Universality limits for power bounded measures

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joint work with B.Eichinger and M.Lukic

Introduction

Setting (until we say differently):

- μ is a positive Borel measure on $\mathbb{R}$ with

$$\forall n \in \mathbb{N}\colon \int_{\mathbb{R}} |t|^n \, d\mu(t) < \infty,$$

  and such that the corresponding moment problem is determinate.
- $\mathbb{C}[z]_n := \{p \mid p \text{ polynomial}, \deg p \leq n\}$.

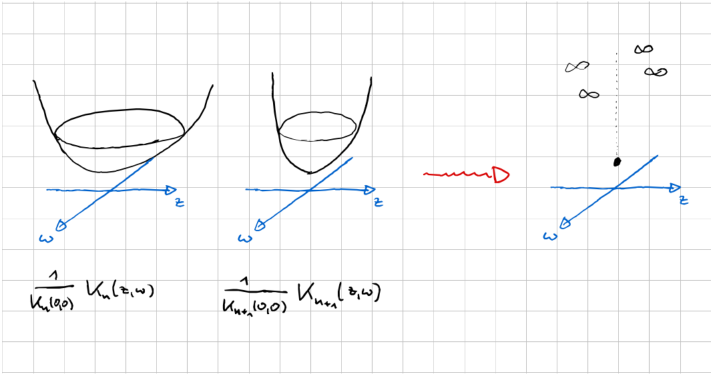
The space $\langle \mathbb{C}[z]_n, (.,.)_{L^2(\mu)} \rangle$ is a reproducing kernel Hilbert space.

Definition

Let $K_n(z,w)$ be the reproducing kernel of $\langle \mathbb{C}[z]_n, (.,.)_{L^2(\mu)} \rangle$. Then K_n is called the *Christoffel-Darboux kernel*.

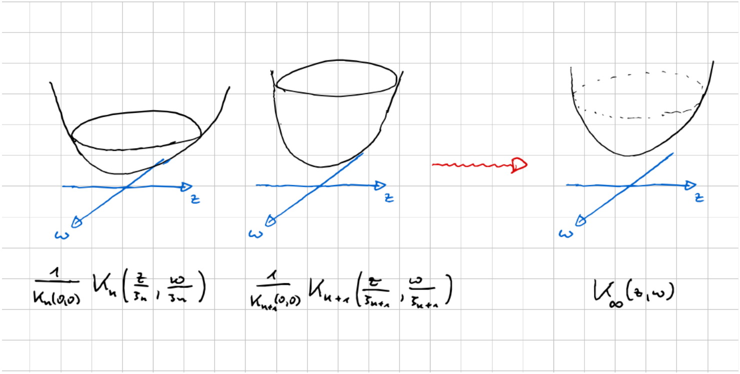
What happens if we send $n \to \infty$?

...things explode...



What happens if we send $n \to \infty$
and in the same time zoom into the vicinity of 0 ?

...if we are lucky a meaningful limit may exists...



Definition

Let $\tau_n > 0$, $n \in \mathbb{N}$. We say that a *rescaling limit exists* with rate τ_n, if the limit

$$K_\infty(z,w) = \lim_{n\to\infty} \frac{1}{K_n(0,0)} K_n\Big(\frac{z}{\tau_n}, \frac{w}{\tau_n}\Big)$$

exists locally uniformly on $\mathbb{C} \times \mathbb{C}$ and is not constant.

Basic questions:

- ▷ For which measures μ does a rescaling limit exist ?
- ▷ If it exists: how to find τ_n and how to compute $K_\infty(z,w)$?

Example

- $\mathrm{d}\mu(t) = e^{-V(t)}\,\mathrm{d}t$ (V polynomial with even degree and positive leading coefficient)

  $\Longrightarrow$ rescaling limit exists with $\tau_n = K_n(0,0)$, $K_\infty(z,w) = \frac{\sin(z-\overline{w})}{z-\overline{w}}$.

- $\mathrm{d}\mu(t) = g(t)\mathbb{1}_{[-1,1]}(t)|t|^\alpha\,\mathrm{d}t$ ($\alpha > -1$, g analytic and positive on $[-1,1]$)

  $\Longrightarrow$ rescaling limit exists with $\tau_n = K_n(0,0)^{\frac{1}{1+\alpha}}$. $K_\infty(z,w)$ is expressed with Bessel functions.

- $\mathrm{d}\mu(t) = g(t)\big[\sigma_-\mathbb{1}_{[-1,0)} + \sigma_+\mathbb{1}_{[0,1]}(t)\big]\,\mathrm{d}t$ ($\sigma_\pm \geq 0$, $\sigma_+ + \sigma_- > 0$, g analytic and positive on $[-1,1]$)

  $\Longrightarrow$ rescaling limit exists with $\tau_n = K_n(0,0)$. $K_\infty(z,w)$ is expressed with confluent hypergeometric functions.

Theorem (Eichinger-Lukic-Simanek 2021)

Assume that the nontangential limit

$$\Delta := \lim_{z \hat{\to} 0} \frac{y}{\pi} \int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{(t-x)^2 + y^2}$$

exists, and $0 < \Delta < \infty$.

$\Longrightarrow$ *rescaling limit exists with*

$$\tau_n = K_n(0,0), \quad K_\infty(z,w) = \frac{\sin[\pi\Delta(z-\overline{w})]}{\pi\Delta(z-\overline{w})}.$$

De Branges space viewpoint

chain of de Branges spaces exhausting $L^2(\mu)$

$$\{0\} \subseteq \cdots \subseteq \mathcal{H}\big(K_n(z,w)\big) \subseteq \mathcal{H}\big(K_{n+1}(z,w)\big) \subseteq \cdots \subseteq L^2(\mu)$$

rescaling limit

$$\mathcal{H}\big(K_\infty(z,w)\big)$$

de Branges space

Definition

Let $\omega > -1$ and $\mathcal{H}$ a de Branges space. Then $\mathcal{H}$ is called *homogeneous of order* ω, if

$$\forall a \in (0,1]\colon\ F(z) \mapsto a^{\omega+1} F(az) \text{ is isometry of } \mathcal{H} \text{ into itself}$$

Example

The Paley-Wiener space $\mathcal{H}\big(\frac{\sin(z-\overline{w})}{z-\overline{w}}\big)$ is homogeneous of order $-\frac{1}{2}$.

chain of de Branges spaces exhausting $L^2(\mu)$

$$\{0\} \subseteq \cdots \subseteq \mathcal{H}\big(K_n(z,w)\big) \subseteq \mathcal{H}\big(K_{n+1}(z,w)\big) \subseteq \cdots \subseteq L^2(\mu)$$

rescaling limit

$$\mathcal{H}\big(K_\infty(z,w)\big)$$

homogeneous de Branges space

A homogeneous de Branges space induces a whole chain of spaces.

Theorem (de Branges 1962, Eichinger-Woracek 2024)

Let $\mathcal{H}$ *be a homogeneous de Branges space of order* $\omega > -1$, *and let* $K(z,w)$ *be the reproducing kernel of* $\mathcal{H}$. *Set*

$$K^{[a]}(z,w) := a^{2(\omega+1)} K(az, aw), \quad a > 0.$$

Then

$$\{0\} \subseteq \cdots \subseteq \mathcal{H}\big(K^{[a]}(z,w)\big) \subseteq \cdots \subseteq \mathcal{H}\big(K^{[a']}(z,w)\big) \subseteq \cdots \subseteq L^2(\nu),$$

where ν *is of the form*

$$\mathrm{d}\nu(t) = \big[\sigma_- \mathbb{1}_{(-\infty,0)} + \sigma_+ \mathbb{1}_{(0,\infty)}(t)\big] \cdot |t|^{2\omega+1}\,\mathrm{d}t$$

with certain $\sigma_\pm \geq 0$, $\sigma_+ + \sigma_- > 0$.

Example

The Paley-Wiener space $\mathcal{H}\big(\frac{\sin(z-\overline{w})}{z-\overline{w}}\big)$ is homogeneous of order $-\frac{1}{2}$.

The measure associated to the Paley-Wiener space is the Lebesgue measure, and the induced chain is

$$\{0\} \subseteq \cdots \subseteq \mathcal{H}\Big(\frac{\sin[a(z-\overline{w})]}{a(z-\overline{w})}\Big) \subseteq \cdots \subseteq L^2(\mathrm{d}t).$$

chain of de Branges spaces exhausting $L^2(\mu)$

$$\{0\} \subseteq \cdots \subseteq \mathcal{H}\big(K_n(z,w)\big) \subseteq \mathcal{H}\big(K_{n+1}(z,w)\big) \subseteq \cdots \subseteq L^2(\mu)$$

rescaling limit

$$\mathcal{H}\big(K_\infty(z,w)\big)$$

homogeneous de Branges space

relation between measures

$$L^2(\nu) \supseteq \cdots \supseteq \mathcal{H}\big(K_\infty^{[a]}(z,w)\big) \supseteq \cdots \supseteq \mathcal{H}\big(K_\infty(z,w)\big) \supseteq \cdots \supseteq \{0\}$$

chain of de Branges spaces exhausting $L^2(\mu)$

$$\{0\} \subseteq \cdots \subseteq \mathcal{H}\big(K_n(z,w)\big) \subseteq \mathcal{H}\big(K_{n+1}(z,w)\big) \subseteq \cdots \subseteq L^2(\mu)$$

rescaling limit

chain of homogeneous de Branges spaces exhausting $L^2(\nu)$

Definition

Let μ and ν be positive Borel measures on $\mathbb{R}$. We say that ν is a *tangent measure* of μ at 0, if $\nu \neq 0$ and

$$\exists \epsilon_n > 0, \epsilon_n \to 0 \ \exists c_n > 0\colon \ \mu_n \to \nu \quad w^* \text{ in } C_c(\mathbb{R})'$$

where

$$\mu_n\big((\alpha,\beta)\big) := c_n \mu\big((\epsilon_n \alpha, \epsilon_n \beta)\big), \quad \alpha < \beta.$$

The set of all tangent measures of μ is denoted as $\operatorname{Tan}(\mu)$.

- If $\nu \in \operatorname{Tan}(\mu)$ and $c > 0$, then $c\nu \in \operatorname{Tan}(\mu)$.
- We say that μ *has a unique tangent measure*, if

$$\exists \nu\colon \ \operatorname{Tan}(\mu) = \{c\nu \mid c > 0\}.$$

Theorem (Mattila 2005)

Assume μ has a unique tangent measure. Then:

- $\operatorname{Tan}(\mu) = \{c\nu \mid c > 0\}$ *where ν is either a multiple of the Dirac measure δ_0, or of the form*

$$\mathrm{d}\nu(t) = \big[\sigma_- \mathbb{1}_{(-\infty,0)}(t) + \sigma_+ \mathbb{1}_{(0,\infty)}(t)\big] \cdot |t|^{2\omega+1}\,\mathrm{d}t$$

  with $\sigma_\pm \geq 0$, $\sigma_+ + \sigma_- > 0$, and $\omega > -1$.
- *The function $r \mapsto \big[\mu\big(-\frac{1}{r}, \frac{1}{r}\big)\big]^{-1}$ is regularly varying with index $2(\omega+1)$.*

A function $\ell : (0,\infty) \to (0,\infty)$ is *regularly varying* with index ρ, if it is measurable and

$$\forall s > 0\colon\ \lim_{r\to\infty} \frac{\ell(sr)}{\ell(r)} = s^\rho.$$

The main theorem

Theorem ("$\frac{1}{2}$-variant", Eichinger-Lukic-Woracek 2024)

The following statements are equivalent.

(i) μ *has a unique tangent measure which is not a multiple of* δ_0.

(ii) *There exists a regularly varying function* ℓ*, such that the rescaling limit exists with* $\tau_n = \ell(K_n(0,0))$.

Assume (i) *and* (ii) *hold. Then*

- ℓ *is an asymptotic inverse of* $r \mapsto \big[\mu\big(-\frac{1}{r},\frac{1}{r}\big)\big]^{-1}$.
- *The limit kernel* $K_\infty(z,w)$ *can be computed from the index of* ℓ *and* $\lim_{r\to\infty}\frac{\mu((0,\frac{1}{r}))}{\mu((-\frac{1}{r},0))}$*, which exists in* $[0,\infty]$.
- *The formula for* $K_\infty(z,w)$ *is an expression involving confluent hypergeometric functions.*

Why $\frac{1}{2}$-variant ?

- Our input is a measure μ that has all power moments ...
- ... but we leave this class of measures ...
- ... the output measure ν, which also hosts the limit space $\mathcal{H}(K_\infty(z,w))$, is only power bounded.

More natural: start with a power bounded measure μ

- all involved measures belong to the same class ...
- fits the philosophy of regular variation (behaves asymptotically like *some* power) ...
- fully fits the setting of "unique tangent measure" ...
- fully fits the setting of homogeneous de Branges spaces.

Problem: which de Branges chain in $L^2(\mu)$ to use ?

- ▷ If μ has all power moments, the chain made up of spaces of polynomials is distinguished naturally.
- ▷ If μ is Poisson finite, there is a naturally distinguished chain: the spaces which are invariant under difference quotients.

These instances of naturally distinguished chains share a property which goes directly to the core of de Branges' ordering theorem:

- ▷ The elements of the elements of the chain are entire functions of bounded type in $\mathbb{C}^{\pm}$.

Theorem (Langer-Woracek 2013)

Let μ be power bounded. Then there exists a unique chain of de Branges spaces exhausting $L^2(\mu)$, such that all elements of members of that chain are functions of bounded type in $\mathbb{C}^+$.

Definition

Let μ be power bounded, let $\{\mathcal{H}(K_t(z,w)) \mid t > 0\}$ be the unique chain with bounded type, and let ℓ be regularly varying. We say that a *rescaling limit exists* with rate ℓ, if the limit

$$K_\infty(z,w) = \lim_{t\to\infty} \frac{1}{K_t(0,0)} K_t\Big(\frac{z}{\ell(K_t(0,0))}, \frac{w}{\ell(K_t(0,0))}\Big)$$

exists locally uniformly on $\mathbb{C} \times \mathbb{C}$ and is not constant.

Theorem (“wishful variant”)

The statement of the “$\frac{1}{2}$-variant” holds verbatim for every power bounded measure.

We have not shown this (due to lack of appropriate machinery).

We have shown:

Theorem

The statement of the “$\frac{1}{2}$-variant” holds verbatim for every Poisson finite measure.

Way to the proof

Step 1

Pass to an alternative viewpoint:

- ✿ move along the given chain $\mathscr{C}$ towards $L^2(\mu)$
  & make a limit of weighted rescalings of the kernel functions
- ❀ produce weighted rescalings of the given chain $\mathscr{C}$ and measure μ
  & make a limit of the resulting chains and measures

Step 2

Develop core theory:

relation between measures: (tangent measure)

$L^2(\nu)$

chain of de Branges spaces exhausting $L^2(\mu)$
(containing functions of bounded type)

$$\{0\} \subseteq \cdots \subseteq \mathcal{H}\big(K_t(z,w)\big) \subseteq \cdots \subseteq \mathcal{H}\big(K_s(z,w)\big) \subseteq \cdots \subseteq L^2(\mu)$$

rescaling limit

$$L^2(\nu) \supseteq \cdots \supseteq \mathcal{H}\big(K_\infty^{[a]}(z,w)\big) \supseteq \cdots \supseteq \mathcal{H}\big(K_\infty(z,w)\big) \supseteq \cdots \supseteq \{0\}$$

chain of homogeneous de Branges spaces exhausting $L^2(\nu)$

Thank you

for your attention