TOC

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Two-dimensional canonical systems in limit point case.

$$y'(t) = z\, J\, H(t)\, y(t) \qquad \text{for } t \in [0,\infty)$$

$z \in \mathbb{C}$

$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Hamiltonian

$H \in L^1_{loc}([0,\infty), \mathbb{R}^{2\times 2})$

$H \geq 0$ a.e. $\quad (H \not\equiv 0$ a.e.$)$

$$\int_0^\infty \operatorname{tr} H(t)\, dt = \infty$$

$y : [0,\infty) \to \mathbb{C}^2$

Includes: Schrödinger, Jacobi, Krein-Feller, etc.

In theory (!) every result about canonical systems yields a corresponding result about each concrete class.

▷ Inclusion of limit circle case: canonical system on a finite interval, $H \in L^1([0,1], \mathbb{R}^{2\times 2})$. Set

$$\tilde{H}(t) := \begin{cases} H(t) & \text{if } t \in [0,1] \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } t \in (1,\infty) \end{cases}$$

▷ The operator model: given Hamiltonian H, let

$L^2(H) :=$ certain closed subspace of $L^2(H(t)\,dt)$

$$\operatorname{graph} A_H := \Big\{ (f,g) \in L^2(H) \times L^2(H) \;\Big|\; f' = JHg, \quad (1,0)f(0) = 0 \Big\}$$

A_H is selfadjoint (in general unbounded) and has simple spectrum.

We are interested in the situation that $\sigma(A_H)$ is discrete.

$\boxed{Q1}$ When is $\sigma(A_H)$ discrete ?

$\boxed{Q2}$ If $\sigma(A_H)$ is discrete, how is it $\boxed{\text{distributed}}$?

convergence class $(s>0)$: $\sum_n \frac{1}{|\lambda_n|^s} < \infty$,

finite type $(s>0)$: $\limsup_{n\to\infty} \frac{n}{|\lambda_n|^s} < \infty$,

or similar (minimal type, other comparison functions)

We do not (and can not) discuss spectral asymptotics $(\lambda_n = n^s(1+o(1)))$.

▷ $\sigma(A_H)$ discrete $\Rightarrow$ $\exists$ nonzero constant in $L^2(H)$.
w.l.o.g. we may assume that $\begin{pmatrix}1\\0\end{pmatrix} \in L^2(H)$.

Theorem 1: $H = \begin{pmatrix} h_1 & h_3 \\ h_3 & h_2 \end{pmatrix}$ with $\int_0^\infty h_1(t)\,dt < \infty$.
Then $\sigma(A_H)$ is discrete, if and only if

$$\lim_{t\to\infty} \left(\int_t^\infty h_1(s)\,ds \cdot \int_0^t h_2(s)\,ds \right) = 0.$$

▷ Discreteness is independent of the off-diagonal (essential step in the proof).

▷ For $h_3 = 0$ this is Krein's criterion for discreteness of the spectrum of a string.

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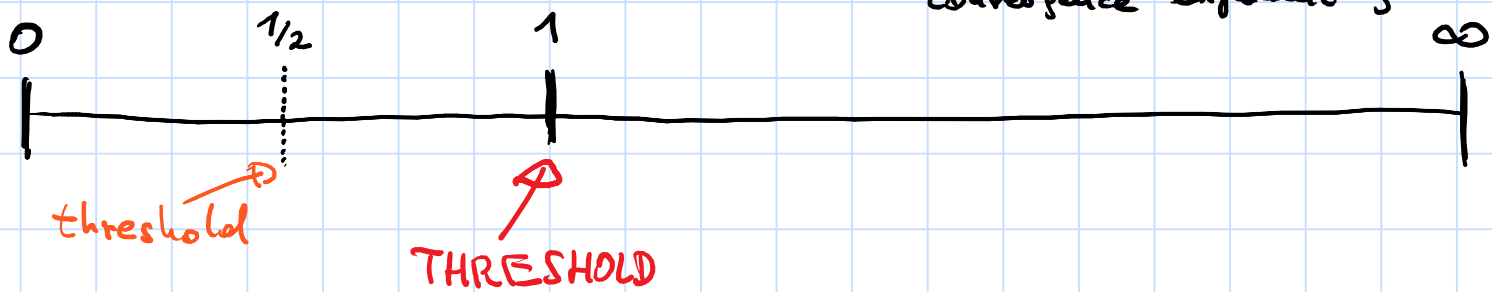


Around $\rho = 1$ the behaviour changes drastically.

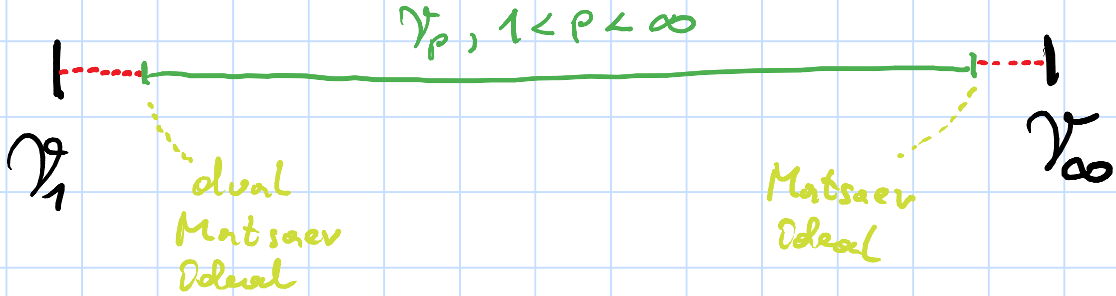
operator theoretic reason : symmetrically normed ideals

function theoretic reason : entire functions of bounded type

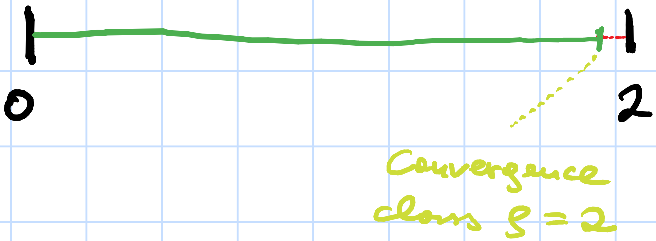
Around $\rho = \frac{1}{2}$ the behaviour changes slightly.

(analytic reasons)

I. dense spectrum : operator theory works for



II. sparse spectrum : function theory / analysis works for



▷ An example Let $\alpha > 1$, $\beta_1, \beta_2 \in \mathbb{R}$, and

$$h_1(t) = \begin{cases} \frac{1}{t^\alpha} & , t > 1 \\ 0 & , t \in [0,1] \end{cases} \qquad h_2(t) = \begin{cases} (\log t)^{\beta_1} (\log \log t)^{\beta_2} & , t > 3 \\ 1 & , t \in [0,3] \end{cases}$$

(i) $\alpha < 2 \Rightarrow$ convergence exponent of $(\lambda_n)_{n=1}^\infty$ is ∞.

(ii) $\alpha > 2 \Rightarrow$ convergence exponent is 1 (and convergence class 1).

(iii) $\alpha = 2$. Let $1 < \rho < \infty$, then

$$\sum_n \frac{1}{|\lambda_n|^\rho} < \infty \iff \beta_1 < -\frac{2}{\rho} \vee \left(\beta_1 = -\frac{2}{\rho} \wedge \beta_2 < -\frac{2}{\rho}\right)$$

$$\limsup_{n\to\infty} \frac{n}{|\lambda_n|^\rho} < \infty \iff \beta_1 < -\frac{2}{\rho} \vee \left(\beta_1 = -\frac{2}{\rho} \wedge \beta_2 \leq 0\right)$$

$$\lim_{n\to\infty} \frac{n}{|\lambda_n|^\rho} = 0 \iff \beta_1 < -\frac{2}{\rho} \vee \left(\beta_1 = -\frac{2}{\rho} \wedge \beta_2 < 0\right)$$

▷ $\mathcal{H}$ separable Hilbert space. A s.n.-ideal is $\langle \mathcal{I}, \|.\|_{\mathcal{I}} \rangle$ where

(i) $\mathcal{I} \subseteq \mathcal{B}(\mathcal{H})$ ideal ($\neq \{0\}, \mathcal{B}(\mathcal{H})$), $\|.\|_{\mathcal{I}}$ complete norm

(ii) $\|ATB\|_{\mathcal{I}} \leq \|A\| \cdot \|T\|_{\mathcal{I}} \cdot \|B\|$ for $T \in \mathcal{I}$, $A, B \in \mathcal{B}(\mathcal{H})$

(iii) $\|T\|_{\mathcal{I}} = \|T\|$ for T 1-dimensional

Examples: Schatten-von Neumann classes $\mathcal{S}_p$, $1 \leq p < \infty$

Orlicz classes, Lorentz ideals, etc.

▷ Calkin correspondence: $T \in \mathcal{S}_\infty \longmapsto (s_n(T))_{n=1}^{\infty}$

s-numbers

s.n.-ideals $\longleftrightarrow$ symmetric Banach sequence spaces

▷ density of a sequences is encoded in membership in $\mathcal{I}$.

▷ $\mathcal{I}$ s.n.-ideal is fully symmetric, if

$\forall T \in \mathcal{I}, A \in \mathcal{B}(\mathcal{H})$:

$$(s_n(A))_{n=1}^{\infty} \prec\!\!\prec (s_n(T))_{n=1}^{\infty} \Longrightarrow A \in \mathcal{I} \wedge \|A\|_{\mathcal{I}} \leq \|T\|_{\mathcal{I}}$$

Hardy-Littlewood majorisation

$\forall N \in \mathbb{N}: \sum_{n=1}^{N} s_n(A) \leq \sum_{n=1}^{N} s_n(T)$

▷ $\mathcal{I}$ s.n.-ideal has the Matsaev property, if

$\forall T$ Volterra operator : $\operatorname{Re} T \in \mathcal{I} \Longrightarrow T \in \mathcal{I}$

▷ For $H = \begin{pmatrix} h_1 & h_3 \\ h_3 & h_2 \end{pmatrix}$ set $H_{diag} = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}$

<u>Theorem 2:</u> Let $H = \begin{pmatrix} h_1 & h_3 \\ h_3 & h_2 \end{pmatrix}$ with $\int_0^\infty h_1(t)\,dt < \infty$.

Let $\mathcal{I} \subseteq \mathfrak{S}_\infty$ be a s.n.-ideal which is fully symmetric and has the Matsaev property. Then

$$A_H^{-1} \in \mathcal{I} \iff A_{H_{diag}}^{-1} \in \mathcal{I}.$$

▷ A theorem of Aleksandrov, Janson, Peller, Rochberg (in a slightly extended form) yields a sequential characterization of membership in an ideal.

▷ Given H with $\int_0^\infty h_1 < \infty$, let $0 = c_0 < c_1 < c_2 < \dots$ be such that $\int_{c_{n-1}}^{c_n} h_1(t)\,dt = 2^{-n} \int_0^\infty h_1(t)\,dt$. Set

$$\alpha_n := 2^{-\frac{n}{2}} \left(\int_{c_{n-1}}^{c_n} h_2(t)\,dt \right)^{1/2} = \sqrt{\int_{c_{n-1}}^{c_n} h_1(t)\,dt \cdot \int_{c_{n-1}}^{c_n} h_2(t)\,dt}$$

<u>Theorem 3:</u> Let $\mathcal{I}$ be a s.n.-ideal which is fully symmetric and has the Matsaev property. Then

$$A_H^{-1} \in \mathcal{I} \iff (\alpha_n)_{n=1}^\infty \in \mathcal{I}$$

▷ Convergence class conditions

Let $M : [0,\infty) \to [0,\infty)$ be

(i) continuous, increasing, convex, $M(0)=0$, $M(1)=1$, $\lim_{x\to\infty} \frac{M(x)}{x} = \infty$

(ii) $\limsup_{x\to 0} M(2x)/M(x) < \infty$

(iii) $\lim_{x\to 0} \frac{1}{\log x} \cdot \log \left[\limsup_{y\to 0} M(xy)/M(y) \right] > 1$

E.g.: $M(x) \approx x^{\delta} |\log x|^{\alpha_1} \, |\log \log x|^{\alpha_2}$ at 0 with $\delta > 1$.

Corollary: Let H with $\int_0^\infty h_1 < \infty$ and $\sigma(A_H)$ discrete, and let $\lambda_1, \lambda_2, \ldots$ be the eigenvalues of A_H (nondecreasing modulus).

Then

$$\sum_n M\left(\frac{1}{|\lambda_n|}\right) < \infty \iff$$

$$\int_0^\infty M\left(\sqrt{\int_t^\infty h_1(s)\,ds \cdot \int_0^t h_2(s)\,ds}\right) \cdot \frac{h_1(t)}{\int_t^\infty h_1(s)\,ds}\, dt < \infty$$

D Type conditions

Let $g : [0,\infty) \to (0,\infty)$ be regularly varying with order $\rho > 1$, i.e., g measurable and $\lim_{x\to\infty} g(yx)/g(x) = y^\rho$ for $y > 0$.

<u>Corollary</u>: Let H with $\int_0^\infty h_1 < \infty$ and $\sigma(A_H)$ discrete, and let $\lambda_1, \lambda_2, \ldots$ be the eigenvalues of A_H (nondecreasing modulus).

Again let

$$\alpha_n := \sqrt{\int_{c_{n-1}}^{c_n} h_1 \cdot \int_{c_{n-1}}^{c_n} h_2} \quad \text{where} \quad \int_{c_{n-1}}^{c_n} h_1 = 2^{-n} \int_0^\infty h_1 ,$$

and let $(\alpha_n^*)_{n=1}^\infty$ be the nonincreasing rearrangement. Then

$$\limsup_{n\to\infty} \frac{n}{g(|\lambda_n|)} < \infty \iff \limsup_{n\to\infty} \frac{n}{g(1/\alpha_n^*)} < \infty$$

$$\lim_{n\to\infty} \frac{n}{g(|\lambda_n|)} = 0 \iff \lim_{n\to\infty} \frac{n}{g(1/\alpha_n^*)} = 0$$