = F(w)$$

Theorem

A function $K : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is the reproducing kernel of a de Branges space, if and only if

$$K(w,z) = \frac{B(z)\overline{A(w)} - A(z)\overline{B(w)}}{z - \overline{w}}$$

where

- A, B *entire, real-valued along the real axis, no common zeroes,*
- $\operatorname{Im} \frac{B(z)}{A(z)} > 0$ *for all* $z \in \mathbb{C}^+$.

Definition

If A, B have the properties listed in the previous theorem, write $\mathcal{H}(A, B)$ for the de Branges space with kernel

$$K(w, z) = \frac{B(z)\overline{A(w)} - A(z)\overline{B(w)}}{z - \overline{w}}$$

The spaces $\mathcal{H}$ and the pairs (A, B) as in the theorem are not in a one-to-one correspondence, but almost.

Example (Paley-Wiener spaces cont.)

Let $a > 0$. Then $\mathcal{PW}_a = \mathcal{H}(A, B)$ with

$$A(z) := \cos(az), \quad B(z) := \sin(az)$$

The reproducing kernel is

$$K(w, z) = \frac{\sin\left[a(z - \overline{w})\right]}{z - \overline{w}}$$

Homogeneity in de Branges spaces

Example (Paley-Wiener spaces cont.)

Fix $a > 0$. If $F \in \mathcal{P}W_a$ and $c \in (0,1]$, consider $G(z) := F(cz)$. Then $G \in \mathcal{P}W_a$ and

$$\begin{aligned} \|G\|^2_{\mathcal{P}W_a} &= \int_{\mathbb{R}} |G(x)|^2 \, \mathrm{d}x = \int_{\mathbb{R}} |F(cx)|^2 \, \mathrm{d}x \\ &= \frac{1}{c} \int_{\mathbb{R}} |F(y)|^2 \, \mathrm{d}y = \frac{1}{c} \cdot \|F\|^2_{\mathcal{P}W_a} \end{aligned}$$

Hence, the map

$$F(z) \mapsto c^{\frac{1}{2}} F(cz)$$

induces an isometry of $\mathcal{P}W_a$ into itself.

Example (Paley-Wiener spaces cont.)

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Are there also other spaces with this – or a similar – property ?

Example (Spaces given by Bessel functions)

Fix $a > 0$ and $p > -\frac{1}{2}$. Set $\nu_1 := p - \frac{1}{2}, \nu_2 := p + \frac{1}{2}$, and define

$$A(z) := 2^{\nu_1}\Gamma(\nu_2) z^{-\nu_1} a^{-\nu_1} J_{\nu_1}(az)$$
$$B(z) := 2^{\nu_1}\Gamma(\nu_2) z^{-\nu_1} a^{\nu_2} J_{\nu_2}(az)$$

Then for each $c \in (0, 1]$ the map

$$F(z) \mapsto c^{p+\frac{1}{2}} F(cz)$$

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- For $p = 0$ we recover the Paley-Wiener spaces: The Bessel functions $J_{-\frac{1}{2}}(z), J_{\frac{1}{2}}(z)$ are just the trigonometric functions $\cos z, \sin z$ multiplied with some constants.

Definition

Let $p > -\frac{1}{2}$. We call a de Branges space $\mathcal{H}$ *homogeneous with power* p, if for each $c \in (0, 1]$ the map

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  One cannot avoid it, somewhere there occurs a shift by $\frac{1}{2}$ in the exponent. We chose to let it be here.

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- Why do we write the exponent as $p + \frac{1}{2}$?
  One cannot avoid it, somewhere there occurs a shift by $\frac{1}{2}$ in the exponent. We chose to let it be here.
- Why $p > -\frac{1}{2}$?
  For $p < -\frac{1}{2}$ such spaces do not exist. For $p = -\frac{1}{2}$ there is only one, namely $\mathcal{H} = \operatorname{span}\{1\}$.

Definition

Let $\mathrm{Hol}(\mathbb{C})$ be the set of entire functions. For $p \in \mathbb{R}$ define

$$\odot_p : \begin{cases} \mathbb{R}^+ \times \mathrm{Hol}(\mathbb{C}) & \to & \mathrm{Hol}(\mathbb{C}) \\ F(z) & \mapsto & c^p F(cz) \end{cases}$$

- The map $\odot_p$ is a group action of $\mathbb{R}^+$ on $\mathrm{Hol}(\mathbb{C})$, and continuous w.r.t. locally uniform convergence.

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Lemma

Let $p > -\frac{1}{2}$. Let $\mathcal{H}$ be a de Branges space, and choose A, B such that $\mathcal{H} = \mathcal{H}(A, B)$. Then $\mathcal{H}$ is homogeneous with power p, if and only if

$$\forall c \in (0, 1]\colon\ \mathcal{H}(c \odot_p A, c \odot_p B) \subseteq \mathcal{H} \text{ isometrically}$$

TASK:

Determine all homogeneous de Branges spaces and describe their structure.

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Most of that was done by de Branges in

L. de Branges, *Homogeneous and periodic spaces of entire functions*, Duke Math. J. 29 (1962), pp. 203–224.

What we add to this:

- ▷ We give an explicit parameterisation of all homogeneous spaces.
- ▷ We correct a mistake in de Branges' paper.
- ▷ We give explicit formulas also for the connection to quantities related with the structure of the space.

Structure theory I

De Branges subspaces

The structure of a de Branges space is very well described by the set of all its *de Branges - subspaces*.

Definition

Let $\mathcal{H}$ be a de Branges space. We write

$$\mathrm{Sub}(\mathcal{H}) := \{\mathcal{H}' \mid \mathcal{H}' \text{ de Branges space, } \mathcal{H}' \subseteq \mathcal{H} \text{ isometrically}\}$$

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$$\mathrm{Sub}(\mathcal{H}) := \{\mathcal{H}' \mid \mathcal{H}' \text{ de Branges space}, \ \mathcal{H}' \subseteq \mathcal{H} \text{ isometrically}\}$$

Theorem

For every de Branges space $\mathcal{H}$ the set $\mathrm{Sub}(\mathcal{H})$ is a chain. It can be parameterised as

$$\{0\} \subsetneq \cdots \subsetneq \underbrace{\mathcal{H}_a}_{a \in (0,1] \setminus \{\dots\}} \subsetneq \cdots \subsetneq \mathcal{H}_1 = \mathcal{H}$$

Example (Paley-Wiener spaces cont.)

Consider the Paley-Wiener space $\mathcal{P}W_1$. Its chain of de Branges subspaces is

$$\mathrm{Sub}(\mathcal{P}W_1) = \{\mathcal{P}W_a \mid a \in (0,1]\}$$

Remember also that $\mathcal{P}W_a = \mathcal{F}(L^2(-a,a))$.

chain of de Branges subspaces $(a \in (0,1])$

$$\{0\} \subsetneq \cdots \subsetneq \mathcal{H}(a \odot_p A, a \odot_p B) \subsetneq \cdots \subsetneq \underset{\mathcal{H}}{\overset{\parallel}{\mathcal{H}(A,B)}}$$

Theorem

Let $p > -\frac{1}{2}$. Let $\mathcal{H}$ be a homogeneous de Branges space, and choose A, B such that $\mathcal{H} = \mathcal{H}(A,B)$. Then

$$\operatorname{Sub}(\mathcal{H}) = \big\{\mathcal{H}(a \odot_p A, a \odot_p B) \mid a \in (0,1]\big\}$$

For all $b \in [0,1]$ (here we set $\mathcal{H}(0 \odot_p A, 0 \odot_p B) := \{0\}$)

$$\mathcal{H}(b \odot_p A, b \odot_p B) = \bigcap_{a>b} \mathcal{H}(a \odot_p A, a \odot_p B)$$

$$\mathcal{H}(b \odot_p A, b \odot_p B) = \operatorname{Clos} \bigcup_{a<b} \mathcal{H}(a \odot_p A, a \odot_p B)$$

Structure theory II

Measures associated with a space

The norm of a de Branges space can be computed by integration along the real axis.

Theorem

If $\mathcal{H} = \mathcal{H}(A, B)$, *then*

$$\forall F \in \mathcal{H}\colon \ \|F\|_{\mathcal{H}}^2 = \int_{\mathbb{R}} |F(t)|^2 \cdot \tfrac{\mathrm{d}t}{A(t)^2+B(t)^2}$$

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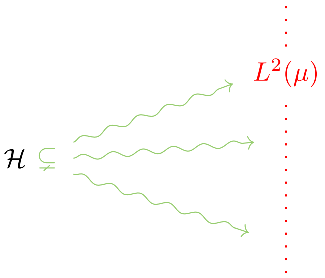
Definition

Let $\mathcal{H}$ be a de Branges space. We write

$$\mathrm{Meas}(\mathcal{H}) := \Big\{\mu \mid \mu \text{ positive Borel measures on } \mathbb{R}, \\ \forall F \in \mathcal{H}\colon\ \|F\|_{\mathcal{H}} = \int_{\mathbb{R}} |F(t)|^2 \cdot \mathrm{d}\mu(t)\Big\}$$

If $\mu \in \mathrm{Meas}(\mathcal{H})$, we say that $\mathcal{H} \subseteq L^2(\mu)$ isometrically.

There always exist infinitely many measures containing $\mathcal{H}$ isometrically.



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$$L^2(\mu)$$

$$\{0\} \subsetneq \cdots \subsetneq \underset{\left(a \in (0,1] \setminus \{\ldots\}\right)}{\mathcal{H}_a} \subsetneq \cdots \subsetneq \mathcal{H} \subsetneq$$

Definition

Let $\mathcal{H}$ be a de Branges space and $\mu \in \mathrm{Meas}(\mathcal{H})$. We write

$$\mathrm{Sub}(\mu, \mathcal{H}) := \big\{ \mathcal{H}' \mid \mathcal{H}' \text{ de Branges space}, \\ \mathcal{H} \subseteq \mathcal{H}' \subseteq L^2(\mu) \text{ isometrically} \big\}$$

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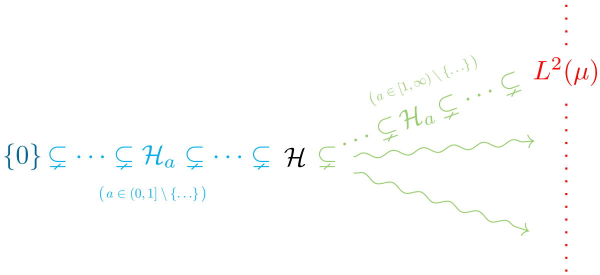
Theorem

For every de Branges space $\mathcal{H}$ and $\mu \in \mathrm{Meas}(\mu)$ the set $\mathrm{Sub}(\mu, \mathcal{H})$ is a chain. It can be parameterised as

$$\begin{array}{c} \mathcal{H}_1 \subsetneq \cdots \subsetneq \mathcal{H}_a \subsetneq \cdots \subsetneq L^2(\mu) \\ \| \qquad\qquad a \in [1,\infty) \setminus \{\ldots\} \qquad\qquad \\ \mathcal{H} \qquad\qquad\qquad\qquad\qquad\qquad\qquad \end{array}$$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$L^2(\mu)$

Example (Paley-Wiener spaces cont.)

Consider the Paley-Wiener space $\mathcal{P}W_1$.

- By definition its norm is given as an integral: we have $\mathcal{P}W_1 \subseteq L^2(\mathrm{d}t)$ where $\mathrm{d}t$ is the Lebesgue measure.
- Other elements of $\mathrm{Meas}(\mathcal{P}W_1)$ are for example

$$\mathrm{d}\mu_\alpha(t) = \frac{\alpha^2 \cos t \sin t}{\alpha^2 \cos^2 t + \sin^2 t}\,\mathrm{d}t$$

  where $\alpha \in \mathbb{R} \setminus \{0\}$, or

$$\mu_0 := \sum_{n\in\mathbb{Z}} \delta_{\pi(n+\frac{1}{2})}$$

  where δ_t denotes the unit point-mass at t.

Example (Paley-Wiener spaces cont.)

Consider the Paley-Wiener space $\mathcal{P}W_1$, the measures $\mu_\alpha \in \mathrm{Meas}(\mathcal{P}W_1)$ we saw.

- $\mathrm{Sub}(\mathrm{d}t, \mathcal{P}W_1) = \{\mathcal{P}W_a \mid a \in [1, \infty)\}$
- For $\alpha \in \mathbb{R} \setminus \{0\}$ the chain $\mathrm{Sub}(\mu_\alpha, \mathcal{P}W_1)$ can be computed (but formulas are too long).
- $\mathrm{Sub}(\mu_0, \mathcal{P}W_1) = \{\mathcal{P}W_1\}$.

Example (Paley-Wiener spaces cont.)

Consider the Paley-Wiener space $\mathcal{P}W_1$, the measures $\mu_\alpha \in \mathrm{Meas}(\mathcal{P}W_1)$ we saw.

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- For $\alpha \in \mathbb{R} \setminus \{0\}$ the chain $\mathrm{Sub}(\mu_\alpha, \mathcal{P}W_1)$ can be computed (but formulas are too long).
- $\mathrm{Sub}(\mu_0, \mathcal{P}W_1) = \{\mathcal{P}W_1\}$.

In general all measures in $\mathrm{Meas}(\mathcal{H})$ are of equal rights. For $\mathcal{P}W_1$, however, the Lebesgue measure is in some sense the "most natural" element of $\mathrm{Meas}(\mathcal{P}W_1)$.

Theorem

Let $p > -\frac{1}{2}$. Let $\mathcal{H}$ be a homogeneous de Branges space, and choose A, B such that $\mathcal{H} = \mathcal{H}(A, B)$. Then there exists a unique measure $\mu \in \mathrm{Meas}(\mathcal{H})$, such that

$$\mathrm{Sub}(\mu, \mathcal{H}) = \{\mathcal{H}(a \odot_p A, a \odot_p B) \mid a \in [1, \infty)\}$$

For all $b \in [1, \infty]$ (here we set $\mathcal{H}(\infty \odot_p A, \infty \odot_p B) := L^2(\mu)$)

$$\mathcal{H}(b \odot_p A, b \odot_p B) = \bigcap_{a>b} \mathcal{H}(a \odot_p A, a \odot_p B)$$

$$\mathcal{H}(b \odot_p A, b \odot_p B) = \mathrm{Clos} \bigcup_{a<b} \mathcal{H}(a \odot_p A, a \odot_p B)$$

A homogeneous de Branges space $\mathcal{H} = \mathcal{H}(A, B)$ has a distinguished measure and chain. For $a > 0$ denote

$$\mathcal{H}_a := \mathcal{H}(a \odot_p A, a \odot_p B)$$

$$L^2(\mu)$$

$$\{0\} \subsetneq \cdots \subsetneq \underset{(a \in (0,1])}{\mathcal{H}_a} \subsetneq \cdots \subsetneq \underset{\substack{\| \\ \mathcal{H}}}{\mathcal{H}_1} \subsetneq \cdots \subsetneq \underset{(a \in [1,\infty))}{\mathcal{H}_a} \subsetneq \cdots \subsetneq L^2(\mu)$$

Parameterisation of homogeneous spaces

Parameters
(P, ψ)
in parameter set
$\mathbb{P}_p \subseteq \mathbb{R}^{2\times 2} \times \mathbb{R}$

$A_{P,\psi}, B_{P,\psi}$ s.t. $\mathcal{H}(A_{P,\psi}, B_{P,\psi})$ is homogeneous

$\mu_{P,\psi}$ s.t. for all $a > 0$ $\mathcal{H}(a \odot_p A_{P,\psi}, a \odot_p B_{P,\psi}) \subseteq L^2(\mu_{P,\psi})$ isometrically

The parameter set

We define a set of parameters depending on the power p.

Definition

If $p > -\frac{1}{2}$ and $p \neq 0$, set

$$\mathbb{P}_p := \Big\{(P,\psi) \in \mathbb{R}^{2\times 2} \times \mathbb{R} \;\Big|\; P \geq 0,\ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\psi \\ 2p \end{pmatrix} \notin \ker P \Big\}$$

Set

$$\begin{aligned} \mathbb{P}_0 := &\Big\{(P,\psi) \in \mathbb{R}^{2\times 2} \times \mathbb{R} \;\Big|\; P \geq 0,\ \ker P = \{0\},\ \psi = 0 \Big\} \\ &\cup \Big\{(P,\psi) \in \mathbb{R}^{2\times 2} \times \mathbb{R} \;\Big|\; P \geq 0,\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \notin \ker P,\ \psi \neq 0 \Big\} \end{aligned}$$

Formulas for $A_{P,\psi}, B_{P,\psi}$

Recall the definition of *confluent hypergeometric* (limit) functions:

$$M(\alpha,\beta,z) := \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\beta)_n} \cdot \frac{z^n}{n!}, \quad {}_0F_1(\beta,z) := \sum_{n=0}^{\infty} \frac{1}{(\beta)_n} \cdot \frac{z^n}{n!},$$

where $\alpha, z \in \mathbb{C}$ and $\beta \in \mathbb{C} \setminus (-\mathbb{N}_0)$. The symbol $(\text{␣})_n$ is the *rising factorial*

$$(\alpha)_0 = 1, \quad (\alpha)_{n+1} = (\alpha)_n(\alpha+n) \quad \text{for } n \in \mathbb{N}_0.$$

Definition

Let $p > -\frac{1}{2}$, and $(P, \psi) \in \mathbb{P}_p$ be given.
Write $P = \begin{pmatrix} \kappa_1 & \kappa_3 \\ \kappa_3 & \kappa_2 \end{pmatrix}$, let $\kappa \in \mathbb{C}$ be a square root of $\det P$, and set $\sigma := 2p\kappa_3 - \psi\kappa_1$ and $\alpha := \frac{\sigma}{2i\kappa} + p$ if $\kappa \neq 0$.

- If $\det P \neq 0$, set

$$A_{P,\psi}(z) = e^{i\kappa z}\Big[\tfrac{1}{2}M(\alpha, 2p+1, -2i\kappa z) + \tfrac{1}{2}M(\alpha+1, 2p+1, -2i\kappa z) - \tfrac{\kappa_3}{2p+1} z M(\alpha+1, 2p+2, -2i\kappa z)\Big],$$

$$B_{P,\psi}(z) = e^{i\kappa z}\tfrac{\kappa_1}{2p+1} z M(\alpha+1, 2p+2, -2i\kappa z).$$

- If $\det P = 0$, set

$$A_{P,\psi}(z) = {}_0F_1(2p+1, -\sigma z) - \tfrac{\kappa_3}{2p+1} z\, {}_0F_1(2p+2, -\sigma z),$$

$$B_{P,\psi}(z) = \tfrac{\kappa_1}{2p+1} z\, {}_0F_1(2p+2, -\sigma z).$$

Theorem

Let $p > -\frac{1}{2}$.

- *For each* $(P, \psi) \in \mathbb{P}_p$ *the space* $\mathcal{H}(A_{P,\psi}, B_{P,\psi})$ *is well-defined and homogeneous with power* p.
- *For each de Branges space* $\mathcal{H}$ *which is homogeneous with power* p *there exists* $(P, \psi) \in \mathbb{P}_p$ *such that* $\mathcal{H} = \mathcal{H}(A_{P,\psi}, B_{P,\psi})$.
- *Let* $(P, \psi), (\tilde{P}, \tilde{\psi}) \in \mathbb{P}_p$ *and write* $P = \begin{pmatrix} \kappa_1 & \kappa_3 \\ \kappa_3 & \kappa_2 \end{pmatrix}$, $P = \begin{pmatrix} \tilde{\kappa}_1 & \tilde{\kappa}_3 \\ \tilde{\kappa}_3 & \tilde{\kappa}_2 \end{pmatrix}$. *Then* $\mathcal{H}(A_{P,\psi}, B_{P,\psi}) = \mathcal{H}(A_{\tilde{P},\tilde{\psi}}, B_{\tilde{P},\tilde{\psi}})$, *if and only if*

$$\kappa_1 = \tilde{\kappa}_1, \quad \det P = \det \tilde{P}, \quad \psi - \tilde{\psi} = \frac{2p}{\kappa_1}\big[\kappa_3 - \tilde{\kappa}_3\big].$$

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$$\kappa_1 = \tilde{\kappa}_1, \quad \det P = \det \tilde{P}, \quad \psi - \tilde{\psi} = \frac{2p}{\kappa_1}\big[\kappa_3 - \tilde{\kappa}_3\big].$$

- The spaces associated with the Bessel functions (with $a = 1$) are recovered by the parameter $(I, 0)$.

Formulas for $\mu_{P,\psi}$

Definition

Let $p > -\frac{1}{2}$, and $(P, \psi) \in \mathbb{P}_p$ be given. Write $P = \begin{pmatrix} \kappa_1 & \kappa_3 \\ \kappa_3 & \kappa_2 \end{pmatrix}$, let $\kappa \in \mathbb{C}$ be the nonnegative square root of $\det P$, and set $\sigma := 2p\kappa_3 - \psi\kappa_1$. Let $\mu_{P,\psi} \ll \mathrm{d}t$, with

$$\frac{d\mu_{P,\psi}}{\mathrm{d}t}(t) = \begin{cases} \mu_+(P,\psi) \cdot |t|^{2p} & \text{if } t > 0, \\ \mu_-(P,\psi) \cdot |t|^{2p} & \text{if } t < 0, \end{cases}$$

$$\mu_+(P,\psi) := \begin{cases} \frac{2^{2p}\kappa^{2p+1}|\Gamma(\frac{\sigma}{2i\kappa}+p+1)|^2}{\kappa_1\Gamma(2p+1)^2} \cdot e^{\pi\frac{\sigma}{2\kappa}} & \text{if } \det P \neq 0, \\ \frac{\pi\sigma^{2p+1}}{\kappa_1\Gamma(2p+1)^2} & \text{if } \det P = 0, \sigma > 0, \\ 0 & \text{if } \det P = 0, \sigma < 0, \end{cases}$$

$$\mu_-(P,\psi) := \begin{cases} \frac{2^{2p}\kappa^{2p+1}|\Gamma(\frac{\sigma}{2i\kappa}+p+1)|^2}{\kappa_1\Gamma(2p+1)^2} \cdot e^{-\pi\frac{\sigma}{2\kappa}} & \text{if } \det P \neq 0, \\ 0 & \text{if } \det P = 0, \sigma > 0, \\ \frac{\pi|\sigma|^{2p+1}}{\kappa_1\Gamma(2p+1)^2} & \text{if } \det P = 0, \sigma < 0. \end{cases}$$

Theorem

Let $p > -\frac{1}{2}$.

- *For each* $(P, \psi) \in \mathbb{P}_p$, *the measure* $\mu_{P,\psi}$ *is the unique measure such that*

$$\forall a > 0\colon\ \mathcal{H}(a \odot_p A_{P,\psi}, a \odot_p B_{P,\psi}) \subseteq L^2(\mu_{P,\psi}) \text{ isometrically}$$

- *Let* $(\mu_+, \mu_-) \in [0, \infty)^2 \setminus \{(0,0)\}$. *Then there exists* $(P, \psi) \in \mathbb{P}_p$ *such that* $\mu_+ = \mu_+(P, \psi)$ *and* $\mu_- = \mu_-(P, \psi)$.
- *Let* $(P, \psi), (\tilde{P}, \tilde{\psi}) \in \mathbb{P}_p$, *and write* $P = \begin{pmatrix} \kappa_1 & \kappa_3 \\ \kappa_3 & \kappa_2 \end{pmatrix}$ *and* $P = \begin{pmatrix} \tilde{\kappa}_1 & \tilde{\kappa}_3 \\ \tilde{\kappa}_3 & \tilde{\kappa}_2 \end{pmatrix}$. *Then* $\mu_{P,\psi} = \mu_{\tilde{P},\tilde{\psi}}$, *if and only if*

$$\kappa_1^{-\frac{2}{1+2p}} \det P = \tilde{\kappa}_1^{-\frac{2}{1+2p}} \det \tilde{P}$$

$$\kappa_1^{\frac{2p}{1+2p}} \psi - \tilde{\kappa}_1^{\frac{2p}{1+2p}} \tilde{\psi} = 2p\Big(\kappa_1^{-\frac{1}{1+2p}} \kappa_3 - \tilde{\kappa}_1^{-\frac{1}{1+2p}} \tilde{\kappa}_3\Big)$$