I. The Hamburger Power Moment Problem

Question: Let $s_0, s_1, s_2, \ldots \in \mathbb{R}$. Describe the set

$$\mathcal{M} := \left\{ \mu \;\middle|\; \begin{array}{l} \mu \text{ positive Borel measure on } \mathbb{R} \\ \forall n : \quad s_n = \int_{\mathbb{R}} t^n \, d\mu(t) \end{array} \right\}$$

Answer: $\mathcal{M}$ is either $\emptyset$ or $|\mathcal{M}| = 1$ or $|\mathcal{M}| = \infty$.

Addition: If $|\mathcal{M}| = \infty$, there exist A, B, C, D entire functions such that

$$\int_{\mathbb{R}} \frac{1}{t-z} \, d\mu(t) = \frac{A(z)\,\tau(z) + B(z)}{C(z)\,\tau(z) + D(z)}, \quad z \in \mathbb{C}^+,$$

gives bijection between $\mathcal{M}$ and $\mathcal{N}_0 \cup \{\infty\}$ (Herglotz functions).

Definition: Assume $|\mathcal{M}| = \infty$. The matrix

$$\mathcal{W}(z) := \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}$$

is called the <u>Nevanlinna matrix</u> of the moment problem.

Operatortheoretic interpretation: Assume $\mu \in \mathcal{M}$, and let $p_n(t)$ be the orthonormal polynomial in $L^2(\mu)$, $\deg p_n = n$, for $n = 0, 1, 2, \ldots$. These polynomials are independent of the choice of μ. They satisfy a three term recurrence

$$z p_n(z) = b_n p_{n+1}(z) + a_n p_n(z) + b_{n-1} p_{n-1}(z), \quad n = 0, 1, 2, \ldots$$

with $a_n \in \mathbb{R}$, $b_n > 0$ $(b_{-1} := 0)$. The Jacobi operator in ℓ^2

$$J := \begin{pmatrix} a_0 & b_0 & & \\ b_0 & a_1 & b_1 & \\ & b_1 & a_2 & \ddots \\ & & \ddots & \ddots \end{pmatrix}$$

is closed symmetric with deficiency index $(1,1)$ if $|\mathcal{M}| = \infty$, and selfadjoint of $|\mathcal{M}| = 1$.

The Nevanlinna matrix $W(z)$ is Krein's u-resolvent matrix for the operator J and the element $u := 1$.

Unitarily equivalent model: Define $(\ell_n)_{n=0}^{\infty}$, $(\phi_n)_{n=0}^{\infty}$ recursively by

$$\ell_0 := 1, \quad \phi_0 := \frac{\pi}{2}, \quad \phi_{-1} := 0, \qquad \phi_{n+1} - \phi_n \in [0, \pi),$$

$$\frac{1}{b_n} = \sin(\phi_{n+1} - \phi_n)\sqrt{\ell_{n+1}\ell_n},$$

$$a_n = -\frac{1}{\ell_n}\Big[\cot(\phi_{n+1} - \phi_n) + \cot(\phi_n - \phi_{n-1})\Big].$$

Define the Hamburger Hamiltonian by

$$x_0 := 0, \quad x_j := \sum_{n=0}^{j} \ell_j,$$

$$H(t) := \begin{pmatrix} \cos^2\phi_j & \cos\phi_j \sin\phi_j \\ \cos\phi_j \sin\phi_j & \sin^2\phi_j \end{pmatrix}, \quad t \in [x_j, x_{j+1}).$$

We have $L := \sum_{j=0}^{\infty} l_j < \infty$ since $|\mathcal{M}| = \infty$.

The minimal operator of the canonical system

$$y'(t) = z J H(t) y(t), \quad t \in [0, L],$$

is unitarily equivalent to J.

II. Growth and spectral density

Theorem (M. Riesz 1923): The functions A, B, C, D are of minimal exponential type, i.e.,

$$\limsup_{r \to \infty} \frac{1}{r} \log \left(\max_{|z|=r} \| \omega(z) \| \right) = 0$$

(or A, B, C, D in place of ω ... all grow with the same speed)

Question: What is the "exact growth" of

$$M(r) := \max_{|z|=r} \| \omega(z) \| \quad ?$$

"Equivalently": What is the asymptotic density of the spectrum of some selfadjoint extension of the Jacobi operator?

What means "exact growth" (in this talk):

Definition: $f : [1,\infty) \to (0,\infty)$ is <u>regularly varying</u> with <u>index</u> $\rho \in \mathbb{R}$, if f is measurable and

$$\forall \lambda \in (0,\infty): \quad \lim_{r\to\infty} \frac{f(\lambda r)}{f(r)} = \lambda^\rho$$

Example:

- $f(r) := r^\rho$ with $\rho \in \mathbb{R}$
- $f(r) := r^\rho (\log r)^{\rho_1} (\log\log r)^{\rho_2} \cdot \ldots \cdot (\log \cdots \log r)^{\rho_m}$ with $\rho, \rho_1, \ldots, \rho_m \in \mathbb{R}$.

Measuring growth:

$$\limsup_{r\to\infty} \frac{1}{f(r)} \log\left(\max_{|z|=r} \|W(z)\|\right) \begin{cases} = 0 \\ \in (0,\infty) \\ = \infty \end{cases}$$

What means "equivalently":

Assume f regularly varying with index $\rho \in (0,1)$

$$n(r) := \# \{ x \in (-r, r) \mid A(x) = 0 \}.$$

Then

$$\limsup_{r \to \infty} \frac{1}{f(r)} \log \left(\max_{|z|=r} |A(z)| \right) \begin{Bmatrix} = 0 \\ \in (0,\infty) \\ = \infty \end{Bmatrix}$$

$$\Longleftrightarrow \quad \limsup_{r \to \infty} \frac{1}{f(r)} n(r) \begin{Bmatrix} = 0 \\ \in (0,\infty) \\ = \infty \end{Bmatrix}$$

($\rho \in \{0,1\}$ is more complicated).

Question: What is the "exact growth" of

$$M(r) := \max_{|z|=r} \| W(z) \| \quad ?$$

Answer...... many facets.

In particular: in terms of which parameters is the answer formulated.

(1) The moment sequence s_n

(2) The orthogonal polynomials P_n

(3) The Jacobi parameters a_n, b_n

(4) The Hamiltonian parameters l_n, ϕ_n

III. Some old and new theorems

A formula for the growth of $\log M(r)$ up to universal constants can be given in terms of

(1) s_n : C. Berg, R. Szwarc 2014

(4) l_n, ϕ_n : M. Langer, J. Reiffenstein, H. Woracek (in preparation)

The formulas are complicated and can be applied directly in only very few cases. For example

Application of (1) : I. Bochkov 2021

Application of (4) : J. Reiffenstein (in preparation)

Theorem (M.S. Livšic 1939): Let $\mathfrak{f}$ be regularly varying with order $\varrho \in (0,1)$. Assume $|\mathcal{M}| = \infty$. Then

$$\limsup_{r\to\infty} \frac{1}{\mathfrak{f}(r)} \log\Big(\max_{|z|=r} \|W(z)\|\Big) \geq \frac{1}{e\varrho} \limsup_{n\to\infty} \frac{[\mathfrak{f}^-(n)]^{\varrho}}{(s_{2n})^{\varrho/2n}}$$

In particular, the order of the entire functions A, B, C, D is $\geq \varrho$ if the right side is > 0.

Here $\mathfrak{f}^-$ denotes an asymptotic inverse of $\mathfrak{f}$, i.e.,

$$\lim_{r\to\infty} \frac{(\mathfrak{f}\circ\mathfrak{f}^-)(r)}{r} = \lim_{r\to\infty} \frac{(\mathfrak{f}^-\circ\mathfrak{f})(r)}{r} = 1.$$

▷ $\mathfrak{f}(r) = r^{\varrho}$: $\mathfrak{f}^-(r) = r^{1/\varrho}$

▷ $\mathfrak{f}(r) = r^{\varrho}(\log r)^{\lambda}$: $\mathfrak{f}^-(r) = \text{const}\cdot r^{1/\varrho}(\log r)^{-\lambda/\varrho}$

Theorem (Yu. M. Berezanskii 1956): Assume that

- $\sum_{n=0}^{\infty} \frac{1}{b_n} < \infty$ (Carleman condition .... necessary for $|\mathcal{M}| = \infty$)
- $(\log b_n)_{n=n_0}^{\infty}$ is concave or convex (regularity of off-diagonal)
- $\left(\frac{a_n}{b_n}\right)_{n=0}^{\infty} \in \ell^1$ (smallness of diagonal)

Then $|\mathcal{M}| = \infty$, and the order of A, B, C, D is equal to the convergence exponent of $\left(\frac{1}{b_n}\right)_{n=0}^{\infty}$, i.e.,

$$\rho = \inf \left\{ \rho > 0 \;\middle|\; \sum_{n=0}^{\infty} \frac{1}{b_n^{\alpha}} < \infty \right\}.$$

If $\rho \in (0,1)$, moreover, $\log M(r) \lesssim r^{\frac{1}{\rho}}$.

$\exists c, r_0 > 0 \; \forall r \geq r_0 : \log M(r) \leq c\, r^{\frac{1}{\rho}}$

Consider b_n, a_n with a power asymptotic

$$b_n = n^{\beta}\left(x_0 + \frac{x_1}{n} + \frac{x_2}{n^2} + O\left(\frac{1}{n^{2+\varepsilon}}\right)\right),$$

$$a_n = n^{\alpha}\left(y_0 + \frac{y_1}{n} + \frac{y_2}{n^2} + O\left(\frac{1}{n^{2+\varepsilon}}\right)\right),$$

with $\alpha, \beta \in \mathbb{R}$, $x_0 > 0$, $y_0 \neq 0$, $x_1, x_2, y_1, y_2 \in \mathbb{R}$.

Case 1: $\alpha < \beta$, $\quad 1 < \beta$

Case 2: $\alpha = \beta$, $|y_0| < 2x_0$, $\quad 1 < \beta$

Case 3: $\alpha = \beta$, $|y_0| = 2x_0$, $\quad \frac{3}{2} < \beta < \frac{2x_1}{x_0} - \frac{2y_1}{y_0}$

Case 4: $\alpha = \beta$, $|y_0| = 2x_0$, $\quad \beta = \frac{2x_1}{x_0} - \frac{2y_1}{y_0}$

and $2 < \beta < \frac{3}{2} + \frac{2z_2}{x_0}$ where

$$z_2 := x_0\left(\frac{2x_2}{x_0} - \frac{2y_2}{y_0} + \frac{y_1}{y_0} - \frac{2x_1y_1}{x_0y_0} + \frac{2y_1^2}{y_0^2}\right)$$

R. Pruckner 2020

R. Pruckner, J. Reiffenstein, H. Woracek (submitted)

J. Reiffenstein (in preparation)

Theorem: $|\mathcal{M}| = \infty \iff$ one of cases 1–4 takes place.

Cases 1, 2, case 3 with $\Delta > 2$: $\log M(r) \asymp r^{\frac{1}{\Delta}}$

Case 3 with $\Delta < 2$: $\log M(r) \asymp r^{\frac{1}{2(\Delta-1)}}$

Case 3 with $\Delta = 2$, case 4: order $= \frac{1}{\Delta}$

The order of A, B, C, D is $= \frac{1}{\Delta}$ in the first and third line, and $= \frac{1}{2(\Delta-1)}$ in the middle.

$\exists\, c_1, c_2, r_0 > 0 \ \forall r \geq r_0$:
$c_1 r^{\frac{1}{\Delta}} \leq \log M(r) \leq c_2 r^{\frac{1}{\Delta}}$

Consider e_n, ϕ_n "comparable" to regularly varying functions in the sense that

$$e_n \asymp d_e(n), \quad |\sin(\phi_{n+1} - \phi_n)| \asymp d_\phi(n), \qquad (*)$$

where d_e, d_ϕ are nonincreasing, regularly varying, and $\int_1^\infty d_e(t)\,dt < \infty$ (which means that $\sum_{n=0}^\infty e_n < \infty$, equivalently that $|M| = \infty$). Let $-\delta_e, -\delta_\phi$ be the indices of d_e, d_ϕ.

R. Pruckner, J. Reiffenstein, H. Woracek (submitted)
J. Reiffenstein (in preparation)

Theorem: Assume (*) holds.

Case $\delta_e + \delta_\phi > 2$: $\log M(r) \asymp \left[\dfrac{1}{d_e d_\phi}\right]^{-}(r)$.

$$= \frac{1}{\delta_e + \delta_\phi}.$$

Case $\delta_e + \delta_\phi < 2$: $\left[\dfrac{1}{d_e d_\phi}\right]^{-}(r) \lesssim \log M(r) \lesssim r \displaystyle\int_{\left[\frac{d_\phi}{d_e}\right]^{-}(r)}^{\infty} d_e(t)\,dt$

Both bounds are sharp.

The order of A, B, C, D is $\in \left[\dfrac{1}{\delta_e + \delta_\phi}, \dfrac{1 - \delta_\phi}{\delta_e - \delta_\phi}\right]$.

Case $\delta_e + \delta_\phi = 2$, $(\delta_e, \delta_\phi) \neq (1,1)$: order $= \dfrac{1}{\delta_e + \delta_\phi}$

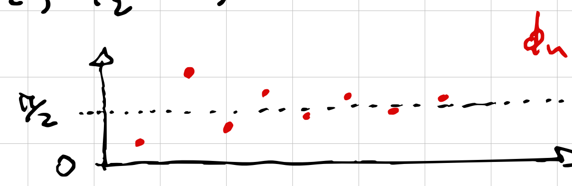
Example : $d_e(t) = \dfrac{1}{t \log^2 t}$, $d_\phi(t) = \dfrac{1}{t^{1/2}}$ $\Longrightarrow$

$$\left[\frac{1}{d_e d_\phi}\right]^{\sim}(r) \asymp \frac{r^{2/3}}{(\log r)^{4/3}}, \quad \underbrace{r \int^{\infty} d_e(t)\,dt}_{\left[\frac{d_\phi}{d_e}\right]^{\sim}(r)} \asymp \frac{r}{\log r}$$

Let $\ell_n := \dfrac{1}{n \log^2 n}$ $\quad (\ell_0 = \ell_1 = 1)$.

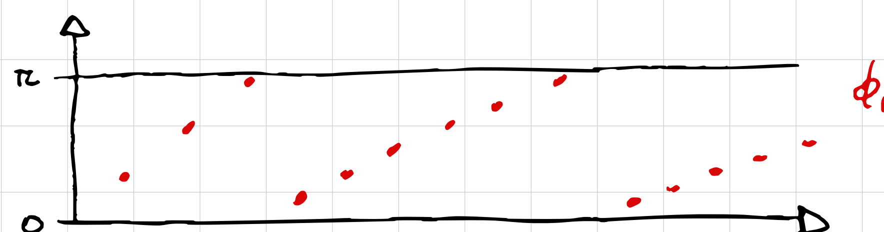
▷ $\phi_n := \dfrac{\pi}{2} + \dfrac{(-1)^n}{\sqrt{n}}$ $\quad (\phi_0 = 0,\ \phi_1 = \frac{\pi}{2},\ \phi_2 = 0)$

$\Longrightarrow$ $\log M(r) \asymp \dfrac{r^{2/3}}{(\log r)^{4/3}}$



▷ $\phi_n := \sqrt{n} \mod \pi$

$\Longrightarrow$ $\log M(r) \asymp \dfrac{r}{\log r}$



Theorem (R. Pruckner, H. Woracek 2022): Assume $\gtrsim$ holds in (*). Then

$$\log M(r) \gtrsim \left[\frac{1}{d_e d_\phi}\right]^{-}(r).$$

The order of A, B, C, D is $\geq \frac{1}{\delta_e + \delta_\phi}$.

Theorem (R. Pruckner, J. Reiffenstein, H. Woracek (submitted)):

Let $\delta_e, \delta_\phi, \gamma_e, \gamma_\phi \geq 0$, $\psi \in \mathbb{R}$, with $\delta_e + \delta_\phi, \gamma_e + \gamma_\phi > 0$. Assume

▷ $l_n \lesssim n^{-\delta_e}$, $|\sin(\phi_{n+1} - \phi_n)| \lesssim n^{-\delta_\phi}$,

▷ $\sum_{n=N+1}^{\infty} l_n \lesssim N^{-\gamma_e}$, $\sum_{n=N+1}^{\infty} l_n \sin^2(\phi_n - \psi) \lesssim N^{-\gamma_\phi}$.

Case		$\log M(r)$ is $\lesssim$	order is $\leq$
$\delta < 1+\gamma$		$r^{\frac{1}{1+\gamma}} (\log r)^{\frac{\gamma}{1+\gamma}}$	$\frac{1}{1+\gamma}$
$\delta = 1+\gamma$		$r^{1/\delta}$	$1/\delta$
$\delta > 1+\gamma$,	$\delta > 2$	$r^{1/\delta}$	$1/\delta$
	$\delta = 2$	$r^{1/2} \log r$	$1/\delta$
	$\delta < 2$, $\delta_e \leq 1+\gamma$	$r^{(2-\delta+\gamma)/(2-\delta+2\gamma)}$	$(2-\delta+\gamma)/(2-\delta+2\gamma)$
	$\delta_e > 1+\gamma$	$r^{(1-\delta_\phi)/(\delta_e-\delta_\phi)}$	$(1-\delta_\phi)/(\delta_e-\delta_\phi)$