Indefinite Canonical Systems. Theory and Examples

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joint work with Michael Kaltenbäck,
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CONTENTS

- Review of the positive definite theory
- Some examples of canonical systems
- Indefinite analogue of canonical systems
- A short account on the literature

REVIEW OF THE POSITIVE DEFINITE THEORY

Hamiltonians and canonical systems

A *Hamiltonian* is a function

- $H : [\sigma_0, \sigma_1) \to \mathbb{R}^{2\times 2}$ defined a.e., measurable;
- $H(t) \geq 0$, $H \in L^1_{loc}((\sigma_0, \sigma_1))$;
- $\int_{\sigma_0}^{\sigma_0+\epsilon} \operatorname{tr} H(t)\, dt < \infty$ (initial value problem);
- H does not vanish on any set of positive measure.

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The *canonical system* with Hamiltonian H is the differential equation

$$y'(x) = z \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} H(x) y(x),\; x \in [\sigma_0, \sigma_1)\,.$$

limit circle case vs. limit point case

A Hamiltonian H is said to be in the

- *limit circle case* if $\int_{\sigma_1-\epsilon}^{\sigma_1} \operatorname{tr} H(t)\,dt < +\infty$;
- *limit point case* if $\int_{\sigma_1-\epsilon}^{\sigma_1} \operatorname{tr} H(t)\,dt = +\infty$.

Summary

Hamiltonian

$$H(t)$$

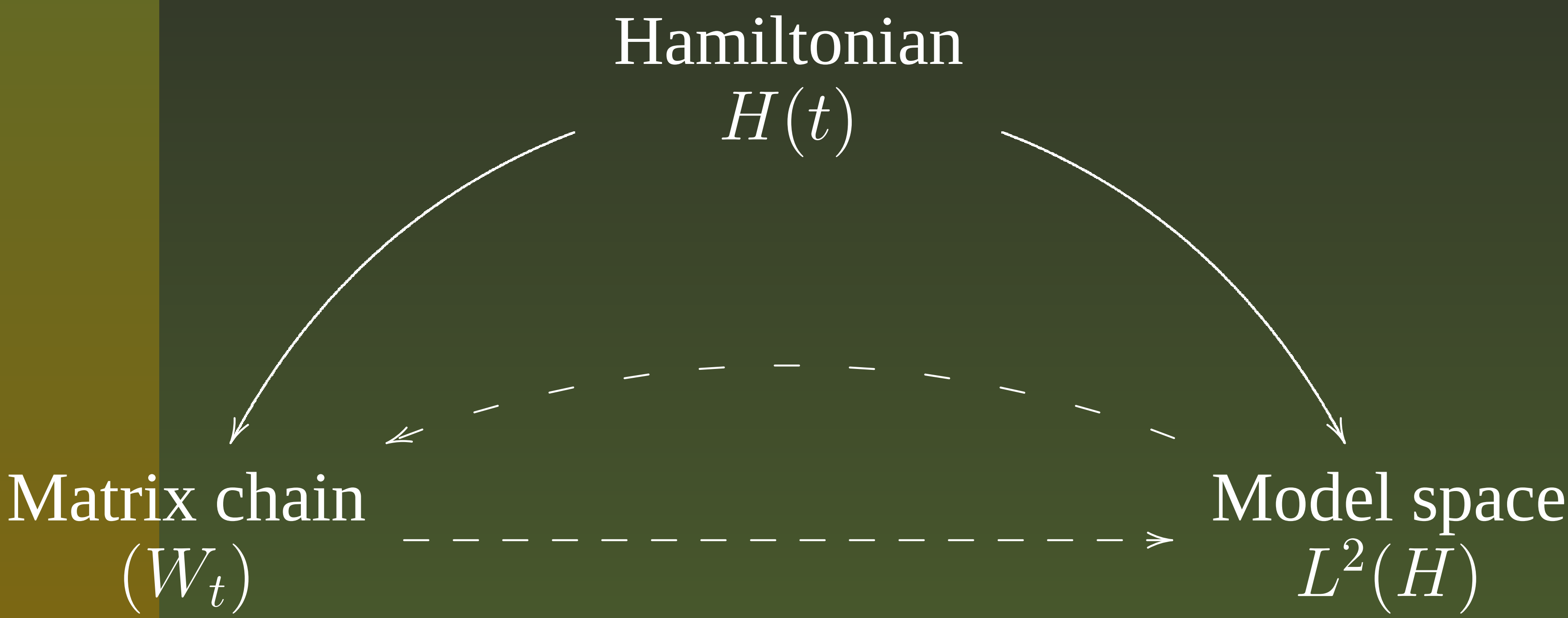
Summary

Hamiltonian
$H(t)$

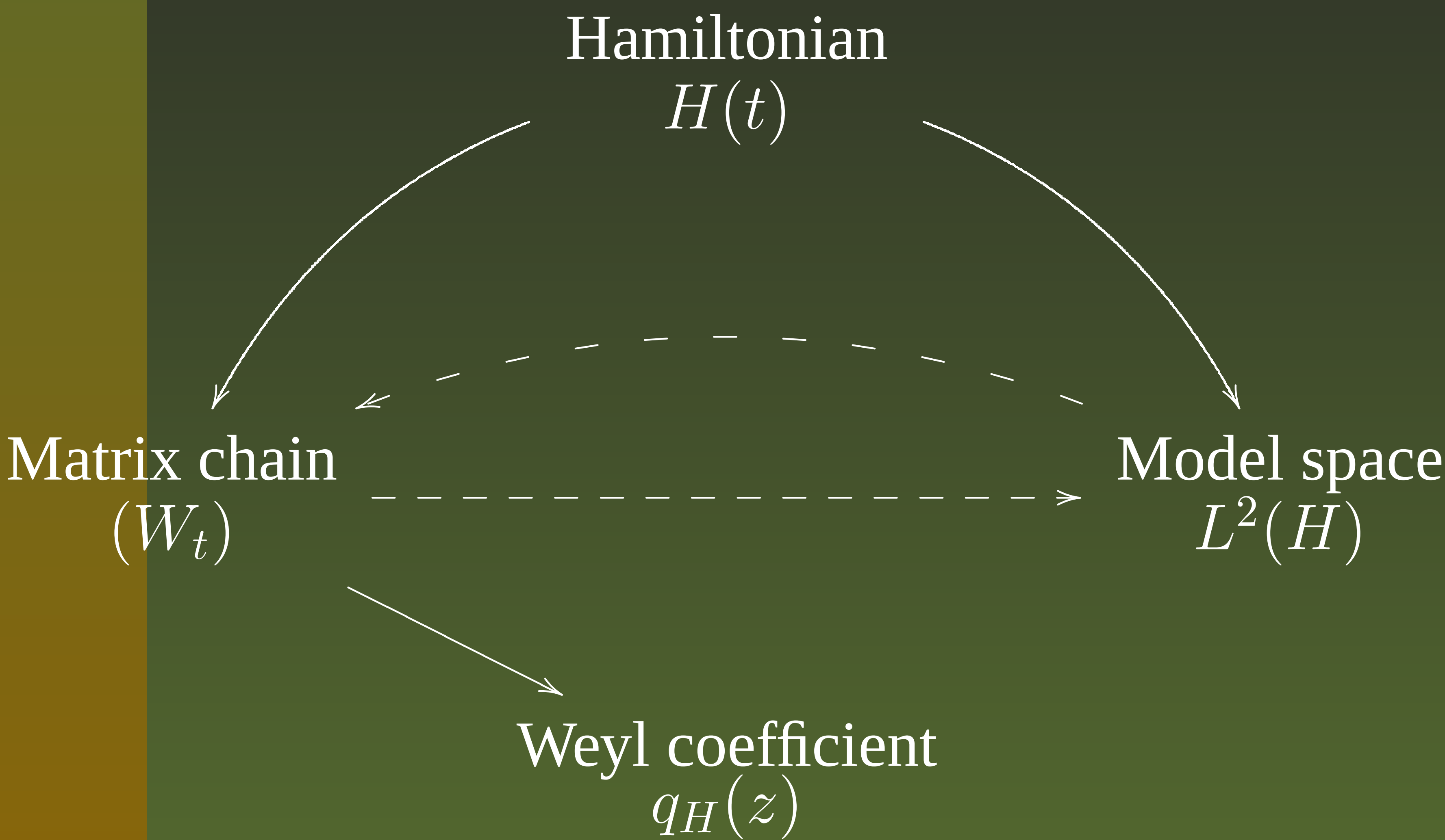
Matrix chain
(W_t)

Model space
$L^2(H)$

Summary



Summary (limit point case)



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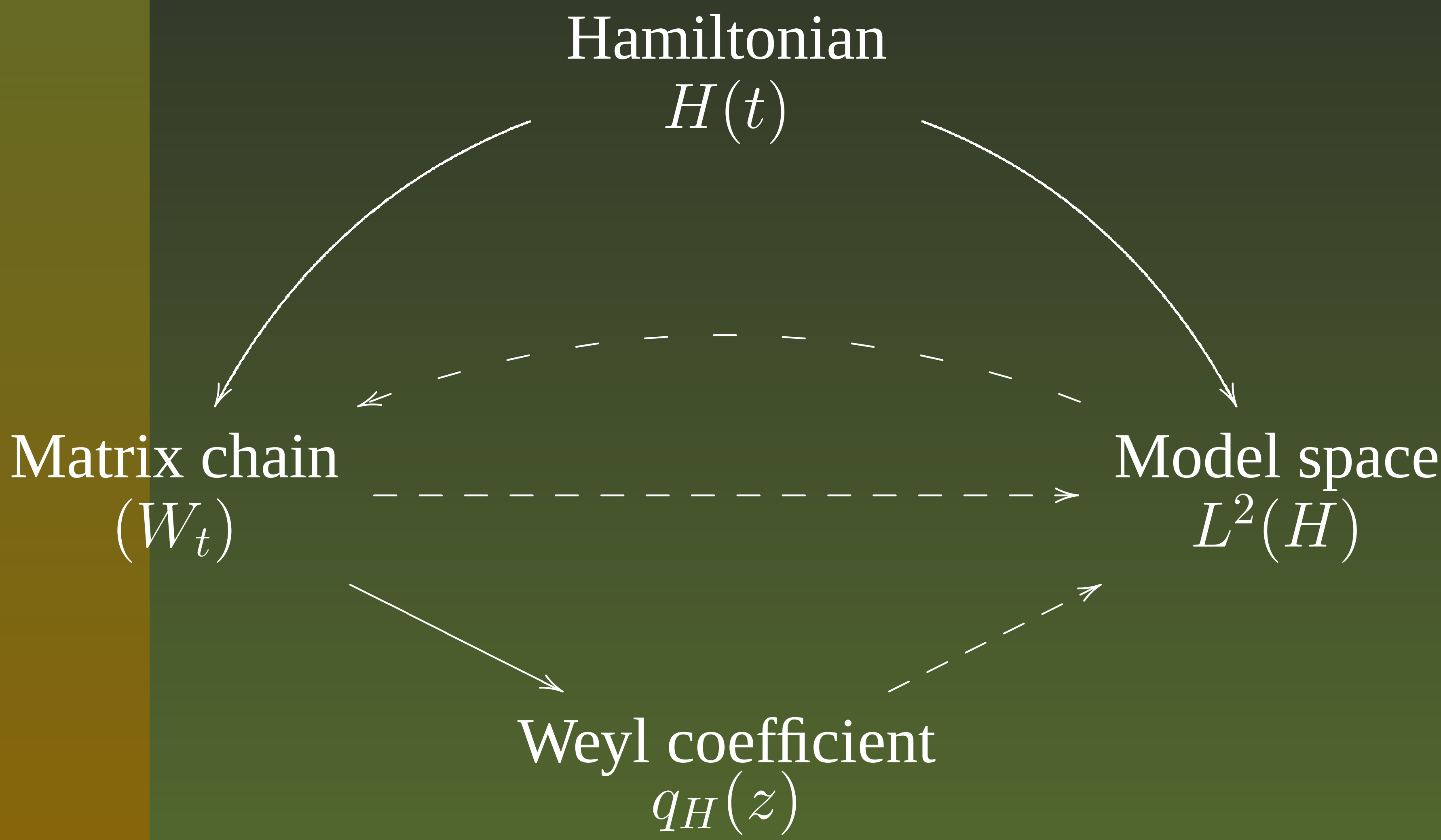

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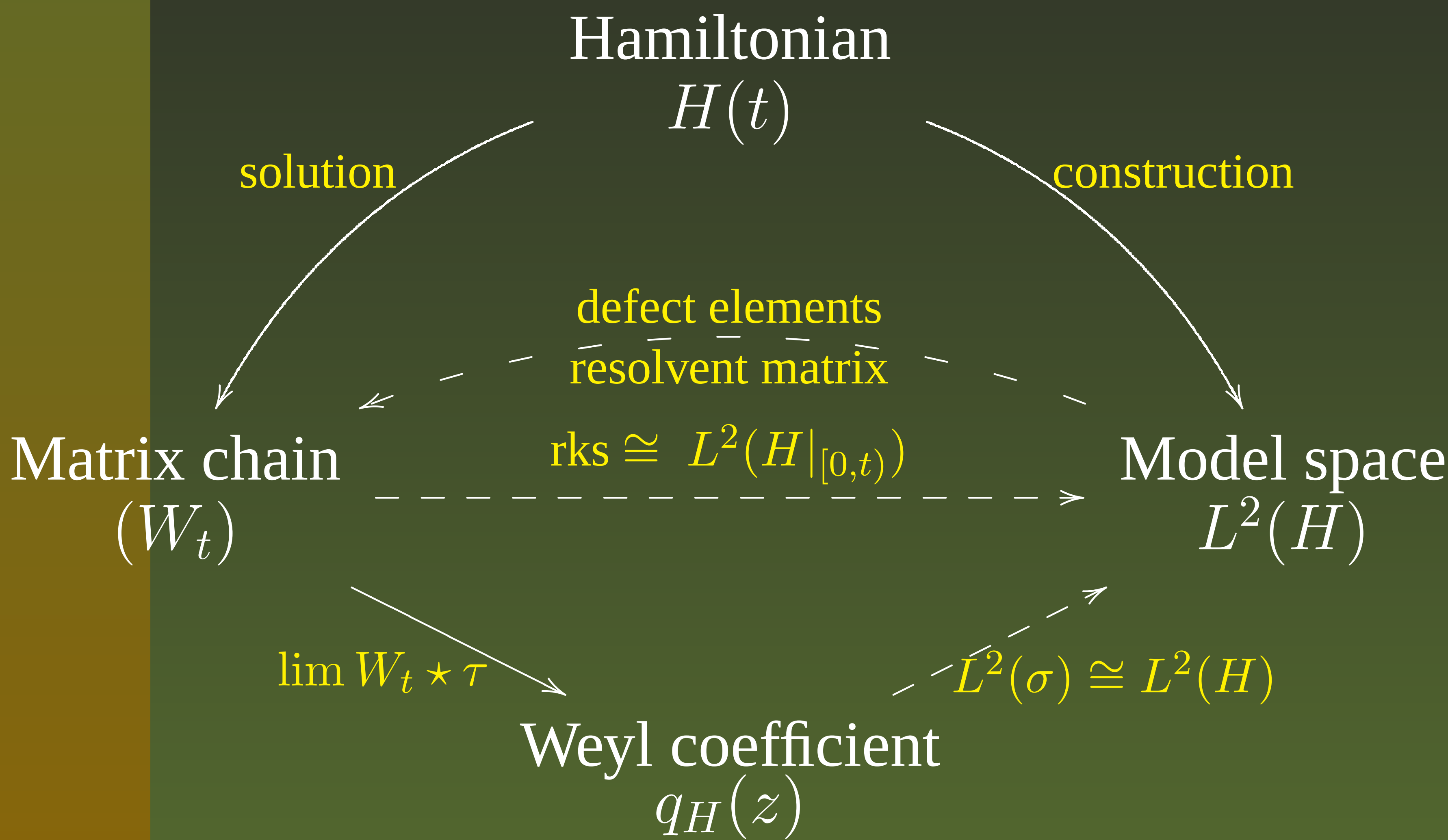


The Inverse Spectral Theorem

The assignment

$$H(t) \mapsto q_H(z)$$

yields a bijection between the set of all Hamiltonians (up to reparameterization) and the Nevanlinna class $\mathcal{N}_0$.

Some Examples of Canonical Systems

Positive definite functions

Let $a \in (0, \infty)$. A function $f : (-2a, 2a) \to \mathbb{C}$ is called positive definite, if $f(-t) = \overline{f(t)}$ and if the kernel

$$K_f(s,t) = f(t-s),\ s,t \in (-a,a)\,,$$

is positive definite. The set of all continuous positive definite functions on the interval $(-2a, 2a)$ is denoted by $\mathcal{P}_{0,a}$.

Positive definite functions

Continuation problem: Let $f \in \mathcal{P}_{0,a}$. Do there exist continuations $\tilde{f} \in \mathcal{P}_{0,\infty}$?

Positive definite functions

Continuation problem: Let $f \in \mathcal{P}_{0,a}$. Do there exist continuations $\tilde{f} \in \mathcal{P}_{0,\infty}$?

Solution: There exists either a unique continuation or infinitely many continuations. In the second case the set of all continuations is parameterized by

$$i \int_0^\infty e^{izt} \tilde{f}(t)\, dt = W_f(z) \star \tau(z)$$

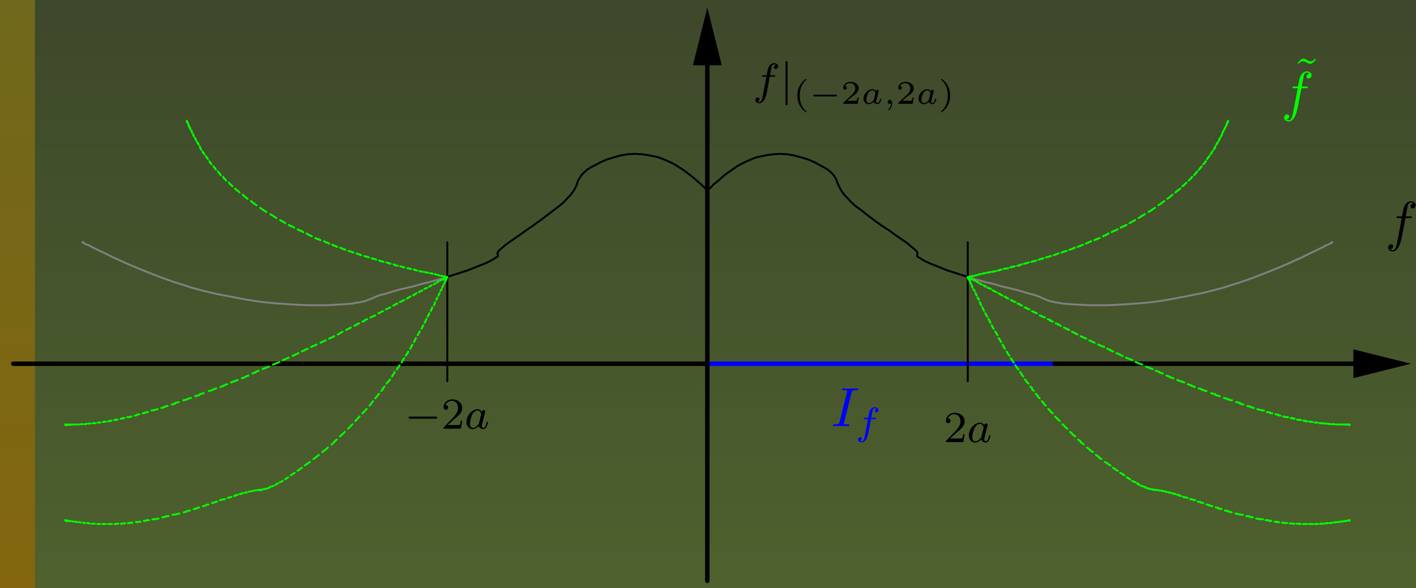
where W_f is a certain entire 2×2-matrix function and the parameter τ runs through the Nevanlinna class $\mathcal{N}_0$.

Pos.def. functions & can. systems

Let $f \in \mathcal{P}_{0,\infty}$. Assume that the set

$$I_f := \left\{ a > 0 : \ f|_{(-2a,2a)} \text{ has infinitely many continuations} \right\}$$

is nonempty.



Pos.def. functions & can. systems

Then the family

$$W_t(z) := \begin{cases} \begin{pmatrix} 1 & 0 \\ -(f(0)^{-1} + t)z & 1 \end{pmatrix} & , \ t \in [-f(0)^{-1}, 0) \\ W_{f|_{(-2t,2t)}}(z) & , \ t \in I_f \end{cases}$$

is the matrix chain of a certain Hamiltonian H_f.

The Bessel equation

The Bessel equation is the eigenvalue problem with singular endpoint 0

$$-u''(x) + \frac{\nu^2 - \frac{1}{4}}{x^2} u(x) = \lambda u(x), \; x > 0$$

Here ν is a parameter $\nu > \frac{1}{2}$ and λ is the eigenvalue parameter.

Bessel equation & can. systems

Rewriting this equation as a first-order-system, making a substitution in the independent variable, and setting $\alpha := 2\nu - 1$, $\lambda = z^2$, yields an equation of the form of a canonical system with

$$H_\alpha(x) = \begin{pmatrix} x^\alpha & 0 \\ 0 & x^{-\alpha} \end{pmatrix}$$

In order that H_α is integrable at 0, we need that $\alpha < 1$, i.e. $\nu < 1$. In this case the matrix chain $(W_{\alpha,t})_{t\in[0,\infty)}$ and the Weyl coefficient q_{H_α} can be computed explicitly:

Bessel equation & can. systems

Let $\alpha \in (0,1)$. Then

$$W_{\alpha,t}(z) =$$

$$= \begin{pmatrix} 2^{\nu_1}\Gamma(\nu) z^{-\nu_1} t^{-\nu_1} J_{\nu_1}(tz) & 2^{\nu_1}\Gamma(\nu) z^{-\nu_1} t^{\nu} J_{\nu}(tz) \\ -2^{-\nu}\Gamma(-\nu_1) z^{\nu} t^{-\nu_1} J_{-\nu_1}(tz) & 2^{-\nu}\Gamma(-\nu_1) z^{\nu} t^{\nu} J_{-\nu}(tz) \end{pmatrix}$$

with $\nu_1 := \frac{\alpha-1}{2} = \nu - 1$, and

$$q_{H_\alpha}(z) = c_\alpha z^{-\alpha}$$

with $c_\alpha := \frac{2^\alpha}{\pi}\Gamma(\nu)^2 \sin \nu e^{i\nu\pi}$.

Indefinite Analogue of Canonical Systems

Indefinite can.systems / Motivation

	POSITIVE DEFINITE	
CONCEPTS	Hamiltonian Matrix chain Boundary triplet Nevanlinna class $\mathcal{N}_0$	

Indefinite can.systems / Motivation

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Indefinite can.systems / Motivation

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Indefinite can.systems / Motivation

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CONCEPTS	Hamiltonian Matrix chain Boundary triplet Nevanlinna class $\mathcal{N}_0$	generalized Nev.class $\mathcal{N}_\kappa$
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EXAMPLES	Positive def.functions Bessel equation $\alpha < 1$ Moment problems Strings	

Indefinite can.systems / Motivation

	POSITIVE DEFINITE	INDEFINITE
CONCEPTS	Hamiltonian	
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	Boundary triplet	B.triplet in Pontryagin space
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I.S.T.	$\{\text{Hamiltonians}\} \leftrightsquigarrow \mathcal{N}_0$	
EXAMPLES	Positive def.functions	
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Indefinite can.systems / Motivation

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EXAMPLES	Positive def.functions	Hermitian indef.functions
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Indefinite can.systems / Motivation

	POSITIVE DEFINITE	INDEFINITE
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General Hamiltonians

A *general Hamiltonian* consists of the data

$$\sigma_0, \ldots, \sigma_{n+1} \in \mathbb{R} \cup \{\pm\infty\},\ \sigma_0 < \sigma_1 < \ldots < \sigma_{n+1},$$

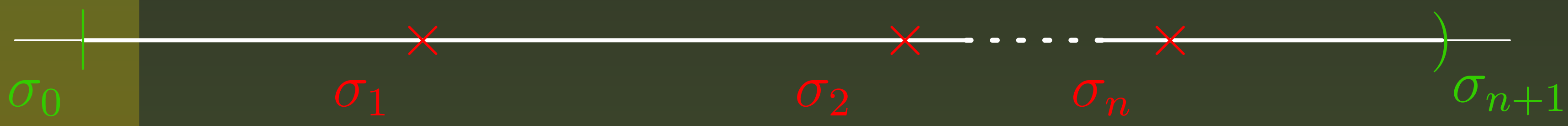
$$H_i : (\sigma_i, \sigma_{i+1}) \to \mathbb{R}^{2\times 2},\ i = 0, \ldots, n,$$

$$E \subseteq \bigcup_{i=0}^{n} (\sigma_i, \sigma_{i+1}) \cup \{\sigma_0, \sigma_{n+1}\} \text{ finite}$$

$$d_{i,0}, \ldots, d_{i,2\Delta_i - 1} \in \mathbb{R}, \quad \ddot{o}_i \in \mathbb{N}_0, b_{i,1}, \ldots, b_{i,\ddot{o}_i+1} \in \mathbb{R}$$

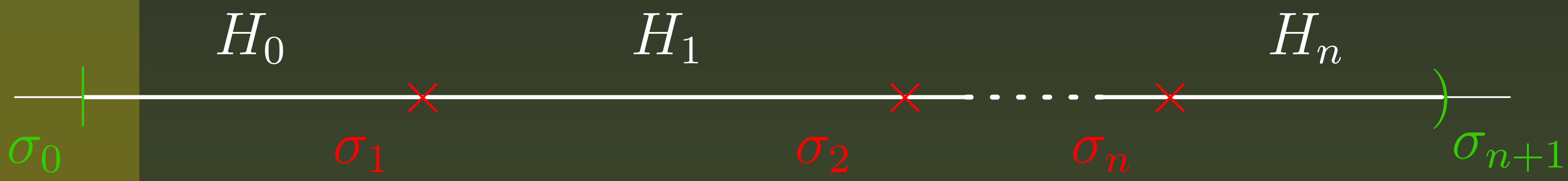
subject to certain conditions.

General Hamiltonians



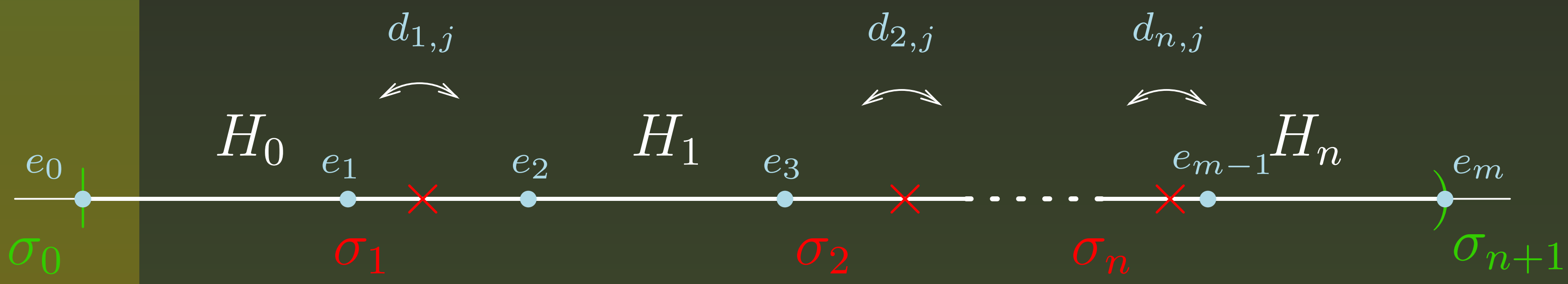
- σ_0 = starting point
- σ_{n+1} = endpoint
- $\sigma_1, \ldots, \sigma_n$ = singularities

General Hamiltonians



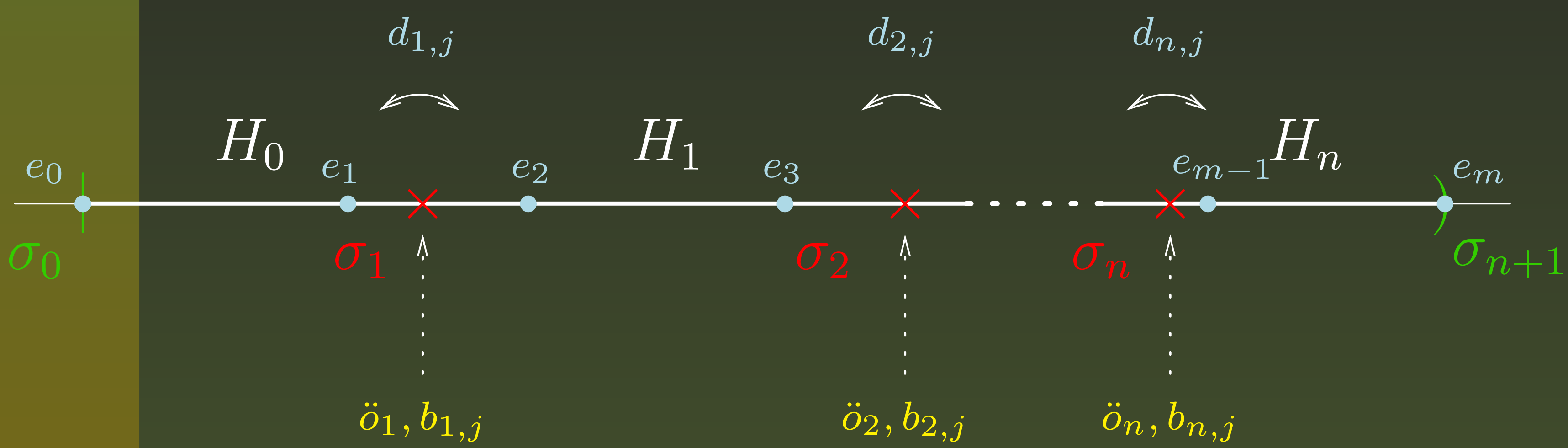
- $H_0, \ldots, H_n =$ Hamiltonians, not integrable at $\sigma_1, \ldots, \sigma_n$ ($\sigma_1, \ldots, \sigma_n$ singularities)
- H_0 integrable at 0 (initial value problem)
- H_n integrable/not at σ_{n+1} (limit circle/point case)
- growth of H_i towards singularity is restricted

General Hamiltonians



- $d_{i,j}$ = interface conditions at a singularity
- E quantitative measurement of ‘local at a singularity’

General Hamiltonians



- $\ddot{o}_i, b_{i,j}$ = contribution concentrated in the singularity

Maximal chains of matrices

Axiomization of ‘fundamental solution’

- $W_0 = I$ and $W_t \in \bigcup_{\kappa\geq 0} \mathcal{M}_\kappa$ for $t \in [\sigma_0, \sigma_1) \cup (\sigma_1, \sigma_2) \cup \ldots \cup (\sigma_n, \sigma_{n+1})$
- $W_s^{-1} W_t \in \bigcup_{\kappa\geq 0} \mathcal{M}_\kappa$ and
$$\operatorname{ind}_- W_t = \operatorname{ind}_- W_s + \operatorname{ind}_- W_s^{-1} W_t$$
- If $W \in \bigcup_{\kappa\geq 0} \mathcal{M}_\kappa$, $W^{-1} W_t \in \bigcup_{\kappa\geq 0} \mathcal{M}_\kappa$ and
$$\operatorname{ind}_- W_t = \operatorname{ind}_- W + \operatorname{ind}_- W^{-1} W_t$$
then $W = W_t$ for some t
- some technical conditions

Theory of indefinite can.systems

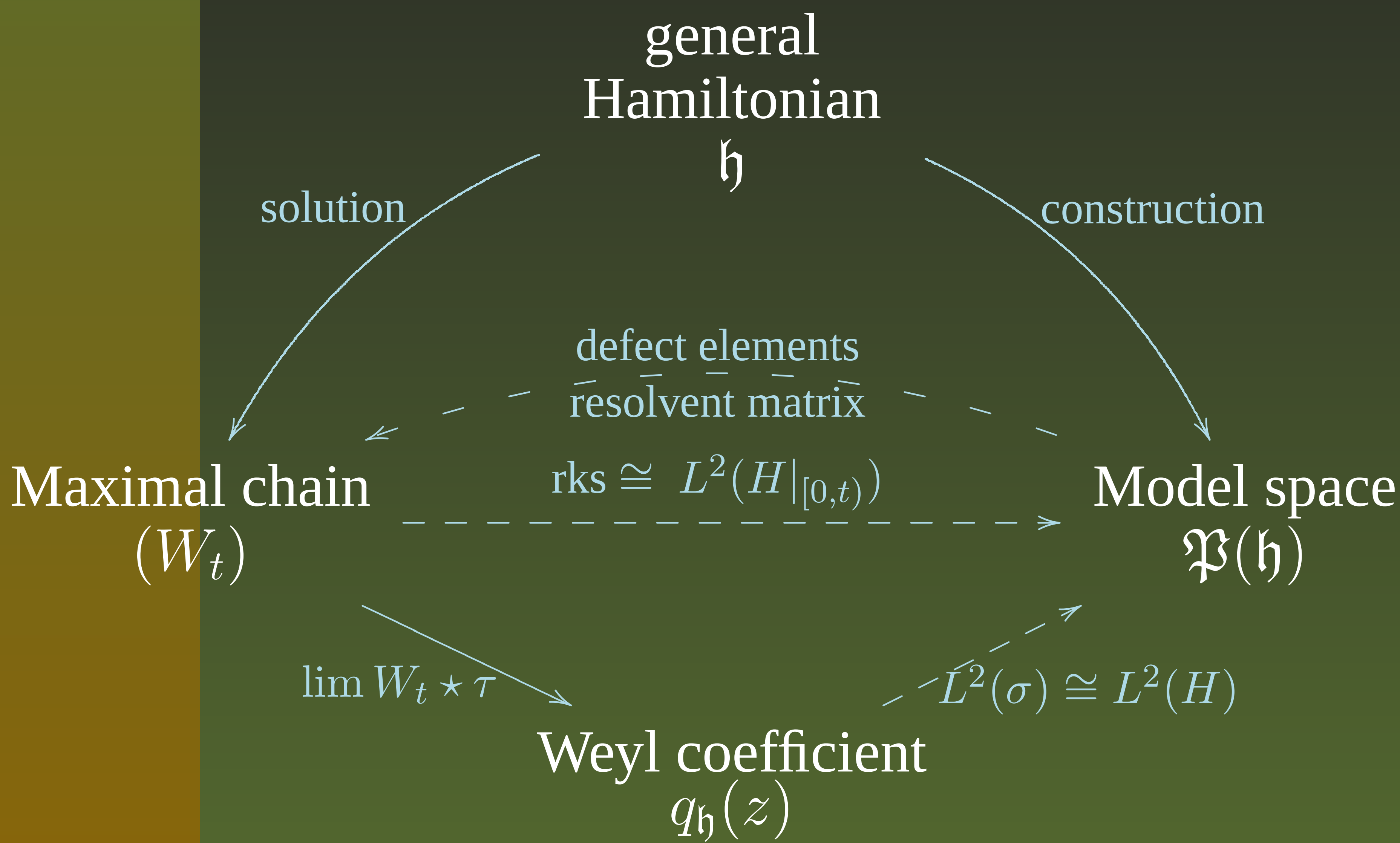
general
Hamiltonian
$\mathfrak{h}$

Maximal chain
(W_t)

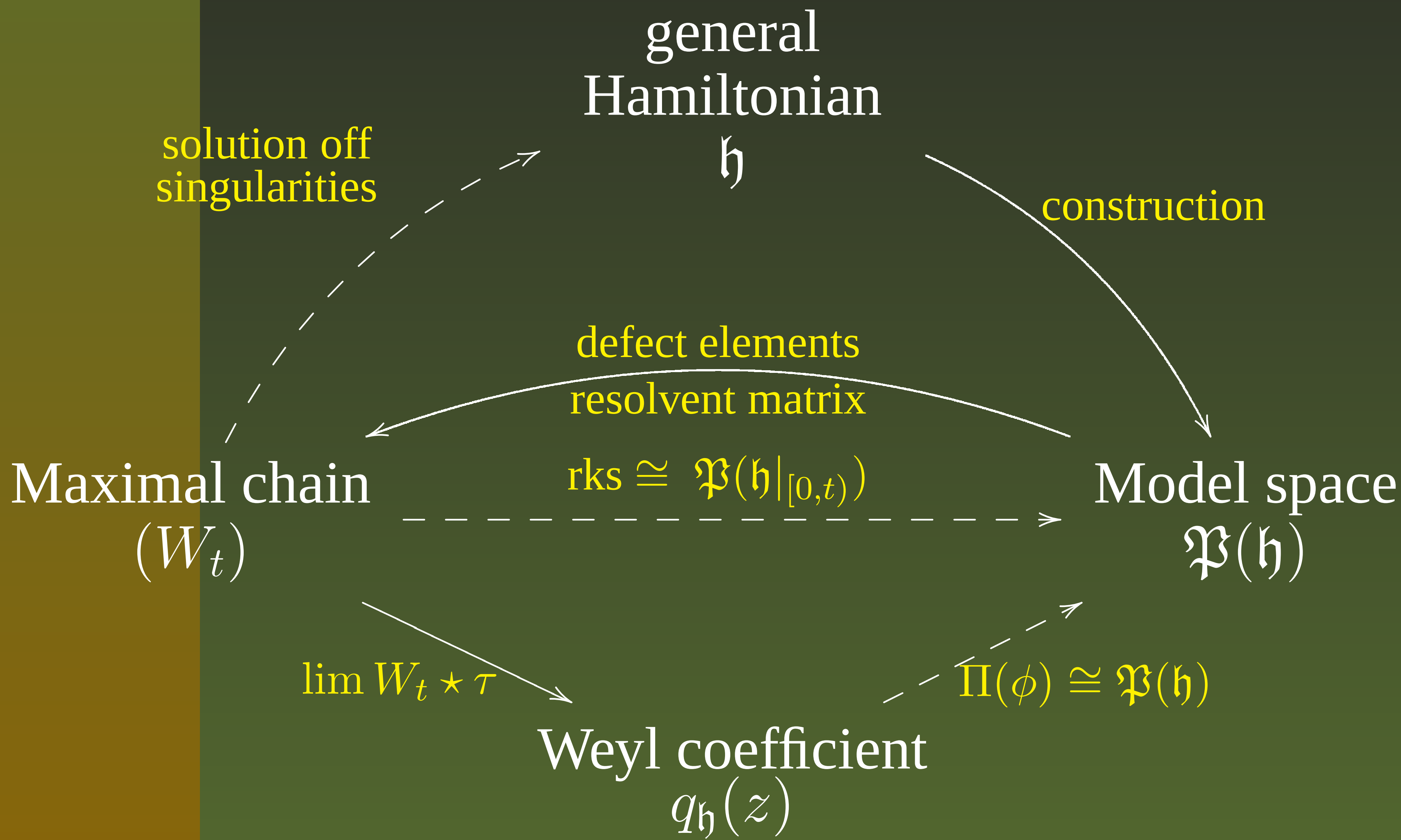
Model space
$\mathfrak{P}(\mathfrak{h})$

Weyl coefficient
$q_{\mathfrak{h}}(z)$

Theory of indefinite can.systems



Theory of indefinite can.systems



The Inverse Spectral Theorem

The assignment

$$\mathfrak{h} \mapsto q_{\mathfrak{h}}(z)$$

yields a bijection between the set of all general Hamiltonians (up to reparameterization) and $\bigcup_{\kappa \geq 0} \mathcal{N}_\kappa$.

Fitting our examples

In our examples we had obtained generalized Nevanlinna function which seemed to be the ‘Weyl coefficient of the underlying indefinite canonical system’, namely:

- $q_f(z) = \frac{i}{z^2} - \frac{1}{z} \in \mathcal{N}_1$ from the hermitian indefinite function $f(t) = 1 - |t|$.
- $q_\alpha(z) = c_\alpha z^{-\alpha} \in \mathcal{N}_{\kappa(\alpha)}$ where $\kappa(\alpha) := \left[\frac{\alpha+1}{2}\right]$, from the Bessel equation with parameter $\alpha \in \mathbb{R}^+ \setminus \{1, 3, 5, \dots\}$.

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In order to fit these examples, we have to find the general Hamiltonian whose Weyl coefficient is q_f or q_α, and see how it is related to the 'Hamiltonians' H_f and H_α.

A Short Account on the Literature

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THE END

The matrix chain (W_t)

Let H be a Hamiltonian defined on $[\sigma_0, \sigma_1)$. Then W_t, $t \in [\sigma_0, \sigma_1)$, denotes the unique solution of the initial value problem

$$\frac{d}{dt} W_t(z) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = z W_t(z) H(t),\ x \in [\sigma_0, \sigma_1),$$

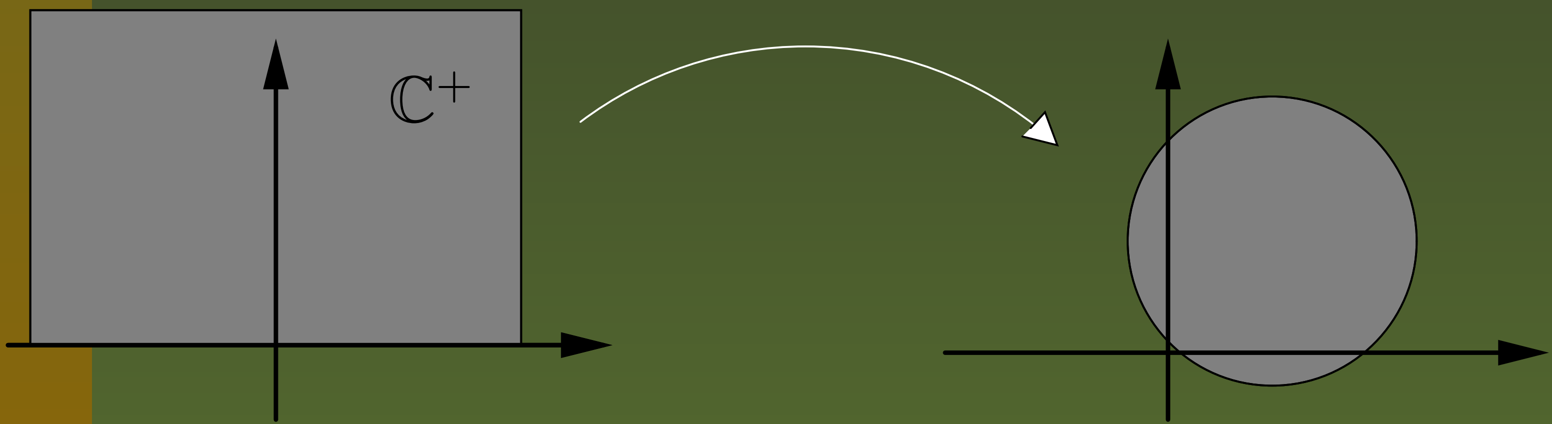
$$W_0(z) = I .$$

The Weyl coefficient $q_H(z)$

For $W = (w_{ij})_{i,j=1}^2 \in \mathbb{C}^{2\times 2}$ and $\tau \in \mathbb{C}$ denote

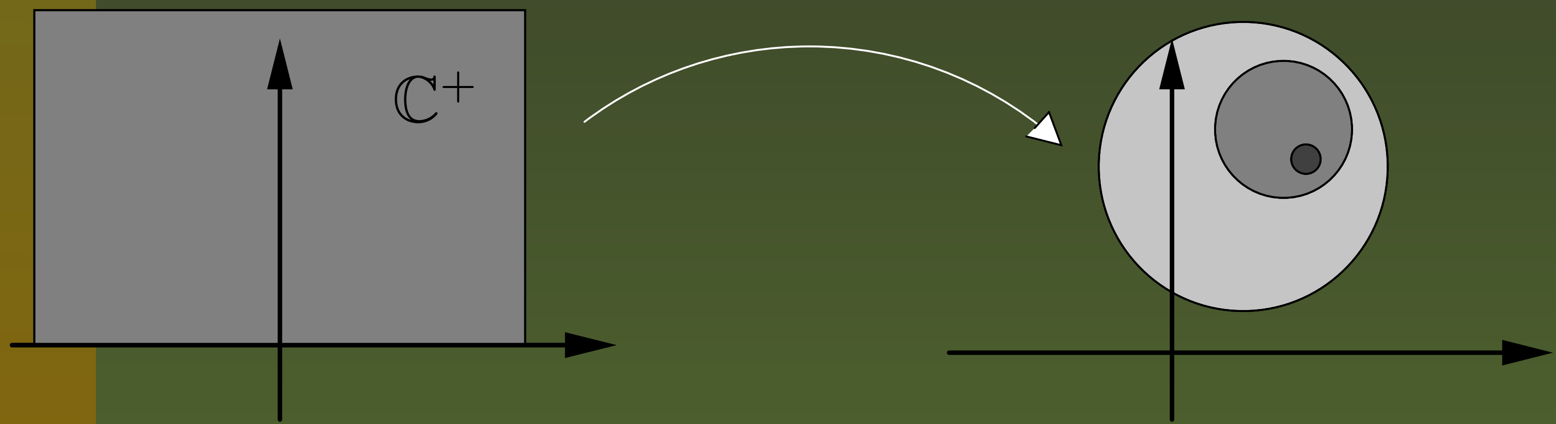
$$W \star \tau := \frac{w_{11}\tau + w_{12}}{w_{21}\tau + w_{22}}$$

The assignment $\tau \mapsto W \star \tau$ maps the upper half plane to some (general) disk:



The Weyl coefficient $q_H(z)$

Let $(W_t)_{t\in[\sigma_0,\sigma_1)}$ be the matrix chain associated with the Hamiltonian H. The assignments $\tau \mapsto W_t \star \tau$ map $\mathbb{C}^+$ to a nested sequence of disks contained in $\mathbb{C}^+$. The disk $W_t \star \mathbb{C}^+$ is contained in the upper half plane and its radius is $[\int_{\sigma_0}^t \operatorname{tr} H(x)\,dx]^{-1}$.



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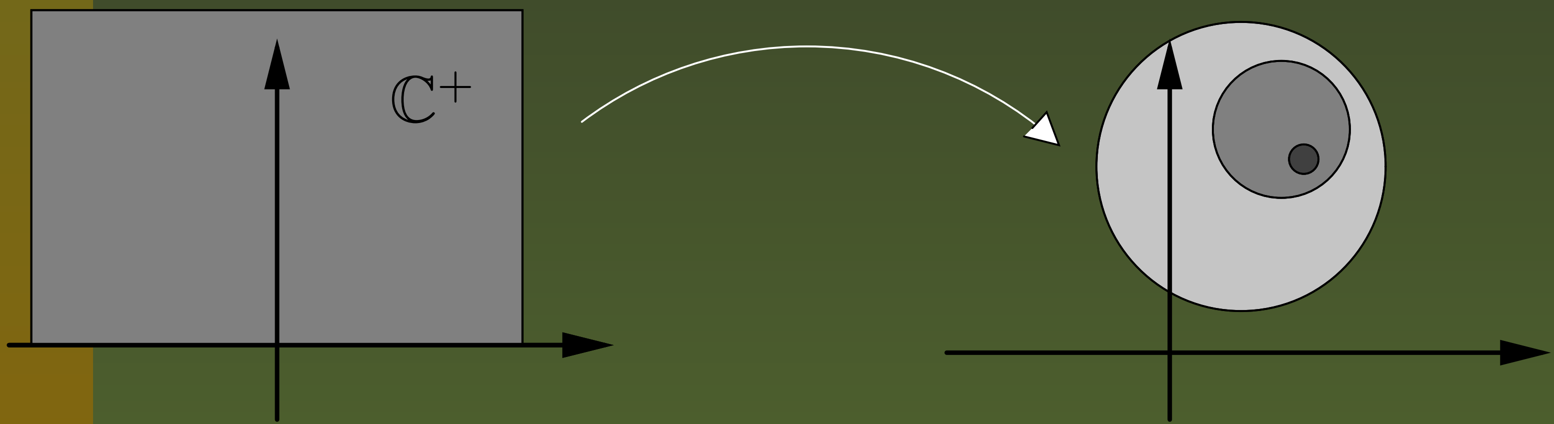
Let $(W_t)_{t\in[\sigma_0,\sigma_1)}$ be the matrix chain associated with the Hamiltonian H. The assignments $\tau \mapsto W_t \star \tau$ map $\mathbb{C}^+$ to a nested sequence of disks contained in $\mathbb{C}^+$. The disk $W_t \star \mathbb{C}^+$ is contained in the upper half plane and its radius is $[\int_{\sigma_0}^t \operatorname{tr} H(x)\, dx]^{-1}$.



Thus the limit $q_H(z) := \lim_{t\nearrow\sigma_1} W_t(z) \star \tau$ exists, does not depend on $\tau \in \mathbb{C}^+$, and belongs to $\mathcal{N}_0$.

The model space $L^2(H)$

Supressing some technicalities which arise from ‘indivisible intervals’, we have

$$L^2(H) := \left\{ f : (\sigma_0, \sigma_1) \to \mathbb{C}^2 : \int_{\sigma_0}^{\sigma_1} f(t)^T H(t) f(t)\, dt < \infty \right\}$$

$$T_{max}(H) := \left\{ (f; g) \in L^2(H)^2 : f \text{ absolutely continuous}, \right.$$
$$\left. f(t) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} H(t) g(t), \text{ a.e.} \right\}$$

$$\Gamma(H)(f; g) := f(\sigma_0),\ (f; g) \in T_{max}(H)$$

W_t from defect elements

Let $y_1(z,x) = (y_1(z,x)_2, y_1(z,x)_2)^T$ and $y_2(z,x) = (y_2(z,x)_2, y_2(z,x)_2)^T$ be the elements of $\ker(T_{max}(H|_{(\sigma_0,t)} - z)$, such that $y_1(z,\sigma_0) = (1,0)^T$ and $y_2(z,\sigma_0) = (0,1)^T$. Then

$$W_t(z) = \begin{pmatrix} y_1(z,t)_1 & y_1(z,t)_2 \\ y_2(z,t)_1 & y_2(z,t)_2 \end{pmatrix}$$

W_t as resolvent matrix

Consider

$$S_1 := \Big\{(x;y) \in T_{max}(H|_{(\sigma_0,t)}) : \\ \pi_{l,1}\Gamma(H|_{(\sigma_0,t)})(x;y) = 0, \pi_r\Gamma(H|_{(\sigma_0,t)})(x;y) = 0\Big\}$$

$$u : (x;y) \mapsto \pi_{l,2}\Gamma(H|_{(\sigma_0,t)})(x;y),\ (x;y) \in T_{max}(H|_{(\sigma_0,t)})$$

Then S_1 is symmetric with defect 1 and $u|_{S_1^*}$ is continuous. The matrix function W_t is a u-resolvent matrix of S_1.

The reproducing kernel space of W_t

The kernel

$$K_{W_t}(w,z) := \frac{W_t(z)\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}W_t(w)^* - \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}}{z-\overline{w}}$$

is positive definite, thus generates a reproducing kernel Hilbert space $\mathfrak{K}(W_t)$. The elements of $\mathfrak{K}(W_t)$ are entire 2-vector-functions.

The operator $\mathcal{S}(W_t)$ of multiplication by z is a symmetry with defect 2. The map $\Gamma(W_t) : f \mapsto f(0)$ is a boundary map for $\mathcal{S}(W_t)$.

The reproducing kernel space of W_t

The boundary triplet $\langle L^2(H|_{(\sigma_0,t)}), T_{min}(H|_{(\sigma_0,t)}), \Gamma(H|_{(\sigma_0,t)})\rangle$ is isomorphic to $\langle \mathfrak{K}(W_t), \mathcal{S}(W_t), \Gamma(W_t)\rangle$. The isomorphism of $L^2(H|_{(\sigma_0,t)})$ to $\mathfrak{K}(W_t)$ is given by

$$f(x) \mapsto \int_{\sigma_0}^{t} W_x(z)H(x)f(x)\,dx\,.$$

The Fourier transform

The Weyl coefficient admits an integral representation of the form

$$q_H(z) = a + bz + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) d\sigma(t)\,.$$

Assuming $b = 0$, the map

$$f(x) \mapsto \int_{\sigma_0}^{\sigma_1} (0,1) W_x(z) H(x) f(x)\, dx$$

is an isomorphism of $L^2(H)$ onto $L^2(\sigma)$.

The class $\mathcal{N}_0$

Denote by $\mathcal{N}_0$ the set of all functions τ, which are analytic in $\mathbb{C} \setminus \mathbb{R}$, satisfy $\tau(\overline{z}) = \overline{\tau(z)}$, and are such that the Nevanlinna kernel

$$Q_\tau(w, z) := \frac{\tau(z) - \overline{\tau(w)}}{z - \overline{w}}$$

is nonnegative definite. This means that each of the quadratic forms

$$q_\tau\big(\xi_1, \ldots, \xi_m\big) := \sum_{i,j=1}^{m} Q_f(z_j, z_i)\xi_i\overline{\xi_j}$$

is nonnegative definite.

The class $\mathcal{N}_0$

A more classical approach to this class of functions is given by the following result:

A function τ which is analytic in $\mathbb{C} \setminus \mathbb{R}$ and satisfies $\tau(\overline{z}) = \overline{\tau(z)}$ belongs to the class $\mathcal{N}_0$, if and only if it maps the open upper half plane into the closed upper half plane.

Hermitian indefinite functions

Let $a \in (0, \infty)$. A function $f : (-2a, 2a) \longrightarrow \mathbb{C}$ is called hermitian indefinite, if $f(-t) = \overline{f(t)}$ and if the kernel

$$K_f(s,t) = f(t-s), \; s,t \in (-a,a)\,,$$

has a finite number of negative squares. The set of all continuous hermitian indefinite functions with κ negative squares on the interval $(-2a, 2a)$ is denoted by $\mathcal{P}_{\kappa,a}$.

Hermitian indefinite functions

Continuation problem: Let $f \in \mathcal{P}_{\kappa_0,a}$. Do there exist continuations $\tilde{f} \in \mathcal{P}_{\kappa,\infty}$?

Clearly, for the existence of a continuation $\tilde{f} \in \mathcal{P}_{\kappa,\infty}$, it is necessary that $\kappa \geq \kappa_0$.

Hermitian indefinite functions

Solution: There exists a number $\Delta(f) \in \mathbb{N} \cup \{0, \infty\}$:

- If $\Delta(f) = 0$, then f has infinitely many continuations in each of the classes $\mathcal{P}_{\kappa,\infty}$, $\kappa \geq \kappa_0$.
- If $0 < \Delta(f) < \infty$, then f has a unique continuation in $\mathcal{P}_{\kappa_0,\infty}$, no continuations in $\mathcal{P}_{\kappa,\infty}$ with $\kappa_0 < \kappa < \kappa_0 + \Delta(f)$, and infinitely many continuations in each of the classes $\mathcal{P}_{\kappa,\infty}$, $\kappa \geq \kappa_0 + \Delta(f)$.
- If $\Delta(f) = \infty$, then f has a unique continuation in $\mathcal{P}_{\kappa_0,\infty}$, and no continuations in any of the classes $\mathcal{P}_{\kappa,\infty}$, $\kappa > \kappa_0$.

Hermitian indefinite functions

Assume that $\Delta(f) < \infty$. Then there exists an entire 2×2-matrix function W_f such that the formula

$$i \int_0^\infty e^{izt} \tilde{f}(t)\, dt = W_f(z) \star \tau(z)$$

parameterizes the continuations of f in $\bigcup_{\kappa \geq \kappa_0} \mathcal{P}_{\kappa,\infty}$. Thereby continuations $\tilde{f} \in \mathcal{P}_{\kappa,\infty}$ correspond to parameters τ in the class $\mathcal{K}_{\kappa-\kappa_0}^{\Delta(f)}$. If $\Delta(f) > 0$, the unique solution in $\mathcal{P}_{\kappa_0,\infty}$ is given by the parameter $\tau = \infty$.

Her.indef. functions & can. systems

An example: The function $f(t) := 1 - |t|$ belongs to $\mathcal{P}_{1,\infty}$. Again consider the restrictions $f|_{(-2t,2t)}$. Then

$$\Delta(f|_{(-2t,2t)}) = \begin{cases} 0 & , \ 0 < t < 1 \text{ or } t > 1 \\ 1 & , \ t = 1 \end{cases}$$

$f|_{(-2t,2t)} \in \mathcal{P}_{0,t}$, $0 < t < 1$, $f|_{(-2t,2t)} \in \mathcal{P}_{1,t}$, $t > 1$, and

$$W_{f|_{(-2t,2t)}}(z) = $$

$$= \begin{pmatrix} \frac{\sin tz - z\cos tz}{(t-1)z} & \left(\frac{1}{z^2} - (t-1)\right)\sin tz - \frac{t\cos tz}{z} \\ \frac{z\cos tz}{t-1} & (t-1)z\sin tz + \cos tz \end{pmatrix}$$

Her.indef. functions & can. systems

The family

$$W_t(z) := \begin{cases} \begin{pmatrix} 1 & 0 \\ -(1+t)z & 1 \end{pmatrix} & , \ t \in [-1,0] \\ W_{f|_{(-2t,2t)}}(z) & , \ t \in (0,1) \cup (1,\infty) \end{cases}$$

satisfies a differential equation of the form of a canonical system with

$$H_f(t) = \begin{cases} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & , \ t \in (-1,0) \\ \begin{pmatrix} (t-1)^2 & 0 \\ 0 & \frac{1}{(t-1)^2} \end{pmatrix} & , \ t \in (0,1) \cup (1,\infty) \end{cases}$$

Her.indef. functions & can. systems

The function H_f is locally integrable on $[-1, 1) \cup (1, \infty)$, but NOT at the point 1. Moreover, $\int_T^\infty \operatorname{tr} H_f(x)\,dx = +\infty$ for $T > 1$, i.e. the 'limit point case' prevails at infinity.

If we formally carry out the construction of the Weyl coefficient, we obtain

$$q_{H_f}(z) = \frac{i}{z^2} - \frac{1}{z}, \; z \in \mathbb{C}^+$$

This function belongs to $\mathcal{N}_1$.

Bessel equation ($\alpha \geq 1$)

The function $H_\alpha(t)$ and the matrices $W_{\alpha,t}(z)$ are well-defined for $\alpha \in \mathbb{R}^+ \setminus \{1, 3, 5, \dots\}$. Moreover, $W_{\alpha,t}$ satisfies a differential equation of the form of a canonical system with H_α. The function H_α is locally integrable on $(0, \infty)$, but NOT at the point 0. Moreover, $\int_T^\infty \operatorname{tr} H_\alpha(x)\, dx = +\infty$ for $T > 0$, i.e. the 'limit point case' prevails at infinity.

If we formally carry out the construction of the Weyl coefficient, we obtain

$$q_{H_\alpha}(z) = c_\alpha z^{-\alpha}, \; z \in \mathbb{C}^+$$

This function belongs to $\mathcal{N}_{\kappa(\alpha)}$ with $\kappa(\alpha) := \left[\frac{\alpha+1}{2}\right]$.

The class $\mathcal{K}_\nu^\Delta$

For $\nu, \Delta \in \mathbb{N}_0$, denote by $\mathcal{K}_\nu^\Delta$ the set of all functions τ, which are meromorphic in $\mathbb{C} \setminus \mathbb{R}$, satisfy $\tau(\overline{z}) = \overline{\tau(z)}$, and are such that the maximal number of negative squares of quadratic forms

$$q_\tau\big(\xi_1, \dots, \xi_m; \eta_0, \dots, \eta_{\Delta-1}\big) :=$$

$$= \sum_{i,j=1}^{m} \frac{\tau(z_i) - \overline{\tau(z_j)}}{z_i - \overline{z_j}} \xi_i \overline{\xi_j} + \sum_{k=0}^{\Delta-1} \sum_{i=1}^{m} \mathrm{Re}(z_i^k \xi_i \overline{\eta_k})$$

is ν. Note that $\mathcal{K}_\nu^\Delta = \mathcal{N}_\nu$.

The class $\mathcal{N}_\kappa$

For $\kappa \in \mathbb{N}_0$, denote by $\mathcal{N}_\kappa$ the set of all functions τ, which are meromorphic in $\mathbb{C} \setminus \mathbb{R}$, satisfy $\tau(\overline{z}) = \overline{\tau(z)}$, and are such that the Nevanlinna kernel

$$Q_f(w, z) := \frac{\tau(z) - \overline{\tau(w)}}{z - \overline{w}}$$

has κ negative squares. This means that the maximal number of negative squares of quadratic forms

$$q_\tau(\xi_1, \dots, \xi_m) = \sum_{i,j=1}^{m} Q_f(z_j, z_i)\xi_i\overline{\xi_j}$$

is equal to κ.

The Weyl coefficient $q_{\mathfrak{h}}(z)$

The limit

$$q_{\mathfrak{h}}(z) := \lim_{t \nearrow \sigma_n} W_t(z) \star \tau$$

exists as a meromorphic function locally uniformly on $\mathbb{C} \setminus \mathbb{R}$ and does not depend on $\tau \in \mathbb{C}^+$.

The maximal chain (W_t)

The matrix function W_t, $t \in \bigcup_{i=1}^{n}(\sigma_{i-1}, \sigma_i)$, is for every $i = 1, \ldots, n$ a solution of the differential equation

$$\frac{d}{dt} W_t(z) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = z W_t(z) H_i(t), \; x \in (\sigma_{i-1}, \sigma_i)$$

On the interval $[\sigma_0, \sigma_1)$ it is uniquely determined by its initial value $W_{\sigma_0} = I$.

The reproducing kernel space of W_t

The kernel

$$K_{W_t}(w,z) := \frac{W_t(z)\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}W_t(w)^* - \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}}{z-\overline{w}}$$

has a finite number of negative squares, thus generates a reproducing kernel Pontryagin space $\mathfrak{K}(W_t)$. The elements of $\mathfrak{K}(W_t)$ are entire 2-vector-functions.

The operator $\mathcal{S}(W_t)$ of multiplication by z is a symmetry with defect 2. The map $\Gamma(W_t) : f \mapsto f(0)$ is a boundary map for $\mathcal{S}(W_t)$.

The reproducing kernel space of W_t

There exists an isomorphism Φ_t of the boundary triplets $\langle \mathfrak{K}(W_t), \mathcal{S}(W_t), \Gamma(W_t)\rangle$ and $\langle \mathfrak{P}^2(\mathfrak{h}|_{(\sigma_0,t)}), S(\mathfrak{h}|_{(\sigma_0,t)}), \Gamma(\mathfrak{h}|_{(\sigma_0,t)})\rangle$

If $J := [s_-, s_+] \subseteq (\sigma_{i-1}, \sigma_i)$, then the map

$$\lambda_J : f(x) \mapsto \int_{s_-}^{s_+} W_x(z) H(x) f(x)\, dx$$

is an isomorphism of $L^2(H_i|_{[s_-,s_+]})$ onto $\mathfrak{K}(W_{s_+})[-]\mathfrak{K}(W_{s_-})$. We have

$$\begin{array}{ccc} L^2(H_i|_J) & \xrightarrow{\iota_J} & \mathfrak{P}(\mathfrak{h}|_{(\sigma_0,s_+)}) \\ \downarrow \lambda_J & \nearrow \Phi_{s_+} & \\ \mathfrak{K}(W_{s_+})[-]\mathfrak{K}(W_{s_-}) & & \end{array}$$

The Fourier transform

The Weyl coefficient admits a representation of the form

$$q_{\mathfrak{h}}(z) = \phi\left(\frac{1}{t-z}\right)$$

with some distribution on $\overline{\mathbb{R}}$. This distribution generates a Pontryagin space $\Pi(\phi)$. There exists an isomorphism of $\mathfrak{P}(\mathfrak{h})$ onto $\Pi(\phi)$.

W_t as resolvent matrix

For $t \in I$ consider

$$S_1 := \Big\{ (x;y) \in T(\mathfrak{h}|_{(\sigma_0,t)}) : \\ \pi_{l,1}\Gamma(\mathfrak{h}|_{(\sigma_0,t)})(x;y) = 0, \pi_r\Gamma(\mathfrak{h}|_{(\sigma_0,t)})(x;y) = 0 \Big\}$$

$$u : (x;y) \mapsto \pi_{l,2}\Gamma(\mathfrak{h}|_{(\sigma_0,t)})(x;y),\ (x;y) \in T(\mathfrak{h}|_{(\sigma_0,t)})$$

Then S_1 is symmetric with defect 1 and $u|_{S_1^*}$ is continuous. The matrix function W_t is a u-resolvent matrix of S_1.

W_t from defect elements

Let $\phi_z, \psi_z \in \ker(T(\mathfrak{h}|_{(\sigma_0,t)}))$ be such that

$$\pi_l\Gamma(\mathfrak{h}|_{(\sigma_0,t)})(\phi_z; z\phi_z) = \begin{pmatrix}1\\0\end{pmatrix},\ \pi_l\Gamma(\mathfrak{h}|_{(\sigma_0,t)})(\psi_z; z\psi_z) = \begin{pmatrix}0\\1\end{pmatrix}$$

Then

$$W_t(z) = \begin{pmatrix}\pi_r\Gamma(\mathfrak{h}|_{(\sigma_0,t)})(\phi_z; z\phi_z)^T\\ \pi_r\Gamma(\mathfrak{h}|_{(\sigma_0,t)})(\psi_z; z\psi_z)^T\end{pmatrix}$$

The model space $\mathfrak{P}(\mathfrak{h})$

Given a general Hamiltonian $\mathfrak{h}$ we construct an operator model, which is a Pontryagin space boundary triplet

$$\langle \mathfrak{P}(\mathfrak{h}), T(\mathfrak{h}), \Gamma(\mathfrak{h}) \rangle$$

The actual construction is quite involved and too complicated to be elaborated here.

The model space $\mathfrak{P}(\mathfrak{h})$

If $J = [s_-, s_+] \subseteq (\sigma_i, \sigma_{i+1})$, there exists an isometric and homeomorphic embedding

$$\iota_J : L^2(H_i|_J) \longrightarrow \mathfrak{P}(\mathfrak{h})$$

If $J \subseteq J'$, then

$$\begin{array}{ccc} L^2(H_i|_J) & \xrightarrow{\iota_J} & \mathfrak{P}(\mathfrak{h}) \\ {\scriptstyle\subseteq}\big\downarrow & \nearrow_{\iota_{J'}} & \\ L^2(H_i|_{J'}) & & \end{array}$$

Hamiltonian for q_f

The general Hamiltonian made up of the data

$$\sigma_0 = -1, \sigma_1 = 1, \sigma_2 = +\infty, \quad E = \{-1, 0, 2, +\infty\}$$

$$H_0(t) = \begin{cases} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & , \; t \in (-1, 0) \\ \begin{pmatrix} (t-1)^2 & 0 \\ 0 & (t-1)^{-2} \end{pmatrix} & , \; t \in (0, 1) \end{cases}$$

$$H_1(t) = \begin{pmatrix} (t-1)^2 & 0 \\ 0 & (t-1)^{-2} \end{pmatrix}$$

$$\ddot{o}_1 = 1, b_{1,1} = 2, b_{1,2} = 0, \quad d_0 = -2, d_1 = 0$$

has Weyl coefficient q_f.

Hamiltonian for q_α

The general Hamiltonian made up of the data

$$\sigma_0 = -1, \sigma_1 = 0, \sigma_2 = +\infty, \quad E = \{-1, 1, +\infty\}$$

$$H_0(t) = \frac{1}{t^2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \; H_1(t) = \begin{pmatrix} t^\alpha & 0 \\ 0 & t^{-\alpha} \end{pmatrix}$$

$$\ddot{o}_1 = 0, \quad d_0 = \frac{1}{\alpha - 1}, d_1 = 0$$

has Weyl coefficient q_α.

The class $\mathcal{M}_\kappa$

$W \in \mathcal{M}_\kappa$ if

- W is entire 2×2-matrix function
- $W(0) = I$
- The kernel

$$K_W(w,z) := \frac{W(z)\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}W(w)^* - \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}}{z-\overline{w}}$$

has κ negative squares