Indefinite canonical systems whose Weyl coefficient has no finite generalized poles of nonpositive type

Graz, 20. 12. 2022

Preface

This talk is based on joint work with M. Langer which is not very recent, but most recently found an unexpected and relevant application.

The papers are

[Operators and Matrices 7 (2013), 477 - 555]
[Fields Institute Communications (2023), 103 - 200]
preprint around since 2015

The "recent application" will be presented in the talk of B. Eichinger after lunch.

Introduction

Usually, I use as motivation and illustration for the theory the spectral theory of equations with two singular endpoints, for example the Bessel-type equation

$$-u''(x) + \left(\frac{\ell(\ell+1)}{x^2} + V_0(x) \right) u(x) = \lambda u(x), \quad x \in (0,\infty),$$

where $\ell > -\frac{1}{2}$ and $V_0 \in L^1_{loc}(0,\infty)$, $xV_0(x) \in L^1(0,1)$.

Today I want to start from a very different side (which leads to the same theoretical object, namely indefinite canonical systems of a particular form).

De Branges spaces and measures

Definition: A Hermite-Biehler function is an entire function E without zeroes on $\mathbb{C}^+ \cup \mathbb{R}$ which satisfies (we denote $F^\#(z) := \overline{F(\bar{z})}$)

$$\left| \frac{E^\#(z)}{E(z)} \right| < 1 \quad \text{for } z \in \mathbb{C}^+ .$$

We write $\mathcal{HB}$ for the set of all Hermite-Biehler functions.

The de Branges space induced by $E \in \mathcal{HB}$ is

$$\mathcal{H}(E) := \left(H^2(\mathbb{C}^+) \ominus \frac{E^\#}{E} H^2(\mathbb{C}^+) \right) \cdot E .$$

The space $\mathcal{H}(E)$ is a reproducing kernel Hilbert space of entire functions, and its reproducing kernel is

$$K_E(\omega, z) = \frac{i}{2} \frac{1 - \frac{E^\#(z)}{E(z)} \cdot \overline{\left(\frac{E^\#(\omega)}{E(\omega)} \right)}}{z - \bar{\omega}}$$

Definition: Let $\mathcal{H}(E)$ be a de Branges space and μ a positive Borel measure on $\mathbb{R}$. We say that $\mathcal{H}(E)$ is contained isometrically in $L^2(\mu)$, and write $\mathcal{H}(E) \subseteq_i L^2(\mu)$, if

$$\forall F \in \mathcal{H}(E): \quad \|F\| = \int_{\mathbb{R}} |F(t)|^2 \, d\mu(t) .$$

For a given de Branges space $\mathcal{H}(E)$ there exist many measures μ with $\mathcal{H}(E) \subseteq_i L^2(\mu)$, and for every measure μ there exist many de Branges spaces $\mathcal{H}(E)$ with $\mathcal{H}(E) \subseteq_i L^2(\mu)$.

Theorem (de Branges): Let μ be a positive (nonzero) Borel measure on $\mathbb{R}$, and consider on the set

$$\mathbb{B}_\mu := \{ \mathcal{H}(E) \mid \mathcal{H}(E) \subseteq_i L^2(\mu) \}$$

the equivalence relation

$$\mathcal{H}(E_1) \sim \mathcal{H}(E_2) \Leftrightarrow \frac{E_1}{E_2} \text{ bounded type in } \mathbb{C}^+.$$

Then each equivalence class w.r.t. $\sim$ is totally ordered.

These equivalence classes can be described by a differential equation, namely a two-dimensional canonical system. This is an equation of the form

$$\frac{\partial}{\partial t}\left(y_1(t,z), y_2(t,z)\right) J = z\left(y_1(t,z), y_2(t,z)\right) H(t)$$

on an interval of $\mathbb{R}$, where the Hamiltonian $H(t) \in \mathbb{R}^{2\times 2}$ is locally integrable and $H(t) \geq 0$, $H(t) \neq 0$ a.e. on $\mathbb{R}$.

An interval (a,b) is called H-indivisible, if $\ker H(t)$ is constant and nonzero a.e. on (a,b). We denote by $I(H)$ the union of all H-indivisible intervals.

Theorem (de Branges): Let $\mathcal{C}$ be one equivalence class of $\mathcal{B}_\mu$ modulo $\sim$. Then there exists a family $(E(t,z))_{t\in\mathbb{R}}$ and a Hamiltonian $H(t)$, $t\in\mathbb{R}$, with $\operatorname{tr} H = 1$, such that

(i) $(A(t,z), B(t,z))_{t\in\mathbb{R}}$ is a solution of the canonical system with Hamiltonian H,

(ii) $E(t,z) \in \mathcal{HB}$ or is a real constant,

(iii) $\mathcal{C} = \{ \mathcal{H}(E(t,z)) \mid t\in\mathbb{R}\setminus I(H),\ E(t,z) \text{ not constant} \}$

In general all equivalence classes in $\mathcal{B}_\mu$ are of equal rights. Not so if μ is Poisson integrable.

Theorem (de Branges): Assume $\int_{\mathbb{R}} \frac{d\mu(t)}{1+t^2} < \infty$. Then there exists a unique equivalence class $\mathcal{C}$ such the elements of $\mathcal{C}$ are generated by functions E which are themselves of bounded type in $\mathbb{C}^+$.

The Hamiltonian corresponding to this class has the property that $\inf(\mathbb{R}\setminus I(H)) > -\infty$, and Hamiltonians corresponding to other classes do not have that property.

Theorem (de Branges): Let H be a Hamiltonian on $\mathbb{R}$, $\operatorname{tr} H = 1$, with $s := \inf(\mathbb{R}\setminus I(H)) > -\infty$, and let $(A(t,z), B(t,z))$ be the solution of the canonical system with $(A(s,z), B(s,z)) = (1,0)$. Then there exists a unique Poisson integrable measure μ such that

$$\{ \mathcal{H}(E(t,z)) \mid t\in\mathbb{R}\setminus I(H),\ t\neq s \}$$

is an equivalence class in $\mathcal{B}_\mu$.

Why $\int_{\mathbb{R}} \frac{d\mu(t)}{1+t^2} < \infty$?

How about measures growing faster, e.g. power growth

$$\exists \Delta > 0 : \int_{\mathbb{R}} \frac{d\mu(t)}{(1+|t|)^\Delta} < \infty \ ?$$

Theorem (LW): Assume $\exists \Delta > 0 : \int_{\mathbb{R}} \frac{d\mu(t)}{(1+|t|)^\Delta} < \infty$. Then there exists a unique equivalence class $\mathcal{C}$ such the elements of $\mathcal{C}$ are generated by functions E which are themselves of bounded type in $\mathbb{C}^+$.

The Hamiltonian corresponding to this chain has "moderate behaviour towards $-\infty$", and Hamiltonians corresponding to other chains do not have that property.

Theorem (LW): Let H be a Hamiltonian on $\mathbb{R}$, $\operatorname{tr} H = 1$, which has "moderate behaviour towards $-\infty$". Then there exists a unique solution of the canonical system with $\lim_{t\to-\infty} (A(t,z), B(t,z)) = (1,0)$. There exists a unique measure μ with power growth, such that

$$\{ \mathcal{H}(E(t,z)) \mid t \in \mathbb{R} \setminus I(H) \}$$

is an equivalence class in $\mathfrak{B}_\mu$.

Method of proof

Revisit the Poisson integrable setting.

Consider the Cauchy integral of μ (... regularised):

$$q(z) := \int_{\mathbb{R}} \frac{1+tz}{t-z} \, \frac{d\mu(t)}{1+t^2} .$$

This function is analytic on $\mathbb{C}^+$ and $\operatorname{Im} q(z) \geq 0$, $z \in \mathbb{C}^+$. Then there exists a Hamiltonian H on $(0,\infty)$, $\operatorname{tr} H = 1$, such that q is the Weyl coefficient of H. Thus H together with the solution having initial value $(1,0)$ at 0 are the required ones.

Power bounded setting: $\varkappa \in \mathbb{N}$ with $\int_{\mathbb{R}} \frac{d\mu(t)}{(1+t^2)^{1+\varkappa}} < \infty$.

Here Pontryagin space theory comes into play.

Consider the Cauchy integral of μ (... regularised):

$$q(z) := (1+z^2)^{\varkappa} \int_{\mathbb{R}} \frac{1+tz}{t-z} \, \frac{d\mu(t)}{(1+t^2)^{1+\varkappa}} .$$

This function is analytic on $\mathbb{C}^+$ and the Nevanlinna kernel

$$N_q(w,z) := \frac{q(z) - \overline{q(w)}}{z - \overline{w}}$$

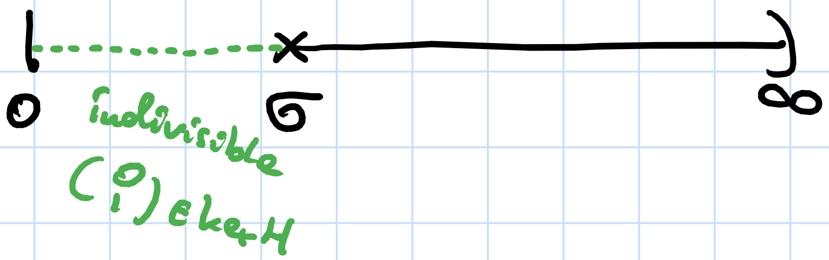
has a finite number of negative squares. Thus there exists an indefinite Hamiltonian $\mathfrak{h}$ such that q is the Weyl coefficient of $\mathfrak{h}$.

Theorem (LW): Let H be an indefinite Hamiltonian. Then the Weyl coefficient q_H is of the form

$$q_H(z) = p(z) + (1+z^2)^{\kappa} \int_{\mathbb{R}} \frac{1+tz}{t-z} \frac{d\mu(t)}{(1+t^2)^{1+\kappa}},$$

polynomial

if and only if H is of the form



Up to a change of variable in t the Hamiltonian from (σ,∞) and the solution with a particular asymptotic behaviour towards σ are the required ones.

Conclusion

Under suitable normalisations of H, we obtain a bijection between all measures with power growth and Hamiltonians having "moderate behaviour towards $-\infty$".

This extends the de Branges correspondence between Poisson integrable measures and Hamiltonians on the half-line.

A lot of results can also be extended, including

- Continuity in both directions w.r.t. appropriate topologies.
- A Fourier transform between $L^2(H)$ and $L^2(\mu)$.
- A bijection of solutions and $\mathbb{C}^2$ by means of regularised boundary values at $-\infty$.

.... etc