Topology

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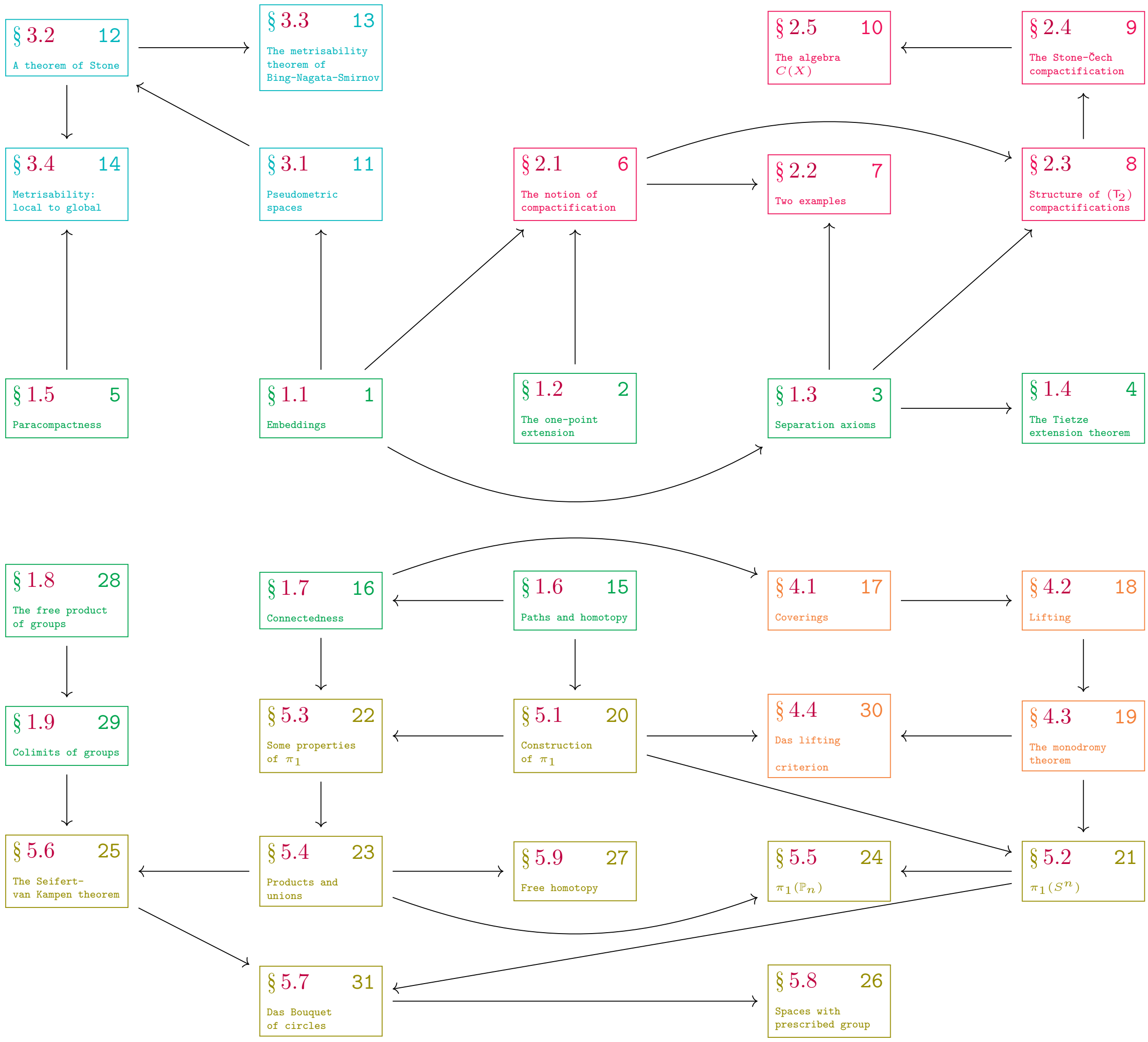

Chapter 1

Miscellaneous

This chapter is of preliminary character. We present several notions and results which build upon the foundations of point-set topology known from the basic courses on analysis. These topics serve as a bridge to the more advanced circles of ideas and theorems investigated in the later chapters of this specialised course.

1.1 Embeddings

Speaking informally, we shall say that a topological space $\langle X, \mathcal{T}\rangle$ is embedded into another one $\langle Y, \mathcal{V}\rangle$, if we can consider it set-theoretically and topologically as a subspace of $\langle Y, \mathcal{V}\rangle$.

Definition 1.1.1. Let $\langle X, \mathcal{T}\rangle$ and $\langle Y, \mathcal{V}\rangle$ be topological spaces, and $\iota\colon X \to Y$ a map. We call ι an *embedding* of $\langle X, \mathcal{T}\rangle$ into $\langle Y, \mathcal{V}\rangle$, if its corestriction $\tilde{\iota}\colon X \to \iota(X)$ is a homeomorphism of $\langle X, \mathcal{T}\rangle$ onto $\langle \iota(X), \mathcal{V}|_{\iota(X)}\rangle$.

Lemma 1.1.2. *Let $\langle X, \mathcal{T}\rangle$ and $\langle Y, \mathcal{V}\rangle$ be topological spaces and $\iota\colon X \to Y$. Then ι is an embedding of $\langle X, \mathcal{T}\rangle$ into $\langle Y, \mathcal{V}\rangle$, if and only if ι is injective and $\mathcal{T}$ is the initial topology of $\mathcal{V}$ induced by the one-element family $\{\iota\}$.*

Proof. In either case, if ι is an embedding or if the stated condition holds, the map ι is injective, and hence $\tilde{\iota}$ is a bijection of X onto $\iota(X)$.

Let $\widetilde{\mathcal{T}}$ be the initial topology on X induced by the one-element family $\{\iota\}$. We have the two diagrams

$$\begin{tikzcd} \langle X,\widetilde{\mathcal{T}}\rangle \arrow[r, "\tilde{\iota}"'] \arrow[rr, bend left, "\iota"] & \langle \iota(X),\mathcal{V}|_{\iota(X)}\rangle \arrow[r, "\subseteq"'] & \langle Y,\mathcal{V}\rangle \end{tikzcd} \quad , \quad \begin{tikzcd} \langle \iota(X),\mathcal{V}|_{\iota(X)}\rangle \arrow[r, "\tilde{\iota}^{-1}"'] \arrow[rr, bend left, "\subseteq"] & \langle X,\widetilde{\mathcal{T}}\rangle \arrow[r, "\iota"'] & \langle Y,\mathcal{V}\rangle \end{tikzcd}$$

The first shows that $\tilde{\iota}$ is continuous, and the second that $\tilde{\iota}^{-1}$ is continuous. Thus $\tilde{\iota}$ is a homeomorphism of $\langle X,\widetilde{\mathcal{T}}\rangle$ onto $\langle\iota(X),\mathcal{V}|_{\iota(X)}\rangle$. We see that $\tilde{\iota}$ is a homeomorphism of $\langle X,\mathcal{T}\rangle$ onto $\langle\iota(X),\mathcal{V}|_{\iota(X)}\rangle$, if and only if $\mathcal{T}=\widetilde{\mathcal{T}}$. ❑

One possibility to construct embeddings is based on the following notions.

Definition 1.1.3.

(i) Let X and Y_i, $i\in I$, be sets and $f_i\colon X\to Y_i$. Then we call the family $\{f_i\,|\,i\in I\}$ *point separating*, if

$$\forall x,y\in X, x\neq y\ \exists i\in I.\ \ f_i(x)\neq f_i(y)$$

(ii) Let $\langle X,\mathcal{T}\rangle$ and $\langle Y_i,\mathcal{V}_i\rangle$, $i\in I$, be topological spaces, and $f_i\colon X\to Y_i$, $i\in I$. Then we call the family $\{f_i\,|\,i\in I\}$ *separating*, if all maps f_i are continuous, it is point separating, and

$$\forall x\in X, A\subseteq X \text{ closed}, x\notin A\ \exists i\in I.\ \ f_i(x)\notin\overline{f_i(A)} \tag{1.1}$$

We start with a corollary of Lemma 1.1.2.

Corollary 1.1.4. *Let X be a set, $\langle Y_i,\mathcal{V}_i\rangle$, $i\in I$, topological spaces, and $f_i\colon X\to Y_i$, $i\in I$, maps. Assume that the family $\{f_i\,|\,i\in I\}$ is point separating, and denote by $\widetilde{\mathcal{T}}$ the initial topology on X induced by the family $\{f_i\,|\,i\in I\}$. Then the product map*

$$\prod_{i\in I} f_i\colon \begin{cases} X & \to & \prod_{i\in I} Y_i \\ x & \mapsto & (f_i(x))_{i\in I}\end{cases}$$

is an embedding of $\langle X,\widetilde{\mathcal{T}}\rangle$ into $\langle\prod_{i\in I}Y_i,\prod_{i\in I}\mathcal{V}_i\rangle$.

Proof. Since $\{f_i\,|\,i\in I\}$ is point separating, the product map is injective. We have the diagram

$$\begin{tikzcd} & & X_i \\ X \arrow[urr, bend left, "f_i"] \arrow[r, "\prod_{i\in I} f_i"'] & \prod_{i\in I} X_i \arrow[ur, "\pi_i"'] & \end{tikzcd}$$

and, by transitivity of building initial topologies, hence $\widetilde{\mathcal{T}}$ is the initial topology induced by the one-element family $\{\prod_{i\in I} f_i\}$. Now apply Lemma 1.1.2. ❑

In the next proposition we give two different hypothesis under which a product map is an embedding of a given space $\langle X, \mathcal{T}\rangle$ into a product.

Proposition 1.1.5. *Let $\langle X, \mathcal{T}\rangle$ and $\langle Y_i, \mathcal{V}_i\rangle$, $i \in I$, be topological spaces, $f_i\colon X \to Y_i$, $i \in I$, be continuous maps, and assume that the family $\{f_i \mid i \in I\}$ is point separating.*

(i) *Assume that (1.1) holds, i.e., that $\{f_i \mid i \in I\}$ is a separating family. Then $\mathcal{T}$ is the initial topology induced by $\{f_i \mid i \in I\}$, and hence the product map $\prod_{i\in I} f_i$ is an embedding of $\langle X, \mathcal{T}\rangle$ into $\langle \prod_{i\in I} Y_i, \prod_{i\in I} \mathcal{V}_i\rangle$.*

(ii) *Assume that $\langle X, \mathcal{T}\rangle$ is compact and all spaces $\langle Y_i, \mathcal{V}_i\rangle$, $i \in I$, are Hausdorff. Then the product map $\prod_{i\in I} f_i$ is an embedding of $\langle X, \mathcal{T}\rangle$ into $\langle \prod_{i\in I} Y_i, \prod_{i\in I} \mathcal{V}_i\rangle$.*

Proof. For the proof of (i), consider a topological space $\langle Y, \mathcal{V}\rangle$, a map $g\colon Y \to X$, and the diagram

$$\begin{tikzcd} & & \langle Y_i, \mathcal{T}_i\rangle \\ \langle Y, \mathcal{V}\rangle \arrow[r, "g"'] \arrow[urr, bend left, "f_i\circ g"] & \langle X, \mathcal{T}\rangle \arrow[ur, "f_i"'] & \end{tikzcd}$$

If g is continuous, then clearly $f_i \circ g$ is continous for every $i \in I$. Assume conversely that $f_i \circ g$ is continuous for all $i \in I$. Let $A \subseteq X$ be closed and $y \notin \overline{g^{-1}(A)}$. Then $g(y) \notin A$, and we find $i \in I$ with $f_i(g(y)) \notin \overline{f_i(A)}$. This means that $y \notin g^{-1}(f_i^{-1}(\overline{f_i(A)}))$. This set is closed and contains $g^{-1}(A)$. Thus $Y \backslash g^{-1}(A)$ is a neighbourhood of y. We conclude that $Y \backslash g^{-1}(A)$ is open.

We come to the proof of (ii). The product map $\prod_{i\in I} f_i$ is continuous and injective, and the product space $\langle \prod_{i\in I} Y_i, \prod_{i\in I} \mathcal{V}_i\rangle$ is Hausdorff. Since (T_2) is inherited by subspaces, the corestriction $\prod_{i\in I} f_i\colon X \to \big(\prod_{i\in I} f_i\big)(X)$ is a continuous bijection from a compact space onto a Hausdorff space. Hence, it is a homeomorphism. ❑

1.2 The one-point extension

Recall the set-theoretic construction of a *disjoint union* of two sets: given two sets A and B, we set

$$A \sqcup B := \{(a,0) \mid a \in A\} \cup \{(b,1) \mid b \in B\} \subseteq (A \cup B) \times \{0,1\}.$$

Then we have the injective maps

$$\iota_A\colon \begin{cases} A & \to & A \sqcup B \\ a & \mapsto & (a,0) \end{cases}, \qquad \iota_B\colon \begin{cases} B & \to & A \sqcup B \\ b & \mapsto & (b,1) \end{cases}.$$

Apparently, $A \sqcup B$ is the union of $\iota_A(A)$ and $\iota_B(B)$, and these two subsets of $A \sqcup B$ are disjoint. Often, one drops explicit notion of ι_A and ι_B and considers A and B as subsets of $A \sqcup B$.

Definition 1.2.1. Let $\langle X, \mathcal{T}\rangle$ be a topological space. Let ∞ be a symbol, and set $\alpha(X) := X \sqcup \{\infty\}$. Denote by ι_α the set-theoretic inclusion map $\iota_\alpha\colon X \to X \sqcup \{\infty\}$ (drop explicit notation of the injection $\iota_{\{\infty\}}\colon \{\infty\} \to \alpha(X)$), and define $\mathcal{T}_\alpha \subseteq \mathcal{P}(\alpha(X))$ as

$$\begin{aligned} \mathcal{T}_\alpha := &\big\{U \subseteq \alpha(X) \mid \infty \notin U, \iota_\alpha^{-1}(U) \text{ open in } X\big\} \\ &\cup \big\{U \subseteq \alpha(X) \mid \infty \in U, X \backslash \iota_\alpha^{-1}(U) \text{ closed and compact in } X\big\}. \end{aligned}$$

Then we call $\langle\alpha(X),\mathcal{T}_\alpha\rangle$ the *one-point extension* of $\langle X,\mathcal{T}\rangle$.

In order to formulate the basic properties of this construction, we recall the following terminology: A topological space $\langle X,\mathcal{T}\rangle$ is called *locally compact*, if every point of X has a compact neighbourhood. A map $\phi\colon X\to Y$ between two topological spaces $\langle X,\mathcal{T}\rangle$ and $\langle Y,\mathcal{V}\rangle$ is called *open*, if it maps open sets to open sets, i.e., if $\phi(O)\in\mathcal{V}$ for all $O\in\mathcal{T}$. It is called an *embedding*, if its corestriction $\tilde{\iota}\colon X\to\iota(X)$ is a homeomorphism of $\langle X,\mathcal{T}\rangle$ onto $\langle\iota(X),\mathcal{V}|_{\iota(X)}\rangle$.

Theorem 1.2.2. *Let $\langle X,\mathcal{T}\rangle$ be a topological space.*

(i) *The one-point extension $\langle\alpha(X),\mathcal{T}_\alpha\rangle$ of $\langle X,\mathcal{T}\rangle$ is a compact topological space.*

(ii) *ι is an embedding and maps open subsets of X to open subsets of $\alpha(X)$.*

(iii) *$\langle\alpha(X),\mathcal{T}_\alpha\rangle$ is (T_2), if and only if $\langle X,\mathcal{T}\rangle$ is (T_2) and locally compact.*

Proof. In order to carry out frequently occurring case distinctions, we denote the two parts in the definition of $\mathcal{T}_\alpha$ as

$$\mathcal{T}_{\alpha,1}:=\big\{U\subseteq\alpha(X)\mid\infty\notin U,\iota_\alpha^{-1}(U)\text{ open in }X\big\},$$
$$\mathcal{T}_{\alpha,2}:=\big\{U\subseteq\alpha(X)\mid\infty\in U,X\backslash\iota_\alpha^{-1}(U)\text{ closed and compact in }X\big\}.$$

Note that also for $U\in\mathcal{T}_{\alpha,2}$ the set $\iota_\alpha^{-1}(U)$ is open in X.

① *We show that $\mathcal{T}_\alpha$ is a topology:* First, $\infty\notin\varnothing$ and $\iota_\alpha^{-1}(\varnothing)=\varnothing$, which is open, and hence $\varnothing\in\mathcal{T}_{\alpha,1}$. Second, $\infty\in\alpha(X)$ and $\iota_\alpha^{-1}(\alpha(X)\backslash\alpha(X))=\varnothing$, which is closed and compact, and hence $\alpha(X)\in\mathcal{T}_{\alpha,2}$.

Let I be a nonempty index set, and U_i, $i\in I$, be a family of elements of $\mathcal{T}_\alpha$. Set $J:=\{i\in I\mid U_i\in\mathcal{T}_{\alpha,2}\}$. If $J=\varnothing$, then we have $\infty\notin\bigcup_{i\in I}U_i$ and

$$\iota_\alpha^{-1}\Big(\bigcup_{i\in I}U_i\Big)=\bigcup_{i\in I}\iota_\alpha^{-1}(U_i)\in\mathcal{T}.$$

If J is nonempty, then $\infty\in\bigcup_{i\in I}U_i$, and we have

$$X\backslash\iota_\alpha^{-1}\Big(\bigcup_{i\in I}U_i\Big)=X\backslash\bigcup_{i\in I}\iota_\alpha^{-1}(U_i)=\bigcap_{i\in I}\big(X\backslash\iota_\alpha^{-1}(U_i)\big).$$

Each of the sets occurring in the intersection is closed in X, hence the intersection is closed. For at least one $i\in I$, in fact for all $i\in J$, the set $X\backslash\iota_\alpha^{-1}(U_i)$ is compact in X, and it follows that the intersection is also compact.

Assume now that I is finite. If $J=I$, then $\infty\in\bigcap_{i\in I}U_i$, and

$$X\backslash\iota_\alpha^{-1}\Big(\bigcap_{i\in I}U_i\Big)=X\backslash\bigcap_{i\in I}\iota_\alpha^{-1}(U_i)=\bigcup_{i\in I}\big(X\backslash\iota_\alpha^{-1}(U_i)\big).$$

All sets occurring in the union are closed and compact, and since I is finite, also their union is closed and compact. If $J\neq I$, we have $\infty\notin\bigcap_{i\in I}U_i$, and

$$\iota_\alpha^{-1}\Big(\bigcap_{i\in I}U_i\Big)=\bigcap_{i\in I}\iota_\alpha^{-1}(U_i).$$

Again, since I is finite, this set is open.

② *We show that* $\langle\alpha(X),\mathcal{T}_\alpha\rangle$ *is compact:* Let $\mathcal{W}\subseteq\mathcal{T}_\alpha$ be an open cover of $\alpha(X)$. Choose $U\in\mathcal{W}$ with $\infty\in U$. The family $\{\iota_\alpha^{-1}(W)\,|\,W\in\mathcal{W}\}$ is an open cover of X, and in particular covers $X\backslash\iota_\alpha^{-1}(U)$. Since $X\backslash\iota_\alpha^{-1}(U)$ is compact, we may extract a finite subcover, say, $\iota_\alpha^{-1}(W_1),\ldots,\iota_\alpha^{-1}(W_n)$. Let $z\in\alpha(X)\backslash U$. Then $z\neq\infty$, and we find $x\in X\backslash\iota_\alpha^{-1}(U)$ with $z=\iota_\alpha(x)$. Since $X\backslash\iota_\alpha^{-1}(U)\subseteq\iota_\alpha^{-1}(W_1)\cup\ldots\cup\iota_\alpha^{-1}(W_n)$ we conclude that

$$\alpha(X)=U\cup\bigcup_{i=1}^{n}W_i,$$

and have found a finite subcover.

③ *We show that* ι_α *is an open embedding:* Clearly, ι_α is injective. We have $\infty\notin\iota_\alpha(X)$ and $\iota_\alpha^{-1}(\iota_\alpha(X))=X\in\mathcal{T}$. Thus, $\iota_\alpha(X)\in\mathcal{T}_{\alpha,1}$. Now let $O\subseteq X$. Then, using that $\iota_\alpha(X)$ is open,

$$\iota_\alpha(O)\in\mathcal{T}_\alpha|_{\iota_\alpha(X)}\ \Leftrightarrow\ \iota_\alpha(O)\in\mathcal{T}_\alpha\ \Leftrightarrow\ \iota_\alpha(O)\in\mathcal{T}_{\alpha,1}\ \Leftrightarrow\ \iota_\alpha^{-1}\big(\iota_\alpha(O)\big)\in\mathcal{T}\ \Leftrightarrow\ O\in\mathcal{T}$$

We see that ι_α maps open subsets of X to open subsets of $\alpha(X)$. Further, since we know that ι_α is injective, it follows that the corestriction of ι_α to a map of X onto $\iota_\alpha(X)$ is a homeomorphism.

④ *We show* (iii)*:* Assume that $\alpha(X)$ is (T_2). The Hausdorff separation axiom is inherited by subspaces and homeomorphic images, hence X is (T_2). Let $x\in X$. Choose disjoint sets $U_x,U_\infty\in\mathcal{T}_\alpha$ with $\iota_\alpha(x)\in U_x$ and $\infty\in U_\infty$. Then $X\backslash\iota_\alpha^{-1}(U_\infty)$ is a compact subset of X, and

$$x\in\iota_\alpha^{-1}(U_x)\subseteq\iota_\alpha^{-1}(\alpha(X)\backslash U_\infty)=X\backslash\iota_\alpha^{-1}(U_\infty).$$

Since $\iota_\alpha^{-1}(U_x)$ is open, $X\backslash\iota_\alpha^{-1}(U_\infty)$ is a neighbourhood of x.

Assume conversely, that X is (T_2) and locally compact. Again using that (T_2) is inherited by homeomorphic images, we obtain that each two different points of $\iota_\alpha(X)$ can be separated by disjoint subsets of $\iota_\alpha(X)$ which are open in the subspace topology $\mathcal{T}_\alpha|_{\iota_\alpha(X)}$. However, since $\iota_\alpha(X)$ is open in $\alpha(X)$, these separating sets are also open in $\alpha(X)$. Now let $x\in\iota_\alpha(X)$ be given. Choose a compact neighbourhood K of $\iota_\alpha^{-1}(x)$. Since X is Hausdorff, K is also closed. Hence, the set $\alpha(X)\backslash\iota_\alpha(K)$, which obviously contains the point ∞, belongs to $\mathcal{T}_{\alpha,2}$. Choose an open neighbourhood $O\subseteq K$ of $\iota_\alpha^{-1}(x)$. Then $\iota_\alpha(O)\in\mathcal{T}_{\alpha,1}$ and contains the point x. Since $O\subseteq K$, we have

$$(\alpha(X)\backslash\iota_\alpha(K))\cap\iota_\alpha(O)=\varnothing.$$

❑

Given a function $f\colon X\to Y$, and a point $y_0\in Y$, we can lift f to the disjoint union $\alpha(X)$ by

$$\tilde f(z):=\begin{cases}f(\iota_\alpha^{-1}(z)) & \text{if } z\in\iota_\alpha(X)\\ y_0 & \text{if } z=\infty\end{cases}. \tag{1.2}$$

Note that this definition ensures that $\tilde f\circ\iota_\alpha=f$:

$$\begin{tikzcd}\alpha(X) \arrow[dr, dashed, "\tilde f"] & \\ X \arrow[u, "\iota_\alpha"] \arrow[r, "f"'] & Y\end{tikzcd}$$

In the context of topological spaces, the question arises whether the extension $\tilde{f}$ is continuous.

Proposition 1.2.3. *Let $\langle X, \mathcal{T}\rangle$ and $\langle Y, \mathcal{V}\rangle$ be topological spaces and $f\colon X \to Y$. Moreover, let $y_0 \in Y$, and consider the function $\tilde{f}\colon \alpha(X) \to Y$ defined by (1.2). Then $\tilde{f}$ is continuous, if and only if f is continuous and*

$$\forall V \in \mathcal{U}(y_0)\ \exists K \subseteq X \text{ closed compact. } f(X\backslash K) \subseteq V. \tag{1.3}$$

Proof.

① *We show the forward implication:* Assume that $\tilde{f}$ is continuous. Since $f = \tilde{f} \circ \iota_\alpha$, the function f is continuous. Let $V \in \mathcal{U}(y_0)$ be given. Choose an open neighbourhood U in $\alpha(X)$ of the point ∞, such that $\tilde{f}(U) \subseteq V$, and set $K := X\backslash\iota_\alpha^{-1}(U)$. Then K is closed and compact in X, and we have

$$f(X\backslash K) = f(\iota_\alpha^{-1}(U)) = \tilde{f}\big(\iota_\alpha(\iota_\alpha^{-1}(U))\big) \subseteq \tilde{f}(U) \subseteq V.$$

② *We show the backward implication:* Assume that f is continuous and satisfies (1.3). Let $O \subseteq Y$ be open. If $y_0 \notin O$, then $\infty \notin \tilde{f}^{-1}(O)$ and $\iota_\alpha^{-1}(\tilde{f}^{-1}(O)) = f^{-1}(O)$ which is open in X. Thus $\tilde{f}^{-1}(O)$ is open in $\alpha(X)$. Consider the case that $y_0 \in O$. Then O is a neighbourhood of y_0 and, according to (1.3), we find a closed and compact set $K \subseteq X$ with $f(X\backslash K) \subseteq O$, i.e., $X\backslash K \subseteq f^{-1}(O)$. We have $\infty \in \tilde{f}^{-1}(O)$ and

$$X\backslash\iota_\alpha^{-1}\big(\tilde{f}^{-1}(O)\big) = X\backslash f^{-1}(O) \subseteq X\backslash(X\backslash K) = K.$$

The set $X\backslash\iota_\alpha^{-1}(\tilde{f}^{-1}(O))$ is closed since f is continuous, and in turn compact since it is contained in the compact set K. Again, we see that $\tilde{f}^{-1}(O)$ is open in $\alpha(X)$.

❏

In the context of Proposition 1.2.3 one also uses the following terminology: if $\langle X, \mathcal{T}\rangle$ and $\langle Y, \mathcal{V}\rangle$ are topological spaces, $f\colon X \to Y$, and $y_0 \in Y$, one says that f *has the limit y_0 at infinity*, if the condition (1.3) holds.

Functions between two topological spaces lift canonically to functions between their one-point extensions.

Definition 1.2.4. Let $\langle X, \mathcal{T}\rangle$ and $\langle Y, \mathcal{V}\rangle$ be topological spaces and $f\colon X \to Y$. Then we define a function $\alpha(f)\colon \alpha(X) \to \alpha(Y)$ as (here $\iota_{\alpha,X}$ and $\iota_{\alpha,Y}$ are the respective inclusion maps)

$$\alpha(f)(x) := \begin{cases} \big(\iota_{\alpha,Y} \circ f \circ \iota_{\alpha,X}^{-1}\big)(x) & \text{if } x \in \iota_{\alpha,X}(X), \\ \infty & \text{if } x = \infty. \end{cases}$$

This definition ensures that we have the diagram

$$\begin{tikzcd}
X \arrow[r, "f"'] \arrow[d, "\iota_{\alpha,X}"'] & Y \arrow[d, "\iota_{\alpha,Y}"] \\
\alpha(X) \arrow[r, dashed, "\alpha(f)"] & \alpha(Y) \\
\{\infty\} \arrow[u, "\subseteq"] \arrow[r, "\mathrm{id}"'] & \{\infty\} \arrow[u, "\subseteq"']
\end{tikzcd}$$

Moreover, observe that passing to the lifting is compactible with composition and identity in the sense that

$$\alpha(f \circ g) = \alpha(f) \circ \alpha(g), \qquad \alpha(\mathrm{id}_X) = \mathrm{id}_{\alpha(X)}\,. \tag{1.4}$$

The question whether $\alpha(f)$ is continuous, is answered by an application of Proposition 1.2.3.

Corollary 1.2.5. *Let $\langle X, \mathcal{T}\rangle$ and $\langle Y, \mathcal{V}\rangle$ be topological spaces, and $f\colon X \to Y$. Then $\alpha(f)$ is continuous, if and only if f is continuous and*

$$\forall A \subseteq Y \text{ closed compact. } f^{-1}(A) \text{ compact} \tag{1.5}$$

Proof. The map $\alpha(f)$ is nothing but the lifting in the sense of (1.2) of the map $\iota_{\alpha_Y} \circ f\colon X \to \alpha(Y)$ using the point ∞. Thus, Proposition 1.2.3 provides us with a characterisation when $\alpha(f)$ is continuous.

Continuity of f occurs in both conditions, the one obtained from Proposition 1.2.3, and the one stated in the present assertion. Hence it is enough to show that for a continuous function f the conditions (1.3) and (1.5) are equivalent.

① *We show that (1.5) implies (1.3):* Let $V \in \mathcal{U}(\infty)$. Choose $U \in \mathcal{T}_\alpha$ with $\infty \in U \subseteq V$. Then the set $A := Y \backslash \iota_\alpha^{-1}(U)$ is closed and compact. Its inverse image $K := f^{-1}(A)$ under f is closed by continuity of f and compact by (1.5). However, $K = X \backslash f^{-1}(\iota_\alpha^{-1}(U))$, and hence

$$(\iota_{\alpha,Y} \circ f)\big(X \backslash K\big) \subseteq U \subseteq V.$$

② *We show that (1.3) implies (1.5):* Let $A \subseteq Y$ be closed and compact. Then $\alpha(Y)\backslash \iota_{\alpha,Y}(A)$ is an open neighbourhood of the point $\infty \in \alpha(Y)$. Choose $K \subseteq X$ closed and compact, such that

$$(\iota_{\alpha,Y} \circ f)(X \backslash K) \subseteq \alpha(Y)\backslash \iota_{\alpha,Y}(A).$$

Since $\iota_{\alpha,Y}$ is injective, applying $\iota_{\alpha,Y}^{-1}$ to this inclusion yields $f(X\backslash K) \subseteq Y \backslash A$, i.e., $X \backslash K \subseteq f^{-1}(Y \backslash A) \subseteq X \backslash f^{-1}(A)$. The set $f^{-1}(A)$ is closed since f is continuous, and in turn it is compact since it is contained in the compact set K.

❏

1.3 Separation axioms

In a topological space there are two natural ways to separate points or subsets: by open sets or by continuous functions. We shall define a (not exhaustive) list of properties, which are called *separation axioms*. Viewed on its own, this looks like just a long list of vocabulary. However, we will see that each of these properties occurs naturally in different contexts. It should also be said that in the literature sometimes terminology is not uniform.

We start with properties of separation by open sets.

Definition 1.3.1. A topological space $\langle X, \mathcal{T}\rangle$ is said to satisfy the separation axiom

▻ (T_0), if

$$\forall x, y \in X, x \neq y\ \exists O_x, O_y \in \mathcal{T}.\ \big(x \in O_x \wedge y \in O_y\big) \wedge \big(y \notin O_x \vee x \notin O_y\big)$$

▻ (T_1), if

$$\forall x, y \in X, x \neq y\ \exists O_x, O_y \in \mathcal{T}.\ \big(x \in O_x \wedge y \in O_y\big) \wedge \big(y \notin O_x \wedge x \notin O_y\big)$$

▻ (T_2) (or is a *Hausdorff space*), if

$$\forall x, y \in X, x \neq y\ \exists O_x, O_y \in \mathcal{T}.\ \big(x \in O_x \wedge y \in O_y\big) \wedge \big(O_x \cap O_y = \varnothing\big)$$

▻ (T_3), if

$$\forall x \in X, B \subseteq X \text{ closed}, x \notin B\ \exists O_x, O_B \in \mathcal{T}.\ \big(x \in O_x \wedge B \subseteq O_B\big) \wedge \big(O_x \cap O_B = \varnothing\big)$$

▻ (T_4), if

$$\forall A, B \subseteq X \text{ closed}, A \cap B = \varnothing\ \exists O_A, O_B \in \mathcal{T}.\ \big(A \subseteq O_A \wedge B \subseteq O_B\big) \wedge \big(O_A \cap O_B = \varnothing\big)$$

The space $\langle X, \mathcal{T}\rangle$ is called

▻ *regular*, if it satisfies (T_1) and (T_3),

▻ *normal*, if it satisfies (T_1) and (T_4).

Let us show that the axioms (T_1), (T_2), and (T_3) are related with closed sets and neighbourhoods.

Lemma 1.3.2. *Let $\langle X, \mathcal{T}\rangle$ be a topological space. Then $\langle X, \mathcal{T}\rangle$ satisfies*

▻ (T_1), *if and only if for every point $x \in X$ the singleton set $\{x\}$ is closed.*

▻ (T_2), *if and only if $\bigcap\{U \in \mathcal{U}(x) \mid U \text{ closed}\} = \{x\}$ for all points $x \in X$.*

▻ (T_3), *if and only if for every point $x \in X$ the set $\{U \in \mathcal{U}(x) \mid U \text{ closed}\}$ forms a base of the neighbourhood filter $\mathcal{U}(x)$.*

Proof.

① Assume that $\langle X, \mathcal{T}\rangle$ is (T_1), and let $x \in X$. For each $y \in X\backslash\{x\}$ we find $O_y \in \mathcal{T}$ with $y \in O_y, x \notin O_y$. We see that $X\backslash\{x\} = \bigcup_{y \in X\backslash\{x\}} O_y$, and hence is open. Conversely, assume singletons are closed, and let $x, y \in X$, $x \neq y$, be given. Then we can use $O_x := X\backslash\{x\}$ and $O_y := X\backslash\{y\}$.

② Assume that $\langle X, \mathcal{T}\rangle$ is (T_2), and let $x \in X$. For each $y \in X\backslash\{x\}$ we find disjoint open sets O_x, O_y with $x \in O_x$ and $y \in O_y$. Then $\overline{O_x}$ is a closed neighbourhood of x and is contained in $X\backslash O_y$. In particular, it does not contain the point y. Conversely, assume that each singleton is the intersection of all closed neighbourhoods, and let $x, y \in X$, $x \neq y$, be given. Choose $U \in \mathcal{U}(x)$ closed with $y \notin U$, and $O \in \mathcal{U}(x)$ open with $O \subseteq U$. Then we can use $O_x := O$ and $O_y := X\backslash U$.

③ Assume that $\langle X, \mathcal{T}\rangle$ is (T_3), let $x \in X$ and $U \in \mathcal{U}(x)$. Choose $O \in \mathcal{T}$ with $x \in O \subseteq U$. The set $B := X\backslash O$ is closed and $x \notin B$. Choose disjoint open sets O_x, O_B with $x \in O_x$ and $B \subseteq O_B$. Then $\overline{O_x}$ is a closed neighbourhood of x, and $\overline{O_x} \subseteq X\backslash O_B \subseteq X\backslash B = O \subseteq U$. Conversely, assume that closed neighbourhoods form a neighbourhood base, let $x \in X$ and $B \subseteq X$ closed with $x \notin B$. Then $X\backslash B$ is a neighbourhood of x, and we can choose a closed neighbourhood U of x with $U \subseteq X\backslash B$. Choose O_x open with $x \in O_x \subseteq U$, and set $O_B := X\backslash U$.

❑

We have already seen that in Hausdorff spaces limits of nets are unique. The next assertion gives a characterisation in a similar direction.

Lemma 1.3.3. *Let $\langle X, \mathcal{T}\rangle$ be a topological space. Then the following statements are equivalent.*

(i) *$\langle X, \mathcal{T}\rangle$ satisfies* (T_2).

(ii) *For every topological space $\langle Y, \mathcal{V}\rangle$, dense subset D of Y, and continuous functions $f, g\colon Y \to X$, it holds that*

$$f|_D = g|_D \;\Rightarrow\; f = g.$$

(iii) *The diagonal $\Delta_X := \{(x,x) \mid x \in X\}$ is closed in the product topology of $X \times X$.*

Proof. We are going to show that "(i)⇒(ii)⇒(iii)⇒(i)".

Assume that (i) holds, and let Y, D, f be as in (ii). Let $y \in Y$ be given, and let $O_f, O_g \subseteq X$ be open sets with $f(y) \in O_f$ and $g(y) \in O_g$. Then $f^{-1}(O_f)$ and $g^{-1}(O_g)$ are open neighbourhoods of y, and hence we find a point $x \in f^{-1}(O_f) \cap f^{-1}(O_g) \cap D$. Since $f(x) = g(x)$, it follows that $O_f \cap O_g \neq \varnothing$. Since X is (T_2), it follows that $f(y) = g(y)$.

Assume that (ii) holds, and let $(z, w) \in \overline{\Delta_X}$. Consider the space $Y := \Delta_X \cup \{(z,w)\}$ endowed with the subspace topology of the product topology. Then Δ_X is dense in Y. Let $\pi_1, \pi_2\colon X \times X \to X$ be the canonical projections onto the first and second, respectively component. Then $\pi_1|_{\Delta_X} = \pi_2|_{\Delta_X}$, and the present assumption (ii) implies that $\pi_1|_Y = \pi_2|_Y$. It follows that $z = w$.

Assume that (iii) holds, and let $x, y \in X$, $x \neq y$. Then $(x,y) \notin \overline{\Delta_X}$, and hence we find open sets $O_x, O_y \subseteq X$ with $(x,y) \in O_x \times O_y$ and $(O_x \times O_y) \cap \Delta_X = \varnothing$. The latter just means that $O_x \cap O_y = \varnothing$. ❑

Next, some properties of separation by continuous function.

Definition 1.3.4. A topological space $\langle X, \mathcal{T}\rangle$ is said to satisfy the separation axiom

▻ $(\mathsf{T}_{2\frac{1}{2}})$, if

$$\forall x, y \in X, x \neq y\ \exists f\colon X \to [0,1] \text{ continuous. } f(x) = 1 \wedge f(y) = 0$$

▻ $(\mathsf{T}_{3\frac{1}{2}})$, if

$$\forall x \in X, B \subseteq X \text{ closed}, x \notin B\ \exists f\colon X \to [0,1] \text{ continuous. } f(x) = 1 \wedge f(B) \subseteq \{0\}$$

▻ $(\mathsf{T}_{4\frac{1}{2}})$, if

$$\forall A, B \subseteq X \text{ closed}, A \cap B = \varnothing\ \exists f\colon X \to [0,1] \text{ continuous. } f(A) \subseteq \{1\} \wedge f(B) \subseteq \{0\}$$

The space $\langle X, \mathcal{T}\rangle$ is called

▻ *completely regular*, if it is (T_1) and $(\mathsf{T}_{3\frac{1}{2}})$.

While the axiom $(\mathsf{T}_{2\frac{1}{2}})$ is seldomly used, and $(\mathsf{T}_{4\frac{1}{2}})$ is redundant, the axiom $(\mathsf{T}_{3\frac{1}{2}})$ is most important.

Remark 1.3.5. The axioms defined above are related among each other as

$$\begin{tikzcd}
 & & & (\mathsf{T}_3) & (\mathsf{T}_4) \\
(\mathsf{T}_0) & (\mathsf{T}_1) \arrow[l, Rightarrow] & (\mathsf{T}_2) \arrow[l, Rightarrow] & (\mathsf{T}_{3\frac{1}{2}}) \arrow[u, Rightarrow] & (\mathsf{T}_{4\frac{1}{2}}) \arrow[u, Leftrightarrow] \\
 & & (\mathsf{T}_{2\frac{1}{2}}) \arrow[u, Rightarrow, dashed] & \substack{\text{regular} \\ =(\mathsf{T}_1)\wedge(\mathsf{T}_3)} \arrow[ul, Rightarrow] \arrow[uu, Rightarrow, dashed, bend left] & \substack{\text{normal} \\ =(\mathsf{T}_1)\wedge(\mathsf{T}_4)} \arrow[l, Rightarrow] \arrow[uu, Rightarrow, dashed, bend right] \arrow[dl, Rightarrow] \\
 & & & \substack{\text{completely} \\ \text{regular} \\ =(\mathsf{T}_1)\wedge(\mathsf{T}_{3\frac{1}{2}})} \arrow[u, Rightarrow] \arrow[ul, Rightarrow] \arrow[uu, Rightarrow, dashed, bend right] &
\end{tikzcd}$$

The only significant result is the downwards implication that (T_4) implies $(\mathsf{T}_{4\frac{1}{2}})$: this is Urysohn's Lemma.

The dashed implications are trivial. The two implications going left from (T_2) are clear from the definitions. The two implications going diagonally left and up, and the fact that normal implies regular, follow since (T_1) means that singleton sets are closed. For passing from normal to completely regular, use in addition that (T_4) implies $(\mathsf{T}_{4\frac{1}{2}})$. The four upwards implications follow by using $f^{-1}([0, \frac{1}{2}))$ and $f^{-1}((\frac{1}{2}, 1])$ as separating open sets.

Next, let us show that most separation axioms are inherited by (sufficiently rich) initial constructions. Recall that, for sets X and Y_i, $i \in I$, a family $\{f_i \mid i \in I\}$ of functions $f_i\colon X \to Y_i$ is called *point separating*, if

$$\forall x, y \in X, x \neq y \;\exists i \in I.\; f_i(x) \neq f_i(y)$$

Proposition 1.3.6. *Let X be a set, $\langle Y_i, \mathcal{V}_i\rangle$, $i \in I$, topological spaces, $f_i\colon X \to Y_i$, $i \in I$, maps, and let $\mathcal{T}$ be the initial topology on X induced by the functions f_i, $i \in I$.*

(i) *The separation axioms (T_3) and $(\mathsf{T}_{3\frac{1}{2}})$ are inherited by $\mathcal{T}$. More precisely: if all spaces $\langle Y_i, \mathcal{V}_i\rangle$, $i \in I$, satisfy (T_3), then also $\langle X, \mathcal{T}\rangle$ satisfies (T_3) (and the same with $3\frac{1}{2}$ in place of 3).*

(ii) *Assume that $\{f_i \mid i \in I\}$ is point separating. Then the separation axioms (T_0), (T_1), (T_2), and $(\mathsf{T}_{2\frac{1}{2}})$, are inherited by $\mathcal{T}$.*

It should be said immediately that (T_4) is not inherited (not even to as simple initial constructions as subspaces), see Example 1.3.8 below.

Proof of Proposition 1.3.6.

① *We show that (T_3) is inherited:* Let $x \in X$. A neighbourhood base of x w.r.t. the initial topology is given by

$$\Big\{\bigcap_{l=1}^{n} f_{i_l}^{-1}(U_{i_l}) \,\Big|\, n \in \mathbb{N}, i_1, \ldots, i_n \in I, U_{i_l} \in \mathcal{U}(f_{i_l}(x))\Big\}.$$

Since the closed neighbourhoods form a neighbourhood base in each of the spaces $\langle Y_i, \mathcal{V}_i \rangle$, we also obtain a base of $\mathcal{U}(x)$ by

$$\Big\{ \bigcap_{l=1}^{n} f_{i_l}^{-1}(U_{i_l}) \,|\, n \in \mathbb{N}, i_1, \ldots, i_n \in I, U_{i_l} \in \mathcal{U}(f_{i_l}(x)) \text{ closed} \Big\}.$$

The elements of this set are all closed in $\langle X, \mathcal{T} \rangle$.

② *We show that* $(\mathsf{T}_{3\frac{1}{2}})$ *is inherited:* Let $x \in X$, $A \subseteq X$ closed with $x \notin A$. The set $O := X \backslash A$ is an open neighbourhood of x, and hence we find a finite subset $J \subseteq I$ and sets $O_i \in \mathcal{V}_i$, $i \in J$, such that

$$x \in \bigcap_{i \in J} f_i^{-1}(O_i) \subseteq O.$$

For each $i \in J$ choose a continuous function $g_i \colon Y_i \to [0,1]$ such that $g_i(f_i(x)) = 1$ and $g_i(Y_i \backslash O_i) = \{0\}$. Set

$$g := \prod_{i \in J} (g_i \circ f_i).$$

Since J is finite, g is a well-defined continuous function of X into $[0,1]$. Clearly, $g(x) = 1$. Consider a point $y \in X \backslash O$. There must exist $i \in J$ with $y \notin f_i^{-1}(O_i)$. Then $g_i(f_i(y)) = 0$, and hence $g(y) = 0$.

③ *We show that* (T_0)*,* (T_1)*, and* (T_2)*, is inherited:* Let $x, y \in X$, $x \neq y$. Since the family $\{f_i \,|\, i \in I\}$ is point separating, we find $i \in I$ with $f_i(x) \neq f_i(y)$. Now choose $O_{f_i(x)}, O_{f_i(y)} \in \mathcal{V}_i$ according to the respective separation axiom (T_0), (T_1), or (T_2). Then $O_x := f_i^{-1}(O_{f_i(x)})$ and $O_y := f_i^{-1}(O_{f_i(y)})$ have the required properties.

④ *We show that* $(\mathsf{T}_{2\frac{1}{2}})$ *is inherited:* Let $x, y \in X$, $x \neq y$, and choose $i \in I$ such that $f_i(x) \neq f_i(y)$. Then we find a continuous function $g \colon Y_i \to [0,1]$ with $g(f_i(x)) = 1$ and $g(f_i(y)) = 0$. The function $g \circ f_i$ has the required properties.

❑

The following, easy to prove but important, characterisation of completely regular spaces is known as the *Tychonoff embedding theorem*.

Theorem 1.3.7. *A topological space $\langle X, \mathcal{T} \rangle$ is completely regular, if and only if there exists a set I and an embedding $\iota \colon X \to [0,1]^I$ (where the cube $[0,1]^I$ is endowed with the product topology).*

Proof. Assume that $\langle X, \mathcal{T} \rangle$ is completely regular, and consider the family $C(X,[0,1])$ of all continuous functions of X into $[0,1]$. We show that this family is separating. Let $x \in X$, $A \subseteq X$ closed with $x \notin A$, then there exists $f \in C(X,[0,1])$ with $f(x) = 1$ and $f(A) = \{0\}$. Clearly, $f(x) \notin \overline{f(A)}$. Since $\langle X, \mathcal{T} \rangle$ is (T_1), singleton sets are closed, and thus $C(X,[0,1])$ is also point separating. By Proposition 1.1.5 (i), the product map $\prod_{f \in C(X,[0,1])} f$ is an embedding of X into $\prod_{f \in C(X,[0,1])} [0,1]$.

Conversely, it is enough to remember Lemma 1.1.2 and the previous proposition. ❑

We can now give an example that the property to be normal is not inherited by subspaces. In fact, every example of a completely regular but not normal space will establish this: by Tychonoff's embedding theorem and Tychonoff's product theorem, every completely regular space is homeomorphic to a subspace of a compact Hausdorff, and hence normal, space.

Example 1.3.8. Consider the nonnegative integers $\mathbb{N}$ with the discrete topology, let I be an uncountable set, and consider the product $X := \mathbb{N}^I$ endowed with the product topology. By Proposition 1.3.6, X is completely regular. Our aim is to show that X is not normal.

① *We define disjoint closed subsets of X:* For each $m \in \mathbb{N}$ set

$$A_m := \big\{(x_i)_{i\in I} \in X \,|\, \forall n \in \mathbb{N}\backslash\{m\}.\ |\{i \in I \,|\, x_i = n\}| \leqslant 1\big\}.$$

The negation of the formula defining A_m reads as

$$\exists n \in \mathbb{N}\backslash\{m\}\ \exists i, j \in I, i \neq j.\ x_i = x_j = n,$$

and hence we can write

$$X\backslash A_m = \bigcup_{n\in\mathbb{N}\backslash\{m\}} \bigcup_{\substack{i,j\in I\\ i\neq j}} \big(\pi_i^{-1}(\{n\}) \cap \pi_j^{-1}(\{n\})\big).$$

This shows that A_m is closed. Next, note that

$$A_m \subseteq \big\{(x_i)_{i\in I} \,|\, x_i = m \text{ for all but at most countably many } i \in I\big\}. \tag{1.6}$$

Since I is uncountable, the sets on the right side of (1.6) are pairwise disjoint, and hence also the sets A_m, $m \in \mathbb{N}$, are pairwise disjoint.

② *We make an inductive construction:* Let $m \in \mathbb{N}$ and $O \subseteq X$ open with $O \supseteq A_m$. We are going to construct an increasing sequence $(n_l)_{l\in\mathbb{N}}$ of numbers $n_l \in \mathbb{N}$, and a sequence $(j_k)_{k\in\mathbb{N}}$ of pairwise different indices $j_k \in I$, such that $(n_{-1} := -1)$

$$\forall l \in \mathbb{N}.\ \bigcap_{0\leqslant k\leqslant n_{l-1}} \pi_{j_k}^{-1}(\{k\}) \ \cap \bigcap_{n_{l-1}<k\leqslant n_l} \pi_{j_k}^{-1}(\{m\}) \subseteq O.$$

Let $l \in \mathbb{N}$, and assume that n_{l-1} and j_k, $0 \leqslant k \leqslant n_{l-1}$, have already been constructed. The point $(x_i)_{i\in I}$ defined by

$$x_i := \begin{cases} k & \text{if } i = j_k, 0 \leqslant k \leqslant n_{l-1} \\ m & \text{otherwise} \end{cases}$$

belongs to A_m, and hence to O. Choose indices $i_1, \ldots, i_N \in I$ and open sets $U_1, \ldots, U_N \subseteq \mathbb{N}$, such that

$$(x_i)_{i\in I} \in \bigcap_{h=1}^{N} \pi_{i_h}^{-1}(U_h) \subseteq O.$$

Without loss of generality, we can assume that the indices $i_1, \ldots, i_N$ are pairwise different (otherwise combine sets with the same index).

We arrange those indices i_h which do not already appear in $\{j_k \,|\, 0 \leqslant k \leqslant n_{l-1}\}$ in a sequence $j_{n_{l-1}+1}, \ldots, j_{n_l}$. If such indices do not exist, set $n_l := n_{l-1} + 1$ and pick some $j_{n_l} \in I\backslash\{j_k \,|\, 0 \leqslant k \leqslant n_{l-1}\}$. Then

$$\bigcap_{0\leqslant k\leqslant n_{l-1}} \pi_{j_k}^{-1}(\{k\}) \ \cap \bigcap_{n_{l-1}<k\leqslant n_l} \pi_{j_k}^{-1}(\{m\}) \subseteq \bigcap_{h=1}^{N} \pi_{i_h}^{-1}(U_h) \subseteq O.$$

③ *We show that for different m, m' the sets A_m and $A_{m'}$ cannot be separated:* Let O_m and $O_{m'}$ be open sets with $A_m \subseteq O_m$ and $A_{m'} \subseteq O_{m'}$. Let $(n_l)_{l\in\mathbb{N}}$ and $(j_k)_{k\in\mathbb{N}}$ be the sequences constructed in the previous step for the set O_m, and set

$$y_i := \begin{cases} k & \text{if } i = j_k, k \in \mathbb{N} \\ m' & \text{otherwise} \end{cases}$$

Then the element $(y_i)_{i\in I}$ belongs to $A_{m'}$, and hence to $O_{m'}$. Choose a finite set $J \subseteq I$ with

$$(y_i)_{i\in I} \in \bigcap_{i\in J} \pi_i^{-1}(\{y_i\}) \subseteq O_{m'}.$$

Since J is finite, we find $l \in \mathbb{N}$ with

$$J \cap \{j_k \mid k \in \mathbb{N}\} = J \cap \{j_k \mid 0 \leqslant k \leqslant n_l\}.$$

Now set

$$z_i := \begin{cases} k & \text{if } i = j_k, 0 \leqslant k \leqslant n_l \\ m & \text{if } i = j_k, n_l < k \leqslant n_{l+1} \\ m' & \text{otherwise} \end{cases}$$

Then, clearly,

$$(z_i)_{i\in I} \in \bigcap_{0\leqslant k\leqslant n_l} \pi_k^{-1}(\{j_k\}) \ \cap \bigcap_{n_l<k\leqslant n_{l+1}} \pi_{j_k}^{-1}(\{m\}) \subseteq O_m.$$

Since $J \cap \{j_k \mid n_l < k \leqslant n_{l+1}\} = \varnothing$, we have $z_i = y_i$ for all $i \in J$, and hence

$$(z_i)_{i\in I} \in \bigcap_{i\in J} \pi_i^{-1}(\{y_i\}) \subseteq O_{m'}.$$

1.4 The Tietze extension theorem

Tietze's theorem is the following characterisation of (T_4)-spaces.

Theorem 1.4.1. *A topological space $\langle X, \mathcal{T}\rangle$ satisfies (T_4), if and only if the following property holds:*

$$\forall A \subseteq X \text{ closed}, f\colon A \to [-1,1] \text{ continuous } \exists F\colon X \to [-1,1] \text{ continuous}.\ F|_A = f. \quad (1.7)$$

Thereby continuity of f is understood w.r.t. the subspace topology $\mathcal{T}|_A$.

In the proof of the forward implication (which is probably the more significant part of the theorem) we use Urysohn's Lemma in an equivalent form.

Lemma 1.4.2. *Let $\langle X, \mathcal{T}\rangle$ be a (T_4)-space, $A \subseteq X$ closed, and $f\colon A \to [-1,1]$ continuous. Then there exists $g\colon X \to [-\frac{1}{3}, \frac{1}{3}]$ with $|f(x) - g(x)| \leqslant \frac{2}{3}$ for all $x \in A$.*

Proof. Consider the sets

$$C := f^{-1}\big([-1,-\tfrac{1}{3}]\big), \quad D := f^{-1}\big([\tfrac{1}{3},1]\big).$$

These sets are closed in A. Since A is closed in X, they are also closed in X. Clearly, $C \cap D = \varnothing$, and Urysohn's Lemma (after composing with the affine map $t \mapsto \frac{2}{3}t - \frac{1}{3}$) provides us with a continuous function $g\colon X \to [-\frac{1}{3},\frac{1}{3}]$ such that $g(C) = \{-\frac{1}{3}\}$ and $g(D) = \{\frac{1}{3}\}$. Distinguishing the cases that $x \in C$, $x \in D$, or $x \in A \backslash (C \cup D)$, shows that g satisfies the required estimate. ❑

Proof of Theorem 1.4.1.

① *We show the backwards implication:* Assume that (1.7) holds, and let $A, B \subseteq X$ be disjoint. The set $C := A \cup B$ is closed in X. We have $A = C \cap (X \backslash B)$, and hence A is open in C. In the same way, B is open in C. Hence, the function $f\colon C \to [-1,1]$ defined as

$$f(x) := \begin{cases} 1 & \text{if } x \in A, \\ -1 & \text{if } x \in B, \end{cases}$$

is continuous. Let $F\colon X \to [-1,1]$ be a continuous extension of f. Then $O_A := F^{-1}((0,1])$ and $O_B := F^{-1}([-1,0))$ are open, disjoint, and contain A and B, respectively.

② *Proof of the forward implication (inductive construction):* Assume that $\langle X, \mathcal{T} \rangle$ is (T_4), and that A and f are given. We use induction on n to define a sequence $(g_n)_{n \in \mathbb{N}}$ of continuous functions $g_n\colon X \to [-1,1]$, with

$$\forall x \in X.\ |g_n(x)| \leqslant 1 - \Big(\frac{2}{3}\Big)^{n+1} \quad \text{and} \quad \forall x \in A.\ |g_n(x) - f(x)| \leqslant \Big(\frac{2}{3}\Big)^{n+1}$$

Applying Lemma 1.4.2 with the function f yields g_0. Assume g_n has already been constructed. Applying Lemma 1.4.2 with the function $(\frac{3}{2})^{n+1}(f - g_n)$ gives a function $h_{n+1}\colon X \to [-\frac{1}{3},\frac{1}{3}]$ with

$$\Big|\Big(\frac{3}{2}\Big)^{n+1}\big(f(x) - g_n(x)\big) - h_{n+1}(x)\Big| \leqslant \frac{2}{3} \quad \text{for } x \in A.$$

Set $g_{n+1} := g_n + \big(\frac{2}{3}\big)^{n+1} h_{n+1}$, then $|g_{n+1}(x) - f(x)| \leqslant \big(\frac{2}{3}\big)^{n+2}$ for all $x \in A$ and

$$|g_{n+1}| \leqslant |g_n| + \Big(\frac{2}{3}\Big)^{n+1}|h_{n+1}| \leqslant \Big(1 - \Big(\frac{2}{3}\Big)^{n+1}\Big) + \Big(\frac{2}{3}\Big)^{n+1} \cdot \frac{1}{3} = 1 - \Big(\frac{2}{3}\Big)^{n+2}.$$

③ *Proof of the forward implication (conclusion):* The series

$$\sum_{n=0}^{\infty} \Big(\frac{2}{3}\Big)^{n+1} h_{n+1}$$

is absolutely and uniformly convergent, hence represents a continuous function on X, and is bounded by $\frac{2}{3}$. Set

$$g := g_0 + \sum_{n=0}^{\infty} \Big(\frac{2}{3}\Big)^{n+1} h_{n+1},$$

then g maps X continuously into $[-1,1]$, and for each $x \in A$ it holds that

$$g(x) = \lim_{n \to \infty} g_n(x) = f(x).$$

❑

We can deduce a variant for functions mapping into open intervals (we give a formulation for the interval $\mathbb{R}$).

Corollary 1.4.3. *Let $\langle X, \mathcal{T}\rangle$ be (T_4), let $A \subseteq X$ closed, and $f\colon A \to \mathbb{R}$ continuous. Then there exists a continuous function $F\colon X \to \mathbb{R}$ with $F|_A = f$.*

Proof. Let $\phi\colon \mathbb{R} \to (-1,1)$ be an increasing bijection, and consider the function $g := \phi \circ f$. Tietze's theorem provides us with a continuous function $G\colon X \to [-1,1]$ with $G|_A = g$. Set $B := G^{-1}(\{1,-1\})$, then B is closed and $A \cap B = \varnothing$. Urysohn's Lemma gives us a continuous function $H\colon X \to [0,1]$ with $H(A) = 1$ and $H(B) = 0$. The function $G \cdot H$ maps X continuously into $[-1,1]$, and satisfies $(GH)(x) = g(x)$ for all $x \in A$. If $x \in X\backslash B$, then $|(GH)(x)| < 1$ by the definition of B, and if $x \in B$ then $(GH)(x) = 0$. Hence, $(GH)(X) \subseteq (-1,1)$. The function $F := \phi^{-1} \circ (GH)\colon X \to \mathbb{R}$ is a continuous extension of f. ❑

Let us now present two quick application of Tietze's theorem.

Corollary 1.4.4. *If a metric space X has the property that every continuous real-valued function on X is bounded, then it is compact.*

Proof. We use contraposition. Assume X is not compact. Then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X which has no convergent subsequence. Without loss of generality we can assume that the points x_n are pairwise different. For every point $x \in X$ there exists $r > 0$ such that $(U_r(x)\backslash\{x\}) \cap \{x_n \mid n \in \mathbb{N}\} = \varnothing$, since otherwise, we could inductively construct a convergent subsequence. In particular, the set $A := \{x_n \mid n \in \mathbb{N}\}$ is closed, and the subspace topology $\mathcal{T}|_A$ is the discrete topology on A.

Let $f\colon A \to \mathbb{R}$ be the function defined as $f(x_n) := n$. Tietze's theorem provides us with a continuous extension $F\colon X \to \mathbb{R}$. This function is clearly unbounded. ❑

Corollary 1.4.5. *There exists a continuous and surjective function of $[0,1]$ onto $[0,1]^2$ (a function with this property is also called a* Peano curve*).*

Proof. We consider the two functions

$$g\colon \begin{cases} \{0,1\}^{\mathbb{N}} & \to \quad [0,1] \\ (a_n)_{n\in\mathbb{N}} & \mapsto \quad \sum_{n=0}^{\infty} \frac{2}{3^{n+1}} a_n \end{cases}$$

$$h\colon \begin{cases} \{0,1\}^{\mathbb{N}} & \to \quad [0,1]^2 \\ (a_n)_{n\in\mathbb{N}} & \mapsto \quad \Big(\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} a_{2n}, \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} a_{2n+1}\Big) \end{cases}$$

Here $\{0,1\}^{\mathbb{N}}$ is endowed with the product topology of the discrete topology on $\{0,1\}$.

We use an estimate of the tails of the defining sum to show that g is injective. Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be different sequences, and set $N := \min\{n \in \mathbb{N} \mid a_n \neq b_n\}$. Then

$$\Big|\sum_{n=0}^{\infty} \frac{2}{3^{n+1}} a_n - \sum_{n=0}^{\infty} \frac{2}{3^{n+1}} b_n\Big| \geqslant \frac{2}{3^{N+1}} \underbrace{|a_N - b_N|}_{=1} - \sum_{n=N+1}^{\infty} \frac{2}{3^{n+1}} \underbrace{|a_n - b_n|}_{\leqslant 1} \geqslant \frac{1}{3^{N+1}}.$$

The similar estimates of tails (or the bounded convergence theorem) show that g and h are continuous. Moreover, h is surjective, since every real number has dyadic representation.

The corestriction of g to a map $\tilde{g}\colon \{0,1\}^{\mathbb{N}} \to g(\{0,1\}^{\mathbb{N}})$ is a continuous bijection. Its domain is compact by Tychonoff's theorem, and its codomain is Hausdorff. We conclude that $\tilde{g}$ is a homeomorphism, and that $g(\{0,1\}^{\mathbb{N}})$ is compact and hence closed in $[0,1]$.

Tietze's theorem gives a continuous function $F\colon [0,1] \to [0,1]^2$ which extends $h \circ g^{-1}$. This extension is, in particular, surjective. ❑

1.5 Paracompactness

In order to introduce the notion of paracompact topological spaces, we need some terminology.

Definition 1.5.1.

(i) Let X be a set and $\mathcal{F}, \mathcal{G} \subseteq \mathcal{P}(X)$. Then $\mathcal{F}$ is called a *refinement* of $\mathcal{G}$, if

$$\forall F \in \mathcal{F}\ \exists G \in \mathcal{G}.\ F \subseteq G.$$

(ii) Let $\langle X, \mathcal{T} \rangle$ be a topological space, and $\mathcal{F} \subseteq \mathcal{P}(X)$. Then $\mathcal{F}$ is called *locally finite*, if every point $x \in X$ has a neighbourhood which intersects at most finitely many members of $\mathcal{F}$.

Definition 1.5.2. A topological space $\langle X, \mathcal{T} \rangle$ is called *paracompact*, if every open cover of X has a locally finite refinement which is again an open cover of X.

Remark 1.5.3. The definition of compactness, that every open cover of the space has a finite subcover, can be formulated equivalently as: a topological space $\langle X, \mathcal{T} \rangle$ is compact, if and only if every open cover of X has a finite refinement which is again an open cover of X. This makes it obvious that paracompactness generalises compactness, and that paracompactness can be thought of as a localised version of compactness.

Paracompact Hausdorff spaces automatically have a stronger separation property.

Proposition 1.5.4. *Let $\langle X, \mathcal{T} \rangle$ be a paracompact Hausdorff space. Then $\langle X, \mathcal{T} \rangle$ is* (T_4).

The essence of the proof is the following lemma.

Lemma 1.5.5. *Let $\langle X, \mathcal{T} \rangle$ be paracompact, and let $A, B \subseteq X$ be closed and disjoint. If*

$$\forall x \in A\ \exists U_x, V_x \in \mathcal{T}.\ \big(x \in U_x \wedge B \subseteq V_x\big) \wedge \big(U_x \cap V_x = \emptyset\big)$$

then

$$\exists O_A, O_B \in \mathcal{T}.\ \big(A \subseteq O_A \wedge B \subseteq O_B\big) \wedge \big(O_A \cap O_B = \emptyset\big)$$

Proof. The family $\mathcal{G} := \{U_x \mid x \in A\} \cup \{X \backslash A\}$ is an open cover of X. Choose a locally finite open cover $\mathcal{F}$ of X which is a refinement of $\mathcal{G}$, and set

$$O_A := \bigcup \big\{F \in \mathcal{F} \mid F \cap A \neq \emptyset\big\}.$$

Clearly, the set O_A is open and contains A.

For $y \in B$ choose an open neighbourhood W_y of y which intersects at most finitely many elements of $\mathcal{F}$, and consider the finite set

$$\mathcal{F}_y := \{F \in \mathcal{F} \mid F \cap A \neq \emptyset \wedge F \cap W_y \neq \emptyset\}.$$

An element of $\mathcal{F}_y$ is not contained in $X \backslash A$. Hence, we can choose, for each $y \in B$ and $F \in \mathcal{F}_y$, an element $x(y, F) \in A$ such that $F \subseteq U_{x(y,F)}$. Set

$$O_B := \bigcup_{y \in B} \Big(W_y \cap \bigcap_{F \in \mathcal{F}_y} V_{x(y,F)} \Big).$$

Since $\mathcal{F}_y$ is finite, O_B is open. We have $y \in W_y$ for all $y \in B$, and all sets V_x contain B, hence $B \subseteq O_B$.

It remains to show that O_A and O_B are disjoint. To this end, we show that

$$\forall F \in \mathcal{F}, F \cap A \neq \emptyset\ \forall y \in B.\ F \cap W_y \cap \bigcap_{G \in \mathcal{F}_y} V_{x(y,G)} = \emptyset.$$

Let $F \in \mathcal{F}$, $F \cap A \neq \emptyset$, and $y \in B$. If $F \cap W_y = \emptyset$, the above intersection is clearly empty. Otherwise, if $F \cap W_y \neq \emptyset$, we have $F \in \mathcal{F}_y$ and hence $F \subseteq U_{x(y,F)}$. However, $U_{x(y,F)} \cap V_{x(y,F)} = \emptyset$, and again the intersection is empty. ❑

Proof of Proposition 1.5.4. Let $A, B \subseteq X$ be closed and disjoint. Since $\langle X, \mathcal{T} \rangle$ is (T_2), the hypothesis of Lemma 1.5.5 is satisfied with the two sets A and $\{y\}$ for each fixed $y \in B$. The conclusion of the lemma now ensures that its hypothesis is satisfied with the two sets B and A. ❑

An essential feature of paracompactness is that it ensures existence of partitions of unity. This, in turn, allows to prove global theorems from local ones.

In the following we denote the *support* of a complex-valued function f defined on some topological space as $\operatorname{supp} f$, i.e.,

$$\operatorname{supp} f := \overline{f^{-1}(\mathbb{C} \backslash \{0\})}.$$

Definition 1.5.6. Let $\langle X, \mathcal{T} \rangle$ be a topological space. A *partition of unity* is a family $\mathcal{E} \subseteq C(X, [0,1])$ of functions with

(i) the family $\{\operatorname{supp} f \mid f \in \mathcal{E}\} \subseteq \mathcal{P}(X)$ is locally finite,

(ii) $\forall x \in X.\ \sum_{f \in \mathcal{E}} f(x) = 1$

Let $\mathcal{F} \subseteq \mathcal{P}(X)$ be an open cover of X, and $\mathcal{E}$ be a partition of unity. Then $\mathcal{E}$ is called *subordinate to* $\mathcal{F}$, if $\{\operatorname{supp} f \mid f \in \mathcal{E}\}$ is a refinement of $\mathcal{F}$.

Note here that by (i) every point $x \in X$ has a neighbourhood U such that only finitely many functions from $\mathcal{E}$ do not vanish identically on U. Hence, the sum in (ii) is a well-defined and continuous function of X into $[0, \infty)$.

Theorem 1.5.7. *Let $\langle X, \mathcal{T} \rangle$ be a topological space which satisfies (T_4), and let $\mathcal{G} \subseteq \mathcal{P}(X)$ be a locally finite open cover of X. Then there exists a partition of unity subordinate to $\mathcal{G}$.*

In the proof we use existence of a strong form of refinements: Let $\mathcal{G} \subseteq \mathcal{P}(X)$. A family $\mathcal{F} = \{O_U \,|\, U \in \mathcal{G}\}$ is called a *shrinking* of $\mathcal{G}$, if $\overline{O_U} \subseteq U$ for all $U \in \mathcal{G}$. Moreover, we call a family $\mathcal{F} \subseteq \mathcal{P}(X)$ *point finite*, if every point $x \in X$ belongs to most finitely many members of $\mathcal{F}$.

Lemma 1.5.8. *Let $\langle X, \mathcal{T}\rangle$ be a topological space which satisfies (T_4), and let $\mathcal{G} \subseteq \mathcal{P}(X)$ be a point finite open cover of X. Then $\mathcal{G}$ has a shrinking which is again an open cover.*

Proof. Let an open cover $\mathcal{G}$ of X be given, and choose a well-ordering $\preceq$ of the set $\mathcal{G}$. Using induction on U, we construct open sets O_U, $U \in \mathcal{G}$, with

$$\forall U \in \mathcal{G}.\ X \backslash \Big(\bigcup_{W \prec U} O_W \cup \bigcup_{V \succ U} V \Big) \subseteq O_U \subseteq \overline{O_U} \subseteq U \tag{1.8}$$

Let $U \in \mathcal{G}$ be given, and assume that open sets O_W have already been constructed for all $W \prec U$, such that the predicate in (1.8) is true for all those indices.

We show that

$$X \backslash \Big(\bigcup_{W \prec U} O_W \cup \bigcup_{V \succ U} V \Big) \subseteq U. \tag{1.9}$$

Let x belong to the set on the left side, and set

$$\mathcal{G}_x := \{V \in \mathcal{G} \,|\, x \in V\}.$$

This set is nonempty since $\mathcal{G}$ is a covering, finite since $\mathcal{G}$ is point-finite, and contained in $\{V \in \mathcal{G} \,|\, V \preceq U\}$. If $\max \mathcal{G}_x \prec U$, then the inductive hypothesis yields $x \in O_{\max \mathcal{G}_x}$ which is a contradiction. Hence, $\max \mathcal{G}_x = U$, i.e., $x \in U$.

The set on the left side of (1.9) is closed, and (T_4) implies that there exists O_U open with the property required by (1.8).

By construction, the family $\{\overline{O_U} \,|\, U \in \mathcal{G}\}$ is a refinement of $\mathcal{G}$. It remains to show that $\{O_U \,|\, U \in \mathcal{G}\}$ is a covering of X. Let $x \in X$, and set $U := \max \mathcal{G}_x$. If $x \in O_W$ for some $W \prec U$, we are done. Otherwise, (1.8) shows that $x \in O_U$. ❑

Proof of Theorem 1.5.7. Let $\mathcal{G}$ be a locally finite open cover of X. Choose a shrinking $\mathcal{F} = \{O_U \,|\, U \in \mathcal{G}\}$ of $\mathcal{G}$. By (T_4), we find W_U open with $\overline{O_U} \subseteq W_U \subseteq \overline{W_U} \subseteq U$, and Urysohn's Lemma provides us with a continuous function $f_U \colon X \to [0,1]$ satisfying

$$f_U(X \backslash W_U) = \{0\}, \quad f_U(\overline{O_U}) = \{1\}.$$

Then for every $U \in \mathcal{G}$ it holds that $\operatorname{supp} f_U \subseteq \overline{W_U} \subseteq U$. We see that $\{\operatorname{supp} f_U \,|\, U \in \mathcal{G}\}$ is locally finite. Hence, the sum $\sum_{U \in \mathcal{G}} f_U$ is a well-defined and continuous function of X into $[0, \infty)$. Since every point $x \in X$ belongs to at least one of the sets O_U, we have $\sum_{U \in \mathcal{G}} f_U(x) \geqslant 1$ for all $x \in X$. Set

$$g_U := \Big(\sum_{V \in \mathcal{G}} f_V \Big)^{-1} \cdot f_U,$$

Then $\{g_U \,|\, U \in \mathcal{G}\}$ is a partition of unity subordinate to $\mathcal{G}$. ❑

As a corollary we obtain that paracompactness indeed corresponds to existence of many partitions of unity.

Recall that a topological space $\langle X, \mathcal{T}\rangle$ is said to satisfy the *separation axiom* (T_1), if

$$\forall x, y \in X, x \neq Y\ \exists O_x, O_y \in \mathcal{T}.\ \big(x \in O_x \wedge y \in O_y\big) \wedge \big(y \notin O_x \wedge x \notin O_y\big)$$

Moreover, recall that in a (T_1)-space singleton sets are closed: Given $x \in X$, choose for each $y \in X$, $y \neq x$, an open neighbourhood O_y according to above formula. Then $X\backslash\{x\} = \bigcup_{y \in X\backslash\{x\}} O_y$.

Corollary 1.5.9. *Let $\langle X, \mathcal{T}\rangle$ be a topological space which satisfies (T_1). Then $\langle X, \mathcal{T}\rangle$ is paracompact and Hausdorff, if and only if for every open cover $\mathcal{G}$ of X there exists a partition of unity subordinate to $\mathcal{F}$.*

Proof. To show the forward implication, let an open cover $\mathcal{G}$ be given. Choose a locally finite open cover $\mathcal{F}$ which is a refinement of $\mathcal{G}$. By Proposition 1.5.4, $\langle X, \mathcal{T}\rangle$ is (T_4), and by Theorem 1.5.7 we find a partition of unity which is subordinate to $\mathcal{F}$ and hence to $\mathcal{G}$.

We come to the backward implication. Let $\mathcal{G}$ be an open cover of X. Choose a partition of unity $\mathcal{E}$ subordinate to $\mathcal{G}$. Then $\{f^{-1}((0,1]) \mid f \in \mathcal{E}\}$ is a locally finite open cover of X which is a refinement of $\mathcal{G}$. It remains to show that (T_1) upgrades to (T_2). Let $x, y \in X$, $x \neq y$, be given. Choose a partition of unity $\mathcal{E}$ subordinate to the open cover $\{X\backslash\{x\}, X\backslash\{y\}\}$. Choose $f \in \mathcal{E}$ with $f(x) > 0$. Then $\operatorname{supp} f$ cannot be contained in $X\backslash\{x\}$, and hence $f(y) = 0$.

$$O_x := f^{-1}\Big(\Big(\frac{f(x)}{2}, 1\Big]\Big), \quad O_y := f^{-1}\Big(\Big[0, \frac{f(x)}{2}\Big)\Big).$$

These two sets are open, disjoint, and $x \in O_x$ and $y \in O_y$. ❏

1.6 Paths and homotopy

Functions are often defined by case distinction, and it is important to know that such functions are continuous. We refer to the following simple and familiar result as the *gluing lemma*.

Lemma 1.6.1. *Let $\langle X, \mathcal{T}\rangle$ and $\langle Y, \mathcal{V}\rangle$ be topological spaces. Let $\{M_i \mid i \in I\}$ be a cover of X, let $f_i\colon M_i \to Y$, $i \in I$, be continuous functions, and assume that*

$$\forall i, j \in I.\ \ f_i|_{M_i \cap M_j} = f_j|_{M_i \cap M_j}$$

Let $f\colon X \to Y$ be the unique function with

$$\forall i \in I.\ \ f|_{M_i} = f_i.$$

Assume that either (i) *or* (ii) *holds:*

(i) *All sets M_i are open.*

(ii) *I is finite and all sets M_i are closed.*

Then f is continuous.

Proof. For $N \subseteq Y$ we have

$$f^{-1}(N) = \bigcup_{i \in I} f_i^{-1}(N).$$

Assume that assumption (i) holds. Let $N \subseteq Y$ be open. Then $f_i^{-1}(N)$ is open in M_i. Since M_i is open in X, also $f_i^{-1}(N)$ is open in X. Thus $f^{-1}(N)$ is open in X.

Assume that assumption (ii) holds. Let $N \subseteq Y$ be closed. Then $f_i^{-1}(N)$ is closed in M_i. Since M_i is closed in X, also $f_i^{-1}(N)$ is closed in X. Since I is finite, $f^{-1}(N)$ is closed in X. ❏

Definition 1.6.2. Let $\langle X, \mathcal{T}\rangle$ be a topological space. A *path* in $\langle X, \mathcal{T}\rangle$ is a continuous map $f\colon [a,b] \to X$ where $a, b \in \mathbb{R}$, $a < b$. The point $f(a)$ is called the *initial point* of the path f, and $f(b)$ is its *terminal point*.

Paths whose terminal and initial points fit together can be concatenated, and one can move through a path in reversed direction.

Definition 1.6.3. Let $\langle X, \mathcal{T}\rangle$ be a topological space.

(i) Let $f\colon [a,b] \to X$ and $g\colon [b,c] \to X$ be paths in X with $f(b) = g(b)$. Then we define the *multiplication* of f and g as the function $f \cdot g\colon [a,c] \to X$ with

$$(f \cdot g)|_{[a,b]} = f \ \wedge \ (f \cdot g)|_{[b,c]} = g.$$

(ii) Let $f\colon [a,b] \to X$ be a path. Then we define the *reversed* path as the function $f^{-1}\colon [-b,-a] \to X$ defined by

$$f^{-1}(t) := f\big(-t\big) \text{ for } t \in [-b,-a].$$

By the gluing lemma the multiplication of two paths is again a path. For further reference we state a couple of obvious properties.

Lemma 1.6.4.

(i) *Multiplication of paths is associative whenever all products are defined: if $f\colon [a,b] \to X$, $g\colon [b,c] \to X$, $h\colon [c,d] \to X$ are paths with $f(b) = g(b)$ and $g(c) = h(c)$, then*

$$(f \cdot g) \cdot h = f \cdot (g \cdot h).$$

(ii) *Let $\langle Y, \mathcal{V}\rangle$ be a topological space, and $\Phi\colon X \to Y$ continuous. If f, g are paths in X whose multiplication is defined, then $\Phi \circ f$ and $\Phi \circ g$ are paths in Y whose multiplication is defined, and*

$$\Phi \circ (f \cdot g) = (\Phi \circ f) \cdot (\Phi \circ g).$$

For every path f in X we have

$$\Phi \circ f^{-1} = (\Phi \circ f)^{-1}.$$

(iii) *Let $f\colon [a,b] \to X$ and $g\colon [b,c] \to X$ be paths with $f(b) = g(b)$, let $\phi\colon [a',b'] \to [a,b]$ and $\psi\colon [b',c'] \to [b,c]$ be paths with $\phi(b') = \psi(b')$ (and therefore equal to b). Then*

$$(f \cdot g) \circ (\phi \cdot \psi) = (f \circ \phi) \cdot (g \circ \psi).$$

Proof. Associativity holds since for both sides the restriction to $[a,b]$ is f, the restriction to $[b,c]$ is g, and the restriction to $[c,d]$ is h. Items (ii) and (iii) are seen by unfolding the definitions and making case distinctions. ❑

Moving through a path at a different speed does not change the image set. We make this precise by introducing a relation on paths.

Definition 1.6.5. Let $\langle X, \mathcal{T}\rangle$ be a topological space, and let $f\colon [a,b] \to X$ and $g\colon [c,d] \to X$ be paths in X. We say that f and g are *reparameterisations* of each other, if there exists an increasing bijection $\phi\colon [a,b] \to [c,d]$ with $f = g \circ \phi$.

If f and g are reparameterisations of each other we write $f \sim_r g$.

Note that for each path $f\colon [a,b] \to X$ and each interval $[c,d]$ there exists a reparameterisation g of f which is defined on the interval $[c,d]$. For example, use $f \circ \phi$ where ϕ is the affine map with $\phi(c) = a$ and $\phi(d) = b$. Hence, when considering paths, one can often restrict considerations to paths defined on some fixed interval, e.g. on $[0,1]$. Moreover, note that a monotone bijection between intervals is automatically continuous.

The relation of reparameterisation is well-behaved and compatible with algebraic operations.

Proposition 1.6.6. *Let $\langle X, \mathcal{T}\rangle$ be a topological space.*

(i) *Reparameterisation is an equivalence relation.*

(ii) *Let f_1, g_1 and f_2, g_2 be two pairs of paths whose multiplication is defined. If $f_1 \sim_r f_2$ and $g_1 \sim_r g_2$, then also $f_1 \cdot g_1 \sim_r f_2 \cdot g_2$.*

(iii) *Let f, g be paths. If $f \sim_r g$, then also $f^{-1} \sim_r g^{-1}$.*

(iv) *Let $\langle Y, \mathcal{V}\rangle$ be a topological space and $\Phi\colon X \to Y$ continuous. Further, let f, g be paths in X. If $f \sim_r g$, then $\Phi \circ f \sim_r \Phi \circ g$.*

Proof. Item (i) follows from

$$f \circ \mathrm{id}_{[a,b]} = f, \quad f = g \circ \phi \;\Rightarrow\; g = f \circ \phi^{-1}, \quad f = g \circ \phi \wedge g = h \circ \psi \;\Rightarrow\; f = h \circ (\psi \circ \phi).$$

To prove item (ii), denote the domains of f_j and g_j as $[a_j, b_j]$ and $[b_j, c_j]$, respectively, and let $\phi\colon [a_1,b_1] \to [a_2,b_2]$ and $\psi\colon [b_1,c_1] \to [b_2,c_2]$ be increasing bijections with $f_1 = f_2 \circ \phi$ and $g_1 = g_2 \circ \psi$. Then the product $\phi \cdot \psi$ is an increasing bijection of $[a_1,c_1]$ onto $[a_2,c_2]$, and

$$(f_1 \cdot g_1) = (f_2 \circ \phi) \cdot (g_2 \circ \psi) = (f_2 \cdot g_2) \circ (\phi \cdot \psi).$$

For the proof of (iii), let $f\colon [a,b] \to X$ and $g\colon [c,d] \to X$ be paths, $\phi\colon [a,b] \to [c,d]$ an increasing bijection, and assume that $f = g \circ \phi$. Then $\psi(t) := -\phi(-t)$ is an increasing bijection of $[-b,-a]$ onto $[-d,-c]$, and

$$f^{-1}(t) = f(-t) = (g \circ \phi)(-t) = g^{-1}(-\phi(-t)) = (g^{-1} \circ \psi)(t).$$

Finally, item (iv) follows since $f = g \circ \phi$ implies $\Phi \circ f = (\Phi \circ g) \circ \phi$. ❑

A relation between paths which is of a different kind is homotopy. Two paths are homotopic, if they can be continuously deformed into each other. At the present stage we introduce a variant of this notion which also fixes the inital and terminal points.

Definition 1.6.7. Let $\langle X, \mathcal{T}\rangle$ be a topological space, and let $f, g\colon [a,b] \to X$ be paths in X with $f(a) = g(a)$ and $f(b) = g(b)$. We say that f and g are *FEP-homotopic* (here "FEP" stands for "fixed end point"), if there exists a continuous function $H\colon [a,b] \times [0,1] \to X$ with

$$\forall t \in [a,b].\; H(t,0) = f(t) \;\wedge\; H(t,1) = g(t) \tag{1.10}$$

$$\forall s \in [0,1].\; H(a,s) = f(a) \;\wedge\; H(b,s) = f(b) \tag{1.11}$$

If f and g are FEP-homotopic we write $f \approx g$, and every function H as above is called a *FEP-homotopy* from f to g. The functions $h_s \colon t \mapsto H(t,s)$ with $s \in [0,1]$ are called the *intermediate paths* of the homotopy.

The relation (1.10) says that H is a deformation of f into g, and (1.11) says that all intermediate paths of H have the same initial point and the same terminal point.

Proposition 1.6.8. *Let $\langle X, \mathcal{T}\rangle$ be a topological space.*

(i) *FEP-homotopy is an equivalence relation.*

(ii) *Let $f_1, f_2 \colon [a,b] \to X$ and $f_2, g_2 \colon [b,c] \to X$ be paths with $f_1(b) = g_1(b)$ and $f_2(b) = g_2(b)$. If $f_1 \approx f_2$ and $g_1 \approx g_2$, then also $f_1 \cdot g_1 \approx f_2 \cdot g_2$.*

(iii) *Let $f, g \colon [a,b] \to X$ be paths. If $f \approx g$, then also $f^{-1} \approx g^{-1}$.*

(iv) *Let $\langle Y, \mathcal{V}\rangle$ be a topological space and $\Phi \colon X \to Y$ continuous. Further, let $f, g \colon [a,b] \to X$ be paths. If $f \approx g$, then $\Phi \circ f \approx \Phi \circ g$.*

(v) *Let $f, g \colon [a,b] \to X$ be paths. If $f \sim_r g$, then also $f \approx g$.*

Proof.

① *We show that $\approx$ is reflexive, symmetric, and transitive:* For $f \approx f$ use $H(t,s) := f(t)$. If H is a FEP-homotopy with $H(t,0) = f(t)$ and $H(t,1) = g(t)$, then $K(t,s) := H(t, 1-s)$ is a FEP-homotopy with $K(t,0) = g(t)$ and $K(t,1) = f(t)$. Assume we have FEP-homotopies H and K with $H(t,0) = f(t)$, $H(t,1) = g(t)$, $K(t,0) = g(t)$, $K(t,1) = h(t)$. By the gluing lemma

$$L(t,s) := \begin{cases} H(t,2s) & \text{if } t \in [a,b], s \in [0,\frac{1}{2}] \\ K(t,2s-1) & \text{if } t \in [a,b], s \in [\frac{1}{2},1] \end{cases}$$

is continuous. Clearly, it is a FEP-homotopy from f to h.

② *We show compatibility with multiplying and reversing paths:* Let H be a FEP-homotopy from f_1 to f_2 and K a FEP-homotopy from g_1 to g_2. By the gluing lemma

$$L(t,s) := \begin{cases} H(t,s) & \text{if } t \in [a,b], s \in [0,1] \\ K(t,s) & \text{if } t \in [b,c], s \in [0,1] \end{cases}$$

is continuous. Clearly, it is a FEP-homotopy from $f_1 \cdot g_1$ to $f_2 \cdot g_2$. Let H be a FEP-homotopy from f to g. Then $K(t,s) := H(-t,s)$ is a FEP-homotopy from f^{-1} to g^{-1}.

③ *We prove item* (iv)*:* Let $H \colon [a,b] \times [0,1] \to X$ be a FEP-homotopy from f to g. Then $\Phi \circ H \colon [a,b] \times [0,1] \to Y$ is a FEP-homotopy from $\Phi \circ f$ to $\Phi \circ g$.

④ *We show that $\approx$ is "larger" than $\sim_r$:* Let $\phi \colon [a,b] \to [a,b]$ be an increasing bijection with $f = g \circ \phi$. Then

$$H(t,s) := g\big((1-s)\phi(t) + st\big) \text{ for } t \in [a,b], s \in [0,1],$$

is a FEP-homotopy from f to g.

❑

1.7 Connectedness

Definition 1.7.1. Let $\langle X, \mathcal{T}\rangle$ be a topological space.

(i) A *separation* of $\langle X, \mathcal{T}\rangle$ is a pair (U, V) of subsets of X with

$$U, V \in \mathcal{T}\backslash\{\varnothing\} \ \wedge \ U \cap V = \varnothing \ \wedge \ U \cup V = X.$$

The space $\langle X, \mathcal{T}\rangle$ is called *connected*, if there exists no separation of $\langle X, \mathcal{T}\rangle$.

(ii) The space $\langle X, \mathcal{T}\rangle$ is called *pathwise connected*, if for each two points $x, y \in X$ there exists a path in X with initial point x and terminal point y.

To make a connection between connectedness and pathwise connectedness, we need a basic example.

Example 1.7.2. Let $a, b \in \mathbb{R}$ with $a < b$. Then the interval $[a, b]$ is connected (where $[a, b]$ is endowed with the subspace topology of the euclidean topology).

To see this, consider $U, V \subseteq [a, b]$ be open and disjoint with $U \cup V = [a, b]$. One of these sets contains the point a, for definiteness assume that $a \in U$. Set

$$c := \sup\{t \in [a, b] \mid [a, t) \subseteq U\}.$$

Note that $c > a$ since U is open, and

$$[a, c) = \bigcup_{t\in[a,c)} [a, t) \subseteq U.$$

Since U is also closed, it follows that $c \in U$. Assume that we had $c < b$. Then $V \neq \varnothing$ and $c = \inf V$. Since V is closed, it follows that $c \in V$, a contradiction. Thus $c = b$. We see that $U = [a, b]$ and $V = \varnothing$, and hence that (U, V) is not a separation.

Lemma 1.7.3. *If $\langle X, \mathcal{T}\rangle$ is pathwise connected, then $\langle X, \mathcal{T}\rangle$ is connected.*

Proof. Assume towards a contradiction that (U, V) is a separation of X. Choose $x \in U$ and $y \in V$, and a path $f\colon [a, b] \to X$ with initial point x and terminal point y. Then $(f^{-1}(U), f^{-1}(V))$ is a separation of $[a, b]$, and this contradicts Example 1.7.2. ❑

Example 1.7.4. Let $\langle Z, \mathcal{T}\rangle$ be a topological vector space, and X a convex subset of Z. Then $\langle X, \mathcal{T}|_X\rangle$ is pathwise connected.

To see this, let $x, y \in X$. Then the whole line segment $\{(1-t)x + ty \mid t \in [0, 1]\}$ belongs to X. Since algebraic operations are continuous, the function

$$f\colon \begin{cases} [0,1] & \to & X \\ \quad t & \mapsto & (1-t)x + ty \end{cases}$$

is continuous, i.e., a path. Its initial point is x and its terminal point is y.

Definition 1.7.5. Let $\langle X, \mathcal{T}\rangle$ be a topological space, and let Y be a subset of X.

(i) We say that Y is a *connected subset of* X, if $\langle Y, \mathcal{T}|_Y\rangle$ is connected.

(ii) We say that Y is a *pathwise connected subset of* X, if $\langle Y, \mathcal{T}|_Y\rangle$ is pathwise connected.

Connectedness of subsets is related with another notion of separation.

Definition 1.7.6. Let $\langle X, \mathcal{T}\rangle$ be a topological space. Two subsets $A, B \subseteq X$ are called *separated in* X, if

$$A \cap \overline{B} = \overline{A} \cap B = \varnothing.$$

Lemma 1.7.7. *Let $\langle X, \mathcal{T}\rangle$ be a topological space and $Y \subseteq X$.*

(i) *Let $A, B \subseteq X$ be two nonempty disjoint sets with $A \cup B = Y$. Then (A, B) is a separation of $\langle Y, \mathcal{T}|_Y\rangle$, if and only if A and B are separated in X.*

(ii) *Y is a connected subset of $\langle X, \mathcal{T}\rangle$, if and only if it cannot be written as a union of two nonempty sets which are separated in X.*

Proof. We proof (i). First, note that (A, B) is a separation of $\langle Y, \mathcal{T}|_Y\rangle$ if and only if $A, B \in \mathcal{T}|_Y$. Assume that B is open in $\langle Y, \mathcal{T}|_Y\rangle$, and choose $O_B \in \mathcal{T}$ with $B = Y \cap O_B$. Then $A \subseteq X\backslash O_B$, and hence also $\overline{A} \subseteq X\backslash O_B$. In particular, $\overline{A} \cap B = \varnothing$. Conversely, assume that $\overline{A} \cap B = \varnothing$. Then $B = Y \cap (X\backslash\overline{A})$, and hence belongs to $\mathcal{T}|_Y$. The same arguments show that $A \in \mathcal{T}|_Y$ if and only if $A \cap \overline{B} = \varnothing$ holds.

Item (ii) is an immediate consequence of (i). ❑

Connectedness and pathwise connectedness are inherited by several constructions.

Theorem 1.7.8.

(i) *Let $\langle X, \mathcal{T}\rangle$ be a topological space, Y a connected subset of X, and $Z \subseteq X$ with $Y \subseteq Z \subseteq \overline{Y}$. Then Z is a connected subset of X.*

(ii) *Let $\langle X, \mathcal{T}\rangle$ and $\langle Y, \mathcal{V}\rangle$ be topological spaces, and $f\colon X \to Y$ a surjective and continuous map. If $\langle X, \mathcal{T}\rangle$ is connected, then $\langle Y, \mathcal{V}\rangle$ is connected.*

(iii) *Let $\langle X, \mathcal{T}\rangle$ be a topological space, and let A_i, $i \in I$, be a family of subsets of X with*

$$\bigcup_{i\in I} A_i = X, \quad \bigcap_{i\in I} A_i \neq \varnothing.$$

If all A_i are connected subsets of X, then $\langle X, \mathcal{T}\rangle$ is connected.

(iv) *Let $\langle X_i, \mathcal{T}_i\rangle$, $i \in I$, be a family of topological spaces, and consider the product $X := \prod_{i\in I} X_i$ endowed with the product topology $\mathcal{T}$ of the topologies $\mathcal{T}_i$. Then $\langle X, \mathcal{T}\rangle$ is connected, if and only if all $\langle X_i, \mathcal{T}_i\rangle$, $i \in I$, are connected.*

The statements in (ii)–(iv) *also hold when "connected" is everywhere replaced by "pathwise connected".*

Proof.

▻ *Item* (i)*:* We use contraposition. Assume that A and B are nonempty sets which are separated in X and satisfy $A \cup B = Z$. Then $\overline{Y} \cap (X \backslash \overline{A}) \supseteq B \neq \varnothing$, and hence also $Y \cap (X \backslash \overline{A}) \neq \varnothing$. In the same way, we obtain $Y \cap (X \backslash \overline{B}) \neq \varnothing$. Since $Y \cap A = Y \cap (X \backslash \overline{B})$ and $Y \cap B = Y \cap (X \backslash \overline{A})$, we found a separation of $\langle Y, \mathcal{T}|_Y \rangle$:

$$Y = Y \cap Z = (Y \cap A) \cup (Y \cap B).$$

▻ *Item* (ii)*; case "connected":* Assume (U, V) is a separation of Y. Then $(f^{-1}(U), f^{-1}(V))$ is a separation of X.

▻ *Item* (ii)*; case "pathwise connected":* Let $y_1, y_2 \in Y$. Choose $x_1, x_2 \in X$ with $f(x_1) = y_1$ and $f(x_2) = y_2$, and choose a path ϕ in X with initial point x_1 and terminal point x_2. Then $f \circ \phi$ is a path in Y with initial point y_1 and terminal point y_2.

▻ *Item* (iii)*; case "connected":* We use contraposition. Assume that (U, V) is a separation of $\langle X, \mathcal{T} \rangle$. Then U and V are separated in X. One of U and V must intersect $\bigcap_{i \in I} A_i$ since this intersection is nonempty. For definiteness assume that $U \cap \bigcap_{i \in I} A_i \neq \varnothing$. Since the sets A_i, $i \in I$, cover all of X, we find $i \in I$ with $A_i \cap V \neq \varnothing$. Then we can write $A_i = (A_i \cap U) \cup (A_i \cap V)$, and the sets $A_i \cap U$ and $A_i \cap V$ are nonempty and separated in X.

▻ *Item* (iii)*; case "pathwise connected":* Let $x_1, x_2 \in X$ be given, and choose $i_1, i_2 \in I$ with $x_1 \in A_{i_1}$ and $x_2 \in A_{i_2}$. Choose a point $z \in \bigcap_{i \in I} A_i$, and paths ϕ_1, ϕ_2 in A_{i_1} and A_{i_2}, respectively, with initial point z and terminal points x_1 and x_2, respectively. Then $\phi_1^{-1} \cdot \phi_2$ is a path in X with initial point x_1 and terminal point x_2.

▻ *Item* (iv)*; case "connected":* The forward implication follows from the already proved item (ii), since projections are continuous and surjective. The reverse implication requires an argument. Before we start, note that if some X_i is empty then also X is empty, and there is nothing to prove. Hence we may assume throughout that all X_i are nonempty.

Consider first the case of the product of two connected spaces, say X, Y. Fix $y \in Y$, and consider the cross-shaped sets

$$A_x := \pi_X^{-1}(\{x\}) \cup \pi_Y^{-1}(\{y\}) \text{ for } x \in X.$$

The subspace $\pi_X^{-1}(\{x\})$ of $X \times Y$ is homeomorphic to Y via π_Y, and $\pi_Y^{-1}(\{y\})$ is homeomorphic to X via π_X. Hence, both of these subspaces are connected. Their intersection contains the point (x, y), and we conclude that A_x is connected. Clearly, we have $X \times Y = \bigcup_{x \in X} A_x$ and $\bigcap_{x \in X} A_x \supseteq \pi_Y^{-1}(\{y\}) \neq \varnothing$, and conclude that $X \times Y$ is connected.

Using induction, it follows immediately that the product of finitely many connected spaces is again connected.

We turn to the general case. Assume that all spaces $\langle X_i, \mathcal{T}_i \rangle$ are connected. Fix elements $z_i \in X_i$, and set

$$A_J := \bigcap_{j \in I \backslash J} \pi_j^{-1}(\{z_j\}) \text{ for } J \subseteq I \text{ finite}.$$

The product map $\prod_{j \in J} \pi_j$ is a homeomorphism of A_J onto $\prod_{j \in J} X_j$. By the already establish "finite case", all sets A_J are connected. Set

$$A := \bigcup_{\substack{J \subseteq I \\ J \text{ finite}}} A_J.$$

Clearly, $(z_i)_{i\in I} \in \bigcap_{i\in I} A_i$, and we obtain that A is connected.

We are going to show that A is dense in X, and this will conclude the proof by the already established item (i). Consider a nonempty open set O in X, and choose a set U of the form

$$U = \bigcap_{l=1}^{n} \pi_{i_l}^{-1}(O_{i_l}),$$

where O_{i_l} is open in X_{i_l} and nonempty, such that $U \subseteq O$. Then $U \cap A_J \neq \varnothing$ where $J := \{i_1, \ldots, i_n\}$.

$\rhd$ *Item* (iv)*; case "pathwise connected":* Again the forward implication holds since projections are continuous and surjective. Conversely, let $(x_i)_{i\in I}, (y_i)_{i\in I} \in X$. Choose paths $\phi_i : [0,1] \to X_i$ with $\phi_i(0) = x_i$ and $\phi_i(1) = y_i$. Then the product map

$$\phi(t) := \big(\phi_i(t)\big)_{i\in I}$$

is a path in X and satisfies $\phi(0) = (x_i)_{i\in I}$ and $\phi(1) = (y_i)_{i\in I}$.

❑

The following statement is a simple corollary but exhibits an important concept.

Corollary 1.7.9. *Let $\langle X, \mathcal{T}\rangle$ be a topological space. The relations defined as*

$$x \simeq_c y :\Leftrightarrow \exists A \text{ connected subset of } X.\ x, y \in A$$
$$x \simeq_{pc} y :\Leftrightarrow \exists A \text{ pathwise connected subset of } X.\ x, y \in A$$

are equivalence relations. We have $\simeq_{pc} \subseteq \simeq_c$ and

$$x \simeq_{pc} y \;\Leftrightarrow\; \exists f \text{ path in } X \text{ with initial point } x \text{ and terminal point } y \tag{1.12}$$

Proof. Singleton sets are obviously connected and pathwise connected. Hence the relations $\simeq_c$ and $\simeq_{pc}$ are reflexive. Symmetry is built in the definition. Transitivity follows from Theorem 1.7.8 (iii). Moreover, the inclusion $\simeq_{pc} \subseteq \simeq_c$ follows from Lemma 1.7.3.

The forward implication in (1.12) is clear, and for the backward implication we use Theorem 1.7.8 (ii) which implies that every path is connected. ❑

The equivalence classes of $\simeq_c$ are called the *connected components* of $\langle X, \mathcal{T}\rangle$, and the equivalence classes of $\simeq_{pc}$ the *path-components* of $\langle X, \mathcal{T}\rangle$. Since $\simeq_{pc} \subseteq \simeq_c$, each connected component is a disjoint union of certain path-components.

Proposition 1.7.10. *Let $\langle X, \mathcal{T}\rangle$ be a topological space. Each connected component of $\langle X, \mathcal{T}\rangle$ is a connected subset of X, and every connected subset of X is entirely contained in one connected component. In particular, $\langle X, \mathcal{T}\rangle$ is connected, if and only if it has only one connected component.*

The same holds when "connected" and "connected component" is everywhere replaced by "pathwise connected" and "path-component", respectively.

Proof. Let $A \subseteq X$ be connected. If $x, y \in A$, then $x \simeq_c y$, and hence x and y belong to the same connected component of X. We see that A lies entirely in one connected component.

Let C be a connected component of X. Fix $z \in C$. Then for every $x \in C$ we have $x \simeq_c z$ and thus find a connected set A with $x, z \in A$. It follows that

$$C = \bigcup \{A \mid A \text{ connected}, z \in A\},$$

and Theorem 1.7.8 (iii) implies that C is connected.

The very same arguments apply with a pathwise connected set and a path-component. ❑

Definition 1.7.11. A topological space $\langle X, \mathcal{T}\rangle$ is called

(i) *locally connected*, if for every point $x \in X$ the set of all connected neighbourhoods of x forms a neighbourhood base of x.

(ii) *locally pathwise connected*, if for every point $x \in X$ the set of all pathwise connected neighbourhoods of x forms a neighbourhood base of x.

Proposition 1.7.12. *A topological space $\langle X, \mathcal{T}\rangle$ is locally connected, if and only if for every open subset U of X all connected components of $\langle U, \mathcal{T}|_U\rangle$ are open.*

The same holds when "locally connected" and "connected component" is replaced by "locally pathwise connected" and "path-component", respectively.

Proof. Assume first that $\langle X, \mathcal{T}\rangle$ is locally connected. Let $U \subseteq X$ be open, let C be a connected component of $\langle U, \mathcal{T}|_U\rangle$, and $x \in C$. Since U is a neighbourhood of x, we find a connected neighbourhood V of x with $V \subseteq U$. Since $\mathcal{T}|_V$ equals $(\mathcal{T}|_U)|_V$, V is a connected subset of $\langle U, \mathcal{T}|_U\rangle$. Since $V \cap C \neq \varnothing$, it follows that $V \subseteq C$. Thus C is a neighbourhood of x.

For the converse implication, let $x \in X$ and $U \subseteq X$ open with $x \in U$. Let C be the connected component of $\langle U, \mathcal{T}|_U\rangle$ with $x \in C$. Since C is open in $\langle U, \mathcal{T}|_U\rangle$ and U is open in X, it follows that C is open in X and hence a neighbourhood of x. Moreover, we know that C is connected.

The same argument applies word by word to the case of path-components. ❑

Corollary 1.7.13. *If $\langle X, \mathcal{T}\rangle$ is locally pathwise connected, then $\simeq_c = \simeq_{pc}$. In particular, a connected and locally pathwise connected space is pathwise connected.*

Proof. By Proposition 1.7.12 all path-components are open. Let C be a connected component of X. Then C is the disjoint union of certain path-components, say $C = \bigcup_{i\in I} P_i$. If I has more than one element, we obtain a separation of C. Namely, choose $j \in I$, then

$$C = P_j \cup \Big(\bigcup_{\substack{i\in I\\ i\neq j}} P_i\Big).$$

This contradicts the fact that C is connected. We conclude that $|I| = 1$, i.e., C is a path-component. ❑

1.8 The free product of groups

We discuss an algebraic construction: the *free product* of a family of groups.

Theorem 1.8.1. *Let G_i, $i \in I$, be a family of groups. There exists a tuple $\langle G, (\gamma_i)_{i\in I}\rangle$ where G is a group and $\gamma_i\colon G_i \to G$ are homomorphisms, with the following property:*

▷ *For every tuple $\langle H, (\phi_i)_{i\in I}\rangle$ where H is a group and $\phi_i\colon G_i \to H$ are homomorphisms,* *there exists a unique homomorphism* $\Phi\colon G \to H$ *with* $\phi_i = \Phi \circ \gamma_i$ *for all* $i \in I$*.*

$$\begin{tikzcd}
\vdots \\
G_i \arrow[dr, "\gamma_i"] \arrow[drrr, bend left, "\phi_i"] \\
\vdots & G \arrow[rr, dashed, "\Phi"] && H \\
G_j \arrow[ur, "\gamma_j"'] \arrow[urrr, bend right, "\phi_j"'] \\
\vdots
\end{tikzcd} \tag{1.13}$$

The group G *with these properties is unique up to isomorphism, and we will denote it as* $\star_{i\in I} G_i$*.*

Uniqueness follows immediately from (1.13).

Proof of Theorem 1.8.1; uniqueness. Assume we have $\langle G, (\gamma_i)_{i\in I}\rangle$ and $\langle G', (\gamma_i')_{i\in I}\rangle$ which both have the stated property. Then there exist Φ, Φ' with

$$\begin{tikzcd}
\vdots \\
G_i \arrow[dr, "\gamma_i"] \arrow[drrr, bend left, "\gamma_i'"] \\
\vdots & G \arrow[rr, dashed, "\Phi"] && G' \\
G_j \arrow[ur, "\gamma_j"'] \arrow[urrr, bend right, "\gamma_j'"'] \\
\vdots
\end{tikzcd}
\qquad
\begin{tikzcd}
\vdots \\
G_i \arrow[dr, "\gamma_i'"] \arrow[drrr, bend left, "\gamma_i"] \\
\vdots & G' \arrow[rr, dashed, "\Phi'"] && G \\
G_j \arrow[ur, "\gamma_j'"'] \arrow[urrr, bend right, "\gamma_j"'] \\
\vdots
\end{tikzcd}$$

From this, we obtain

$$\begin{tikzcd}
\vdots \\
G_i \arrow[dr, "\gamma_i"] \arrow[drrr, bend left, "\gamma_i"] \\
\vdots & G \arrow[rr, dashed, bend left, "\Phi'\circ\Phi"] \arrow[rr, dashed, bend right, "\mathrm{id}_G"'] && G \\
G_j \arrow[ur, "\gamma_j"'] \arrow[urrr, bend right, "\gamma_j"'] \\
\vdots
\end{tikzcd}
\qquad
\begin{tikzcd}
\vdots \\
G_i \arrow[dr, "\gamma_i'"] \arrow[drrr, bend left, "\gamma_i'"] \\
\vdots & G' \arrow[rr, dashed, bend left, "\Phi\circ\Phi'"] \arrow[rr, dashed, bend right, "\mathrm{id}_{G'}"'] && G' \\
G_j \arrow[ur, "\gamma_j'"'] \arrow[urrr, bend right, "\gamma_j'"'] \\
\vdots
\end{tikzcd}$$

and uniqueness implies $\Phi' \circ \Phi = \mathrm{id}_G$ and $\Phi \circ \Phi' = \mathrm{id}_{G'}$. ❑

The existence part of Theorem 1.8.1 relies on the construction of the monoid of words. Recall that a *word* w over a (nonempty) *alphabet* A is a finite tuple $a_1 a_2 \cdots a_n$ of elements of A. The elements a_i in this tuple are called the *letters* of the word w. The number n of letters of w is called the *length* of the word. We denote the set of all words over the alphabet A as A^*. Formally, thus,

$$A^* := \bigcup_{n\in\mathbb{N}} A^n.$$

A particular role is played by the *empty word*: this is the unique element of A^0,and we denote it as ε. The set A^* becomes a monoid, when endowed with the binary operation of *concatenation*: given $w = a_1 \cdots a_n$ and $v := b_1 \cdots b_m$, set

$$w \cdot v := a_1 \cdots a_n b_1 \cdots b_m.$$

Formally, concatenation is first defined as map

$$\cdot\colon \begin{cases} A^n \times A^m & \to & A^{n+m} \\ \big((a_1,\dots,a_n),(b_1,\dots,b_m)\big) & \mapsto & (a_1,\dots,a_n,b_1,\dots,b_m) \end{cases}$$

and then glued together to a map $\cdot\colon A^* \times A^* \to A^*$. Clearly, this operation is associative and the empty word is a unit element.

Proof of Theorem 1.8.1; existence. Denote by A the disjoint union of the sets G_i, $i \in I$. Formally, thus,

$$A := \bigcup_{i\in I}(G_i \times \{i\}).$$

Moreover, denote the multiplication in G_i as $\cdot_i\colon G_i \times G_i \to G_i$, and let e_i be the unit element of G_i.

① *We define an equivalence relation on A^*:* First, unit elements can be skipped. Set

$$a_1 \cdots a_n \sim_1 b_1 \cdots b_m :\Leftrightarrow$$
$$m = n-1 \ \wedge\ \exists k \in \{1,\dots,n\}, i \in I.\ a_k = e_i \ \wedge\ b_l = \begin{cases} a_l & \text{if } l < k \\ a_{l+1} & \text{if } l \geqslant k \end{cases}$$

Second, two neighbouring elements which belong to the same group can be multiplied out. Set

$$a_1 \cdots a_n \sim_2 := b_1 \cdots b_m :\Leftrightarrow$$
$$m = n-1 \ \wedge\ \exists k \in \{1,\dots,n-1\}, i \in I.\ a_k, a_{k+1} \in G_i \ \wedge\ b_l = \begin{cases} a_l & \text{if } l < k \\ a_k \cdot_i a_{k+1} & \text{if } l = k \\ a_{l+1} & \text{if } l > k \end{cases}$$

Now let $\Theta \subseteq A^* \times A^*$ be the smallest equivalence relation containing $\sim_1 \cup \sim_2$.

② *Θ is compatible with concatenation:* Inspecting the definitions, it is clear that

$$\forall w_1, w_2, v \in A^*.\ w_1 \sim_1 w_2 \ \Rightarrow\ w_1 \cdot v \sim_1 w_2 \cdot v,$$

$$\forall w_1, w_2, v \in A^*.\ w_1 \sim_2 w_2 \ \Rightarrow\ w_1 \cdot v \sim_2 w_2 \cdot v,$$

and the same for concatenation with v from the right.

We thus obtain a monoid A^*/Θ by representantwise definition

$$w/_\Theta \cdot v/_\Theta := (w \cdot v)/_\Theta \text{ for } w, v \in A^*.$$

The unit element of this monoid is $\varepsilon/_\Theta$.

③ *A^*/Θ is a group:* We have to prove existence of inverses. Let $w = a_1 \cdots a_n$ be given. Then (inverse elements a_k^{-1} are computed in the respective group to which the letter a_k belongs)

$$\begin{aligned}(a_1 \cdots a_n) \cdot (a_n^{-1} \cdots a_1^{-1}) &= a_1 \cdots a_n a_n^{-1} \cdots a_1^{-1} \sim_2 a_1 \cdots a_{n-1}(a_n a_n^{-1}) a_{n-1}^{-1} \cdots a_1^{-1} \\ &\sim_1 a_1 \cdots a_{n-1} a_{n-1}^{-1} \cdots a_1^{-1} \sim_2 \ldots \sim_1 a_1 a_1^{-1} \sim_1 \varepsilon\end{aligned}$$

and the same for the product in reversed order.

④ *Definition of γ_i:* For $i \in I$ denote by $\delta_i\colon G_i \to A^*$ the map assigning to an element the corresponding one-letter word. I.e., $\delta_i(a) := a$, where the "a" on the left side is an element of G_i and "a" on the right side is the one-letter word. Further, let $\pi\colon A^* \to A^*/\Theta$ be the canonical projection. Then we define

$$\gamma_i := \pi \circ \delta_i.$$

For each $i \in I$ and elements $a, b \in G_i$, we have $a \cdot_i b \sim_2 ab$, and hence

$$\gamma_i(a \cdot_i b) = \pi(ab) = \pi(a \cdot b) = \pi(a) \cdot \pi(b) = \gamma_i(a) \cdot \gamma_i(b),$$

i.e., γ_i is a homomorphism.

⑤ *We prove existence of Φ:* Denote by e the unit element of the group H. First we work on the level of words, and construct $\Phi_0\colon A^* \to H$. Let $w = a_1 \cdots a_n$ with $n \geqslant 1$ be given, and let $i_k \in I$ be the indices with $a_k \in G_{i_k}$. Then we set

$$\Phi_0(w) := \phi_{i_1}(a_1) \cdot \ldots \cdot \phi_{i_n}(a_n),$$

where multiplications takes place in the group H. Moreover, $\Phi_0(\varepsilon) := e$. Since multiplication in H is associative, Φ_0 is a homomorphism of monoids. It is built in the definition of Φ_0 that the diagram

$$\begin{tikzcd}
\vdots \\
G_i \arrow[dr, "\delta_i"] \arrow[drr, bend left, "\phi_i"] \\
\vdots & A^* \arrow[r, dashed, "\Phi_0"] & H \\
G_j \arrow[ur, "\delta_j"'] \arrow[urr, bend right, "\phi_j"'] \\
\vdots
\end{tikzcd}$$

commutes.

If $w \sim_1 v$, and k is as in the definition of $\sim_1$, then $\phi_{i_k}(a_{i_k}) = e$, and hence $\Phi_0(w) = \Phi_0(v)$. If $w \sim_2 v$, and k is as in the definition of $\sim_2$, then $i_k = i_{k+1}$ and $\phi_{i_k}(a_k) \cdot \phi_{i_{k+1}}(a_{k+1}) = \phi_{i_k}(a_k a_{k+1})$. Hence, also in this case $\Phi_0(w) = \Phi_0(v)$. Thus Φ_0 factors to a homomorphism $\Phi\colon A^*/_\Theta \to H$, and we have

$$\begin{tikzcd}
\vdots \\
G_i \arrow[dr, "\delta_i"] \arrow[drrr, bend left, "\phi_i"] \\
\vdots & A^* \arrow[r, "\pi"] \arrow[rr, bend right, "\Phi_0"'] & A^*/_\Theta \arrow[r, dashed, "\Phi"] & H \\
G_j \arrow[ur, "\delta_j"'] \arrow[urrr, bend right, "\phi_j"'] \\
\vdots
\end{tikzcd}$$

⑥ *We prove uniqueness of Φ:* Assume we have some homomorphism $\Phi'\colon G \to H$ which makes (1.13) commute. Every word $w = a_1 \cdots a_n$ is the concatenation of its letters, i.e., $w = a_1 \cdot \ldots \cdot a_n$. Using (1.13), we obtain

$$\Phi'(w/_\Theta) = \Phi'(a_1/_\Theta) \cdot \ldots \cdot \Phi'(a_n/_\Theta) = \phi_1(a_1) \cdot \ldots \cdot \phi_n(a_n) = \Phi(w/_\Theta).$$

❑

Lemma 1.8.2. *Let G_i, $i \in I$, be a family of groups.*

(i) *The free product $\star_{i\in I} G_i$ is generated by $\bigcup_{i\in I} \gamma_i(G_i)$.*

(ii) *All maps γ_i are injective. Let us deduce this from the universal property (1.13).*

(iii) *Denote by e the unit element of $\star_{i\in I} G_i$. Then*

$$\forall i, j \in I, i \neq j.\ \gamma_i(G_i) \cap \gamma_j(G_j) = \{e\} \tag{1.14}$$

Proof. Item (i) is clear from the construction of G as a factor of the monoid of words, since each word is the product of its letters.

We come to the proof of (ii). Fix $i \in I$ and for all $j \neq i$ let $1\colon G_j \to G_i$ be the constant homomorphism mapping everything to the unit element. Then we find Φ with

$$\begin{tikzcd}
\vdots & & \\
G_i \arrow[dr, "\gamma_i"] \arrow[drr, bend left, "\mathrm{id}_{G_i}"] & & \\
\vdots & \star_{i\in I} G_i \arrow[r, dashed, "\Phi"] & G_i \\
G_j \arrow[ur, "\gamma_j"'] \arrow[urr, bend right, "1"'] & & \\
\vdots & &
\end{tikzcd}$$

To see (iii), we use that the groups G_i can be embedded into their direct product. Let $\phi_j\colon G_j \to \prod_{i\in I} G_i$ be the map (e_i is the unit element of G_i)

$$\phi_j(a) := (b_i)_{i\in I} \text{ with } b_i := \begin{cases} a & \text{if } i = j \\ e_i & \text{otherwise} \end{cases}$$

Then ϕ_j is injective and a homomorphism. Let $\Phi\colon \star_{i\in I} G_i \to \prod_{i\in I} G_i$ be the homomorphism with $\Phi \circ \gamma_j = \phi_j$ for all $j \in I$. Since

$$\forall i, j \in I, i \neq j.\ \phi_i(G_i) \cap \phi_j(G_j) = \{(e_i)_{i\in I}\},$$

injectivity of the maps ϕ_i implies (1.14). ❑

The choice of the name "free product" is justified by the following fact. Recall here that a group G is called *free with basis B*, if $B \subset G$ and for every group H and map $\phi\colon B \to H$ there exists a unique homomorphism $\Phi\colon G \to H$ with $\Phi|_B = \phi$. The elements of B are also called the *generators* of G. For example, the free group with one generator is just $\mathbb{Z}$ (with basis $\{1\}$).

Lemma 1.8.3. *Let G_i, $i \in I$, be a family of groups, and assume that G_i is free with basis B_i. Then $\star_{i\in I} G_i$ is free with basis $\bigcup_{i\in I} B_i$. Note here that this is a disjoint union.*

In particular, we can write every free group (with some basis B) as the free product of $|B|$ copies of $\mathbb{Z}$.

Proof. Set $B := \bigcup_{i\in I} \gamma_i(B_i)$. Let H be a group and $\phi\colon B \to H$.

We show that ϕ has an extension to a homomorphism. Consider the maps $\phi \circ (\gamma_i|_{B_i})\colon B_i \to H$, let $\phi_i\colon G_i \to H$ be the homomorphism with $\phi_i|_{B_i} = \phi \circ (\gamma_i|_{B_i})$, and let $\Phi\colon \star_{i\in I} \to H$ be the homomorphism with $\Phi \circ \gamma_i = \phi_i$. Then $\Phi|_B = \phi$.

We show uniqueness. Assume that $\Phi'\colon \star_{i\in I} \to H$ is any homomorphism with $\Phi'|_B = \phi$. The map $\Phi' \circ \gamma_i\colon G_i \to H$ is a homomorphism, and satisfies $(\Phi' \circ \gamma_i)|_{B_i} = \phi \circ (\gamma_i|_{B_i})$. Hence, we must have $\Phi' \circ \gamma_i = \phi_i$, and in turn $\Phi' = \Phi$. ❑

1.9 Colimits of groups

Having available the free product of a family of groups G_i, we can make a more general construction where relations between the groups G_i are permitted. To explain this in a structured way, some vocabulary is needed. We deal with diagrams of groups of a very particular form (and not with the general situation where arbitrary diagrams are permitted).

Definition 1.9.1. A *diagram of groups* is a triple $\langle I, G_J, \iota_{J,j}\rangle$ where

▻ I is a nonempty set,

▻ G_J are groups where J runs through all subsets of I with $1 \leqslant |J| \leqslant 2$,

▻ $\iota_{J,j}\colon G_J \to G_{\{j\}}$ are homomorphisms where $J \subseteq I$ with $|J| \leqslant 2$ and $j \in J$.

$$\begin{tikzcd}
 & G_{\{i\}} \\
G_{\{i,j\}} \arrow[ur, dotted, "\iota_{\{i,j\},i}"] \arrow[dr, dotted, "\iota_{\{i,j\},j}"'] & \\
 & G_{\{j\}}
\end{tikzcd}$$

Definition 1.9.2. A *cone* over a diagram $\langle I, G_J, \iota_{J,j}\rangle$ is a tuple $\langle G, \gamma_i\rangle$ where

▻ G is a group,

▻ $\gamma_i\colon G_{\{i\}} \to G$ are homomorphisms where $i \in I$,

▻ $\forall i, j \in I.\ \gamma_i \circ \iota_{\{i,j\},i} = \gamma_j \circ \iota_{\{i,j\},j}$

$$\begin{tikzcd}
 & G_{\{i\}} \arrow[dr, "\gamma_i"] & \\
G_{\{i,j\}} \arrow[ur, dotted, "\iota_{\{i,j\},i}"] \arrow[dr, dotted, "\iota_{\{i,j\},j}"'] & & G \\
 & G_{\{j\}} \arrow[ur, "\gamma_j"'] &
\end{tikzcd}$$

Definition 1.9.3. A *colimit* of a diagram $\langle I, G_J, \iota_{J,j}\rangle$ is a cone $\langle G, \gamma_i\rangle$ with the following property:

▻ For every cone $\langle H, \phi_i\rangle$ over the diagram $\langle I, G_J, \iota_{J,j}\rangle$, there exists a unique homomorphism $\Phi\colon G \to H$ with $\phi_i = \Phi \circ \gamma_i$ for all $i \in I$.

$$\begin{tikzcd}
 & G_{\{i\}} \arrow[dr, "\gamma_i"] \arrow[drr, bend left, "\phi_i"] & & \\
G_{\{i,j\}} \arrow[ur, dotted, "\iota_{\{i,j\},i}"] \arrow[dr, dotted, "\iota_{\{i,j\},j}"'] & & G \arrow[r, dashed, "\Phi"] & H \\
 & G_{\{j\}} \arrow[ur, "\gamma_j"'] \arrow[urr, bend right, "\phi_j"'] & &
\end{tikzcd}$$

Let us give an example of a colimit.

Example 1.9.4. Let G be a group and N_i, $i \in I$, be normal subgroups of G. Denote by N the smallest normal subgroup of G which contains $\bigcup_{i\in I} N_i$. Moreover, let

$$p_i\colon G \to G/_{N_i}, \qquad \bar{p}_i\colon G/_{N_i} \to G/_N, \qquad p\colon G \to G/_N$$

be the canonical projections. We claim that $\langle G/_N, \bar{p}_i\rangle$ is a colimit of the diagram

$$\begin{tikzcd}
 & G/_{N_i} \\
G \arrow[ur, dotted, "p_i"] \arrow[dr, dotted, "p_j"'] & \vdots \\
 & G/_{N_j}
\end{tikzcd}$$

To establish this claim, consider an arbitrary cone $\langle H, \phi_i\rangle$ over this diagram. Then the map $\phi_i \circ p_i\colon G \to H$ does not depend on $i \in I$. Hence, its kernel contains all N_i and thus it factors through $G/_N$, i.e., we find a homomorphism $\Phi\colon G/_N \to H$ with $\phi_i \circ p_i = \Phi \circ p$.

$$\begin{tikzcd}
 & G/_{N_i} \arrow[dr, "\bar{p}_i"] \arrow[drr, bend left, "\phi_i"] & & \\
G \arrow[ur, dotted, "p_i"] \arrow[dr, dotted, "p_j"'] \arrow[rr, dashed, "p"] & & G/_N \arrow[r, dashed, "\Phi"] & H \\
 & G/_{N_j} \arrow[ur, "\bar{p}_j"'] \arrow[urr, bend right, "\phi_j"'] & &
\end{tikzcd}$$

Since p_i is surjective, the relation $\phi_i \circ p_i = \Phi \circ p = \Phi \circ \bar{p}_i \circ p_i$ implies that $\phi_i = \Phi \circ \bar{p}_i$.

Assume we have another homomorphism $\Phi'\colon G/N \to H$ with $\phi_i = \Phi' \circ \bar{p}_i$. Then

$$\Phi' \circ p = \Phi' \circ \bar{p}_i \circ p_i = \phi_i \circ p_i = \Phi \circ \bar{p}_i \circ p_i = \Phi \circ p,$$

and surjectivity of p implies that $\Phi' = \Phi$.

The following basic result says that colimits always exist and are essentially unique. This result is neither specific for groups nor for the particular form of the diagrams considered; it holds in a much more general context (which we do not touch upon).

Theorem 1.9.5.

(i) *Let $\langle I, G_J, \iota_{J,j}\rangle$ be a diagram of groups. Then there exists a colimit of $\langle I, G_J, \iota_{J,j}\rangle$.*

(ii) *Let $\langle I, G_J, \iota_{J,j}\rangle$ and $\langle I, G'_J, \iota'_{J,j}\rangle$ be two diagrams with the same index set I, let $\langle G, \gamma_i\rangle$ be a colimit of the first and $\langle G', \gamma'_i\rangle$ a colimit of the second.*

Assume that we have isomorphisms $\mu_J\colon G_J \to G'_J$ for all $J \subseteq I$, $1 \leqslant |J| \leqslant 2$, with

$$\forall J \subseteq I, |J| = 2\ \forall j \in J.\ \iota'_{J,j} \circ \mu_J = \mu_{\{j\}} \circ \iota_{J,j}$$

Then there exists an isomorphism $\mu\colon G \to G'$ with

$$\forall i \in I.\ \mu \circ \gamma_i = \gamma'_i \circ \mu_{\{i\}}$$

$$\begin{tikzcd}
& & G_{\{i\}} \arrow[r, "\gamma_i"] \arrow[dd, "\mu_{\{i\}}"] & G \arrow[dd, dashed, "\mu"] \\
G_{\{i,j\}} \arrow[urr, dotted, "\iota_{\{i,j\},i}"] \arrow[r, dotted, "\iota_{\{i,j\},j}"] \arrow[dd, "\mu_{\{i,j\}}"'] & G_{\{j\}} \arrow[urr, "\gamma_j"'] \arrow[dd, "\mu_{\{j\}}"] & & \\
& & G'_{\{i\}} \arrow[r, "\gamma'_i"] & G' \\
G'_{\{i,j\}} \arrow[urr, dotted, "\iota'_{\{i,j\},i}"] \arrow[r, dotted, "\iota'_{\{i,j\},j}"'] & G'_{\{j\}} \arrow[urr, "\gamma'_j"'] & &
\end{tikzcd}$$

Existence of colimits is a consequence of our knowledge about free products. To motivate this, let us explain that a free product is actually an example of a colimit.

Example 1.9.6. Let G_i, $i \in I$, be groups, and consider the diagram

$$\begin{tikzcd}
& G_i \\
\{1\} \arrow[ur, dotted] \arrow[dr, dotted] & \vdots \\
& G_j
\end{tikzcd} \tag{1.15}$$

where the dotted maps are the unique homomorphisms from the trivial group into G_i. Then every group G with arbitrary homomorphisms $\phi_i\colon G_i \to G$ forms a cone over this diagram. In particular, $\langle \star_{i\in I} G_i, \gamma_i\rangle$ is a cone over (1.15). The universal property of the free product ensures that it is actually a colimit.

Proof of Theorem 1.9.5, existence. Let a diagram $\langle I, G_J, \iota_{J,j}\rangle$ be given. In contrast to the above example, the free product will not form a cone over this diagram. We pass to a factor in order to enforce the required commutation relations. Set

$$L_0 := \Big\{ (\gamma_i \circ \iota_{\{i,j\},i})(a)(\gamma_j \circ \iota_{\{i,j\},j})(a)^{-1} \,\big|\, i,j \in I, i \neq j, a \in G_{\{i,j\}} \Big\},$$

and let N be the smallest normal subgroup of $\star_{i\in I} G_i$ which contains L_0. Moreover, denote by $p\colon G \to G/_N$ the canonical projection. The definition of L_0 is made such that

$$(p \circ \gamma_i) \circ \iota_{\{i,j\},i} = (p \circ \gamma_j) \circ \iota_{\{i,j\},j},$$

and thus $\langle \big(\star_{i\in I} G_i\big)/_N, p\circ\gamma_i\rangle$ is a cone over $\langle I, G_J, \iota_{J,j}\rangle$.

Let $\langle H, \phi_i\rangle$ be some cone over $\langle I, G_J, \iota_{J,j}\rangle$. By the universal property of the free product we find $\Phi\colon \star_{i\in I} G_i \to H$ with $\phi_i = \Phi \circ \gamma_i$.

$$\begin{tikzcd}
& G_{\{i\}} \arrow[dr, dashed, "\gamma_i"] \arrow[drr, bend left, "p\circ\gamma_i"] \arrow[drrr, bend left, "\phi_i"] & & & \\
G_{\{i,j\}} \arrow[ur, dotted, "\iota_{\{i,j\},i}"] \arrow[dr, dotted, "\iota_{\{i,j\},j}"] & & \star_{i\in I} G_i \arrow[r, dashed, "p"] \arrow[rr, dashed, bend right, "\Phi"'] & \big(\star_{i\in I} G_i\big)/_N & H \\
& G_{\{j\}} \arrow[ur, dashed, "\gamma_j"] \arrow[urr, bend right, "p\circ\gamma_j"'] \arrow[urrr, bend right, "\phi_j"'] & & &
\end{tikzcd}$$

We have

$$\Phi \circ \gamma_i \circ \iota_{\{i,j\},i} = \phi_i \circ \iota_{\{i,j\},i} = \phi_j \circ \iota_{\{i,j\},j} = \Phi \circ \gamma_j \circ \iota_{\{i,j\},j},$$

and hence $L_0 \subseteq \ker \Phi$. Thus Φ factors, and we find $\Phi' \colon \left(\star_{i\in I} G_i\right)\big/_N \to H$ with $\Phi = \Phi' \circ p$. Using this, it follows that

$$\Phi' \circ p \circ \gamma_i = \Phi \circ \gamma_i = \phi_i,$$

and we see that Φ' is a required fill-in. To show uniqueness, assume we have Φ'' which also satisfies $\Phi'' \circ p \circ \gamma_i = \phi_i$. Since the union $\bigcup_{i\in I} \gamma_i(G_i)$ generates the free product, it follows that $\Phi'' \circ p = \Phi' \circ p$. Now surjectivity of p implies that $\Phi'' = \Phi'$.

We conclude that

$$\left\langle \left(\star_{i\in I} G_i\right)\big/_N, p \circ \gamma_i \right\rangle$$

is a colimit of the diagram $\langle I, G_J, \iota_{J,j} \rangle$. ❑

Uniqueness of colimits is a simple argument using the defining universal property of a colimit.

Proof of Theorem 1.9.5, uniqueness.

① We have

$$\gamma_i' \circ \mu_{\{i\}} \circ \iota_{\{i,j\},i} = \gamma_i' \circ \iota'_{\{i,j\},i} \circ \mu_{\{i,j\}} = \gamma_j' \circ \iota'_{\{i,j\},j} \circ \mu_{\{i,j\}} = \gamma_j' \circ \mu_{\{j\}} \circ \iota_{\{i,j\},j},$$

hence $\langle G', \gamma_i' \circ \mu_{\{i\}} \rangle$ is a cone over the diagram $\langle I, G_J, \iota_{J,j} \rangle$. Thus there exists a fill-in $\mu \colon G \to G'$ with $\gamma_i' \circ \mu_{\{i\}} = \mu \circ \gamma_i$.

Changing the roles of primed and unprimed groups and maps provides us with a fill-in $\nu \colon G' \to G$ satisfying $\gamma_i \circ \mu_{\{i\}}^{-1} = \nu \circ \gamma_i'$.

② Trivially, $\langle G, \gamma_i \rangle$ is a cone over $\langle I, G_J, \iota_{J,j} \rangle$. The corresponding fill-in $G \to G$ obviously is id_G. We have

$$\nu \circ \mu \circ \gamma_i = \nu \circ \gamma_i' \circ \mu_{\{i\}} = \gamma_i \circ \mu_{\{i\}}^{-1} \circ \mu_{\{i\}} = \gamma_i,$$

and uniqueness of the fill-in implies that $\nu \circ \mu = \mathrm{id}_G$.

Changing the roles of primed and unprimed groups and maps yields $\mu \circ \nu = \mathrm{id}_{G'}$.

❑

Chapter 2

Compactifications

Compact spaces, in particular compact Hausdorff spaces, are a very well-behaved class of topological spaces. While products of compact spaces are again compact by Tychonoff's Theorem, subspaces of compact spaces are in general not anymore compact. If a topological space is (homeomorphic to) a subspace of a compact space, or even a compact Hausdorff space, this outer structure can be useful for many purposes.

2.1 The notion of compactification

Speaking informally, a compactification of a topological space X is a compact space which contains X as a subspace and is not superficially large. Naturally, we thereby think up to homeomorphisms.

Definition 2.1.1. Let $\langle X, \mathcal{T}\rangle$ be a topological space.

(i) A *compactification* of $\langle X, \mathcal{T}\rangle$ is a triple $\langle Y, \mathcal{V}, \iota\rangle$ where $\langle Y, \mathcal{V}\rangle$ is a compact topological space and ι is an embedding of $\langle X, \mathcal{T}\rangle$ onto a dense subspace of $\langle Y, \mathcal{V}\rangle$.

(ii) A (T_2)-*compactification* of $\langle X, \mathcal{T}\rangle$ is a compactification $\langle Y, \mathcal{V}, \iota\rangle$ of $\langle X, \mathcal{T}\rangle$ with $\langle Y, \mathcal{V}\rangle$ being Hausdorff.

Remark 2.1.2. The requirement that $\iota(X)$ is dense can always be achieved. Assume $\langle X, \mathcal{T}\rangle$ is a topological space, $\langle Y, \mathcal{V}\rangle$ is a compact topological space, and $\iota\colon X \to Y$ is an embedding. Then we set $Y' := \overline{Y}$, endow Y' with the subspace topology of Y, and let $\iota'\colon X \to Y'$ be the corestriction of ι. Then Y' is compact, ι' is an embedding, and the image of X under ι' is dense in Y'.

Thus, if we have some embedding of $\langle X, \mathcal{T}\rangle$ into a compact space $\langle Y, \mathcal{V}\rangle$, we automatically obtain a compactification. Clearly, if $\langle Y, \mathcal{V}\rangle$ is (T_2), then this will be a (T_2)-compactification.

On the collection of all compactifications of a fixed topological space, we have a natural notion of morphisms. Namely, again speaking informally, continuous maps which leave X pointwise fixed.

Definition 2.1.3. Let $\langle X, \mathcal{T}\rangle$ be a topological space, and let $\langle Y, \mathcal{V}, \iota_Y\rangle$ and $\langle Z, \mathcal{W}, \iota_Z\rangle$ be compactifications of $\langle X, \mathcal{T}\rangle$. A *morphism*

$$\langle Y, \mathcal{V}, \iota_Y\rangle \xrightarrow{\phi} \langle Z, \mathcal{W}, \iota_Z\rangle$$

is a map $\phi\colon Y \to Z$ which is $\mathcal{V}$-to-$\mathcal{W}$–continuous and satisfies $\phi \circ \iota_Y = \iota_Z$:

$$\begin{tikzcd} & X \arrow[dl, "\iota_Y"'] \arrow[dr, "\iota_Z"] & \\ Y \arrow[rr, "\phi"] & & Z \end{tikzcd}$$

Observe that the composition of morphisms is again a morphism, and that for every compactification $\langle Y, \mathcal{V}, \iota\rangle$ the identity map id_Y is a morphism from $\langle Y, \mathcal{V}, \iota\rangle$ to itself. Thus we naturally have the notion of isomorphism: a morphism $\phi\colon Y \to Z$ is an isomorphism, if there exists a morphism in the reverse direction $\psi\colon Z \to Y$, such that $\phi \circ \psi = \mathrm{id}_Z$ and $\psi \circ \phi = \mathrm{id}_Y$.

Spelled out concretely, an *isomorphism* between two compactifications $\langle Y, \mathcal{V}, \iota_Y\rangle$ and $\langle Z, \mathcal{W}, \iota_Z\rangle$ of a topological space $\langle X, \mathcal{T}\rangle$, is a $\mathcal{V}$-to-$\mathcal{W}$–homeomorphism $\phi\colon Y \to Z$ with $\phi \circ \iota_Y = \iota_Z$.

The central question which poses itself is:

> What can one say about the "structure" of the collection of all compactifications (or (T_2)-compactifications) of a given topological space $\langle X, \mathcal{T}\rangle$?

Some more concrete instances of this vague question could be: does there exist a compactification, is it unique, is there a largest or smallest one, etc. Of course, in all these questions we think up to isomorphism.

We can easily answer the question for existence of a compactification affirmatively. This is a corollary of Theorem 1.2.2.

Corollary 2.1.4. *Every topological space has a compactification.*

Proof. Let $\langle X, \mathcal{T}\rangle$ be given, and consider its one-point extension $\langle \alpha(X), \mathcal{T}_\alpha\rangle$. This is a compact space, and the inclusion map ι_α is an embedding. Thus, $\langle X, \mathcal{T}\rangle$ has the compactification $\langle \overline{\iota_\alpha(X)}, \mathcal{T}_\alpha|_{\iota_\alpha(X)}, \iota_\alpha\rangle$. ❑

In this context it is worth to observe that

$$\iota_\alpha(X) \text{ not dense} \;\Leftrightarrow\; \iota_\alpha(X) \text{ closed} \;\Leftrightarrow\; \{\infty\} \text{ open in } \alpha(X) \;\Leftrightarrow\; \langle X, \mathcal{T}\rangle \text{ is compact}$$

The first two equivalences follow since $\alpha(X) \backslash \iota_\alpha(X) = \{\infty\}$, and the last equivalence follows from the definition of $\mathcal{T}_\alpha$.

Definition 2.1.5. Let $\langle X, \mathcal{T}\rangle$ be a non-compact topological space. Then $\langle \alpha(X), \mathcal{T}_\alpha, \iota_\alpha\rangle$ is called the *one-point compactification* or *Alexandroff compactification* of $\langle X, \mathcal{T}\rangle$.

Remark 2.1.6. If $\langle X, \mathcal{T}\rangle$ and $\langle Y, \mathcal{V}\rangle$ are homeomorphic non-compact topological spaces, then their one-point compactifications are isomorphic.

This holds since a homeomorphism $\phi\colon X \to Y$ certainly satisfies (1.5) and thus lifts to a homeomorphism $\alpha(\phi)\colon \alpha(X) \to \alpha(Y)$. That $\alpha(\phi)$ satisfies $\alpha(\phi)\circ\iota_X = \iota_Y$ holds by definition.

Let us now explain in two different situations that there is no hope for uniqueness. There even is the counterintuitive curiosity that a compact space has many non-isomorphic compactifications. Note here that a compact space $\langle X, \mathcal{T}\rangle$ obviously can be considered as its own compactification: $\langle X, \mathcal{T}, \mathrm{id}_X\rangle$.

Example 2.1.7. Let X be a nonempty set, and consider the indiscrete topology $\mathcal{T} := \{\varnothing, X\}$ on X. Then $\langle X, \mathcal{T}\rangle$ is compact. Let Y be a set whose cardinality is larger or equal than the cardinality of X, and choose an injective map $\iota\colon X \to Y$. We endow Y also with the indiscrete topology $\mathcal{V} := \{\varnothing, Y\}$. The subspace topology $\mathcal{V}|_{\iota(X)}$ is the indiscrete topology on $\iota(X)$, and hence ι is a homeomorphism of X onto $\iota(X)$. Since $\iota(X) \neq \varnothing$, it is dense in Y. It follows that $\langle Y, \mathcal{V}, \iota\rangle$ is a compactification of $\langle X, \mathcal{T}\rangle$.

The base sets of isomorphic compactifications must in particular have the same cardinality. We see that for each cardinality $\geqslant |X|$ there exists at least one compactification, and all these compactifications are non-isomorphic.

A short argument shows that this curiosity cannot occur under presence of the Hausdorff separation axiom.

Lemma 2.1.8. *If $\langle X, \mathcal{T}\rangle$ is compact and Hausdorff, then $\langle X, \mathcal{T}, \mathrm{id}_X\rangle$ is, up to isomorphism, the only* (T_2)*-compactification of $\langle X, \mathcal{T}\rangle$.*

Proof. Assume we have a (T_2)-compactification $\langle Y, \mathcal{V}, \iota_Y\rangle$ of a compact Hausdorff space $\langle X, \mathcal{T}\rangle$. Then $\iota(X)$ is compact in $\langle Y, \mathcal{V}\rangle$, and hence closed. However, it is also dense, and therefore $\iota(X) = Y$. As a bijective and continuous map between two compact Hausdorff spaces, ι is a homeomorphism. The validity of $\iota\circ\mathrm{id}_X = \iota$ is trivial. ❑

The major interest of course is to investigate compactifications of spaces which are not already compact themselves. And in the non-compact setting, uniqueness fails even for very well-behaved spaces.

Example 2.1.9. Consider the real numbers $\mathbb{R}$ endowed with the euclidean topology $\mathcal{E}$. We present two different compactifications of $\langle \mathbb{R}, \mathcal{E}\rangle$.

① Denote by $\mathbb{S}^1$ the unit circle in the plane, and let $\mathbb{S}^1$ be endowed with the subspace topology $\mathcal{V}_1$ of the euclidean topology on $\mathbb{R}^2 \cong \mathbb{C}$. Moreover, let $\iota_1 : \mathbb{R} \to \mathbb{S}^1$ be defined by the formula

$$\iota_1(t) := \exp\big(2i\arctan t\big) \text{ for } t \in \mathbb{R},$$

where arctan denotes the branch with values in $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Then ι_1 is a bijection of $\mathbb{R}$ onto $\mathbb{S}^1\backslash\{-1\}$, is continuous, and maps open intervals to open arcs. Hence, ι_1 it is a homeomorphism of $\mathbb{R}$ onto $\mathbb{S}^1\backslash\{-1\}$. Clearly, $\mathbb{S}^1$ is a compact Hausdorff space, and $\mathbb{S}^1\backslash\{-1\}$ is dense in $\mathbb{S}^1$. Hence, $\langle \mathbb{S}^1, \mathcal{V}_1, \iota_1\rangle$ is a (T_2)-compactification of $\langle \mathbb{R}, \mathcal{E}\rangle$.

② The second example is quite similar to the first. Denote by $\mathbb{C}_r$ the open right half-plane, consider the half-circle $\mathbb{S}^1 \cap \overline{\mathbb{C}_r}$, and let $\mathcal{V}_2$ be the subspace topology on $\mathbb{S}^1 \cap \overline{\mathbb{C}_r}$ of the euclidean topology in the plane. Moreover, let $\iota_2 \colon \mathbb{R} \to \mathbb{S}^1 \cap \overline{\mathbb{C}_r}$ be defined by the formula

$$\iota_2(t) := \exp\big(i \arctan t\big) \text{ for } t \in \mathbb{R}.$$

For the same reasons as above, ι_2 is a homeomorphism of $\mathbb{R}$ onto $\mathbb{S}^1 \cap \mathbb{C}_r$, and it follows that $\langle \mathbb{S}^1 \cap \overline{\mathbb{C}_r}, \mathcal{V}_2, \iota_2 \rangle$ is a (T_2)-compactification of $\mathbb{R}$.

Since $|\mathbb{S}^1 \backslash \iota_1(\mathbb{R})| = 1$ and $|(\mathbb{S}^1 \cap \overline{\mathbb{C}_r}) \backslash \iota_2(\mathbb{R})| = 2$, the compactifications in ① and ② cannot be isomorphic. And, as one may expect, $\mathbb{S}^1 \cap \overline{\mathbb{C}_r}$ is larger than $\mathbb{S}^1$ in the sense that there exists a surjective morphism ϕ from $\langle \mathbb{S}^1 \cap \overline{\mathbb{C}_r}, \mathcal{V}_2, \iota_2 \rangle$ onto $\langle \mathbb{S}^1, \mathcal{V}_1, \iota_1 \rangle$: namely, define $\phi \colon \mathbb{S}^1 \cap \overline{\mathbb{C}_r} \to \mathbb{S}^1$ as

$$\phi(z) := \begin{cases} \iota_1 \circ \iota_2^{-1}(z) & \text{if } z \in \iota_2(\mathbb{R}), \\ -1 & \text{if } z \in \{i, -i\}. \end{cases}$$

If $(z_n)_{n \in \mathbb{N}}$ is a sequence in Y_2 which converges to i or $-i$, then $\phi(z_n)$ converges to -1, hence ϕ is continuous. The property that $\phi \circ \iota_2 = \iota_1$ is built in the definition.

The construction in these examples is ad-hoc and depends on an auxiliary structure. A simple, yet extremely important, way to construct compactifications intrinsically is by using separating families of maps.

Lemma 2.1.10. *Let $\langle X, \mathcal{T} \rangle$ be a topological space, and let $f_i \colon X \to Y_i$, $i \in I$, be a separating family of maps into compact spaces $\langle Y_i, \mathcal{V}_i \rangle$, $i \in I$. Set $Y := \overline{(\prod_{i \in I} f_i)(X)}$, where the closure is understood w.r.t. the product topology $\prod_{i \in I} \mathcal{V}_i$, let $\mathcal{V}$ be the subspace topology on Y of $\prod_{i \in I} \mathcal{V}_i$, and let $\iota \colon X \to Y$ be the corestriction of $\prod_{i \in I} f_i$. Then $\langle Y, \mathcal{V}, \iota \rangle$ is a compactification of $\langle X, \mathcal{T} \rangle$.*

Proof. By Proposition 1.1.5 (i), ι is an embedding, and by the definition of Y, $\iota(X)$ is dense in Y. Tychonoff's Theorem ensures that $\langle Y, \mathcal{V} \rangle$ is compact. ❑

Let us revisit Example 2.1.9 and show that both compactifications constructed there could also be obtained by means of Lemma 2.1.10.

Example 2.1.11.

① Let $h \colon \mathbb{R} \to [0, 1]$ be the piecewise linear and continuous function defined as

$$h(x) := \max\{1 - |x|, 0\} \text{ for } x \in \mathbb{R},$$

and for $n \in \mathbb{N}$ and $q \in \mathbb{Q}$ let $h_{n,q}$ be the rescaled and shifted function

$$h_{n,q}(x) := h\big(n(x - q)\big) \text{ for } x \in \mathbb{R}.$$

These functions satisfy

$$h_{n,q}(x) \begin{cases} = 0 & \text{if } |x - q| \geqslant \frac{1}{n}, \\ \in \big[\frac{1}{2}, 1\big] & \text{if } |x - q| \leqslant \frac{1}{2n}, \end{cases}$$

and hence $\{h_{n,q} \mid n \in \mathbb{N}, q \in \mathbb{Q}\}$ is a separating family.

We extend the functions $h_{n,q}$ to $\mathbb{S}^1$ by setting

$$\tilde{h}_{n,q}(z) := \begin{cases} (h_{n,q} \circ \iota_1^{-1})(z) & \text{if } z \in \mathbb{S}^1 \backslash \{-1\}, \\ 0 & \text{if } z = -1. \end{cases}$$

This definition ensures that $\tilde{h}_{n,q} \circ \iota_1 = h_{n,q}$. The function $\tilde{h}_{n,q}$ is continuous, since its restrictions to $\mathbb{S}^1 \backslash \{-1\}$ and $\mathbb{S}^1 \backslash \iota_1(\operatorname{supp} h_{n,q})$ are both continuous, and these two sets are open and cover $\mathbb{S}^1$. Moreover, the family $\{\tilde{h}_{n,q} \,|\, n \in \mathbb{N}, q \in \mathbb{Q}\}$ is point separating. By Proposition 1.1.5 (ii), the product map $\tilde{\phi} := \prod_{n \in \mathbb{N}, q \in \mathbb{Q}} \tilde{h}_{n,q}$ is an embedding. We also see that $\tilde{\phi}(\mathbb{S}^1)$ is a compact, and hence closed, subset of $\prod_{n \in \mathbb{N}, q \in \mathbb{Q}} [0,1]$.

Let $\langle Y, \mathcal{V}, \iota \rangle$ be the compactficiation of $\mathbb{R}$ obtained from the separating family $\{h_{n,q} \,|\, n \in \mathbb{N}, q \in \mathbb{Q}\}$ by means of Lemma 2.1.10, i.e., ι is the product map $\iota = \prod_{n \in \mathbb{N}, q \in \mathbb{Q}} h_{n,q}$ and Y is the closure of $\iota(\mathbb{R})$. It follows that

$$Y = \overline{\iota(\mathbb{R})}^{\prod[0,1]} = \overline{\tilde{\phi}(\iota_1(\mathbb{R}))}^{\prod[0,1]} = \overline{\tilde{\phi}(\iota_1(\mathbb{R}))}^{\tilde{\phi}(\mathbb{S}^1)} = \tilde{\phi}\big(\overline{\iota_1(\mathbb{R})}\big) = \tilde{\phi}(\mathbb{S}^1).$$

Putting together, $\tilde{\phi}$ is a homeomorphism of $\mathbb{S}^1$ onto Y, and $\tilde{\phi} \circ \iota_1 = \iota$. This means that it is an isomorphism from $\langle \mathbb{S}^1, \mathcal{V}_1, \iota_1 \rangle$ to $\langle Y, \mathcal{V}, \iota \rangle$.

② We proceed in exactly the same way, only adding one function which distinguishes "left from right": set

$$h(x) := \max\big\{0, \min\{x, 1\}\big\} \text{ for } x \in \mathbb{R}.$$

The family $\{h\} \cup \{h_{n,q} \,|\, n \in \mathbb{N}, q \in \mathbb{Q}\}$ is separating.

The functions $h_{n,q}$ and h can be extended to continuous functions of $\mathbb{S}^1 \cap \overline{\mathbb{C}_r}$ into $[0,1]$ by means of

$$\hat{h}(z) := \begin{cases} (h \circ \iota_2^{-1})(z) & \text{if } z \in \mathbb{S}^1 \cap \mathbb{C}_r, \\ 1 & \text{if } z = i, \\ 0 & \text{if } z = -i, \end{cases} \qquad \hat{h}_{n,q}(z) := \begin{cases} (h_{n,q} \circ \iota_2^{-1})(z) & \text{if } z \in \mathbb{S}^1 \cap \mathbb{C}_r, \\ 0 & \text{if } z \in \{i, -i\}, \end{cases}$$

so that $\hat{h} \circ \iota_2 = h$ and $\hat{h}_{n,q} \circ \iota_2 = \iota$. The product map $\hat{\phi} := \hat{h} \times \prod_{n \in \mathbb{N}, q \in \mathbb{Q}} \hat{h}_{n,q}$ is an embedding of $\mathbb{R}$ into $[0,1] \times \prod_{n \in \mathbb{N}, q \in \mathbb{Q}} [0,1]$, and we obtain in the same way as above that it yields an isomorphism from $\langle \mathbb{S}^1 \cap \overline{\mathbb{C}_r}, \mathcal{V}_2, \iota_2 \rangle$ to the compactification $\langle \hat{Y}, \hat{\mathcal{V}}, \hat{\iota} \rangle$ obtained from the separating family $\{h\} \cup \{h_{n,q} \,|\, n \in \mathbb{N}, q \in \mathbb{Q}\}$.

From the presently elaborated point of view, it is also clear that $\langle \hat{Y}, \hat{\mathcal{V}}, \hat{\iota} \rangle$ can be mapped onto $\langle Y, \mathcal{V}, \iota \rangle$ with a surjective morphism. The first of these two compactifications is constructed from a larger family of continuous functions, and we can just use the projection of $[0,1] \times \prod_{(n,q) \in \mathbb{N} \times \mathbb{Q}} [0,1]$ onto the factor $\prod_{(n,q) \in \mathbb{N} \times \mathbb{Q}} [0,1]$.

2.2 Two examples

We have already seen one very special compactification of a topological space $\langle X, \mathcal{T} \rangle$, namely its Alexandroff compactification $\langle \alpha(X), \mathcal{T}_\alpha, \iota_\alpha \rangle$. If $\langle X, \mathcal{T} \rangle$ is completely regular, we obtain from Tychonoff's embedding theorem (Theorem 1.3.7) another very special compactification. Namely, if

$$\iota := \prod_{f \in C(X,[0,1])} f\colon X \to \prod_{f \in C(X,[0,1])} [0,1]$$

is the embedding constructed in Tychonoff's theorem, we set $\beta(X) := \overline{\iota(X)}$, let $\mathcal{T}_\beta$ be the subspace topology of the product topology, and ι_β the corestriction of ι. Recall here Remark 2.1.2, and that the product is compact by Tychonoff's product theorem. This compactification is called the *Stone-Čech compactification* of $\langle X, \mathcal{T}\rangle$.

In this section we give two quick applications, one of them using the Stone-Čech compactification and the other using the Alexandroff compactification.

The first is a description of the dual space of ℓ^∞. Recall that

$$\ell^\infty := \big\{(a_n)_{n\in\mathbb{N}} \mid \sup_{n\in\mathbb{N}} |a_n| < \infty\big\}, \qquad \|(a_n)_{n\in\mathbb{N}}\|_\infty := \sup_{n\in\mathbb{N}} |a_n| \text{ for } (a_n)_{n\in\mathbb{N}} \in \ell^\infty,$$

is a complete normed space, and that its *dual space* $(\ell^\infty)'$ is the space of all continuous linear functions of ℓ^∞ into $\mathbb{C}$.

Here we will use, without giving its proof, the *Riesz-Markov-Kakutani representation theorem*.

Theorem 2.2.1 (Riesz-Markov representation theorem)**.** *Let $\langle X, \mathcal{T}\rangle$ be a compact Hausdorff space. Denote by $\langle C(X), \|\cdot\|_\infty\rangle$ the Banach space of all complex-valued continuous functions on X endowed with the* supremum norm*, i.e.,*

$$C(X) := \big\{f\colon X \to \mathbb{C} \mid f \textit{ continuous}\big\}, \qquad \|f\|_\infty := \sup_{x\in X} |f(x)| \textit{ for } f \in C(X),$$

and by $\langle \mathscr{M}(X), \|\cdot\|\rangle$ the Banach space of all regular Borel measures on X endowed with the total variation norm*, i.e.,*

$$\mathscr{M}(X) := \big\{\mu \mid \mu \textit{ regular Borel measure on } X\big\}, \qquad \|\mu\| := |\mu|(X) \textit{ for } \mu \in \mathscr{M}(X).$$

Then the formula

$$\mu \mapsto \Big(f \mapsto \int_X f \,\mathrm{d}\mu\Big) \textit{ for } \mu \in \mathscr{M}(X), f \in C(X),$$

establishes an isometric isomorphism of $\langle \mathscr{M}(X), \|\cdot\|\rangle$ onto the dual space $\langle C(X), \|\cdot\|_\infty\rangle'$.

Our aim is to prove the following fact.

Theorem 2.2.2. *The dual space $(\ell^\infty)'$ is isometrically isomorphic to the space $\mathscr{M}(\beta(\mathbb{N}))$ of all regular Borel measures on the Stone-Čech compactification $\beta(\mathbb{N})$ where $\mathbb{N}$ is endowed with the discrete topology.*

In the proof we use a general property of the Stone-Čech compactification, which we shall now explain. Denote by $C_b(X)$ the set of all complex-valued continuous and bounded functions on X. Clearly, $C_b(X)$ is a subalgebra of $C(X)$. When endowed with the supremum norm

$$\|f\|_\infty := \sup_{x\in X} |f(x)| \text{ for } f \in C_b(X),$$

it becomes a complete normed space.

Proposition 2.2.3. *Let $\langle X, \mathcal{T}\rangle$ be a completely regular space. Then the map*

$$\iota_\beta^*\colon \begin{cases} C(\beta(X)) & \to & C_b(X) \\ f & \mapsto & f \circ \iota_\beta \end{cases}$$

is an isometric and bijective algebra homomorphism.

Proof. The fact that ι_β^* is an algebra homomorphism is clear since the algebra operations are defined as pointwise sum and product. Moreover, clearly, $\|\iota_\beta^*(f)\|_\infty \leqslant \|f\|_\infty$ for all $f \in C(\beta(X))$. For the reverse inequality, note that $\iota_\beta(X)$ is dense in $\beta(X)$ and that the map $x \mapsto |f(x)|$ is continuous.

It remains to show that ι_β^* is surjective. To achieve this, we set, for $\gamma > 0$,

$$L_\gamma\colon \begin{cases} \mathbb{C} & \to & \mathbb{C} \\ \xi & \mapsto & \frac{1}{2\gamma}\xi + \frac{1}{2} \end{cases}$$

Then L_γ is invertible, in fact $L_\gamma^{-1}(\xi) = 2\gamma(\xi - \frac{1}{2})$.

Given a real-valued function $f \in C_b(X)$, define a map $\phi_f\colon \prod_{f\in C(X,[0,1])}[0,1] \to \mathbb{C}$ as

$$\phi_f := \begin{cases} \pi_f & \text{if } f(X) \subseteq [0,1] \\ L_{\|f\|_\infty}^{-1} \circ \pi_{L_{\|f\|_\infty}\circ f} & \text{if } f(X) \nsubseteq [0,1] \end{cases}$$

Clearly, ϕ_f is continuous. For all $f \in C_b(X)$ and $x \in X$, it holds that

$$\phi_f(\iota_\beta(x)) = \left\{ \begin{array}{ll} \pi_f(\iota_\beta(x)) & \text{if } f(X) \subseteq [0,1] \\ L_{\|f\|_\infty}^{-1}\big((L_{\|f\|_\infty} \circ f)(x)\big) & \text{if } f(X) \nsubseteq [0,1] \end{array} \right\} = f(x)$$

and this means that $\iota_\beta^*(\phi_f|_{\beta(X)}) = f$. Given $f \in C_b(X)$ arbitrary, decompose f into real- and imaginary parts. ❑

The asserted description of $(\ell^\infty)'$ is a corollary.

Proof of Theorem 2.2.2. We have $\ell^\infty = C_b(\mathbb{N})$. ❑

The second application is an extension of the *Stone-Weierstraß Theorem* to locally compact Hausdorff spaces instead of compact Hausdorff spaces. To illustrate the scope of this extension, note that $\mathbb{R}$ is locally compact but not compact.

For a non-compact topological space $\langle X, \mathcal{T}\rangle$, we denote by $C_0(X)$ the set of all real-valued continuous functions on X which have the limit 0 at infinity, i.e.,

$$\forall \epsilon > 0\ \exists K \subseteq X \text{ closed compact } \forall x \in X\backslash K.\ |f(x)| < \epsilon$$

Clearly, $C_0(X)$ is a subalgebra of $C_b(X)$, and is closed w.r.t. $\|\cdot\|_\infty$. We also see that $C_0(X)$ is a complete normed space when endowed with the supremum norm.

Theorem 2.2.4. *Let $\langle X, \mathcal{T}\rangle$ be a non-compact locally compact Hausdorff space, and let $\mathcal{A} \subseteq C_0(X)$. If $\mathcal{A}$ is a subalgebra which separates points and vanishes at no point of X, then $\mathcal{A}$ is dense in $C_0(X)$.*

Proof. Consider the one-point compactification $\langle \alpha(X), \mathcal{T}_\alpha, \iota \rangle$ of $\langle X, \mathcal{T} \rangle$, and set $\mathcal{B} := (\iota^*)^{-1}(\mathcal{A})$. Then $\mathcal{B}$ is a subalgebra of $C(\alpha(X))$, and all functions $g \in \mathcal{B}$ satisfy $g(\infty) = 0$. The linear span $\mathcal{C} := \operatorname{span}(\mathcal{B} \cup \{1\})$ is again a subalgebra of $C(\alpha(X))$, which clearly does not vanish at any point of $\alpha(X)$.

Let us show that $\mathcal{C}$ separates points of $\alpha(X)$. Let $x, y \in \alpha(X)$ with $x \neq y$ be given. If $x, y \in \iota(X)$, then we find $f \in \mathcal{A}$ with $f(\iota^{-1}(x)) \neq f(\iota^{-1}(y))$. Let $g \in C(\alpha(X))$ be the extension of f, i.e., $\iota^*(g) = f$. Then $g \in \mathcal{B} \subseteq \mathcal{C}$, and

$$g(x) = f(\iota^{-1}(x)) \neq f(\iota^{-1})(y) = g(y).$$

Assume now that $x \in \iota(X)$ and $y = \infty$. Since $\mathcal{A}$ does not vanish at any point of X, we find $f \in \mathcal{A}$ with $f(\iota^{-1}(x)) \neq 0$. Let again $g \in C(\alpha(X))$ be the extension of f. Then $g \in \mathcal{B} \subseteq \mathcal{C}$ and $g(x) \neq 0$ while $g(\infty) = 0$.

We conclude from the version of the Stone-Weierstraß Theorem for compact and Hausdorff spaces that $\mathcal{C}$ is dense in $C(\alpha(X))$. Let $f \in C_0(X)$ be given. Its extension $g \in C(\alpha(X))$ can be approximated by a sequence $(g_n)_{n\in\mathbb{N}}$ of functions $g_n \in \mathcal{C}$. Since $g(\infty) = 0$, also the sequence $(g_n - g_n(\infty) \cdot 1)_{n\in\mathbb{N}}$ converges to g. These functions belong to $\mathcal{C}$ and vanish at the point ∞, hence they belong to $\mathcal{B}$. The map $\iota^*|_{\{f \in C(\alpha(X)) \,|\, f(\infty)=0\}}$ is isometric w.r.t. the supremum norm on $C(\alpha(X))$ and $C_0(X)$, respectively. Hence, it follows that

$$\lim_{n\to\infty} \iota^*\big(g_n - g_n(\infty) \cdot 1\big) = f$$

w.r.t. $\|\cdot\|_\infty$. ❑

Remark 2.2.5. The corresponding variant of the Stone-Weierstraß Theorem for the algebra $C_0(X, \mathbb{C})$, where X is a locally compact Hausdorff space, follows similar as in the compact case: Given $\mathcal{A}$, consider the algebra

$$\mathcal{B} := \{f \in \mathcal{A} \,|\, f(X) \subseteq \mathbb{R}\},$$

Then $\mathcal{B}$ is dense in $C_0(X, \mathbb{R})$.

2.3 Structure of (T_2)-compactifications

In essence we have already established for which spaces (T_2)-compactifications exist, namely by means of Lemma 2.1.10 (or Theorem 1.3.7) and Proposition 1.3.6.

Proposition 2.3.1. *A topological space $\langle X, \mathcal{T} \rangle$ has a (T_2)-compactification, if and only if it is completely regular.*

Proof. If $\langle X, \mathcal{T} \rangle$ is completely regular, we can separate points from closed sets and from points (since points are closed) with continuous functions into $[0, 1]$. This shows that the family

$$C(X, [0,1]) := \big\{f : X \to [0,1] \,|\, f \text{ continuous}\big\}$$

is separating. Now Lemma 2.1.10 provides a (T_2)-compactification of $\langle X, \mathcal{T} \rangle$.

Conversely, since compact Hausdorff spaces are completely regular, and this property is inherited by subspaces and homeomorphic images, existence of a (T_2)-compactification implies that $\langle X, \mathcal{T} \rangle$ is completely regular. ❑

In this section we further investigate the totality of (T_2)-compactifications of a given topological space. In view of the above it is not surprising that separating families of functions play a crucial role. Also keep in mind the procedures carried out in Example 2.1.11 and Section 2.2; these may serve as a model for much of what will be done in this and the following section.

Definition 2.3.2. Let $\langle X, \mathcal{T}\rangle$ be a topological space. We denote by

(i) $\mathbb{K}(\langle X, \mathcal{T}\rangle)$, the class of all (T_2)-compactifications of $\langle X, \mathcal{T}\rangle$;

(ii) $\mathbb{F}(\langle X, \mathcal{T}\rangle)$, the set of all separating families contained in $C(X, [0,1])$.

We already know when separating families exist.

Remark 2.3.3. We have $\mathbb{F}(\langle X, \mathcal{T}\rangle) \neq \varnothing$, if and only if $\langle X, \mathcal{T}\rangle$ is completely regular.

To see this, remember the Tychonoff embedding theorem: in its proof we have shown that

$$\langle X, \mathcal{T}\rangle \text{ completely regular} \;\Rightarrow\; C(X,[0,1]) \text{ separating} \;\Rightarrow\; \exists I\ \exists \iota\colon X \to [0,1]^I \text{ embedding} \;\Rightarrow\; \langle X, \mathcal{T}\rangle \text{ completely regular}$$

On $\mathbb{F}(\langle X, \mathcal{T}\rangle)$ we have the partial order given by set-theoretic inclusion. On $\mathbb{K}(\langle X, \mathcal{T}\rangle)$ we define a relation as

$$\langle Y, \mathcal{V}, \iota\rangle \sqsubseteq \langle Z, \mathcal{W}, \kappa\rangle :\Leftrightarrow \big\{\phi\colon Z \to Y \,|\, \phi \text{ morphism from } \langle Z, \mathcal{W}, \kappa\rangle \text{ to } \langle Y, \mathcal{V}, \iota\rangle\big\} \neq \varnothing$$

This relation is reflexive and transitive, since we always have the identity map as a morphism, and morphisms can be composed. To better understand $\sqsubseteq$, we observe the following properties of morphisms between (T_2)-compactifications.

Lemma 2.3.4. *Let $\langle X, \mathcal{T}\rangle$ be a topological space, and let $\langle Y, \mathcal{V}, \iota\rangle$ and $\langle Z, \mathcal{W}, \kappa\rangle$ be two (T_2)-compactfications of $\langle X, \mathcal{T}\rangle$. Then*

$$\Big|\big\{\phi\colon Y \to Z \,|\, \phi \textit{ morphism from } \langle Y, \mathcal{V}, \iota\rangle \textit{ to } \langle Z, \mathcal{W}, \kappa\rangle\big\}\Big| \leqslant 1.$$

Assume that $\langle Y, \mathcal{V}, \iota\rangle \xrightarrow{\phi} \langle Z, \mathcal{W}, \kappa\rangle$ is a morphism. Then

(i) *ϕ is surjective;*

(ii) *If ϕ is injective, then ϕ is an isomorphism;*

(iii) *ϕ maps $\iota(X)$ onto $\kappa(X)$ and $Y\backslash\iota(X)$ onto $Z\backslash\kappa(X)$.*

Proof. The action of a morphism is uniquely determined on the set $\iota(X)$, and since $\langle Z, \mathcal{W}\rangle$ is Hausdorff, thus also on $\overline{\iota(X)}$. This set, however, equals all of Y.

Let ϕ be given. The image $\phi(Y)$ is a compact subset of Z, and since Z is Hausdorff thus also closed. It contains $\kappa(X)$, and therefore equals all of Z. Since $\langle Y, \mathcal{V}\rangle$ is compact and $\langle Z, \mathcal{W}\rangle$ is Hausdorff, ϕ being bijective implies that ϕ is a homeomorphism, and hence an isomorphism between the compactifications.

We come to the proof of (iii). First, $\phi \circ \iota = \kappa$ implies that $\phi(\iota(X)) = \kappa(X)$. We show that $\phi^{-1}(\kappa(X)) \subseteq \iota(X)$. Since ϕ is surjective, the second equality will follow from this. Let $y \in Y$ and $x \in X$ with $\phi(y) = \kappa(x)$. Choose a net $(x_i)_{i\in I}$ in X with $\lim_{i\in I} \iota(x_i) = y$. By continuity of ϕ, then $\lim_{i\in I} \phi(\iota(x_i)) = \phi(y) = \kappa(x)$. Since $\phi \circ \iota = \kappa$, and κ is a homeomorphism onto its image, this implies that $\lim_{i\in I} x_i = x$. Using continuity of ι, and that Y is Hausdorff (hence limits are unique), we conclude that

$$y = \lim_{i\in I} \iota(x_i) = \iota(x).$$

❑

Corollary 2.3.5. *Let $\langle X, \mathcal{T}\rangle$ be a topological space. Using the representantwise definition, the relation $\sqsubseteq$ induces a partial order on isomorphy classes of* (T_2)*-compactifications of $\langle X, \mathcal{T}\rangle$.*

Proof. First, we show that by the definition via representants indeed a relation on isomorphy classes is well-defined. Assume that $\langle Y_1, \mathcal{V}_1, \iota_1\rangle$, $\langle Y_2, \mathcal{V}_2, \iota_2\rangle$, and $\langle Z_1, \mathcal{W}_1, \kappa_1\rangle$, $\langle Z_2, \mathcal{W}_2, \kappa_2\rangle$ are two pairs of isomorphic (T_2)-compactifications of $\langle X, \mathcal{T}\rangle$, and let $\sigma\colon Y_1 \to Y_2$ and $\tau\colon Z_1 \to Z_2$ be the corresponding isomorphisms. If $\phi\colon Z_1 \to Y_1$ is a morphism, then $\sigma\circ\phi\circ\tau^{-1}\colon Z_2 \to Y_2$ is also a morphism.

Reflexivity and transitivity are clearly inherited. We have to show antisymmetry. Assume that $\langle Y, \mathcal{V}, \iota\rangle$ and $\langle Z, \mathcal{W}, \kappa\rangle$ are two (T_2)-compactfications of $\langle X, \mathcal{T}\rangle$, and $\phi\colon Z \to Y$ and $\psi\colon Y \to Z$ are morphisms. Then $\phi\circ\psi\colon Y \to Y$ also is a morphism, and by uniqueness of morphisms thus equal to id_Y. Therefore, ψ is injective, and hence an isomorphism. ❑

By virtue of Lemma 2.1.10, one can assign to each separating family $F \subseteq C(X, [0,1])$ a (T_2)-compactification of $\langle X, \mathcal{T}\rangle$.

Definition 2.3.6. Let $\langle X, \mathcal{T}\rangle$ be a topological space. For a separating family $F \subseteq C(X, [0,1])$, we denote by ꝁ(F) the (T_2)-compactification of $\langle X, \mathcal{T}\rangle$ given by the set $X_F := \overline{\left(\prod_{f\in F} f\right)(X)}$ endowed with the subspace topology $\mathcal{T}_F$ of the product topology on $[0,1]^F$, and the embedding $\iota_F := \prod_{f\in F} f$.

Usually we think of a product $\prod_{f\in F}[0,1]$ as "tuples of numbers" $(x_f)_{f\in F}$ with $x_f \in [0,1]$. Sometimes it is practical to adopt the set-theoretic viewpoint and consider the product $\prod_{f\in F}[0,1]$ as the set $[0,1]^F$ of all functions from F to $[0,1]$.

Remark 2.3.7. Let $\langle X, \mathcal{T}\rangle$ be a topological space, and $F \subseteq C(X, [0,1])$ a separating family. Then

▷ ι_F is the corestriction of the point evaluation map

$$e_x\colon \begin{cases} X & \to & [0,1]^F \\ x & \mapsto & \Big(f \mapsto f(x) \text{ for } f \in F\Big) \end{cases}$$

Taking the viewpoint of products as tuples, e_x is the unique map with (π_f is the canoncial projection onto the f-th component)

$$\begin{tikzcd} X \arrow[d, dashed, "e_x"'] \arrow[dr, "f"] & \\ \prod\limits_{f\in F}[0,1] \arrow[r, "\pi_f"', "(f\in F)"'] & {[0,1]} \end{tikzcd}$$

▷ $\mathcal{T}_F$ is the restriction of the topology of pointwise convergence,

---

Also an assignment in the direction reverse to $\mathscr{k}$ can be defined. This relies on a general construction.

Definition 2.3.8. Let $\langle Z, \mathcal{W}\rangle$ be a fixed topological space. We have an assignment which maps

▷ a topological space $\langle X, \mathcal{T}\rangle$ to the set $C(X,Z)$ of all $\mathcal{T}$-to-$\mathcal{W}$–continuous functions of X into Z,

▷ a continuous map $f\colon X \to Y$ from a topological space $\langle X, \mathcal{T}\rangle$ to another one $\langle Y, \mathcal{V}\rangle$ to the map

$$f^*\colon \begin{cases} C(Y,Z) & \to \quad C(X,Z) \\ \qquad \phi & \mapsto \quad \phi \circ f \end{cases}$$

This assignment clearly is compatible with composition and identity in the sense that

$$(g \circ f)^* = f^* \circ g^*, \quad (\mathrm{id}_X)^* = \mathrm{id}_{C(X,Z)}\,.$$

Definition 2.3.9. Let $\langle X, \mathcal{T}\rangle$ be a topological space. For a (T_2)-compactification $\langle Y, \mathcal{V}, \iota\rangle$, we denote

$$\mathscr{f}(\langle Y, \mathcal{V}, \iota\rangle) := \iota^*(C(Y,[0,1])).$$

Lemma 2.3.10. *Let $\langle X, \mathcal{T}\rangle$ be a topological space and $\langle Y, \mathcal{V}, \iota\rangle$ a (T_2)-compactification. Then $\mathscr{f}(\langle Y, \mathcal{V}, \iota\rangle)$ is a separating family.*

Proof. Since $\langle X, \mathcal{T}\rangle$ is Hausdorff, it is enough to show that $\mathscr{f}(\langle Y, \mathcal{V}, \iota\rangle)$ separates points from closed sets in the required strong way. Let $x \in X$ and $A \subseteq X$ closed with $x \notin A$. Then $\iota(x) \notin \iota(A)$ since ι is injective, and $\iota(A)$ is closed in the subspace topology of $\iota(X)$ since ι is an embedding. Choose $B \subseteq Y$ closed with $B \cap \iota(X) = \iota(A)$, then $\iota(x) \notin B$. A compact Hausdorff space is normal and hence completely regular. Thus we can choose $f\colon Y \to [0,1]$ continuous with $f(\iota(x)) = 1$ and $f(B) \subseteq \{0\}$. This implies

$$(f \circ \iota)(x) = 1, \quad (f \circ \iota)(A) \subseteq \{0\},$$

and we found a continuous function on X with the required separation property. ❑

Observe the following fact.

Remark 2.3.11. Let $\langle Y, \mathcal{V}, \iota\rangle$ be a (T_2)-compactification of $\langle X, \mathcal{T}\rangle$. Then the map $\iota^*\colon C(Y,[0,1]) \to C(X,[0,1])$ is injective. The corestriction of ι^* to a map from $C(Y,[0,1])$ to $\mathscr{f}(\langle Y, \mathcal{V}, \iota\rangle)$ is thus bijective.

To show this, let $g_1, g_2 \in C(Y,[0,1])$ with $g_1 \circ \iota = g_2 \circ \iota$ be given. This equality means that $g_1|_{\iota(X)} = g_2|_{\iota(X)}$, and since $\iota(X)$ is dense in Y and $[0,1]$ is Hausdorff, we find that $g_1 = g_2$, cf. Lemma 1.3.3.

We can now say quite a lot about the structure of (T_2)-compactifications.

Theorem 2.3.12. *Let $\langle X, \mathcal{T}\rangle$ be a topological space.*

(i) *The assignments $\mathpzc{k}\colon \mathbb{F}(\langle X,\mathcal{T}\rangle) \to \mathbb{K}(\langle X,\mathcal{T}\rangle)$ and $\mathpzc{f}\colon \mathbb{K}(\langle X,\mathcal{T}\rangle) \to \mathbb{F}(\langle X,\mathcal{T}\rangle)$ are monotone.*

(ii) $\forall F \subseteq C(X,[0,1])$ *separating.* $(\mathpzc{f}\circ\mathpzc{k})(F) \supseteq F$.

(iii) $\forall \langle Y,\mathcal{V},\iota\rangle$ (T_2)*-compactification of* $\langle X,\mathcal{T}\rangle$. $(\mathpzc{k}\circ\mathpzc{f})(\langle Y,\mathcal{V},\iota\rangle) \cong \langle Y,\mathcal{V},\iota\rangle$.

Proof.

① *We prove monotonicity of $\mathpzc{k}$:* Let $F, G \subseteq C(X,[0,1])$ be two separating families, and assume that $F \subseteq G$. Let $\rho\colon [0,1]^G \to [0,1]^F$ be the restriction map, i.e., $\rho(g) := g|_F$. Then ρ is continuous w.r.t. pointwise convergence, and

$$\begin{tikzcd} & X \arrow[dl, "\iota_F"'] \arrow[dr, "\iota_G"] & \\ {[0,1]^F} & & {[0,1]^G} \arrow[ll, "\rho"] \end{tikzcd}$$

By continuity, $\rho\big(\overline{\iota_G(X)}\big) \subseteq \overline{\iota_F(X)}$. Hence the restriction of ρ to a map from X_G to X_F is a morphism, i.e., $\mathpzc{k}(F) \sqsubseteq \mathpzc{k}(G)$.

② *We prove monotonicity of $\mathpzc{f}$:* Assume that $\langle Y,\mathcal{V},\iota\rangle \sqsubseteq \langle Z,\mathcal{W},\kappa\rangle$, and let ϕ be the morphism $\phi\colon Z \to Y$. Then

$$\begin{tikzcd} & X \arrow[dl, "\iota"'] \arrow[dr, "\kappa"] & \\ Y & & Z \arrow[ll, "\phi"] \end{tikzcd} \qquad \text{hence} \qquad \begin{tikzcd} & C(X,[0,1]) & \\ C(Y,[0,1]) \arrow[ur, "\iota^*"] \arrow[rr, "\phi^*"'] & & C(Z,[0,1]) \arrow[ul, "\kappa^*"'] \end{tikzcd}$$

This shows that

$$\mathpzc{f}(\langle Y,\mathcal{V},\iota\rangle) = \iota^*\big(C(Y,[0,1])\big) = \kappa^*\big(\phi^*(C(Y,[0,1]))\big) \subseteq \kappa^*\big(C(Z,[0,1])\big) = \mathpzc{f}(\langle Z,\mathcal{W},\kappa\rangle).$$

③ *We show that $(\mathpzc{f}\circ\mathpzc{k})(F) \supseteq F$:* For every $f \in F$ we have

$$\begin{tikzcd} X \arrow[r, "\iota_F"] \arrow[dr, "f"'] & X_F \arrow[r, "\subseteq"] \arrow[d, dashed, "\pi_f|_{X_F}"] & \prod_{f\in F}[0,1] \arrow[dl, "\pi_f"] \\ & {[0,1]} & \end{tikzcd}$$

The projection π_f, and hence also its restriction, is continuous. We see that

$$f = \iota_F^*(\pi_f|_{X_F}) \in \mathpzc{f}\big(\mathpzc{k}(F)\big).$$

④ *We show that there exists an injective morphism from $\langle Y,\mathcal{V},\iota\rangle$ to $(\mathpzc{k}\circ\mathpzc{f})(\langle Y,\mathcal{V},\iota\rangle)$ (and recall Lemma 2.3.4* (ii)*):* Set $F := \mathpzc{f}(\langle Y,\mathcal{V},\iota\rangle)$. Since ι^* is injective, we can consider its inverse $(\iota^*)^{-1}\colon F \to C(Y,[0,1])$. Set $\phi = \big[\prod_{f\in F}(\iota^*)^{-1}f\big]$, explicitly this is

$$\phi\colon \begin{cases} Y & \to \quad \prod_{f\in F}[0,1] \\ y & \mapsto \quad \big([(\iota^*)^{-1}f](y)\big)_{f\in F} \end{cases}$$

Then ϕ is continuous. Since $\langle Y, \mathcal{V}\rangle$ is compact Hausdorff, the family $C(Y, [0,1])$ is point separating. Thus, ϕ is injective. By the definition of ι^*, we have $[(\iota^*)^{-1} f] \circ \iota = f$, and this shows that we have the diagram

$$\begin{tikzcd} & X \arrow[dl, "\iota"'] \arrow[dr, "\iota_F"] & \\ Y \arrow[rr, "\phi"'] & & \prod_{f\in F}[0,1] \end{tikzcd}$$

By continuity, $\phi(Y) = \phi\big(\overline{\iota(X)}\big) \subseteq \overline{\iota_F(X)} = X_F$. Hence, the restriction of ϕ to a map from Y to X_F is an injective morphism.

❑

Let us remark that Theorem 2.3.12 (iii) implies that the totality of isomorphy classes of (T_2)-compactifications of a fixed topological space $\langle X, \mathcal{T}\rangle$ forms a set. Moreover, we have the immediate

Corollary 2.3.13. *Let $\langle X, \mathcal{T}\rangle$ be a topological space. Two (T_2)-compactfications $\langle Y, \mathcal{V}, \iota\rangle$ and $\langle Z, \mathcal{W}, \kappa\rangle$ of $\langle X, \mathcal{T}\rangle$ are isomorphic, if and only if $\iota^*(C(Y,[0,1])) = \kappa^*(C(Z,[0,1]))$.*

2.4 The Stone-Čech compactification

By Theorem 2.3.12, the set of isomorphy classes of (T_2)-compactifications of a completely regular space $\langle X, \mathcal{T}\rangle$ contains a largest element. Namely, the class containing $\text{ꝁ}(C(X,[0,1]))$: for every (T_2)-compactification $\langle Y, \mathcal{V}, \iota\rangle$, we have

$$\langle Y, \mathcal{V}, \iota\rangle = \text{ꝁ}\big(\text{ꞙ}(\langle Y, \mathcal{V}, \iota\rangle)\big) \sqsubseteq \text{ꝁ}(C(X,[0,1])).$$

Definition 2.4.1. Let $\langle X, \mathcal{T}\rangle$ be a completely regular topological space. The (T_2)-compactification $\text{ꝁ}(C(X,[0,1]))$ is called the *Stone-Čech compactification*, and we will denote it as $\langle \beta(X), \mathcal{T}_\beta, \iota_\beta\rangle$.

The construction "$\beta(\cdot)$" exists not only on the level of spaces, but also on the level of maps.

Theorem 2.4.2. *Let $\langle X, \mathcal{T}\rangle$ and $\langle Y, \mathcal{V}\rangle$ be completely regular topological spaces, and $f\colon X \to Y$ be a continuous map. Then there exists a unique continuous map $\beta(f)\colon \beta(X) \to \beta(Y)$, such that*

$$\begin{tikzcd} \beta(X) \arrow[r, dashed, "\beta(f)"] & \beta(Y) \\ X \arrow[u, "\iota_{\beta,X}"] \arrow[r, "f"'] & Y \arrow[u, "\iota_{\beta,Y}"'] \end{tikzcd} \tag{2.1}$$

Passing from f to $\beta(f)$ is compatible with composition and identity in the sense that

$$\beta(g \circ f) = \beta(g) \circ \beta(f), \quad \beta(\mathrm{id}_X) = \mathrm{id}_{\beta(X)}\,. \tag{2.2}$$

Proof. For $\phi \in C(Y,[0,1])$ we have $f^*(\phi) \in C(X,[0,1]) = \iota^*_{\beta,X}(C(\beta(X),[0,1]))$, and hence can apply $(\iota^*_{\beta,X})^{-1}$. Consider the product map

$$\Psi := \prod_{\phi \in C(Y,[0,1])} (\iota^*_{\beta,X})^{-1}\big(f^*(\phi)\big) \colon \beta(X) \to \prod_{\phi \in C(Y,[0,1])} [0,1].$$

This map is continuous. We have

$$\pi_\phi \circ \Psi \circ \iota_{\beta,X} = (\iota^*_{\beta,X})^{-1}\big(f^*(\phi)\big) \circ \iota_{\beta,X} = f^*(\phi) = \phi \circ f = \pi_\phi \circ \iota_{\beta,Y} \circ f.$$

Since the family of projections π_ϕ, $\phi \in C(Y,[0,1])$, is point separating, we have the diagram

$$\begin{tikzcd} \beta(X) \arrow[r, dashed, "\Psi"] & \prod_{\phi \in C(Y,[0,1])} [0,1] \\ X \arrow[u, "\iota_{\beta,X}"] \arrow[r, "f"'] & Y \arrow[u, "\iota_{\beta,Y}"'] \end{tikzcd}$$

Continuity of Ψ implies that

$$\Psi(\beta(X)) = \Psi\big(\overline{\iota_{\beta,X}(X)}\big) \subseteq \overline{\Psi(\iota_{\beta,X}(X))} = \overline{\iota_{\beta,Y}(f(X))} \subseteq \beta(Y).$$

The corestriction $\beta(f)$ of Ψ to a map of $\beta(X)$ into $\beta(Y)$, thus makes the required diagram (2.1) commute.

Since $\iota_{\beta,X}(X)$ is dense in $\beta(X)$ and $\beta(Y)$ is Hausdorff, there exists at most one map making (2.1) commute. From this (2.2) follows immediately:

$$\begin{tikzcd} \beta(X) \arrow[r, dotted, bend left, "\mathrm{id}_{\beta(X)}"] \arrow[r, dashed, bend right, "\beta(\mathrm{id}_X)"'] & \beta(X) \\ X \arrow[u, "\iota_{\beta,X}"] \arrow[r, "\mathrm{id}_X"'] & X \arrow[u, "\iota_{\beta,X}"'] \end{tikzcd} \qquad \begin{tikzcd} \beta(X) \arrow[r, dashed, "\beta(f)"'] \arrow[rr, dotted, bend left, "\beta(g\circ f)"] & \beta(Y) \arrow[r, dashed, "\beta(g)"'] & \beta(Z) \\ X \arrow[u, "\iota_{\beta,X}"] \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & Y \arrow[u, "\iota_{\beta,Y}"'] \arrow[r, "g"] & Z \arrow[u, "\iota_{\beta,Z}"'] \end{tikzcd}$$

❏

By the definition of the Stone-Čech compactification $\beta(X)$, every continuous map $f\colon X \to [0,1]$ has a continuous extension $\tilde{f}\colon \beta(X) \to [0,1]$ (in the sense that $\tilde{f} \circ \iota_\beta = f$). Moreover, by Corollary 2.3.13, this property characterises $\beta(X)$ up to isomorphism. Using Theorem 2.4.2, we can deduce that in fact a stronger extension property holds true.

Corollary 2.4.3. *Let $\langle X, \mathcal{T}\rangle$ be a completely regular space, and let $\langle Y, \mathcal{V}\rangle$ be a compact Hausdorff space. Then*

$$\iota^*_{\beta,X} \colon C(\beta(X), Y) \to C(X,Y)$$

is bijective.

Proof. Injectivity follows since $\iota_{\beta,X}(X)$ is dense in $\beta(X)$ and Y is Hausdorff. To show surjectivity, let $f \in C(X,Y)$. Then we have the diagram

$$\begin{tikzcd} \beta(X) \arrow[r, "\beta(f)"] & \beta(Y) \\ X \arrow[u, "\iota_{\beta,X}"] \arrow[r, "f"'] & Y \arrow[u, "\cong"', "\iota_{\beta,Y}"] \end{tikzcd}$$

hence $f = \iota^*_{\beta,X}\big(\iota^{-1}_{\beta,Y} \circ \beta(f)\big)$. ❑

The remainder $\beta(X)\backslash\iota_\beta(X)$ in the Stone-Čech compactification is often large with a complicated topological structure. For example, we have the following result.

Proposition 2.4.4. *Let $\langle X, \mathcal{T}\rangle$ be a completely regular non-compact topological space. Then no point in $\beta(X)\backslash\iota_\beta(X)$ possesses a countable neighbourhood base.*

Proof. Let $z \in \beta(X)\backslash\iota_\beta(X)$, and assume towards a contradiction that $\mathcal{U}(z)$ has an at most countable basis. Since $\iota_\beta(X)$ is dense in $\beta(X)$, the singleton set $\{z\}$ cannot be open. Thus $\mathcal{U}(z)$ cannot have a finite basis.

① *We construct a basis of $\mathcal{U}(z)$ with additional properties:* Start from some countable basis $(U_n)_{n\in\mathbb{N}}$ of $\mathcal{U}(z)$ with U_n open and $U_0 = \beta(X)$. The closed neighbourhoods of z form a basis of $\mathcal{U}(z)$. Hence, we find a subsequence $(U_{n_k})_{k\in\mathbb{N}}$ with $n_0 = 0$ and

$$\forall k \in \mathbb{N}.\ \overline{U_{n_{k+1}}} \subseteq U_{n_k}$$

The intersection of all sets U_{n_k} equals $\{z\}$, and every set U_{n_k} must have nonempty intersection with $\iota_\beta(X)$. Hence, we find a further subsequence $(U_{n_{k_l}})_{l\in\mathbb{N}}$ with $k_0 = 0$ and

$$\forall l \in \mathbb{N}.\ \big|\iota_\beta(X) \cap \big(U_{n_{k_l}}\backslash\overline{U_{n_{k_{l+1}}}}\big)\big| \geqslant 2$$

② *We construct a function $f\colon X \to [0,1]$:* Let $(U_n)_{n\in\mathbb{N}}$ be a neighbourhood base of $\mathcal{U}(z)$ with the properties from the previous step. For each $n \in \mathbb{N}$ choose two points $x_n, y_n \in X$, $x_n \neq y_n$, with $\iota_\beta(x_n), \iota_\beta(y_n) \in U_n\backslash\overline{U_{n+1}}$. Choose open sets $O_n \subseteq X$ with

$$x_n \in O_n \subseteq \overline{O_n} \subseteq \iota_\beta^{-1}\big(U_n\backslash\overline{U_{n+1}}\big)\backslash\{y_n\},$$

and choose continuous functions $f_n\colon X \to [0,1]$ with $f_n(x_n) = 1$ and $f_n(X\backslash O_n) = \{0\}$. Now define $f\colon X \to [0,1]$ by

$$f(x) := \begin{cases} f_n(x) & \text{if } n \in \mathbb{N}, x \in \iota_\beta^{-1}\big(U_n\backslash\overline{U_{n+1}}\big) \\ 0 & \text{otherwise} \end{cases} \tag{2.3}$$

By this formula a function f is indeed well-defined, since the sets $\iota_\beta^{-1}\big(U_n\backslash\overline{U_{n+1}}\big)$, $n \in \mathbb{N}$, are pairwise disjoint.

③ *We show that f is continuous:* To achieve this, we show that every point $x \in X$ has an open neighbourhood such that f restricted to this neighbourhood is continuous. Let $x \in X$ be given. If there exists $n \in \mathbb{N}$ such that x belongs to the set $\iota_\beta^{-1}\big(U_n\backslash\overline{U_{n+1}}\big)$, then we are done since this set is open and the restriction of f to it coincides with f_n.

Assume that x belongs to none of those sets. Then, in particular, $x \in \bigcap_{n\in\mathbb{N}}(X\backslash\overline{O_n})$. Since $\bigcap_{n\in\mathbb{N}} U_n = \{z\}$ and $\iota_\beta(x) \neq z$, there exists $m \in \mathbb{N}$ with $\iota_\beta(x) \notin U_m$. We see that

$$x \in \big(X\backslash\iota_\beta^{-1}(\overline{U_{m+1}})\big) \cap \bigcap_{n=0}^{m} \big(X\backslash\overline{O_n}\big). \tag{2.4}$$

This set is open. Consider one of its elements, say y. If in the definition (2.3) of $f(y)$ the second case takes place, $f(y) = 0$. If the first case takes place with some $n \in \mathbb{N}$, then we must have $n \leqslant m$, and $f(y) = f_n(y) = 0$ since $y \notin O_n$. Thus f vanishes identically on the open neighbourhood (2.4) of x.

④ *We derive a contradiction:* Since $(U_n)_{n\in\mathbb{N}}$ is a neighbourhood base of z, we have

$$\lim_{n\to\infty} \iota_\beta(x_n) = \lim_{n\to\infty} \iota_\beta(y_n) = z,$$

and hence also

$$\lim_{n\to\infty} [(\iota_\beta^*)^{-1}(f)](\iota_\beta(x_n)) = \lim_{n\to\infty} [(\iota_\beta^*)^{-1}(f)](\iota_\beta(y_n)) = [(\iota_\beta^*)^{-1}(f)](z).$$

We have reached a contradiction, since $[(\iota_\beta^*)^{-1}(f)](\iota_\beta(x_n)) = f(x_n) = 1$ for all $n \in \mathbb{N}$, whereas $[(\iota_\beta^*)^{-1}(f)](\iota_\beta(y_n)) = f(y_n) = 0$ for all $n \in \mathbb{N}$.

❑

The next result says that the local structure of $\beta(X)$ at a point in $\iota_\beta(X)$ is not more complicated than it is in X.

Proposition 2.4.5. *Let $\langle X, \mathcal{T}\rangle$ be a completely regular non-compact topological space. Define $\Phi\colon \mathcal{P}(X) \to \mathcal{P}(\beta(X))$ as*

$$\Phi(U) := \beta(X)\backslash\overline{\iota_\beta(X\backslash U)} \text{ for } U \in \mathcal{P}(X).$$

Then for each point $x \in X$, the set $\Phi(\mathcal{U}^X(x))$ is a neighbourhood base of $\iota_\beta(x)$ in $\beta(X)$.

Proof. The set $\Phi(U)$ is always open. Let $x \in X$ be given.

We show that $\Phi(\mathcal{U}^X(x)) \subseteq \mathcal{U}^{\beta(X)}(\iota_\beta(x))$. Let $U \in \mathcal{U}^X(x)$, then we find $f \in C(X, [0,1])$ with $f(x) = 1$ and $f(X\backslash U) \subseteq \{0\}$. By the universal property of the Stone-Čech compactification there exists $\tilde{f} \in C(\beta(X), [0,1])$ with $\tilde{f} \circ \iota_\beta = f$. We have $\tilde{f}(\iota_\beta(x)) = 1$ and $\tilde{f}(\iota_\beta(X\backslash U)) = 0$, and hence $\iota_\beta(x) \notin \overline{\iota_\beta(X\backslash U)}$. It follows that $\iota_\beta(x) \in \Phi(U)$, and hence that $\Phi(U) \in \mathcal{U}^{\beta(X)}(\iota_\beta(x))$.

Let $W \in \mathcal{U}^{\beta(X)}(\iota_\beta(x))$. Choose $\tilde{f} \in C(\beta(X), [0,1])$ with $\tilde{f}(\iota_\beta(x)) = 1$ and $\tilde{f}(\beta(x)\backslash W) \subseteq \{0\}$, and set

$$U := (\tilde{f} \circ \iota_\beta)^{-1}\big((\frac{1}{2}, 1]\big).$$

Then U is open and $x \in U$, hence $U \in \mathcal{U}^X(x)$. We show that $\Phi(U) \subseteq W$. Assume that $z \in \beta(X)\backslash W$. Then $\tilde{f}(z) = 0$, and hence $\tilde{f}^{-1}([0, \frac{1}{2})) \in \mathcal{U}^{\beta(X)}(z)$. Let $V \in \mathcal{U}^{\beta(X)}(z)$. Since $\iota_\beta(X)$ is dense in $\beta(X)$,

$$\iota_\beta(X) \cap V \cap \tilde{f}^{-1}\big([0, \frac{1}{2})\big) \neq \varnothing.$$

In other words, there exists $y \in X$ such that $\iota_\beta(y) \in V$ and $(\tilde{f} \circ \iota_\beta)(y) < \frac{1}{2}$. Clearly, $y \in X\backslash U$, and we see that $V \cap \iota_\beta(X\backslash U) \neq \varnothing$. Since V was arbitrary, it follows that $z \in \overline{\iota_\beta(X\backslash U)}$, i.e., $z \notin \Phi(U)$. ❑

Recall that a topological space $\langle X, \mathcal{T}\rangle$ is called *first countable*, if for every point $x \in X$ the neighbourhood filter $\mathcal{U}(x)$ has an at most countable basis.

Corollary 2.4.6. *Two first countable and completely regular spaces $\langle X, \mathcal{T}\rangle$ and $\langle Y, \mathcal{V}\rangle$ are homeomorphic, if and only if their Stone-Čech compactifications are homeomorphic.*

Proof. The forward implication holds due to (2.2). Assume that $\phi\colon \beta(X) \to \beta(Y)$ is a homeomorphism. Then for every point $z \in \beta(X)$ it holds that

$$\mathcal{U}^{\beta(X)}(z) \text{ has an at most countable basis } \Leftrightarrow \qquad \mathcal{U}^{\beta(X)}(\phi(z)) \text{ has an at most countable basis}$$

By the countability assumption on X and Proposition 2.4.5, points in $\iota_\beta(X)$ have a countable neighbourhood base, while by Proposition 2.4.4 points in the remainder $\beta(X)\backslash\iota_\beta(X)$ have no countable neighbourhood base. The same holds for Y, and we conclude that $\phi(\iota_{\beta,X}(X)) = \iota_{\beta,Y}(Y)$. It follows that the map $\iota_{\beta,Y}^{-1} \circ \phi \circ \iota_{\beta,X} : X \to Y$ is a homeomorphism. ❑

This corollary shows a typical trade off: on the one hand it is harder to handle the Stone-Čech compactification of a space than the space itself, on the other hand checking that a map between compact Hausdorff spaces is a homeomorphism is much easier than doing the same for a map between non-compact spaces.

2.5 The algebra $C(X)$

We start with a general basic notion.

Definition 2.5.1. Let X be a set, and $\alpha\colon \mathcal{P}(X) \to \mathcal{P}(X)$. Then α is called a *closure operator*, if

(i) $\forall A \in \mathcal{P}(X).\ A \subseteq \alpha(A)$ (*extensive*)

(ii) $\forall A \in \mathcal{P}(X).\ \alpha(\alpha(A)) = \alpha(A)$ (*idempotent*)

(iii) $\forall A, B \in \mathcal{P}(X).\ \big(A \subseteq B \ \Rightarrow\ \alpha(A) \subseteq \alpha(B)\big)$ (*monotone*)

A closure operator α is called a *topological closure operator*, if furthermore

(iv) $\forall A, B \in \mathcal{P}(X).\ \alpha(A \cup B) = \alpha(A) \cup \alpha(B)$

(v) $\alpha(\varnothing) = \varnothing$

Note that (iv) is stronger than (iii), in fact, monotonicity is equivalent to

$$\forall A, B \in \mathcal{P}(X).\ \alpha(A \cup B) \supseteq \alpha(A) \cup \alpha(B)$$

Moreover, by induction (iv) implies

$$\forall n \in \mathbb{N}\ \forall A_1, \ldots, A_n \in \mathcal{P}(X).\ \alpha\Big(\bigcup_{i=1}^{n} A_i\Big) = \bigcup_{i=1}^{n} \alpha(A_i).$$

It is clear that for a topological space $\langle X, \mathcal{T}\rangle$ the operator α defined by $\alpha(A) := \overline{A}$ is a topological closure operator. Moreover, the closed sets in $\langle X, \mathcal{T}\rangle$ are exactly the fixed points of α.

In the next proposition we show that also a converse holds.

Proposition 2.5.2. *Let X be a set and $\alpha\colon \mathscr{P}(X) \to \mathscr{P}(X)$ a topological closure operator. Then there exists a unique topology $\mathcal{T}$ on X, such that $\alpha(A) = \overline{A}$ for all $A \in \mathscr{P}(X)$. This topology is given as*

$$\mathcal{T} = \big\{O \in \mathscr{P}(X) \,|\, \alpha(X\backslash O) = X\backslash O\big\}. \tag{2.5}$$

Proof.

① *We show that the right side of (2.5) is a topology:* Define $\mathcal{T}$ by (2.5). Since α is extensive we have $\alpha(X) = X$, hence $\varnothing \in \mathcal{T}$. By the property (v) of a topological closure operator, we have $X \in \mathcal{T}$.

Let $n \in \mathbb{N}$ and $O_1, \dots, O_n \in \mathcal{T}$, and set $A_i := X\backslash O_i$ for $i = 1, \dots, n$. Using property (iv) of α, we find

$$\alpha\Big(X\backslash \bigcap_{i=1}^{n} O_i\Big) = \alpha\Big(\bigcup_{i=1}^{n} A_i\Big) = \bigcup_{i=1}^{n} \alpha(A_i) = \bigcup_{i=1}^{n} A_i = X\backslash \bigcap_{i=1}^{n} O_i,$$

and hence $\bigcap_{i=1}^{n} O_i \in \mathcal{T}$.

Let O_i, $i \in I$, be a family of elements of $\mathcal{T}$, and set $A_i := X\backslash O_i$ for $i \in I$. Using that α is extensive and monotone, we find

$$X\backslash \bigcup_{i\in I} O_i \subseteq \alpha\Big(X\backslash \bigcup_{i\in I} O_i\Big) = \alpha\Big(\bigcap_{i\in I} A_i\Big) \subseteq \bigcap_{i\in I} \alpha(A_i) = \bigcap_{i=1}^{n} A_i = X\backslash \bigcup_{i\in I} O_i.$$

② *We show that $\alpha(A)$ is the closure w.r.t. $\mathcal{T}$:* Since α is idempotent, $X\backslash\alpha(A) \in \mathcal{T}$ for all $A \subseteq X$. Moreover, by the definition of $\mathcal{T}$, every $\mathcal{T}$-closed set is a fixed point of α. Since α is extensive and monotone, we find

$$\overline{A} \subseteq \overline{\alpha(A)} = \alpha(A) \subseteq \alpha(\overline{A}) = \overline{A}$$

for all $A \subseteq X$.

③ *Uniqueness:* A topology is uniquely determined by its closed sets. Closed sets are exactly the fixed points of the closure operator $A \mapsto \overline{A}$, and hence there can exist at most one topology with $\alpha(A) = \overline{A}$ for all $A \in \mathscr{P}(X)$.

❏

Example 2.5.3. Let $\langle R, +, \cdot, 1\rangle$ be a commutative ring with unit element. Recall that a subset I of R is called an *ideal* of R if $I + I \subseteq I$ and $R \cdot I \subseteq I$. It is called a *proper ideal*, if $I \neq R$, equivalently, if $1 \notin R$. It is called a *maximal ideal*, if it is a maximal element in the set of all proper ideals. Zorn's lemma implies that every proper ideal is contained in a maximal ideal, in particular there exist maximal ideals. We denote the set of all maximal ideals of R by $\mathscr{M}(R)$.

Define $\alpha\colon \mathscr{P}(\mathscr{M}(R)) \to \mathscr{P}(\mathscr{M}(R))$ as

$$\alpha(M) := \Big\{J \in \mathscr{M}(R) \,|\, J \supseteq \bigcap M\Big\} \text{ for } M \in \mathscr{P}(\mathscr{M}(R)).$$

Here we understand the intersection of the empty set as the whole base set R.

$\rhd$ *Let us show that α is a topological closure operator:* If $J \in M$, then clearly $J \supseteq \bigcap M$. Thus α is extensive. Let $J \in \alpha(\alpha(M))$. Since every element of $\alpha(M)$ contains $\bigcap M$, we obtain

$$J \supseteq \bigcap \alpha(M) \supseteq \bigcap M,$$

and hence $J \in \alpha(M)$. Thus α is idempotent. If $M_1 \subseteq M_2$, then $\bigcap M_1 \supseteq \bigcap M_2$, and we see that α is monotone. Further, since $\bigcap \varnothing = R$, we have $\alpha(\varnothing) = \varnothing$.

It remains to show that $\alpha(M_1 \cup M_2) \subseteq \alpha(M_1) \cup \alpha(M_2)$ for all $M_1, M_2 \in \mathscr{P}(\mathscr{M}(R))$. Let $J \in \mathscr{P}(\mathscr{M}(R))$ with $J \notin \alpha(M_1) \cup \alpha(M_2)$ be given. Choose $x_1 \in (\bigcap M_1) \backslash J$ and $x_2 \in (\bigcap M_2) \backslash J$. Since every maximal ideal is also a prime ideal, it follows that $x_1 x_2 \notin J$. However, clearly, $x_1 x_2 \in \bigcap M_1 \cap \bigcap M_2 = \bigcap (M_1 \cup M_2)$. Thus $J \notin \alpha(M_1 \cup M_2)$.

$\rhd$ By means of Proposition 2.5.2 there exists a unique topology on $\mathscr{M}(R)$ whose closure operator coincides with α. We denote this topology as $\mathcal{T}_{\mathscr{M}}$. Note that the topological space $\langle \mathscr{M}(R), \mathcal{T}_{\mathscr{M}} \rangle$ depends only on the isomorphy class of the ring R.

$\rhd$ *We show that $\langle \mathscr{M}(R), \mathcal{T}_{\mathscr{M}} \rangle$ is compact and* (T_1)*:* For each $I \in \mathscr{M}(R)$ we have

$$\alpha(\{I\}) = \big\{ J \in \mathscr{M}(R) \,|\, J \supseteq I \big\} = \{I\},$$

hence $\{I\}$ is closed. This shows (T_1).

The proof of compactness depends on an algebraic fact. Namely, given $A \subseteq R$, the set

$$\Big\{ \sum_{i=1}^{n} r_i a_i \,|\, n \in \mathbb{N}, a_i \in A, r_i \in R \Big\}$$

is the smallest ideal containing A. As a consequence, for any given family I_l, $l \in L$, of proper ideals, the smallest ideal containing $\bigcup_{l \in L} I_l$ is proper, if and only if for every finite subset $L' \subseteq L$ the smallest ideal containing $\bigcup_{l \in L'} I_l$ is proper.

Now compactness is easy to check. Let M_l, $l \in L$, be a family of closed subsets with $\bigcap_{l \in L} M_l = \varnothing$. Then we have

$$\varnothing = \bigcap_{l \in L} M_l = \bigcap_{l \in L} \alpha(M_l) = \Big\{ J \in \mathscr{M}(R) \,|\, J \supseteq \bigcup_{l \in L} \Big(\bigcap M_l \Big) \Big\},$$

i.e., the smallest ideal containing the union $\bigcup_{l \in L} \big(\bigcap M_l \big)$ is R. Thus we find a finite subset L' of L such that the smallest ideal containing the union $\bigcup_{l \in L'} \big(\bigcap M_l \big)$ is R, which just means that $\bigcap_{l \in L'} \alpha(M_l) = \varnothing$.

To each topological space X we can naturally associated the ring $C(X, \mathbb{R})$, and to each continuous function f between topological spaces the function f^*, cf. Definition 2.3.8. When passing from the topological structure X to the algebraic structure $C(X, \mathbb{R})$ some loss may happen. The next result shows that for compact Hausdorff spaces this is not the case.

Theorem 2.5.4. *Let $\langle X, \mathcal{T} \rangle$ be a compact Hausdorff space. Then $\langle X, \mathcal{T} \rangle$ is homeomorphic to $\langle \mathscr{M}(C(X, \mathbb{R})), \mathcal{T}_{\mathscr{M}} \rangle$.*

Proof.

① *We construct a map $\Phi\colon X \to \mathscr{M}(C(X,\mathbb{R}))$:* Let $x \in X$, and denote by ϕ_x the point evaluation functional

$$\phi_x\colon \begin{cases} C(X,\mathbb{R}) & \to & \mathbb{R} \\ f & \mapsto & f(x) \end{cases}$$

Since $C(X,\mathbb{R})$ contains all constant functions we have $\phi_x(C(X,\mathbb{R})) = \mathbb{R}$, since the algebraic operations on $C(X,\mathbb{R})$ are defined pointwise the map ϕ_x is a ring homomorphism, and since $\mathbb{R}$ is a field we know that $\ker\phi_x$ is a maximal ideal in $C(X,\mathbb{R})$. Thus we may define

$$\Phi\colon \begin{cases} X & \to & \mathscr{M}(C(X,\mathbb{R})) \\ x & \mapsto & \ker\phi_x \end{cases}$$

② *We show that Φ is bijective:* Injectivity is easy to see. Let $x_1, x_2 \in X$ with $x_1 \neq x_2$. Since X is compact (T_2), we find $f \in C(X,[0,1])$ with $f(x_1) = 0$ and $f(x_2) = 1$. This means that $f \in \Phi(x_1)$, but $f \notin \Phi(x_2)$, and we conclude that $\Phi(x_1) \neq \Phi(x_2)$.

To prove surjectivity, let $J \in \mathscr{M}(C(X,\mathbb{R}))$ be given. We show that the family

$$\mathcal{C} := \big\{ f^{-1}(\{0\}) \mid f \in J \big\} \subseteq \mathscr{P}(X)$$

has the finite intersection property. To this end, consider finitely many elements $f_1, \ldots, f_n \in J$. The function $g := f_1^2 + \ldots + f_n^2$ again belongs to J, in particular, is not invertible in $C(X,\mathbb{R})$. Since a function is invertible in $C(X,\mathbb{R})$ if and only if it is zero-free, we find a point $x \in X$ with $g(x) = 0$. This implies that $f_i(x) = 0$ for all $i = 1, \ldots, n$, and we see that $\bigcap_{i=1}^n f_i^{-1}(\{0\}) \neq \varnothing$. Since X is compact, it follows that $\bigcap_{f\in J} f^{-1}(\{0\}) \neq \varnothing$. Choose x in this intersection. Then $J \subseteq \ker\phi_x$, and since J is a maximal ideal, it follows that $J = \ker\phi_x$.

③ *We show that Φ is open:* Let $O \subseteq X$ be open and $x \in O$. Since X is compact (T_2), we find $f \in C(X,\mathbb{R})$ with $f(x) = 1$ and $f(X\backslash O) \subseteq \{0\}$. This says that

$$\Phi(x) \not\supseteq \bigcap_{y \in X\backslash O} \Phi(y),$$

i.e., that $\Phi(x) \notin \alpha(\Phi(X\backslash O))$. We obtain

$$\Phi(x) \in \mathscr{M}(C(X,\mathbb{R}))\backslash\alpha(\Phi(X\backslash O)) \subseteq \mathscr{M}(C(X,\mathbb{R}))\backslash\Phi(X\backslash O) = \Phi(O).$$

For the last equality we used that Φ is bijective. Since $x \in O$ was arbitrary, it follows that $\Phi(O)$ is open.

④ *We show that Φ is continuous:* Let $A \subseteq X$. Then

$$\forall f \in C(X,\mathbb{R}).\ \Big(f(A) \subseteq \{0\} \ \Leftrightarrow\ f(\overline{A}) \subseteq \{0\} \Big)$$

In other words, $\bigcap_{y\in A}\Phi(y) = \bigcap_{y\in\overline{A}}\Phi(y)$. This implies that $\alpha(\Phi(A)) = \alpha(\Phi(\overline{A}))$, and we obtain

$$\Phi(\overline{A}) \subseteq \alpha(\Phi(\overline{A})) = \alpha(\Phi(A)).$$

❑

Note that Step ④ in the above proof could be substituted by using that $\mathcal{M}(C(X,\mathbb{R}))$ is compact.

Corollary 2.5.5. *Let $\langle X, \mathcal{T}\rangle$ and $\langle Y, \mathcal{V}\rangle$ be compact Hausdorff spaces. Then $\langle X, \mathcal{T}\rangle$ and $\langle Y, \mathcal{V}\rangle$ are homeomorphic, if and only $C(X,\mathbb{R})$ and $C(Y,\mathbb{R})$ are isomorphic (as rings).*

Proof. The forward implication is trivial: if $\phi\colon X \to Y$ is a homeomorphism, then $\phi^*\colon C(Y,\mathbb{R}) \to C(X,\mathbb{R})$ is an isomorphism. For the converse, assume that $C(X,\mathbb{R})$ and $C(Y,\mathbb{R})$ are isomorphic. Then $\langle \mathscr{M}(C(X,\mathbb{R})), \mathcal{T}_{\mathscr{M}}\rangle$ and $\langle \mathscr{M}(C(Y,\mathbb{R})), \mathcal{T}_{\mathscr{M}}\rangle$ are homeomorphic. By Theorem 2.5.4, this implies that $\langle X, \mathcal{T}\rangle$ and $\langle Y, \mathcal{V}\rangle$ are homeomorphic. ❑

This result can be lifted to a class of non-compact spaces by passing to Stone-Čech compactifications. For a topological space $\langle X, \mathcal{T}\rangle$ we denote by $C_b(X,\mathbb{R})$ be the set of all real-valued continuous and bounded functions on X. Clearly, $C_b(X,\mathbb{R})$ is a commutative ring with unit element. It also carries the structure of an $\mathbb{R}$-algebra and becomes a Banach space when endowed with the supremum norm (but this will not be used here).

Corollary 2.5.6. *Let $\langle X, \mathcal{T}\rangle$ and $\langle Y, \mathcal{V}\rangle$ be completely regular and first countable spaces. Then $\langle X, \mathcal{T}\rangle$ and $\langle Y, \mathcal{V}\rangle$ are homeomorphic, if and only $C_b(X,\mathbb{R})$ and $C_b(Y,\mathbb{R})$ are isomorphic (as rings).*

Proof. By Corollary 2.4.6 and Corollary 2.5.5, $\langle X, \mathcal{T}\rangle$ and $\langle Y, \mathcal{V}\rangle$ are homeomorphic, if and only if $C(\beta(X),\mathbb{R})$ and $C(\beta(Y),\mathbb{R})$ are isomorphic. Using Corollary 2.4.3, we see that $C(\beta(X),\mathbb{R})$ and $C_b(X,\mathbb{R})$ are isomorphic via $\iota^*_{\beta,X}$ (and the same for Y). ❑

Observe that, e.g., every metric space is completely regular and first countable.

Chapter 3

Metrisability

Let X be a set. Given a metric d on X, a topology $\mathcal{T}_d$ can be constructed by using open d-balls as a basis. Not every topology $\mathcal{T}$ on X arises in this way. In fact, if there exists a metric d such that $\mathcal{T} = \mathcal{T}_d$, then $\mathcal{T}$ must have very strong properties; for example it must be first-countable and parakompakt, in particular, normal. The question arises to characterise those topologies on X, for which there does exist a metric d such that $\mathcal{T} = \mathcal{T}_d$. We present some results answering this question. Thereby we work in the setting of pseudo-metrics, which is a slight (but no essential) generalisation.

3.1 Pseudometric spaces

Definition 3.1.1. Let X be a set. A *pseudo-metric* on X is a map $d\colon X \times X \to \mathbb{R}$ which satisfies

(i) $\forall x, y, z \in X.\ d(x, y) \leqslant d(x, z) + d(y, z)$ (the *triangle inequality*)

(ii) $\forall x \in X.\ d(x, x) = 0$

A *metric* on X is a map $d\colon X \times X \to \mathbb{R}$, which satisfies (i), (ii), and

(iii) $\forall x, y \in X.\ d(x, y) = 0 \ \Rightarrow\ x = y$

A *pseudo-metric space* (or *metric space*) is a pair $\langle X, d\rangle$, where X is a set and d is a pseudo-metric (or metric, respectively).

Given a pseudo-metric on a set X, we denote, for nonempty subsets $A, B \subseteq X$,

$$d(B, A) := \inf\{d(x, y) \mid x \in B, y \in A\}.$$

If B is a singleton set, say $B = \{x\}$, we also write

$$d(x, A) := d(\{x\}, A) = \inf\{d(x, y) \mid y \in A\}.$$

Definition 3.1.2. Let $\langle X, d_X\rangle$ and $\langle Y, d_Y\rangle$ be pseudo-metric spaces and $f\colon X \to Y$. Then the function f is called *d_X-to-d_Y–continuous*, if

$$\forall x \in X\ \forall \epsilon > 0\ \exists \delta > 0\ \forall y \in X.\ d_X(y, x) < \delta \ \Rightarrow\ d_Y(f(y), f(x)) < \epsilon \tag{3.1}$$

It is called *d_X-to-d_Y–uniformly continuous*, if

$$\forall \epsilon > 0\ \exists \delta > 0\ \forall x \in X\ \forall y \in X.\ d_X(y, x) < \delta \ \Rightarrow\ d_Y(f(y), f(x)) < \epsilon$$

It is called *d_X-to-d_Y–contractive*, if

$$\forall x, y \in X.\ d_Y(f(x), f(y)) \leqslant d_X(x, y)$$

If no confusion can occur, we will often drop explicit notation of d_X and d_Y.

Clearly, every uniformly continuous function is also continuous. Further, the composition of continuous (or uniformly continuous) functions is again continuous (or uniformly continuous, respectively), and the identity function $\mathrm{id}_X\colon X \to X$ is uniformly continuous.

The axioms defining a (pseudo-) metric immediately imply some more properties. Moreover, let $d_{|\cdot|}$ be the *euclidean metric* $d_{|\cdot|}(s, t) := |s - t|$ on $\mathbb{R}$.

Lemma 3.1.3. *Let X be a nonempty set, and d a pseudo-metric on X. Then*

(i) $\forall x, y \in X.\ d(x, y) \geqslant 0$

(ii) $\forall x, y \in X.\ d(x, y) = d(y, x)$ *(*symmetry*)*

(iii) $\forall x, y, z \in X.\ |d(x, z) - d(y, z)| \leqslant d(x, y)$ *(the* reverse triangle inequality*)*

(iv) *Let $A \subseteq X$ be nonempty. Then the map $d(\cdot, A)\colon x \mapsto d(x, A)$ is d-to-$d_{|\cdot|}$–contractive.*

Proof.

① *We prove the relations* (i)–(iii)*:* Let $x, y \in X$, then

$$2d(x, y) = d(x, y) + d(x, y) \geqslant d(x, x) = 0,$$

and

$$d(x, y) \leqslant d(x, x) + d(y, x) = d(y, x) \leqslant d(y, y) + d(x, y) = d(x, y).$$

If, moreover, $z \in X$, then

$$d(x, z) \leqslant d(x, y) + d(z, y) = d(x, y) + d(y, z),$$
$$d(y, z) \leqslant d(y, x) + d(z, x) = d(x, y) + d(x, z).$$

② *We show that $d(\cdot, A)$ is contractive:* Let $A \subseteq X$, $A \neq \emptyset$, and let $\epsilon > 0$. If $x, y \in X$ with $d(x,y) < \epsilon$, then $|d(x,z) - d(y,z)| \leqslant d(x,y) < \epsilon$ for all $z \in A$. Hence,

$$\big|\inf\{d(x,z) \,|\, z \in A\} - \inf\{d(y,z) \,|\, z \in A\}\big| \leqslant \epsilon.$$

❑

Every pseudo-metric on a set X induces a topology on X.

Definition 3.1.4. Let X be a nonempty set, and d a pseudo-metric on X.

(i) For each $x \in X$ and $r \geqslant 0$, the *open ball* with *center* x and *radius* r is the set

$$U_r^d(x) := \big\{y \in X \,|\, d(y,x) < r\big\}.$$

The *closed ball* with *center* x and *radius* r is the set

$$B_r^d(x) := \big\{y \in X \,|\, d(y,x) \leqslant r\big\},$$

We shall drop explicit notation of d if no confusion can occur.

(ii) The *topology induced by* d is the set

$$\mathcal{T}_d := \big\{O \subseteq X \,|\, \forall x \in O \;\exists r > 0 \,|\, U_r^d(x) \subseteq O\big\}.$$

The choice of terminology in this definition is justified by the facts elaborated in Proposition 3.1.5 below. There we also see the significance of the axiom (iii) in the definition of a metric: it implies a certain richness of $\mathcal{T}_d$.

Recall that a topological space $\langle X, \mathcal{T}\rangle$ is said to satisfy the *separation axiom* (T_0), if

$$\forall x, y \in X, x \neq y \;\exists O_x, O_y \in \mathcal{T}.\; \big(x \in O_x \wedge y \in O_y\big) \wedge \big(y \notin O_x \vee x \notin O_y\big)$$

Proposition 3.1.5. *Let X be a nonempty set, and d a pseudo-metric on X.*

(i) *$\mathcal{T}_d$ is a topology on X.*

(ii) *Every open ball $U_r(x)$ with $x \in X$ and $r > 0$ is open w.r.t. $\mathcal{T}_d$, and every closed ball $B_r(x)$ with $x \in X$ and $r \geqslant 0$ is closed w.r.t. $\mathcal{T}_d$.*

(iii) *For every subset $A \subseteq X$ we have*

$$\overline{A} = \big\{x \in X \,|\, d(x,A) = 0\big\}.$$

(iv) *Let $\langle \tilde{X}, \tilde{d}\rangle$ be another pseudo-metric space, and $f\colon X \to \tilde{X}$. Then f is d-to-$\tilde{d}$–continuous, if and only if f is $\mathcal{T}_d$-to-$\mathcal{T}_{\tilde{d}}$–continuous.*

(v) *The space $\langle X, \mathcal{T}_d\rangle$ is (T_4).*

(vi) *It holds that*

$$d \text{ is a metric} \;\Leftrightarrow\; \langle X, \mathcal{T}_d\rangle \text{ is } (\mathsf{T}_2) \;\Leftrightarrow\; \langle X, \mathcal{T}_d\rangle \text{ is } (\mathsf{T}_0) \tag{3.2}$$

Proof.

① *We show that $\mathcal{T}_d$ is a topology:* The facts that $\emptyset \in \mathcal{T}_d$, $X \in \mathcal{T}_d$, and that $\mathcal{T}_d$ is invariant under unions, are clear from the definition. Let $O_1, O_2 \in \mathcal{T}_d$, and $x \in O_1 \cap O_2$. Then we can choose $r_1, r_2 > 0$, such that $U_{r_1}(x) \subseteq O_1$ and $U_{r_2}(x) \subseteq O_2$. For $r := \min\{r_1, r_2\}$, it thus holds that $U_r(x) = U_{r_1}(x) \cap U_{r_2}(x) \subseteq O_1 \cap O_2$.

② *We show that open balls belong to $\mathcal{T}_d$:* Let $x \in X$, $r > 0$, and $y \in U_r(x)$. Set $r_1 := r - d(y,x)$, and consider $z \in U_{r_1}(y)$. Then

$$d(z,x) \leqslant d(z,y) + d(x,y) < r_1 + d(x,y) = r,$$

and we see that $U_{r_1}(y) \subseteq U_r(x)$.

③ *We show that closed balls are closed w.r.t. $\mathcal{T}_d$:* Let $x \in X$, $r > 0$, and $y \in X \backslash B_r(x)$. Set $r_1 := d(y,x) - r$, and consider $z \in U_{r_1}(y)$. Then

$$d(z,x) \geqslant d(x,y) - d(z,y) > d(x,y) - r_1 = r,$$

and we see that $U_{r_1}(y) \subseteq X \backslash B_r(x)$.

④ *We show the stated characterisation of the closure:* We have

$$\begin{aligned} d(x,A) = 0 \;\Leftrightarrow\; \forall r > 0\; \exists y \in A.\; d(x,y) < r \;\Leftrightarrow\; \forall r > 0.\; U_r(x) \cap A \neq \emptyset \\ \Leftrightarrow\; \forall O \in \mathcal{T}_d, x \in O.\; O \cap A \neq \emptyset \;\Leftrightarrow\; x \in \overline{A} \end{aligned}$$

⑤ *We show that notions of continuity coincide:* Assume that f is d-to-$\tilde{d}$–continuous, and let $O \in \mathcal{T}_{\tilde{d}}$ and $x \in f^{-1}(O)$ be given. Choose $\epsilon > 0$, such that $U_\epsilon^{\tilde{d}}(f(x)) \subseteq O$, and choose $\delta > 0$ as in the definition (3.1) of metric continuity. Then $f(U_\delta^d(x)) \subseteq U_\epsilon^{\tilde{d}}(f(x)) \subseteq O$, i.e., $U_\delta^d(x) \subseteq f^{-1}(O)$. We see that $f^{-1}(O)$ is open.

Conversely, assume that f is $\mathcal{T}_d$-to-$\mathcal{T}_{\tilde{d}}$–continuous, and let $x \in X$ and $\epsilon > 0$ be given. We have $U_\epsilon^{\tilde{d}}(f(x)) \in \mathcal{T}_{\tilde{d}}$, and by topological continuity thus $f^{-1}(U_\epsilon^{\tilde{d}}(f(x))) \in \mathcal{T}_d$. The point x belongs to this set, and hence we may choose $\delta > 0$ such that $U_\delta^d(x) \subseteq f^{-1}(U_\epsilon^{\tilde{d}}(f(x)))$. This says in other words that $f(U_\delta^d(x)) \subseteq U_\epsilon^{\tilde{d}}(f(x))$, and this is the property in the definition of d-to-$\tilde{d}$–continuity.

⑥ *We show that $\mathcal{T}_d$ is (T_4):* Let $A, B \subseteq X$ be closed and disjoint. If $A = \emptyset$, set $O_A := \emptyset$ and $O_B := X$, then O_A, O_B are open, disjoint, and separate A and B. The case that $B = \emptyset$ is treated analogously.

Assume now that $A, B \neq \emptyset$, and set

$$O_A := \{x \in X \mid d(x,A) < d(x,B)\}, \quad O_B := \{x \in X \mid d(x,B) < d(x,A)\}.$$

Obviously, O_A and O_B are disjoint. Since both functions $d(\cdot, A)$ and $d(\cdot, B)$ are continuous, O_A and O_B are open. Let $x \in A$. Since $X \backslash B$ is open and $x \in X \backslash B$, we find $r > 0$ with $U_r(x) \subseteq X \backslash B$. In other words, $d(x,B) \geqslant r$. On the other hand, clearly, $d(x,A) = 0$, and we see that $x \in O_A$. Since $x \in A$ was arbitrary, we have $A \subseteq O_A$. It follows in the same way that $B \subseteq O_B$.

⑦ *We show the equivalences (3.2):* Assume that d is a metric, and let $x, y \in X$, $x \neq y$. Set $r := \frac{1}{2} d(x,y)$, then $r > 0$. For every $z \in U_r(x)$, it holds that

$$d(z,y) \geqslant d(y,x) - d(z,x) > d(y,x) - \frac{1}{2} d(x,y) = r,$$

and we see that $U_r(x) \cap U_r(y) = \varnothing$. Since $U_r(x)$ and $U_r(y)$ are open neighbourhoods of x and y, respectively, we see that $\langle X, \mathcal{T}_d \rangle$ is (T_2).

The implication "$\Rightarrow$" in the second equivalence is trivial. Assume now that $\langle X, \mathcal{T}_d \rangle$ is (T_0), and let $x, y \in X$, $x \neq y$. Choose $O \in \mathcal{T}_d$ which contains one of the points x, y, but not the other. For definiteness, assume that $x \in O$ and $y \notin O$. Choose $r > 0$, such that $U_r(x) \subseteq O$, then $y \notin U_r(x)$, and hence

$$d(x, y) \geqslant r > 0.$$

❑

We now name the central notion discussed in this chapter.

Definition 3.1.6. Let $\langle X, \mathcal{T} \rangle$ be a topological space. The space $\langle X, \mathcal{T} \rangle$ is called *pseudo-metrisable*, if there exists a pseudo-metric d on X, such that $\mathcal{T} = \mathcal{T}_d$. It is called *metrisable* if there exists a metric d on X, such that $\mathcal{T} = \mathcal{T}_d$.

Pseudo-metrisability is inherited by several topological constructions. For completeness, we provide the proof.

Lemma 3.1.7.

(i) *Let $\langle X, \mathcal{T} \rangle$ be a topological space, $\langle Y, d \rangle$ be a pseudo-metric (or metric) space, and $\phi \colon X \to Y$ be a $\mathcal{T}$-to-$\mathcal{T}_d$–embedding. Then*

$$d_X(x, y) := d(\phi(x), \phi(y)) \quad \text{for } x, y \in X,$$

is a pseudo-metric (or metric, respectively) on X and $\mathcal{T}_{d_X} = \mathcal{T}$.

(ii) *Let $\langle X, d \rangle$ be a pseudo-metric (or metric) space. Then*

$$d^\flat(x, y) := \frac{d(x, y)}{1 + d(x, y)} \quad \text{for } x, y \in X, \tag{3.3}$$

is a pseudo-metric (or metric, respectively) on X and $\mathcal{T}_{d^\flat} = \mathcal{T}_d$.

(iii) *Let $\langle X_i, d_i \rangle$, $i \in \mathbb{N}$, be pseudo-metric (or metric) spaces. Let $X := \prod_{i \in \mathbb{N}} X_i$, let $\pi_i \colon X \to X_i$ be the canonical projections, and $\mathcal{T}$ the product topology of the topologies $\mathcal{T}_{d_i}$. Then*

$$d(x, y) := \sup_{i \in \mathbb{N}} \frac{1}{i} d_i^\flat\big(\pi_i(x), \pi_i(y)\big) \quad \text{for } x, y \in X,$$

is a pseudo-metric (or metric, respectively) on X and $\mathcal{T}_d = \mathcal{T}$.

The same statement holds for a finite family of (pseudo-) metric spaces.

Proof.

① *Proof of* (i)*:* That d_X satisfies the axioms of a pseudo-metric is clear. If d is a metric, injectivity of ϕ implies that also d_X is a metric. Since ϕ is a $\mathcal{T}$-to-$\mathcal{T}_d$–embedding, the set

$$\big\{\phi^{-1}(U_r^d(y)) \,|\, r > 0, y \in \phi(X)\big\} \subseteq \mathscr{P}(X)$$

is a basis for the topology $\mathcal{T}$. We have, by the definition of d_X,

$$\phi^{-1}(U_r^d(\phi(x))) = U_r^{d_X}(x),$$

and since $\{U_r^{d_X}(x) \,|\, r > 0, x \in X\}$ is a basis for $\mathcal{T}_{d_X}$ it follows that $\mathcal{T} = \mathcal{T}_{d_X}$.

② *Proof of* (ii)*:* To start with, note that the function

$$\beta\colon \begin{cases} [0,\infty) & \to & [0,1) \\ t & \mapsto & \frac{t}{1+t} \end{cases}$$

is an increasing bijection. Next, we show that

$$\forall x, y \geqslant 0.\ \beta(x+y) \leqslant \beta(x) + \beta(y) \tag{3.4}$$

To establish this, multiply with the denominators to obtain the equivalent inequality

$$(x+y)(1+x)(1+y) \leqslant (x(1+y) + y(1+x))(1+x+y).$$

the left side equals $(x+y)^2 + (x+y)(1+xy)$, and the right side equal $(x+y)^2 + (x+y)(1+2xy) + 2xy$, hence this inequality holds true.

From (3.4) the triangular inequality for $d^\flat$ follows immediately. The other axioms of a (pseudo-) metric are clearly satisfied by $d^\flat$. Equality of topologies follows since

$$U_r^d(x) = U_{\beta(r)}^{d^\flat}(x) \quad \text{for } r > 0, x \in X.$$

③ That d is a pseudo-metric is clear. If all d_i are metrics, then d also is a metric, since the projections π_i, $i \in I$, are jointly injective. We have to show equality of topologies. On the one hand, given $n \in \mathbb{N}$, $r_1, \ldots, r_n > 0$, and $(x_i)_{i\in\mathbb{N}} \in X$, we have

$$U_r^d\big((x_i)_{i\in I}\big) \subseteq \bigcap_{i=1}^{n} \pi_i^{-1}\big(U_{r_i}^{d_i}(x_i)\big) \quad \text{with } r := \frac{1}{n}\beta\big(\min_{i=1,\ldots,n} r_i\big).$$

On the other hand, given $r > 0$ and $(x_i)_{i\in\mathbb{N}} \in X$, choose $n \in \mathbb{N}$ with $\frac{1}{n+1} < r$, then

$$\bigcap_{i=1}^{n} \pi_i^{-1}\big(U_{\beta^{-1}(r)}^{d_i}(x_i)\big) \subseteq U_r^d\big((x_i)_{i\in I}\big).$$

❑

The following corollary provides a practical way to conclude that a space $\langle X, \mathcal{T}\rangle$ is (pseudo-) metrisable.

Corollary 3.1.8. *Let $\langle X, \mathcal{T}\rangle$ be a topological space. If there exists an at most countable separating family of maps into pseudo-metrisable (metrisable) spaces, then $\langle X, \mathcal{T}\rangle$ is pseudo-metrisable (metrisable, respectively).*

Proof. Assume that $\langle Y_n, \mathcal{V}_n\rangle$, $n \in \mathbb{N}$, are pseudo-metrisable, and that $f_n\colon X \to Y_n$, $n \in \mathbb{N}$, are such that $\{f_n \mid n \in \mathbb{N}\}$ is separating. By Proposition 1.1.5 (i), the product map

$$\prod_{n\in\mathbb{N}} f_n\colon X \to \prod_{n\in\mathbb{N}} Y_n$$

is an embedding of $\langle X, \mathcal{T}\rangle$ into $\langle \prod_{n\in\mathbb{N}} Y_n, \prod_{n\in\mathbb{N}} \mathcal{V}_n\rangle$. Using Lemma 3.1.7 (i) and (iii), we obtain that $\mathcal{T}$ is induced by some pseudo-metric.

If all spaces $\langle Y_n, \mathcal{V}_n\rangle$ are metrisable, Lemma 3.1.7 gives a metric. ❑

Examples of (pseudo-) metric spaces are obtained from (semi-) normed linear spaces. Recall: if X is a linear space (over the scalar field $\mathbb{R}$ or $\mathbb{C}$), and $p\colon X \to [0,\infty)$ is a function, then p is called a *seminorm* on X, if

(i) $\forall x, y \in X.\ p(x+y) \leqslant p(x) + p(y)$ (the *triangle inequality*)

(ii) $\forall x \in X, \alpha \in \mathbb{R}$ or $\mathbb{C}.\ p(\alpha x) = |\alpha| p(x)$ (*homogenity*)

It is called a *norm*, if additionally

(iii) $\forall x \in X.\ p(x) = 0 \ \Rightarrow\ x = 0$

A seminorm p gives rise to a pseudo-metric d_p, namely via

$$d_p(x,y) := p(x-y) \quad \text{for } x, y \in X.$$

Thereby, d_p is a metric if and only if p is a norm. All topological notions of a (semi-) normed spaced are understood w.r.t. this (pseudo-) metric.

Lemma 3.1.7 shows that every topological space $\langle X, \mathcal{T}\rangle$ which is homeomorphic to a subspace of a (semi-) normed space is (pseudo-) metrisable. It is an interesting fact that a certain converse holds.

Lemma 3.1.9. *For every metric space $\langle X, d\rangle$ there exists a normed space $\langle Z, \|\cdot\|\rangle$ and a contractive embedding $\iota\colon X \to Z$. If $d(X\times X)$ is a bounded subset of $[0,\infty)$, the embedding can be chosen to be isometric.*

Proof. If d is not bounded, consider the metric $d^\flat$ from (3.3) Then the identity map id_X is a d-to-$d^\flat$–contractive $\mathcal{T}_d$-to-$\mathcal{T}_{d^\flat}$–homeomorphism. In order to prove the present assertion, it is thus enough to consider the case that d is bounded, and construct an isometric embedding into a normed space. Thus assume throughout the following that d is bounded.

Let $\mathscr{B}(X,\mathbb{R})$ be the linear space of all bounded real-valued functions on X, and endow $\mathscr{B}(X,\mathbb{R})$ with the *supremum norm*

$$\|f\|_\infty := \sup_{x\in X} |f(x)| \quad \text{for } f \in \mathscr{B}(X,\mathbb{R}).$$

Consider the function $\iota\colon X \to \mathscr{B}(X,\mathbb{R})$ defined by

$$[\iota(x)](y) := d(y,x) \quad \text{for } x, y \in X.$$

Then it holds, for each two elements $x_1, x_2 \in X$, that

$$\|\iota(x_1) - \iota(x_2)\|_\infty = \sup_{y\in X} \big|d(y,x_1) - d(y,x_2)\big| = d(x_1,x_2).$$

The second equality follows from the reverse triangle inequality (to show "$\leqslant$") and by setting $y = x_1$ (to show "$\geqslant$"). Being isometric, ι is injective and (its corestriction) has an isometric inverse. In particular, ι is an embedding. ❑

3.2 A theorem of Stone

A (pseudo-) metric space has a richer geometry than an arbitrary topological space. For example, every topology $\mathcal{T}_d$ has, by definition, a basis consisting of open balls, and balls have specific geometric properties.

First recall the, purely set-theoretic, notion of a refinement.[1].

[1]This definition and item (iii) of the following definition repeats in Definition 1.5.1.

Definition 3.2.1. Let X be a set and $\mathcal{F}, \mathcal{G} \subseteq \mathscr{P}(X)$. Then $\mathcal{F}$ is called a *refinement* of $\mathcal{G}$, if

$$\forall F \in \mathcal{F}\ \exists G \in \mathcal{G}.\ F \subseteq G.$$

Now we single out some properties, which turns out to be decisive.

Definition 3.2.2. Let $\langle X, \mathcal{T} \rangle$ be a topological space, and $\mathcal{F} \subseteq \mathscr{P}(X)$. Then $\mathcal{F}$ is called

(i) *discrete*, if every point $x \in X$ has a neighbourhood which intersects at most one member of $\mathcal{F}$;

(ii) *σ-discrete* (or *countably discrete*), if it is a union of at most countably many discrete families.

(iii) *locally finite*, if every point $x \in X$ has a neighbourhood which intersects at most finitely many members of $\mathcal{F}$.

(iv) *σ-locally finite* (or *countably locally finite*), if it is a union of at most countably many locally finite families.

Observe that every discrete family is locally finite, and, correspondingly, that every σ-discrete family is σ-locally finite. Moreover, every family having only one element is discrete, and hence every at most countable family is σ-discrete.

Theorem 3.2.3 (M.H.Stone)**.** *Let X be a set and d a pseudo-metric on X. Then every open cover of X has a σ-discrete and locally finite refinement, which is again an open cover of X.*

Proof. Let $\mathcal{G}$ be an open cover of X.

① *We construct a candidate $\mathcal{F} \subseteq \mathscr{P}(X)$ for the required refinement:* Choose a well-ordering $\preceq$ on the set $\mathcal{G}$. We define families $\mathcal{F}_n \subseteq \mathscr{P}(X)$ for each $n \in \mathbb{N}$ by using induction.

Let $n \in \mathbb{N}$, and assume that families $\mathcal{F}_m$ have already been defined for all $m < n$. First set, for each $U \in \mathcal{G}$,

$$F(U,n) := \Big\{ z \in X \,|\, U_{\frac{3}{2^n}}(z) \subseteq U,\ U = \min\{V \in \mathcal{G} \,|\, z \in V\},\ z \notin \bigcup_{m<n} \Big(\bigcup \mathcal{F}_m \Big) \Big\}.$$

Note here that, since $\mathcal{G}$ is a cover of X, the set $\{V \in \mathcal{G} \,|\, z \in V\}$ is nonempty. Now define

$$\mathcal{F}_n := \Big\{ \bigcup_{z \in F(U,n)} U_{\frac{1}{2^n}}(z) \,|\, U \in \mathcal{G} \Big\}.$$

Our candidate for a refinement of $\mathcal{G}$ with the required properties is $\mathcal{F} := \bigcup_{n \in \mathbb{N}} \mathcal{F}_n$.

② *We show that $\mathcal{F}$ is an open cover and a refinement of $\mathcal{G}$:* Denote

$$O_{U,n} := \bigcup_{z \in F(U,n)} U_{\frac{1}{2^n}}(z) \quad \text{for } U \in \mathcal{G}, n \in \mathbb{N}.$$

Then, clearly, $O_{U,n}$ is open and $O_{U,n} \subseteq U$. Thus $\mathcal{F}$ is an open refinement of $\mathcal{G}$.

Let $x \in X$. Set $U := \min\{V \in \mathcal{G} \,|\, z \in V\}$, and choose $n \in \mathbb{N}$ sufficiently large so that $U_{\frac{3}{2^n}}(x) \subseteq U$. Then either $x \in \bigcup_{m<n} \big(\bigcup \mathcal{F}_m \big)$ or $x \in F(U,n)$. In both cases, $x \in \bigcup \mathcal{F}$.

③ *We show that* $d(O_{U,n}, O_{W,n}) \geqslant \frac{1}{2^n}$ *for* $U \neq W$*:* Let $n \in \mathbb{N}$ and $U, W \in \mathcal{G}$, $U \neq W$, and let $x \in O_{U,n}$ and $y \in O_{W,n}$. Choose $z \in F(U,n)$ and $w \in F(W,n)$ such that $x \in U_{\frac{1}{2^n}}(z)$ and $y \in U_{\frac{1}{2^n}}(w)$. Since $\preceq$ is a well-ordering, we have either $U \prec W$ or $W \prec U$. For definiteness, assume that the first case takes place: $U \prec W$. Since $W = \min\{V \in \mathcal{G} \mid z \in V\}$ we have $w \notin U$, and since $U_{\frac{3}{2^n}}(z) \subseteq U$, therefore $d(w,z) \geqslant \frac{3}{2^n}$. Applying twice the reverse triangle inequality, we find

$$d(x,y) \geqslant d(x,w) - d(y,w) \geqslant \big[d(w,z) - d(x,z)\big] - d(y,w) > \frac{1}{2^n}.$$

As a consequence of the now established fact that $d(O_{U,n}, O_{W,n}) \geqslant \frac{1}{2^n}$ for all $U \neq W$, we see that each family $\mathcal{F}_n$ is discrete. Namely, every open ball with diameter $\frac{1}{2^n}$ can intersect at most one set $O_{U,n}$, $U \in \mathcal{G}$.

④ *We show "*$m \geqslant 1 \wedge U_{\frac{1}{2^{m-1}}}(x) \subseteq O_{U,n} \Rightarrow d\big(x, \bigcup_{l \geqslant m+n} \bigcup \mathcal{F}_l\big) \geqslant \frac{1}{2^m}$*":* Let $l \geqslant m+n$, $y \in \bigcup \mathcal{F}_l$, and choose $W \in \mathcal{G}$ and $w \in F(W,l)$ such that $y \in U_{\frac{1}{2^l}}(w)$. Since $n < l$ we have $w \notin \bigcup \mathcal{F}_n$, and hence also $w \notin O_{U,n}$. By the present assumption, thus $d(w,x) \geqslant \frac{1}{2^{m-1}}$. It follows that

$$d(x,y) \geqslant d(x,w) - d(y,w) > \frac{1}{2^{m-1}} - \frac{1}{2^l} \geqslant \frac{1}{2^{m-1}} - \frac{1}{2^m} = \frac{1}{2^m}.$$

As a consequence of the now established implication, we see that $\mathcal{F}$ is locally finite. Namely, given $x \in X$, choose $n \in \mathbb{N}$ such that $x \in \bigcup \mathcal{F}_n$, $U \in \mathcal{G}$ such that $x \in O_{U,n}$, and $m \geqslant 1$ such that $U_{\frac{1}{2^{m-1}}}(x) \subseteq O_{U,n}$. Then the open ball $U_{\frac{1}{2^{m+n}}}(x)$ intersects non of the elements of $\bigcup_{l \geqslant m+n} \mathcal{F}_l$, and at most one element of each of $\mathcal{F}_l$ for $l < m+n$.

❑

3.3 The metrisability theorem of Bing-Nagata-Smirnov

Recall that a topological space $\langle X, \mathcal{T} \rangle$ is said to satisfy the *separation axiom* (T_3), if

$$\forall x \in X, A \subseteq X \text{ closed}, x \notin A\ \exists O_x, O_A \in \mathcal{T}.\ \big(x \in O_x \wedge A \subseteq O_A\big) \wedge \big(O_x \cap O_A = \varnothing\big)$$

Note that this condition is equivalent to

$$\forall x \in X, A \subseteq X \text{ closed}, x \notin A\ \exists O \in \mathcal{T}.\ x \in O \subseteq \overline{O} \subseteq X \backslash A$$

Remark 3.3.1. Every pseudo-metrisable space $\langle X, d \rangle$ satisfies (T_3). Namely, given $x \in X$ and $A \subseteq X$ closed with $x \notin A$, we know from Proposition 3.1.5 (iii) that $d(x,A) > 0$. The sets

$$O_x := U_{\frac{d(x,A)}{2}}(x), \quad O_y := \Big\{y \in X \mid d(y,A) < \frac{d(x,A)}{2}\Big\},$$

are disjoint open neighbourhoods of x and A.

Theorem 3.3.2. *Let* $\langle X, \mathcal{T} \rangle$ *be a topological space which is* (T_3)*. Then the following statements are equivalent.*

(i) $\langle X, \mathcal{T} \rangle$ *is pseudo-metrisable.*

(ii) *The topology $\mathcal{T}$ has a σ-discrete basis.*

(iii) *The topology $\mathcal{T}$ has a σ-locally finite basis.*

The equivalence "(i)⇔(ii)" is a *metrisability theorem of Bing*, and the equivalence "(i)⇔(iii)" is the *metrisability theorem of Nagata-Smirnov*.

As a corollary we obtain a *metrisability theorem of Urysohn*. Recall here that a topological space $\langle X, \mathcal{T}\rangle$ is called *second countable*, if it has an at most countable basis.

Corollary 3.3.3. *Let $\langle X, \mathcal{T}\rangle$ be a topological space. If $\langle X, \mathcal{T}\rangle$ is second-countable,* (T_3) *and* (T_0)*, then it is metrisable.*

Proof. The topology $\mathcal{T}$ has a countable, and hence σ-discrete, basis. ❏

We come to the proof of Theorem 3.3.2. The implication "(i)⇒(ii)" follows from Stone's theorem.

Proof of Theorem 3.3.2 "(i)⇒(ii)*".* Choose a pseudo-metric d such that $\mathcal{T} = \mathcal{T}_d$. For $n \in \mathbb{N}$ let $\mathcal{G}_n$ be the open cover

$$\mathcal{G}_n := \{U_{2^{-n}}(x) \mid x \in X\}.$$

By Theorem 3.2.3, we find a σ-discrete open cover $\mathcal{F}_n$ which is a refinement of $\mathcal{G}_n$. Set $\mathcal{B} := \bigcup_{n\in\mathbb{N}} \mathcal{F}_n$, then $\mathcal{B}$ is again a σ-discrete open cover.

We show that $\mathcal{B}$ is a basis for $\mathcal{T}$. Let $O \in \mathcal{T}_d$ and $x \in O$. Choose $n \in \mathbb{N}$ with $U_{2^{-n}}(x) \subseteq O$, and choose $F \in \mathcal{F}_{n+1}$ with $x \in F$. Since $\mathcal{F}_{n+1}$ is a refinement of $\mathcal{G}_{n+1}$, we find $y \in X$ such that $F \subseteq U_{2^{-(n+1)}}(y)$. In particular, $d(x,y) < 2^{-(n+1)}$, and hence

$$x \in F \subseteq U_{2^{-(n+1)}}(y) \subseteq U_{2^{-n}}(x) \subseteq O.$$

❏

The implication "(ii)⇒(iii)" is of course trivial. For the proof of "(iii)⇒(i)", we present two lemmata. The first contains a statement which in some sense expresses the essence of local finiteness.

Lemma 3.3.4. *Let $\langle X, \mathcal{T}\rangle$ be a topological space, and $\mathcal{F} \subseteq \mathcal{P}(X)$. If $\mathcal{F}$ is locally finite, then*

$$\overline{\bigcup_{F\in\mathcal{F}} F} = \bigcup_{F\in\mathcal{F}} \overline{F}.$$

Proof. The set on the left side is closed and contains all sets F from the family $\mathcal{F}$, hence it also contains all sets $\overline{F}$, $F \in \mathcal{F}$. This shows that "$\supseteq$" holds. To show the reverse inclusion, let $x \in \overline{\bigcup_{F\in\mathcal{F}} F}$. Choose $U \in \mathcal{U}(x)$ which intersects only finitely many elements of $\mathcal{F}$, say, $F_1, \ldots, F_n$. For every neighbourhood $V \in \mathcal{U}(x)$ with $V \subseteq U$, we have

$$V \cap \bigcup_{i=1}^n F_i = V \cap \bigcup_{F\in\mathcal{F}} F \neq \varnothing,$$

and hence

$$x \in \overline{\bigcup_{i=1}^n F_i} = \bigcup_{i=1}^n \overline{F_i} \subseteq \bigcup_{F\in\mathcal{F}} \overline{F}.$$

❏

Second, we show that under the assumption of (iii), the separation property (T_3) implies (T_4).

Lemma 3.3.5. *Let $\langle X, \mathcal{T}\rangle$ be a topological space. If $\langle X, \mathcal{T}\rangle$ is (T_3) and has a σ-locally finite basis, then it is (T_4).*

Proof. Choose $\mathcal{B}_n \subseteq \mathcal{T}$, $n \in \mathbb{N}$, such that each $\mathcal{B}_n$ is locally finite and that $\mathcal{B} := \bigcup_{n\in\mathbb{N}} \mathcal{B}_n$ is a basis of $\mathcal{T}$.

Let $A, B \subseteq X$ be closed and disjoint. Since $\langle X, \mathcal{T}\rangle$ is (T_3) and $\mathcal{B}$ is a basis, we can choose for each $a \in A$ a set $M_a \in \mathcal{B}$ with

$$a \in M_a \subseteq \overline{M_a} \subseteq X \backslash B.$$

Analogously, choose for each $b \in B$ a set $N_b \in \mathcal{B}$ with $b \in N_b \subseteq \overline{N_b} \subseteq X \backslash A$. Consider the families

$$\mathcal{F}_n := \{M_a \,|\, a \in A, M_a \in \mathcal{B}_n\}, \quad \mathcal{G}_n := \{N_b \,|\, b \in B, N_b \in \mathcal{B}_n\}.$$

As subfamilies of $\mathcal{B}_n$, both are locally finite. Set

$$F_n := \bigcup_{F\in\mathcal{F}_n} F, \quad G_n := \bigcup_{G\in\mathcal{G}_n} G,$$

then

$$\overline{F_n} = \bigcup_{F\in\mathcal{F}_n} \overline{F} \subseteq X\backslash B, \quad \overline{G_n} = \bigcup_{G\in\mathcal{G}_n} \overline{G} \subseteq X\backslash A.$$

Set

$$P_n := F_n \backslash \bigcup_{i=1}^{n} \overline{G_i}, \quad Q_n := G_n \backslash \bigcup_{i=1}^{n} \overline{F_i}.$$

Since a finite union of closed sets is again closed, P_n and Q_n are open. Moreover, clearly, $P_n \cap Q_m = \varnothing$, $n, m \in \mathbb{N}$.

Set

$$O_A := \bigcup_{n\in\mathbb{N}} P_n, \quad O_B := \bigcup_{m\in\mathbb{N}} Q_m.$$

Then O_A and O_B are open and disjoint. If $a \in A$, then we find $n \in \mathbb{N}$ with $M_a \in \mathcal{F}_n$, and hence $a \in P_n$. Thus $A \subseteq O_A$. In the same way, it follows that $B \subseteq O_B$. ❑

The need to have a countable index set in the representation of the basis as union of locally finite families arises in the proof of Lemma 3.3.5 at the point where we conclude that P_n and Q_m are open: an uncountable well-ordered set has infinite beginning sections.

Proof of Theorem 3.3.2 "(iii)⇒(i)". Choose a basis $\mathcal{B}$ of $\mathcal{T}$ which can be written as a union $\mathcal{B} := \bigcup_{n\in\mathbb{N}} \mathcal{B}_n$ of locally finite families $\mathcal{B}_n \subseteq \mathcal{T}$, $n \in \mathbb{N}$. Moreover, note that by the previous lemma $\langle X, \mathcal{T}\rangle$ is (T_4) and thus Urysohn's Lemma is available.

① *Given $(n, m) \in \mathbb{N} \times \mathbb{N}$, we construct a pseudo-metric $d_{n,m}$ on X:* For $U \in \mathcal{B}_m$ set

$$U^* := \bigcup \{V \in \mathcal{B}_n \,|\, \overline{V} \subseteq U\}.$$

Since $\mathcal{B}_n$ is locally finite, we have $\overline{U^*} \subseteq U$. If $U^* \neq \varnothing$ and $U \neq X$, we apply Urysohn's Lemma, and find a continuous function $f_U\colon X \to [0,1]$ such that $f_U(X\backslash U) = \{0\}$ and $f_U(\overline{U^*}) = \{1\}$. If $U^* = \varnothing$ take $f_U := 0$, and if $U = X$ take $f_U := 1$. Now set

$$d_{n,m}(x,y) := \sum_{U\in\mathcal{B}_m} |f_U(x) - f_U(y)| \quad \text{for } x, y \in X. \tag{3.5}$$

Since $\mathcal{B}_m$ is locally finite, every point $x \in X$ has a neighbourhood V_x which intersects only finitely many members of $\mathcal{B}_m$. Given $x, y \in X$, let $U_1, \ldots, U_N$ be all those elements of $\mathcal{B}_m$ which intersect V_x or V_y. It follows that

$$\forall (x,y) \in V_x \times V_y.\ d_{n,m}(x,y) = \sum_{j=1}^{N} |f_{U_j}(x) - f_{U_j}(y)|.$$

Hence, (3.5) defines a continuous function $d_{n,m}\colon X \times X \to \mathbb{R}$. The axioms of a pseudo-metric are obviously fulfilled.

② *We construct a separating family on $\langle X, \mathcal{T}\rangle$:* Consider the topological spaces $\langle X, \mathcal{T}_{d_{n,m}}\rangle$, and the maps $f_{n,m}\colon \langle X,\mathcal{T}\rangle \to \langle X, \mathcal{T}_{d_{n,m}}\rangle$ which all act as the identity function: $f_{n,m}(x) := x$ for all $x \in X$.

Since $d_{n,m}$ is continuous, every open $d_{n,m}$-ball belongs to $\mathcal{T}$. Since the open balls form a basis of $\mathcal{T}_{d_{n,m}}$, it follows that $\mathcal{T}_{d_{n,m}} \subseteq \mathcal{T}$. In other words, the map $f_{n,m}$ is $\mathcal{T}$-to-$\mathcal{T}_{d_{n,m}}$–continuous. Obviously, each map $f_{n,m}$ is injective, in particular, the family $\{f_{n,m} \mid (n,m) \in \mathbb{N} \times \mathbb{N}\}$ is point separating.

Let $x \in X$, and $A \subseteq X$ closed with $x \notin A$. Choose $U \in \mathcal{B}$ with $x \in U \subseteq X\backslash A$, then choose $O \in \mathcal{T}$ with $x \in O \subseteq \overline{O} \subseteq U$, and then $V \in \mathcal{B}$ with $x \in V \subseteq O$. Now choose $n, m \in \mathbb{N}$ with $U \in \mathcal{B}_m$ and $V \in \mathcal{B}_n$. Then $x \in U^*$, and hence

$$d_{n,m}(x,A) = \inf_{y\in A} d_{n,m}(x,y) \geqslant \inf_{y\in A} |\underbrace{f_U(x)}_{=1} - \underbrace{f_U(y)}_{=0}| = 1.$$

Hence, x does not belong to the closure of A w.r.t. $\mathcal{T}_{d_{n,m}}$, i.e., $f_{n,m}(x)$ does not belong to the closure of $f_{n,m}(A)$ in the space $\langle X, \mathcal{T}_{n,m}\rangle$.

We can now apply Corollary 3.1.8 to conclude that $\langle X, \mathcal{T}\rangle$ is pseudo-metrisable. ❑

3.4 Metrisability: local to global

The following result is a *metrisability theorem of Smirnov*.

Theorem 3.4.1. *Let $\langle X, \mathcal{T}\rangle$ be a paracompact Hausdorff space, and assume that every point $x \in X$ has a neighbourhood $U \in \mathcal{U}(x)$ such that $\langle U, \mathcal{T}|_U\rangle$ is metrisable. Then $\langle X, \mathcal{T}\rangle$ is metrisable.*

This theorem can be proven with the usual ways of arguing (juggling with sets and coverings) using the Nagata-Smirnov metrisation theorem. We give a different proof, which does not depend on the Nagata-Smirnov theorem, but uses paracompactness in the form of existence of partitions of unity.

Proof. Choose an open cover $\mathcal{F}$ of X such that for every element $U \in \mathcal{F}$ the space $\langle U, \mathcal{T}|_U\rangle$ is metrisable, and choose a partition of unity $(\psi_i)_{i\in I}$ subordinate to this cover.

① *We construct a family of functions into normed spaces:* For each $i \in I$ we choose $U_i \in \mathcal{F}$ with $\operatorname{supp}\psi_i \subseteq U_i$. According to Lemma 3.1.9, we find for each $i \in I$ an embedding ι_i of $\langle U_i, \mathcal{T}|_{U_i}\rangle$ into a normed space $\langle Z_i, \|\cdot\|_i\rangle$. Set

$$f_i\colon \begin{cases} X & \to & Z_i \times \mathbb{R} \\ x & \mapsto & (\psi_i(x)\iota_i(x), \psi_i(x)) \end{cases} \qquad \text{for } i \in I.$$

Here the product $\psi_i(x)\iota_i(x)$ actually means the function

$$x \mapsto \begin{cases} \psi_i(x)\iota_i(x) & \text{if } x \in U_i, \\ 0 & \text{if otherwise.} \end{cases}$$

We endow the space $Z_i \times \mathbb{R}$ with the sum norm of $\|\cdot\|_i$ and the euclidean norm, denote the corresponding metric by d_i and the corresponding topology by $\mathcal{V}_i$.

② *We prove that $\{f_i \,|\, i \in I\}$ is separating:* Since $\operatorname{supp}\psi_i \subseteq U_i$, the functions f_i are $\mathcal{T}$-to-$\mathcal{V}_i$–continuous. Next, let $x, y \in X$ and assume that $f_i(x) = f_i(y)$ for all $i \in I$. Choose $i \in I$ with $\psi_i(x) > 0$, then it follows that $x, y \in U_i$ and $\iota_i(x) = \iota_i(y)$. Since ι_i is injective, thus $x = y$.

Let $x \in X$ and $A \subseteq X$ closed, and assume that $f_i(x) \in \overline{f_i(A)}$ for all $i \in I$. Choose $i \in I$ with $\psi_i(x) > 0$, i.e., $x \in U_i$. Choose a net $(a_l)_{l\in L}$ of elements $a_l \in A$ such that $\lim_{l\in L} f_i(a_l) = f_i(x)$. Then we have, in particular, $\lim_{l\in L} \psi_i(a_l) = \psi_i(x) > 0$. By passing to a subnet if necessary, we may assume without loss of generality that $\psi_i(a_l) > 0$ for all $l \in L$. It follows that $a_l \in U_i$ for all $l \in L$, and that $\lim_{l\in L} \iota_i(a_l) = \iota_i(x)$. Since ι_i is a homeomorphism onto its image, it follows that $\lim_{l\in L} a_l = x$ in $\langle U_i, \mathcal{T}|_{U_i}\rangle$, and hence in $\langle X, \mathcal{T}\rangle$. This shows that $x \in \overline{A}$.

③ *Embedding into the product:* We denote

$$Z := \prod_{i\in I}(Z_i \times \mathbb{R}), \quad \mathcal{V} := \prod_{i\in I} \mathcal{V}_i.$$

Then, according to Proposition 1.1.5 (i), the product map $f := \prod_{i\in I} f_i$ is an embedding of $\langle X, \mathcal{T}\rangle$ into $\langle Z, \mathcal{V}\rangle$.

④ *The direct sum:* Let $Y := \bigoplus_{i\in I}(Z_i \times \mathbb{R})$ be the direct sum of the linear spaces $Z_i \times \mathbb{R}$, i.e., the subset of their product consisting of all elements with only finitely many nonzero coordinates. This space can be endowed with the sum norm $\|\cdot\|_\Sigma$ of the norms $\|\cdot\|_i$, i.e.,

$$\|(z_i)_{i\in I}\|_\Sigma := \sum_{i\in I} \|z_i\|_i \quad \text{for } (z_i)_{i\in I} \in \bigoplus_{i\in I}(Z_i \times \mathbb{R}).$$

We denote the corresponding metric as d_Σ, and the corresponding topology as $\mathcal{V}_\Sigma$.

The projections $\pi_i|_Y\colon Y \to (Z_i \times \mathbb{R}$ are contractive, in particular, continuous. Thus $\mathcal{V}|_Y \subseteq \mathcal{V}_\Sigma$. In general, this inclusion will be a proper one, but on small subspaces the topologies coincide. Namely, let $J \subseteq I$ be finite, and consider the subspace

$$Y_J := \bigcap_{i\in I\setminus J} \pi_i^{-1}(\{0\}) \subseteq Y.$$

The restriction of $\mathcal{V}_\Sigma$ to Y_J is induced by the restriction of the sum norm. Let $(y_i)_{i\in I} \in Y_J$ and $r > 0$. Then

$$\bigcap_{j\in J} \pi_j^{-1}\big(U^{d_j}_{\frac{r}{|J|}}(y_j)\big) \cap Y_J \subseteq U^{d_\Sigma}_r\big((y_i)_{i\in I}\big) \cap Y_J,$$

and we conclude that $\mathcal{V}_\Sigma|_{Y_J} \subseteq \mathcal{V}|_{Y_J}$.

⑤ *Embedding into the direct sum:* The family $\{\operatorname{supp}\psi_i \mid i \in I\}$ is locally finite. Hence, given $x \in X$, we can choose an open neighbourhood O_x of x, which intersects only finitely many of $\operatorname{supp}\psi_i$, $i \in I$. Set $J := \{i \in I \mid O_x \cap \operatorname{supp}\psi_i \neq \varnothing\}$, then

$$f(O_x) \subseteq Y_J.$$

In particular, $f(X) \subseteq Y$. Since f is $\mathcal{T}$-to-$\mathcal{V}$–continuous and $\mathcal{V}_\Sigma|_{Y_J} \subseteq \mathcal{V}|_{Y_J}$, it follows that $f|_{O_x}$ is $\mathcal{T}|_{O_x}$-to-$\mathcal{V}_\Sigma$–continuous. Since $x \in X$ was arbitrary, we conclude that f is $\mathcal{T}$-to-$\mathcal{V}_\Sigma$–continuous.

Since $f^{-1}\colon f(X) \to X$ is $\mathcal{V}|_{f(X)}$-to-$\mathcal{T}$–continuous, and $\mathcal{V}_\Sigma \supseteq \mathcal{V}|_Y$, we also have that f^{-1} is $\mathcal{V}_\Sigma|_{f(X)}$-to-$\mathcal{T}$–continuous.

Having an embedding of $\langle X, \mathcal{T}\rangle$ into the metrisable space $\langle Y, \mathcal{V}_\Sigma\rangle$, Lemma 3.1.7 (i) yields that $\langle X, \mathcal{T}\rangle$ is metrisable. ❏

Chapter 4

Covering spaces

A covering of a topological space X can be seen as a larger space which can be projected onto X and in which certain loops in X are unfolded. Covering spaces arose in complex analysis in the study of Riemann surfaces. They play an important role in the structure theory of topological spaces due to their relation to the fundamental group of a space.

4.1 Coverings

Loosely speaking, a covering space of some topological space X is a topological space $\tilde{X}$ together with a map projecting $\tilde{X}$ onto X, such that for sufficiently small neighbourhoods U in X the subspace $p^{-1}(U) \subseteq \tilde{X}$ looks like a stack of pancakes hovering above U.

In our definition we include a connectedness assumption. This is done for practical purposes, since many results hold only under such assumptions.

Definition 4.1.1. Let $\langle X, \mathcal{T} \rangle$ be a topological space. A *covering* of $\langle X, \mathcal{T} \rangle$ is a triple $\langle \tilde{X}, \tilde{\mathcal{T}}, p \rangle$, where

(i) $\langle \tilde{X}, \tilde{\mathcal{T}} \rangle$ is a pathwise connected topological space,

(ii) $p \colon \tilde{X} \to X$ is continuous,

(iii) each point $x \in X$ has an open neighbourhood U_x, such that the inverse image $p^{-1}(U_x)$ is the union of some nonempty family of pairwise disjoint nonempty open sets $S_{x,i}$, $i \in I_x$, with the property that for all $i \in I$ the map $p|_{S_{x,i}}$ is a homeomorphism of $S_{x,i}$ onto U_x.

If $\langle \tilde{X}, \tilde{\mathcal{T}}, p\rangle$ is a covering of $\langle X, \mathcal{T}\rangle$, the map p is called the *covering projection* and the set $p^{-1}(\{x\})$ is called the *fiber of* x. Moreover, a set U_x with the properties stated in (iii) is called an *evenly covered neighbourhood of* x and the corresponding sets $S_{x,i}$, $i \in I_x$, from (iii) are called the *sheets over* U_x.

We start with a simple general observation.

Remark 4.1.2. Let $\langle X, \mathcal{T}\rangle$ be a topological space, and $\langle \tilde{X}, \tilde{\mathcal{T}}, p\rangle$ be a covering of $\langle X, \mathcal{T}\rangle$. Then p is surjective, and $\langle X, \mathcal{T}\rangle$ is pathwise connected.

To see that p is surjective, note that for every evenly covered neighbourhood U there exists at least one corresponding sheet S and $p|_S$ is a homeomorphism of S onto U. Hence, $U \subseteq p(\tilde{X})$. The family of all evenly covered neighbourhoods forms an open cover of X, and we see that $X \subseteq p(\tilde{X})$.

As a continuous image of a pathwise connected space, $\langle X, \mathcal{T}\rangle$ is also pathwise connected.

Lemma 4.1.3. *Let $\langle X, \mathcal{T}\rangle$ be a topological space, and $\langle \tilde{X}, \tilde{\mathcal{T}}, p\rangle$ be a covering of $\langle X, \mathcal{T}\rangle$. Then p is open.*

Proof. Let $O \subseteq \tilde{X}$ be open, and let $x \in p(O)$. Choose an evenly covered neighbourhood U of x. Then $O \cap p^{-1}(U) \neq \emptyset$, and we can choose a sheet S over U with $O \cap S \neq \emptyset$. The set $O \cap S$ is open in S, and hence $p(O \cap S)$ is open in U. Since U is open, $p(O \cap S)$ is also open in X. Clearly, $x \in p(O \cap S) \subseteq p(O)$. ❑

Remark 4.1.4. Assume we have a covering $\langle \tilde{X}, \tilde{\mathcal{T}}, p\rangle$ of some topological space $\langle X, \mathcal{T}\rangle$. Let $\sim$ be the kernel of p, let $\pi\colon \tilde{X} \to \tilde{X}/_\sim$ be the canonical projection, and let $\tilde{X}/_\sim$ be endowed with the factor topology (i.e., the final topology from the family $\{\pi\}$). Moreover, let $\tilde{p}\colon \tilde{X}/_\sim \to X$ be the map with

$$\begin{tikzcd} \tilde{X} \arrow[r, "p"] \arrow[d, "\pi"'] & X \\ \tilde{X}/_\sim \arrow[ur, dashed, "\tilde{p}"'] & \end{tikzcd}$$

Then $\tilde{p}$ is a homeomorphism.

This is seen by a general argument which only needs that p is surjective, continuous, and open. Since p is surjective, $\tilde{p}$ is bijective. Since $\tilde{X}/_\sim$ carries the final topology, $\tilde{p}$ is continuous. Let $O \subseteq \tilde{X}/_\sim$ be open. This means that $\pi^{-1}(O) \subseteq \tilde{X}$ is open. Since π is surjective, we obtain that

$$\tilde{p}(O) = \tilde{p}\big(\pi(\pi^{-1}(O))\big) = p\big(\pi^{-1}(O)\big)$$

and the set on the right side is open since p is open.

The following observation is often practical when one intends to prove that a given triple is a covering.

Lemma 4.1.5. *The axiom* (iii) *in the definition of a covering can be replaced by the following two requirements:*

(iii') $p\colon \tilde{X} \to X$ *is open,* (iii") *each point* $x \in X$ *has an open neighbourhood* U_x*, such that the inverse image* $p^{-1}(U_x)$ *is the union of some nonempty family of pairwise disjoint nonempty open sets* $S_{x,i}$*,* $i \in I_x$*, with the property that for all* $i \in I$ *the map* $p|_{S_{x,i}}$ *is a bijection of* $S_{x,i}$ *onto* U_x*.*

To be precise: a triple $\langle \tilde{X}, \tilde{T}, p\rangle$ *is a covering of* $\langle S, \mathcal{T}\rangle$*, if and only if it satisfies Definition 4.1.1* (i),(ii)*, and* (iii')*,* (iii")*.*

Proof. Definition 4.1.1 (iii) implies (iii') by Lemma 4.1.3, and (iii") is obvious. Conversely, we have to check that $p|_{S_{x,i}}$ is a homeomorphism: being a restriction of a continuous map $p|_{S_{x,i}}$ is continuous, and being a restriction of an open map to an open set it is open. ❑

Every pathwise connected space $\langle X, \mathcal{T}\rangle$ has the trivial covering $\langle X, \mathcal{T}, \mathrm{id}_X\rangle$. Let us give some examples of non-trivial coverings.

We start with two examples dealing with spheres: denote by S^n the *n-sphere* (here $\|.\|$ is the euclidean norm)

$$S^n := \left\{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\right\}$$

Unless specified differently, we always endow S^n with the subspace topology $\mathcal{T}_n$ inherited from the euclidean topology of $\mathbb{R}^{n+1}$.

It is often convenient to identify $\mathbb{R}^2$ with the complex number field, and regard S^1 as the unit circle in $\mathbb{C}$

$$S^1 = \left\{z \in \mathbb{C} \mid |z| = 1\right\}.$$

Remark 4.1.6. Let us show that the sphere S^n is pathwise connected.

First, the circle S^1 is the image of the interval $[0, 2\pi]$ under the continuous map $\Phi(\theta) := e^{i\theta}$, and hence pathwise connected.

Let $n \geqslant 2$. We use *spherical coordinates*. This is the map $\Phi\colon [0, 2\pi]^{n-1} \times [0, \pi] \to S^n$ defined as

$$\Phi\colon \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_n \end{pmatrix} \mapsto \begin{pmatrix} \sin\phi_1 \cdot \sin\phi_2 \cdots \sin\phi_n \\ \cos\phi_1 \cdot \sin\phi_2 \cdots \sin\phi_n \\ \cos\phi_2 \cdot \sin\phi_3 \cdots \sin\phi_n \\ \vdots \\ \cos\phi_{n-1} \cdot \sin\phi_n \\ \cos\phi_n \end{pmatrix}$$

It is a continuous surjection of $[0, 2\pi]^{n-1} \times [0, \pi]$ onto S^n.

Proposition 4.1.7. *Let* $k \geqslant 2$*, and let* $p^{(k)}\colon S^1 \to S^1$ *be the map* $p^{(k)}(z) := z^k$*. Then* $\langle S^1, \mathcal{T}_1, p^{(k)}\rangle$ *is a covering of* $\langle S^1, \mathcal{T}_1\rangle$*.*

Proof. We know that S^1 is pathwise connected. Moreover, clearly, the map $p^{(k)}$ is continuous.

① *We show that* $p^{(k)}$ *is open:* The intersection of an open disk in $\mathbb{C}$ with the unit circle is an open arc. Hence, the set of all open arcs $I_{\alpha,\beta} := \{e^{it} \mid \alpha < t < \beta\}$ with $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, is a basis for the subspaces topology $\mathcal{T}_1$.

We have $p^{(k)}(I_{\alpha,\beta}) = I_{k\alpha,k\beta}$, and hence $p^{(k)}$ induces a bijection of the set of all arcs onto itself. In particular, this implies that $p^{(k)}$ maps open sets to open sets.

② *We construct evenly covered neighbourhoods:* Let $x_0 \in S^1$ and write $x_0 = e^{i\theta}$ with some $\theta \in \mathbb{R}$. Let U be the open arc $I_{\theta-\pi,\theta+\pi}$. Then U is an open neighbourhood of x_0. The inverse image $[p^{(k)}]^{-1}(U)$ is the union of the k disjoint open arcs

$$S_l := \Big(\frac{\theta}{k} + \frac{2l-1}{k}\pi, \frac{\theta}{k} + \frac{2l+1}{k}\pi\Big) \text{ where } l = 0, \dots, k-1.$$

For each $l \in \{0, \dots, k-1\}$ the map $p^{(k)}|_{S_l}$ is a bijection of S_l onto U.

❑

The case "$k = 2$" of Proposition 4.1.7 admits an immediate generalisation to higher dimensional spheres. To make the connection, remember Remark 4.1.4.

Proposition 4.1.8. *Let $n \geqslant 2$. Let $\sim \subseteq S^n \times S^n$ be the equivalence relation*

$$x \sim y :\Leftrightarrow (x = y \ \vee \ x = -y)$$

and let $p_n \colon S^n \to S^n/_\sim$ be the canonical projection. Let $S^n/_\sim$ be endowed with the final topology $\mathcal{V}$ induced by $\{p_n\}$.

Then $\langle S^n, \mathcal{T}_n, p_n\rangle$ is a covering of $\langle S^n/_\sim, \mathcal{V}\rangle$.

Proof. We know that S^n is pathwise connected. Moreover, since the topology on $S^n/_\sim$ is defined as the final topology, the map p_n is continuous.

① *We show that p_n is open:* Let $O \subseteq S^n$ be open. Then $p_n^{-1}(p_n(O)) = O \cup (-O)$ is also open, and indeed $p_n(O)$ is open in $S^n/_\sim$.

② *We construct evenly covered neighbourhoods:* Let $x_0 \in S^n$ and consider the point $x_0/_\sim \in S^n/_\sim$. Then $p_n^{-1}(\{x_0/_\sim\}) = \{x_0, -x_0\}$. Set

$$O_+ := \big\{x \in S^n \mid \|x - x_0\| < \frac{1}{2}\big\}, O_- := \big\{x \in S^n \mid \|x + x_0\| < \frac{1}{2}\big\}.$$

Then O_+ and O_- are disjoint, open in S^n, and satisfy $O_- = -O_+$. Set $U := p_n(O_+)$. Then U is an open neighbourhood of $x_0/_\sim$ in $S^n/_\sim$. We have $p_n^{-1}(U) = O_+ \cup O_-$. The restrictions $p_n|_{O_+}$ and $p_n|_{O_-}$ are injective, and thus map O_+ and O_-, respectively, bijectively onto U.

❑

As a third example, we prove a general result which yields coverings. Recall that a *topological group* is a triple $\langle G, \cdot, \mathcal{T}\rangle$, such that $\langle G, \cdot\rangle$ is a group, $\langle G, \mathcal{T}\rangle$ is a topological space, and the algebraic operations

$$\cdot \colon G \times G \to G, \quad .^{-1} \colon G \to G$$

are continuous.

Theorem 4.1.9. *Let $\langle G, \cdot, \mathcal{T}\rangle$ be a pathwise connected topological group, and let $H \subseteq G$ be a normal subgroup of $\langle G, \cdot\rangle$ which is a discrete subspace of $\langle X, \mathcal{T}\rangle$. Denote by p the canonical projection $p \colon G \to G/H$, and consider G/H with the quotient topology $\mathcal{V}$ (i.e., the final topology induced by $\{p\}$). Then $\langle G, \mathcal{T}, p\rangle$ is a covering of $\langle G/H, \mathcal{V}\rangle$.*

Proof.

① *We show that p is open:* Let $O \subseteq G$ be open. Then

$$p^{-1}\big(p(O)\big) = \bigcup_{x \in O} H \cdot x = \bigcup_{x \in O} \bigcup_{y \in H} y \cdot x = \bigcup_{y \in H} y \cdot O.$$

Left-translations $T_y \colon x \mapsto y \cdot x$ are homeomorphisms, and we see that $p^{-1}(p(O))$ is open in G. This means that $p(O)$ is open in G/H.

② *We construct an evenly covered neighbourhood of the unit element* 1 *of G:* Since H is discrete, we can choose an open set $W \subseteq G$ with $W \cap H = \{1\}$. By continuity of the algebraic operations, we find an open neighbourhood V of 1 with $V \cdot V^{-1} \subseteq W$. Now set

$$U_1 := p(V).$$

Then U_1 is an open neighbourhood of $1/_H \in G/H$. We have

$$p^{-1}(U_1) = \bigcup_{y \in H} y \cdot V.$$

The sets $y \cdot V$ are open. Let us show that they are pairwise disjoint: assume we have $y, z \in H$ and $v, w \in V$ with $yv = zw$, then

$$vw^{-1} = y^{-1}z \in VV^{-1} \cap H = \{1\},$$

and hence $y = z$.

Since $H = \ker p$, we have for each $y \in H$ that $p(y \cdot V) = p(V) = U_1$. Since $y \cdot V$ and U_1 are open in G and G/H, respectively, and $p \colon G \to G/H$ is an open map, also $p|_{y \cdot V} \colon y \cdot V \to U_1$ is an open map. Clearly, it is also continuous. Let us show that $p|_{y \cdot V}$ is injective: assume we have $v, w \in V$ with $p(yv) = p(yw)$. Then also $p(v) = p(w)$, and we find

$$vw^{-1} \in H \cap VV^{-1} = \{1\},$$

i.e., $v = w$. Alltogether, $p|_{y \cdot V}$ is a homeomorphism of $y \cdot V$ onto U_1.

③ *Evenly covered neighbourhoods of other points are found by translating:* The factor group G/H is again a topological group, and hence translations in both, G and G/H, are homeomorphisms. The projection p is a homomorphism, in other words, for all $x \in G$

$$p \circ T_x = T_{p(x)} \circ p.$$

Given $x \in G$, set $U_{x/_H} := T_{p(x)}(U_1)$. Then $U_{x/_H}$ is an evenly covered neighbourhood of $x/_H$ (with sheets $xy \cdot V$, $y \in H$).

❑

Example 4.1.10. The additive group $\langle \mathbb{R}, + \rangle$ becomes a topological group when endowed with the euclidean topology $\mathcal{E}$. It has the discrete subgroup $\mathbb{Z}$, and the factor space $\mathbb{R}/\mathbb{Z}$ is homeomorphic to the unit circle S^1 in $\mathbb{R}^2$ endowed with the subspace topology $\mathcal{T}$ of the euclidean topology of $\mathbb{R}^2$. A homeomorphism is given by map $x/_{\mathbb{Z}} \mapsto e^{2\pi i x}$.

The above theorem implies that $\langle \mathbb{R}, \mathcal{E}, p \rangle$, where $p(x) := e^{2\pi i x}$, is a covering of $\langle S^1, \mathcal{T} \rangle$.

We show a structure result about covering spaces.

Proposition 4.1.11. *Let $\langle \tilde{X}, \tilde{\mathcal{T}}, p\rangle$ be a covering of $\langle X, \mathcal{T}\rangle$. Then all fibers $p^{-1}(\{x\})$, $x \in X$, have the same cardinality.*

The cardinality of fibers of a covering is called its *degree*. If the degree of a covering is finite, say n, then one also speaks of an *n-fold covering*. Note that for every evenly covered neighbourhood U the cardinality of the set of sheets lying over U is equal to the degree of the covering.

For example, Proposition 4.1.7 gives a k-fold covering, and the degree of the covering given in Theorem 4.1.9 is equal to the cardinality of the normal subgroup H.

Proof of Proposition 4.1.11. Let $x \in X$. Choose an evenly covered neighbourhood U of x, and let S_i, $i \in I$, be the sheet over U. Consider a point $y \in U$. Then in every sheet S_i we find a unique point $\tilde{y}_i \in S_i$ with $p(\tilde{y}_i) = y$. Hence, a function $\phi\colon I \to p^{-1}(\{y\})$ is well-defined by $\phi(i) := \tilde{y}_i$. This function is injective since different sheets are disjoint, and surjective since the fiber of y is contained in the union of all sheets.

Since a fiber cannot be larger than $\tilde{X}$, we have

$$X = \bigcup_{\substack{\kappa \text{ cardinality} \\ \kappa \leqslant |\tilde{X}|}} \big\{x \in X \mid |p^{-1}(\{x\})| = \kappa\big\}$$

By what we showed above, every set in this union is open. Since X is connected, thus only one of these sets can be nonempty. ❑

4.2 Lifting of continuous functions

The notion of a lifting of a map plays a central role.

Definition 4.2.1. Let $\langle X, \mathcal{T}\rangle$ be a topological space and $\langle \tilde{X}, \tilde{\mathcal{T}}, p\rangle$ a covering of $\langle X, \mathcal{T}\rangle$. Further, let $\langle Y, \mathcal{V}\rangle$ be a topological space and $f\colon Y \to X$ a continuous map.

A continuous map $\tilde{f}\colon Y \to \tilde{X}$ is called a *lifting* of f, if it satisfies $p \circ \tilde{f} = f$.

$$\begin{tikzcd}
 & \langle \tilde{X}, \tilde{\mathcal{T}}\rangle \arrow[d, "p"] \\
\langle Y, \mathcal{V}\rangle \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & \langle X, \mathcal{T}\rangle
\end{tikzcd}$$

Let us show a uniqueness property. Despite being easy to prove, uniqueness of lifting is an important property and often used.

Theorem 4.2.2. *Let $\langle \tilde{X}, \tilde{\mathcal{T}}, p\rangle$ be a covering of $\langle X, \mathcal{T}\rangle$, let $x_0 \in X$ and $\tilde{x}_0$ be an element of the fiber of x_0. Moreover, let $\langle Y, \mathcal{V}\rangle$ be a topological space, $y_0 \in Y$, and $f\colon Y \to X$ be continuous with $f(y_0) = x_0$. If $\langle Y, \mathcal{V}\rangle$ is connected, then there exists at most one lifting $\tilde{f}$ of f with $\tilde{f}(y_0) = \tilde{x}_0$.*

Proof. Assume that $\tilde{f}_1, \tilde{f}_2\colon Y \to \tilde{X}$ are continuous functions with

$$p \circ \tilde{f}_1 = p \circ \tilde{f}_2 = f, \quad \tilde{f}_1(y_0) = \tilde{f}_2(y_0) = \tilde{x}_0.$$

Consider the sets

$$A := \big\{y \in Y \mid \tilde{f}_1(y) = \tilde{f}_2(y)\big\}, \quad B := Y \backslash A.$$

① *We show that A is open:* Let $a \in A$ and choose an evenly covered neighbourhood U of $f(a)$. Let S be the sheet over U with $\tilde{f}_1(a) = \tilde{f}_2(a) \in S$. Then

$$W := \tilde{f}_1^{-1}(S) \cap \tilde{f}_2^{-1}(S)$$

is an open neighbourhood of a in Y. If $y \in W$, then $\tilde{f}_1(y), \tilde{f}_2(y) \in S$ and $p(\tilde{f}_1(y)) = f(y) = p(\tilde{f}_2(y))$. Since $p|_S$ is injective, it follows that $\tilde{f}_1(y) = \tilde{f}_2(y)$, i.e., $y \in A$.

② *We show that B is open:* Let $b \in B$ and choose an evenly covered neighbourhood U of $f(b)$. If $\tilde{f}_1(b)$ and $\tilde{f}_2(b)$ belong to the same sheet S over U, injectivity of $p|_S$ together with $p \circ \tilde{f}_1 = p \circ \tilde{f}_2$ implies that $\tilde{f}_1(b) = \tilde{f}_2(b)$. Thus $\tilde{f}_1(b)$ and $\tilde{f}_2(b)$ lie in different sheets over U, say S_1 and S_2. Set

$$W := \tilde{f}_1^{-1}(S_1) \cap \tilde{f}_2^{-1}(S_2).$$

Then W is an open neighbourhood of b in Y. We have $\tilde{f}_1(W) \subseteq S_1$ and $\tilde{f}_2(W) \subseteq S_2$, and $S_1 \cap S_2 = \varnothing$ implies that $W \subseteq B$.

Since Y is connected and $A \neq \varnothing$, we must have $B = \varnothing$. ❑

Note that the above proof simplifies if $\tilde{X}$ is Hausdorff: in this case, A is closed simply by continuity of $\tilde{f}_1$ and $\tilde{f}_2$.

The question whether a lifting exists is a deep issue which lies at the heart of the theory of covering spaces. In this place, we only consider a particular situation where existence of liftings can be shown.

The following elementary lemma will be used repeatedly.

Lemma 4.2.3. *Let $\langle Y, d\rangle$ be a compact metric space, and let $\{O_i \mid i \in I\}$ be an open cover of Y. Then there exists $\epsilon > 0$, such that every ball $U_\epsilon(y)$ with $y \in Y$ is contained in some set O_i.*

A number ϵ with the property stated in the lemma is called a *Lebesgue number* of the cover $\{O_i \mid i \in I\}$.

Proof of Lemma 4.2.3. For each $x \in Y$ we find $i(x) \in I$ with $x \in O_i$, and then $\epsilon(x) > 0$ with $U_{2\epsilon(x)}(x) \subseteq O_{i(x)}$. The balls $\{U_{\epsilon(x)}(x) \mid x \in Y\}$ form an open cover of Y, and since Y is compact we find $x_1, \ldots, x_n \in Y$ with

$$Y = \bigcup_{i=1}^{n} U_{\epsilon(x_i)}(x_i).$$

Set $\epsilon := \min\{\epsilon(x_1), \ldots, \epsilon(x_n)\}$. Given $y \in Y$, choose $l \in \{1, \ldots, n\}$ with $y \in U_{\epsilon(x_l)}(x_l)$. Then, for each $z \in U_\epsilon(y)$,

$$d(z, x_l) \leqslant d(z, y) + d(y, x_l) < \epsilon + \epsilon(x_l) \leqslant 2\epsilon(x_l).$$

This shows that $U_\epsilon(y) \subseteq U_{2\epsilon(x_l)}(x_l) \subseteq O_{i(x_l)}$. ❑

We give a general result on existence of a diagonal fill-in. It can be seen as proving existence of a lifting with prescribed initial values.

Theorem 4.2.4. *Let $\langle \tilde{X}, \tilde{\mathcal{T}}, p\rangle$ be a covering of $\langle X, \mathcal{T}\rangle$, let $\langle Y, \mathcal{V}\rangle$ be a topological space, let $a, b \in \mathbb{R}$ with $a < b$, and denote*

$$\iota\colon \begin{cases} Y & \to & Y \times [a,b] \\ y & \mapsto & (y,a) \end{cases}$$

Further, let $\tilde{f}\colon Y \to \tilde{X}$ and $F\colon Y \times [a,b] \to X$ be continuous maps with $F \circ \iota = p \circ \tilde{f}$. Then there exists a continuous map $\tilde{F}\colon Y \times [a,b] \to \tilde{X}$ with $\tilde{F} \circ \iota = \tilde{f}$ and $p \circ \tilde{F} = F$.

$$\begin{tikzcd} Y \arrow[r, "\tilde{f}"] \arrow[d, "\iota"'] & \tilde{X} \arrow[d, "p"] \\ Y \times [a,b] \arrow[r, "F"'] \arrow[ur, dashed, "\tilde{F}"] & X \end{tikzcd}$$

If $\langle Y, \mathcal{V}\rangle$ is connected, then $\tilde{F}$ is unique.

If $Y = \varnothing$, there is nothing to prove. Hence, assume throughout that $Y \neq \varnothing$. The uniqueness statement follows immediately from Theorem 4.2.2 since $\tilde{F}$ is a lifting of F with prescribed values at $Y \times \{a\}$. The existence result is the major part of the theorem.

The proof of Theorem 4.2.4 is slightly technical. Before we dive into the details, let us illustrate the nature of the theorem by deducing that paths always can be lifted.

Corollary 4.2.5. *Let $\langle \tilde{X}, \tilde{\mathcal{T}}, p\rangle$ be a covering of $\langle X, \mathcal{T}\rangle$, let $f\colon [a,b] \to X$ be a path in X, set $x_0 := f(a)$, and let $\tilde{x}_0$ be an element of the fiber of x_0. Then there exists a unique lifting $\tilde{f}\colon [a,b] \to \tilde{X}$ of f with $\tilde{f}(a) = \tilde{x}_0$.*

Proof. Let Y be a one-element set, say $Y := \{*\}$, endowed with the discrete topology. Let $\phi\colon Y \to \tilde{X}$ be the map $\phi(*) := \tilde{x}_0$, and $F\colon Y \times [a,b] \to X$ be the map $F(*,t) := f(t)$. Theorem 4.2.4 provides us with a diagonal fill-in

$$\begin{tikzcd} \{*\} \arrow[r, "\phi"] \arrow[d, "\iota"'] & \tilde{X} \arrow[d, "p"] \\ \{*\} \times [a,b] \arrow[r, "F"'] \arrow[ur, dashed, "\tilde{F}"] & X \end{tikzcd}$$

Set $\tilde{f}(t) := \tilde{F}(*,t)$. Then

$$p(\tilde{f}(t)) = p(\tilde{F}(*,t)) = F(*,t) = f(t) \text{ for } t \in [a,b],$$
$$\tilde{f}(a) = \tilde{F}(*,a) = (\tilde{F} \circ \iota)(*) = \phi(*) = \tilde{x}_0.$$

Uniqueness is clear by Theorem 4.2.2. ❑

The main step towards the proof of Theorem 4.2.4 is to establish a local version.

Lemma 4.2.6. *Let data be given as in the theorem. For each point $y \in Y$ there exists an open neighbourhood W_y of y and a continuous function $\tilde{F}_y\colon W_y \times [a,b] \to \tilde{X}$, such that*

$$\begin{tikzcd} W_y \arrow[r, "\tilde{f}|_{W_y}"] \arrow[d, "\iota|_{W_y}"'] & \tilde{X} \arrow[d, "p"] \\ W_y \times [a,b] \arrow[r, "F|_{W_y \times [a,b]}"'] \arrow[ur, dashed, "\tilde{F}_y"] & X \end{tikzcd}$$

Proof. Let $y \in Y$ be fixed. For each $s \in [a,b]$ choose an evenly covered neighbourhood U_s of the point $F(y,s) \in X$. By continuity of F, choose open neighbourhoods $V_s \subseteq Y$ of y and $V_s' \subseteq [a,b]$ of s with

$$F(V_s \times V_s') \subseteq U_s.$$

The family $\{V_s' \mid s \in [a,b]\}$ is an open cover of $[a,b]$. Let $\epsilon > 0$ be a Lebesgue number for this cover. Choose $n \in \mathbb{N}$ with $\frac{b-a}{n} < \epsilon$, and consider the partition of $[a,b]$ given by the points

$$t_k := a + k \cdot \frac{b-a}{n} \text{ for } k = 0, \dots, n.$$

For each $k \in \{1, \dots, n\}$, choose $s_k \in [a,b]$ with $[t_{k-1}, t_k] \subseteq V_{s_k}'$, and set

$$W_y := \bigcap_{k=1}^{n} V_{s_k}.$$

Then W_y is an open neighbourhood of y. This construction of W_y ensures that

$$\forall k \in \{1, \dots, n\}.\ F\big(W_y \times [t_{k-1}, t_k]\big) \subseteq U_{s_k} \tag{4.1}$$

Our aim is to inductively construct continuous functions $\tilde{F}_{y,k}\colon W_y \times [a, t_k] \to \tilde{X}$, $k = 0, \dots, n$, with

$$\begin{tikzcd} W_y \arrow[r, "\tilde{f}|_{W_y}"] \arrow[d, "\iota|_{W_y}"'] & \tilde{X} \arrow[d, "p"] \\ W_y \times [a, t_k] \arrow[r, "F|_{W_y \times [a,t_k]}"'] \arrow[ur, dashed, "\tilde{F}_{y,k}"] & X \end{tikzcd} \tag{4.2}$$

▷ *Base case:* Set

$$\tilde{F}_{y,0}\colon \begin{cases} W_y \times \{a\} & \to & \tilde{X} \\ (z,a) & \mapsto & \tilde{f}(z) \end{cases}$$

Clearly, $\tilde{F}_{y,0}$ is continuous and satisfies $\tilde{F}_{y,0} \circ \iota = \tilde{f}$. By the assumption of the theorem, we have

$$(p \circ \tilde{F}_{y,0})(z,a) = p(\tilde{f}(z)) = (F \circ \iota)(z) = F(z,a) \text{ for } z \in W_y.$$

▷ *Induction step:* Let $k \in \{1, \dots, n\}$ be given, and assume that $\tilde{F}_{y,k-1}$ has already been constructed. We want to continue $\tilde{F}_{y,k-1}$ to a function defined on $W_y \times [a, t_k]$.

Denote

$$\tilde{g}_k(z)\colon \begin{cases} W_y & \to & \tilde{X} \\ z & \mapsto & \tilde{F}_{y,k-1}(z, t_{k-1}) \end{cases}$$

We have

$$(p \circ \tilde{g}_k)(W_y) = (p \circ \tilde{F}_{y,k-1})(W_y \times \{t_{k-1}\}) = F(W_y \times \{t_{k-1}\}) \subseteq U_{s_k},$$

and hence $W_y = \tilde{g}_k^{-1}(p^{-1}(U_{s_k}))$. Denote by $S_{k,i}$, $i \in I$, the sheets over U_{s_k}. Then W_y can be written as the disjoint union of open sets

$$W_y = \bigcup_{i \in I} \tilde{g}_k^{-1}(S_{k,i}).$$

For each $i \in I$ set

$$\tilde{G}_{y,k,i} := \big(p|_{S_{k,i}}\big)^{-1} \circ F|_{\tilde{g}_k^{-1}(S_{k,i}) \times [t_{k-1}, t_k]} \colon \tilde{g}_k^{-1}(S_{k,i}) \times [t_{k-1}, t_k] \to \tilde{X}$$

This function is well-defined by (4.1), and clearly it is continuous and maps into $S_{k,i}$. Let $\tilde{G}_{y,k} \colon W_y \times [t_{k-1}, t_k] \to \tilde{X}$ be the unique function with

$$\forall i \in I.\ \tilde{G}_{y,k}|_{\tilde{g}_k^{-1}(S_{k,i}) \times [t_{k-1}, t_k]} = \tilde{G}_{y,k,i}.$$

By the gluing lemma, $\tilde{G}_{y,k}$ is continuous.

From the definition of $\tilde{G}_{y,k,i}$ we see that

$$\forall i \in I.\ p \circ \tilde{G}_{y,k,i} = F|_{\tilde{g}_k^{-1}(S_{k,i}) \times [t_{k-1}, t_k]} \tag{4.3}$$

in particular,

$$\forall i \in I\ \forall z \in \tilde{g}_k^{-1}(S_{k,i}).\ p\big(\tilde{G}_{y,k,i}(z, t_{k-1})\big) = F(z, t_{k-1}) = p\big(\tilde{F}_{y,k-1}(z, t_{k-1})\big)$$

Since $\tilde{F}_{y,k-1}(z, t_{k-1}) = \tilde{g}_k(z) \in S_{k,i}$ for all $z \in \tilde{g}_k^{-1}(S_{k,i})$, injectivity of $p|_{S_{k,i}}$ implies

$$\forall i \in I.\ \tilde{G}_{y,k}|_{\tilde{g}_k^{-1}(S_{k,i}) \times \{t_{k-1}\}} = \tilde{G}_{y,k,i}|_{\tilde{g}_k^{-1}(S_{k,i}) \times \{t_{k-1}\}} = \tilde{F}_{y,k-1}|_{\tilde{g}_k^{-1}(S_{k,i}) \times \{t_{k-1}\}}$$

Let $\tilde{F}_{y,k} \colon W_y \times [a, t_k] \to \tilde{X}$ be the unique function with

$$\tilde{F}_{y,k}|_{W_y \times [0, t_{k-1}]} = \tilde{F}_{y,k-1}, \quad \tilde{F}_{y,k}|_{W_y \times [t_{k-1}, t_k]} = \tilde{G}_{y,k}.$$

By the gluing lemma, $\tilde{F}_{y,k}$ is continuous. By the inductive hypothesis it satisfies $\tilde{F}_{y,k} \circ \iota = \tilde{f}$, and by the inductive hypothesis and (4.3) it satisfies $p \circ \tilde{F}_{y,k} = F|_{W_y \times [a, t_k]}$.

The function $\tilde{F}_y := \tilde{F}_{y,n}$ satisfies the properties required in the assertion of the lemma. ❑

Passing to the global result is not anymore difficult.

Proof of Theorem 4.2.4. The family $\{W_y \mid y \in Y\}$ is an open cover of Y. We claim that

$$\forall y_1, y_2 \in Y.\ \tilde{F}_{y_1}|_{(W_{y_1} \cap W_{y_2}) \times [a,b]} = \tilde{F}_{y_2}|_{(W_{y_1} \cap W_{y_2}) \times [a,b]} \tag{4.4}$$

Once this claim is established, we may define $\tilde{F}$ as the unique function with

$$\forall y \in Y.\ \tilde{F}|_{W_y \times [a,b]} = \tilde{F}_y$$

and $\tilde{F}$ will be continuous by the gluing lemma and make the required diagram commute since all functions $\tilde{F}_y$ do so.

To establish (4.4), let $y_1, y_2 \in Y$ and $z \in W_{y_1} \cap W_{y_2}$ be given. We have

$$\tilde{F}_{y_1}(z, a) = (\tilde{F}_{y_1} \circ \iota)(z) = \tilde{f}(z) = (\tilde{F}_{y_2} \circ \iota)(z) = \tilde{F}_{y_2}(z, a),$$
$$(p \circ \tilde{F}_{y_1})(z, t) = F(z, t) = (p \circ \tilde{F}_{y_2})(z, t) \text{ for } t \in [a, b].$$

This shows that both paths $t \mapsto \tilde{F}_{y_1}(z, t)$ and $t \mapsto \tilde{F}_{y_2}(z, t)$ are liftings of $t \mapsto F(z, t)$ with initial point $\tilde{f}(z)$. By uniqueness of liftings, they must coincide, i.e., $\tilde{F}_{y_1}(z, t) = \tilde{F}_{y_2}(z, t)$ for all $t \in [a, b]$. ❑

4.3 The monodromy theorem

Another consequence of Theorem 4.2.4 is a topological version of the *monodromy theorem*.

Theorem 4.3.1. *Let $\langle \tilde{X}, \tilde{\mathcal{T}}, p\rangle$ be a covering of $\langle X, \mathcal{T}\rangle$, let $x_0, x_1 \in X$, and let $f, g\colon [a,b] \to X$ be two FEP-homotopic paths in X which both have initial point x_0 and terminal point x_1. Let $\tilde{x}_0$ be in the fiber of x_0, and let $\tilde{f}$ and $\tilde{g}$ be the liftings of f and g, respectively, with inital point $\tilde{x}_0$. Then $\tilde{f}$ and $\tilde{g}$ have the same terminal point and are FEP-homotopic.*

Proof. Let $H\colon [a,b] \times [0,1]$ be a FEP-homotopy from f to g, i.e., H is continuous with

$$\big(H|_{[a,b]\times\{0\}} = f \ \wedge\ H|_{[a,b]\times\{1\}} = g\big) \ \wedge\ \big(H|_{\{a\}\times[0,1]} = x_0 \ \wedge\ H|_{\{b\}\times[0,1]} = x_1\big)$$

To fit notation, we switch variables: set $F(s,t) := H(t,s)$. Moreover, let $\tilde{f}\colon [0,1] \to \tilde{X}$ be the constant function $\tilde{f}(s) := \tilde{x}_0$. Theorem 4.2.4 provides us with a diagonal fill-in

$$\begin{tikzcd}
{[0,1]} \arrow[r, "\tilde{f}"] \arrow[d, "\iota"'] & \tilde{X} \arrow[d, "p"] \\
{[0,1] \times [a,b]} \arrow[r, "F"'] \arrow[ur, dashed, "\tilde{F}"] & X
\end{tikzcd}$$

We switch back the roles of variables: set

$$\tilde{H}(t,s) := \tilde{F}(s,t).$$

Then we have

$$\begin{aligned}
&\tilde{H}(a,s) = \tilde{F}(s,a) = (\tilde{F}\circ\iota)(s) = \tilde{f}(s) = \tilde{x}_0 \text{ for } s \in [0,1],\\
&p\circ\tilde{H}(t,0) = p\circ\tilde{F}(0,t) = F(0,t) = H(t,0) = f(t) \text{ for } t \in [a,b],\\
&p\circ\tilde{H}(t,1) = p\circ\tilde{F}(1,t) = F(1,t) = H(t,1) = g(t) \text{ for } t \in [a,b].
\end{aligned}$$

By uniqueness of lifting, it follows that $\tilde{f}(t) = \tilde{H}(t,0)$ and $\tilde{g}(t) = \tilde{H}(t,1)$ for all $t \in [a,b]$.

Let $\tilde{k}$ be the constant path $\tilde{k}(s) := \tilde{f}(b)$, $s \in [0,1]$. We have

$$(p\circ\tilde{k})(s) = p(\tilde{f}(b)) = p(\tilde{H}(b,0)) = H(b,0) = x_1 = H(b,s) \text{ for } s \in [0,1],$$

i.e., $\tilde{k}$ is a lifting of the path $h\colon s \mapsto H(b,s)$ with initial point $\tilde{f}(b)$. Since $p\circ\tilde{F} = F$, also the path $s \mapsto \tilde{H}(b,s)$ is a lifting of h. It also has initial point $\tilde{H}(b,0) = \tilde{f}(b)$, and uniqueness of lifting implies

$$\tilde{k}(s) = \tilde{H}(b,s) \text{ for } s \in [0,1].$$

In particular, $\tilde{f}(b) = \tilde{H}(b,1) = \tilde{g}(b)$. Clearly, now $\tilde{H}$ is a FEP-homotopy from $\tilde{f}$ to $\tilde{g}$. ❑

Let us give a corollary to illustrate the power of the monodromy theorem. The importance of this discussion is not in the proven statement, but in the concepts occurring in its proof. We will meet the arguments made here again later in a more general and systematic context.

Corollary 4.3.2. *Let $Q(z) = \sum_{k=0}^{n} a_k z^n$ be a polynomial with complex coefficients. If $n \geqslant 1$ and $a_n \neq 0$, then there exists a point $z \in \mathbb{C}$ with $Q(z) = 0$.*

Proof. For $r > 0$ consider the functions

$$H\colon \begin{cases} [0,2\pi]\times[0,1] & \to \quad \mathbb{C} \\ \qquad (t,s) & \mapsto \quad a_n r^n e^{int} + s\sum_{k=0}^{n-1} a_k r^k e^{ikt} \end{cases}$$

$$K\colon \begin{cases} [0,2\pi]\times[0,1] & \to \quad \mathbb{C} \\ \qquad (t,s) & \mapsto \quad Q(s\cdot re^{it}) \end{cases}$$

Then H and K are continuous, and

$$\begin{aligned} &\forall t\in[0,2\pi].\ H(t,0) = a_n r^n e^{int} \wedge H(t,1) = Q(re^{it}) \\ &\forall s\in[0,1].\ H(0,s) = H(2\pi,s) \\ &\forall t\in[0,2\pi].\ K(t,0) = a_0 \wedge K(t,1) = Q(re^{it}) \\ &\forall s\in[0,1].\ K(0,s) = K(2\pi,s) \end{aligned}$$

Now choose r such that

$$r > \max\Big\{1, \frac{1}{|a_n|}\sum_{k=0}^{n-1}|a_k|\Big\}.$$

Then the function H maps into $\mathbb{C}\backslash\{0\}$, and we can rescale and rotate H in such a way that the resulting map $\tilde{H}$ maps into S^1 and keeps endpoints fixed:

$$\tilde{H}(t,s) := \frac{H(t,s)}{|H(t,s)|}\cdot\frac{|H(0,s)|}{H(0,s)} \text{ for } (t,s)\in[0,2\pi]\times[0,1].$$

Clearly, $\tilde{H}$ is continuous.

Assume now that Q has no zeroes. Then the function K maps into $\mathbb{C}\backslash\{0\}$, and we can make the same construction:

$$\tilde{K}(t,s) := \frac{K(t,s)}{|K(t,s)|}\cdot\frac{|K(0,s)|}{K(0,s)} \text{ for } (t,s)\in[0,2\pi]\times[0,1].$$

Plugging together $\tilde{H}$ and $\tilde{K}$, we obtain a FEP-homotopy between the paths $f(t) := e^{int}$ and $g(t) := 1$ for $t\in[0,2\pi]$.

Consider the covering of S^1 given by the space $\mathbb{R}$ with covering projection $p(t) := e^{2\pi it}$. The liftings of f and g with initial point 0 are obviously given as

$$\tilde{f}(t) := \frac{n}{2\pi}t, \quad \tilde{g}(t) := 0 \text{ for } t\in[0,2\pi].$$

By the monodromy theorem the liftings must have the same endpoint, i.e., $n = 0$. ❑

4.4 The lifting criterion

We already saw that coverings are closely connected with fundamental groups. Existence of lifting of paths and homotopies (the monodromy theorem) plays a crucial role when working with π_1. The following theorem is the *lifting criterion*.

Theorem 4.4.1. *Let $\langle X, \mathcal{T}\rangle$ be a topological space and $\langle \tilde{X}, \tilde{\mathcal{T}}, p\rangle$ be a covering of $\langle X, \mathcal{T}\rangle$. Further, let $\langle Y, \mathcal{V}\rangle$ be a pathwise connected topological space and $\phi\colon Y \to X$ a continuous map. Fix a base point $y_0 \in Y$, set $x_0 := \phi(y_0)$, and fix $\tilde{x}_0 \in p^{-1}(\{x_0\})$.*

There exists a lifting $\tilde{\phi}\colon Y \to \tilde{X}$ of ϕ with $\tilde{\phi}(y_0) = \tilde{x}_0$ if and only if

$$\pi_1(\phi)\big(\pi_1(Y, y_0)\big) \subseteq \pi_1(p)\big(\pi_1(\tilde{X}, \tilde{x}_0)\big). \tag{4.5}$$

Necessity of the stated condition (4.5) is clear: if we have a lifting

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, "\tilde{\phi}"] \arrow[r, "\phi"'] & X \end{tikzcd}$$

then we can apply π_1 to obtain

$$\begin{tikzcd} & \pi_1(\tilde{X}, \tilde{x}_0) \arrow[d, "\pi_1(p)"] \\ \pi_1(Y, y_0) \arrow[ur, "\pi_1(\tilde{\phi})"] \arrow[r, "\pi_1(\phi)"'] & \pi_1(X, x_0) \end{tikzcd}$$

and this diagram immediately implies (4.5).

Proof of sufficiency of (4.5). Assume that (4.5) holds. The idea to construct a lifting is obtained by reverse engineering: If we already had a lifting, say $\tilde{\phi}\colon Y \to \tilde{X}$, then for every path f in Y the lifting $\tilde{f}\colon [0,1] \to \tilde{X}$ of the path $\phi \circ f\colon [0,1] \to X$ is given as $\tilde{\phi} \circ f$. In particular, we must have $\tilde{\phi}(f(1)) = \tilde{f}(1)$. Since Y is pathwise connected, every point of Y can be realised as $f(1)$ with suitable f.

① *Definition of a map $\tilde{\phi}$:* For each $y \in Y\backslash\{y_0\}$ choose a path $f_y\colon [0,1] \to Y$ with initial point y_0 and terminal point y. Such paths exist, remember Corollary 1.7.13. Moreover, we set $f_{y_0} := \mathbb{1}_{y_0}$. Let $\tilde{f}_y\colon [0,1] \to \tilde{X}$ be the lifting of the path $\phi \circ f_y\colon [0,1] \to X$ with initial point $\tilde{x}_0$, and define

$$\tilde{\phi}(y) := \tilde{f}_y(1).$$

The fact that $p \circ \tilde{\phi} = \phi$ is built in the definition. We have

$$p(\tilde{\phi}(y)) = p(\tilde{f}_y(1)) = f_y(1) = y.$$

Moreover, since the lifting of the constant path $\mathbb{1}_{y_0}$ with initial point $\tilde{x}_0$ is $\mathbb{1}_{\tilde{x}_0}$, we have

$$\tilde{\phi}(y_0) = \mathbb{1}_{\tilde{x}_0}(1) = \tilde{x}_0.$$

② *We show independence of the choice of f:* Let $y \in Y$ be fixed, and let f and g be two paths in Y which both have initial point y_0 and terminal y. Denote by $\tilde{f}$ and $\tilde{g}$ the liftings of $\phi \circ f$ and $\phi \circ g$, respectively, with initial point $\tilde{x}_0$. Now consider the loop

$$h := f \bullet g^{\leftarrow} \in \mathscr{L}(Y, y_0),$$

and let $\tilde{h}$ be the lifting of h with initial point $\tilde{x}_0$. We split $\tilde{h}$ into the parts lifting f and g, and set $\tilde{h}_f(t) := \tilde{h}(\frac{t}{2})$ and $\tilde{h}_g(t) := \tilde{h}(1 - \frac{t}{2})$. Then $\tilde{h}_f$ is the lifting of f with initial point $\tilde{x}_0$, i.e., $\tilde{h}_f = \tilde{f}$. The path $\tilde{h}_g$ is the lifting of g with initial point $\tilde{h}(1)$.

The inclusion (4.5) guarantees existence of a loop $\tilde{d} \in \mathscr{L}(\tilde{X}, \tilde{x}_0)$, such that

$$\phi \circ h \approx p \circ \tilde{d}.$$

We view $\tilde{d}$ as lifting of the path $p \circ \tilde{d}$ with initial point $\tilde{x}_0$, and recall that $\tilde{h}$ is the lifting of $\phi \circ h$ with the same initial point. By the monodromy theorem the terminal points of the paths $\tilde{d}$ and $\tilde{h}$ coincide, i.e.,

$$\tilde{h}(1) = \tilde{d}(1) = \tilde{x}_0.$$

Returning to the splitting of $\tilde{h}$ in the parts $\tilde{h}_f$ and $\tilde{h}_g$, we now see that $\tilde{h}_g$ is the lifting of g with initial point $\tilde{x}_0$, i.e., $\tilde{h}_g = \tilde{g}$. It follows that

$$\tilde{f}(1) = \tilde{h}_f(1) = \tilde{h}\big(\frac{1}{2}\big) = \tilde{h}_g(1) = \tilde{g}(1).$$

③ *We show that $\tilde{\phi}$ is continuous:* Again fix $y \in Y$. Choose an evenly covered neighbourhood $U \subseteq X$ of the point $\phi(y)$. Then $\phi^{-1}(U)$ is a neighbourhood of y, and hence we find a pathwise connected neighbourhood $V \subseteq Y$ of y with $\phi(V) \subseteq U$.

We use the freedom in the choice of the paths for the definition of $\tilde{\phi}$, which was established in the previous step, to show that $\tilde{\phi}(V)$ is pathwise connected. Given $z \in V$, choose a path $f_{y,z}$ in Y with initial point y and terminal point z. Denote by $\tilde{f}_{y,z}$ the lifing of $\phi \circ f_{y,z}$ with initial point $\tilde{f}_y(1)$. Then $f_y \bullet f_{y,z}$ is a path connecting y_0 with z, and $\tilde{f}_y \bullet \tilde{f}_{y,z}$ is its lifting with initial point $\tilde{x}_0$. Hence, $\tilde{\phi}(z) = \tilde{f}_{y,z}(1)$, and see that $\tilde{f}_{y,z}$ is a path connecting $\tilde{\phi}(y)$ with $\tilde{\phi}(z)$.

Our choice of V, and the fact that $p \circ \tilde{\phi} = \phi$ guarantees that $p(\tilde{\phi}(V)) \subseteq U$, and hence that

$$\tilde{\phi}(V) \subseteq p^{-1}(U) = \bigcup_{i \in I} S_i,$$

where S_i are the sheets over U. Since the sheets are open and pairwise disjoint, it follows that $\tilde{\phi}(V)$ lies entirely in one single sheet, say $\tilde{\phi}(V) \subseteq S_j$ with a particular $j \in I$. Knowing this, we can write

$$\tilde{\phi}(z) = \big((p|_{S_j})^{-1} \circ \phi\big)(z) \text{ for } z \in V,$$

and obtain that $\tilde{\phi}|_V$ is continuous. Since V is a neighbourhood of y, it follows that $\tilde{\phi}$ is continuous at the point y.

❑

Chapter 5

The fundamental group

Seeking a classification of topological spaces (up to homeomorphism), one associates algebraic objects with a topological space which are homeomorphism invariants. We define and study the fundamental group of a topological space. This is the probably simplest construction of the kind, yet already yields a powerful invariant.

5.1 Construction of the fundamental group

A path in a topological space is called a *loop*, if its initial and terminal points are equal. The common initial and terminal point is called the *base point* of the loop. We use the notation $\mathscr{L}(X, x_0)$ for the set of all loops defined on the unit interval and based at x_0:

$$\mathscr{L}(X, x_0) := \big\{ f\colon [0,1] \to X \mid f \text{ continuous}, f(0) = f(1) = x_0 \big\}.$$

The fundamental group is the set of all loops with fixed base point and up to FEP-homotopy.

Definition 5.1.1. Let $\langle X, \mathcal{T} \rangle$ be a topological space and $x_0 \in X$. Then we define

$$\pi_1(X, x_0) := \mathscr{L}(X, x_0) / {\approx}.$$

The set $\pi_1(X, x_0)$ is called the *fundamental group* of $\langle X, \mathcal{T} \rangle$ with base point x_0.

Given a continuous map between two topological spaces, say $\phi\colon X \to Y$, we naturally have a map between loops with corresponding base points, namely

$$\phi \circ . : \begin{cases} \mathscr{L}(X, x_0) & \to & \mathscr{L}(Y, \phi(x_0)) \\ f & \mapsto & \phi \circ f \end{cases}$$

By Proposition 1.6.8 (iv), we have

$$f \approx g \;\Rightarrow\; \phi \circ f \approx \phi \circ g$$

and hence the map $\phi \circ .$ can be pushed to the factors modulo FEP-homotopy.

Definition 5.1.2. If $\langle X, \mathcal{T}\rangle$ and $\langle Y, \mathcal{V}\rangle$ are topological spaces, $x_0 \in X$, $y_0 \in Y$, and $\phi\colon X \to Y$ is a continuous map with $\phi(x_0) = y_0$, then we define $\pi_1(\phi)$ as the unique map with (p denotes the canonical projection)

$$\begin{tikzcd} \mathscr{L}(X, x_0) \arrow[r, "f \mapsto \phi \circ f"] \arrow[d, "p"'] & \mathscr{L}(Y, y_0) \arrow[d, "p"] \\ \pi_1(X, x_0) \arrow[r, dashed, "\pi_1(\phi)"'] & \pi_1(Y, y_0) \end{tikzcd}$$

Written explicitly, this is

$$\big[\pi_1(\phi)\big](f/_{\approx}) := (\phi \circ f)/_{\approx} \text{ for } f/_{\approx} \in \pi_1(X, x_0).$$

Obviously, we have the usual computation rules (whenever the composition is defined and respects base points)

$$\pi_1(\phi \circ \psi) = \pi_1(\phi) \circ \pi_1(\psi), \quad \pi_1(\mathrm{id}_X) = \mathrm{id}_{\pi_1(X, x_0)}\,.$$

We have defined concatenation $f \cdot g$ and reversion f^{-1} of paths as partial operations depending on the domains and endpoints of the involved paths. Our goal is to push these operations to $\pi_1(X, x_0)$. Achieving this requires some technical effort.

We use the notation $\mathscr{P}(X, x_0, x_1)$ for the set of paths in X defined on the unit interval and having initial point x_0 and terminal point x_1:

$$\mathscr{P}(X, x_0, x_1) := \big\{ f\colon [0,1] \to X \mid f \text{ continuous}, f(0) = x_0, f(1) = x_1 \big\}.$$

Note that $\mathscr{L}(X, x_0) = \mathscr{P}(X, x_0, x_0)$.

First, we take care of different domains of paths by introducing suitable reparameterisations. Set

$$\alpha(t) := t - 1, \quad \beta(t) := 2t.$$

Then we denote

$$\blacksquare : \begin{cases} \mathscr{P}(X, x_0, x_1) \times \mathscr{P}(X, x_1, x_2) & \to & \mathscr{P}(X, x_0, x_2) \\ (f, g) & \mapsto & \big[f \cdot (g \circ \alpha)\big] \circ \beta \end{cases}$$

$$.^{\leftarrow} : \begin{cases} \mathscr{P}(X, x_0, x_1) & \to & \mathscr{P}(X, x_1, x_0) \\ f & \mapsto & f^{-1} \circ \alpha \end{cases}$$

Explicitly, this is

$$(f \bullet g)(t) = \begin{cases} f(2t) & \text{if } t \in [0, \frac{1}{2}] \\ g(2t-1) & \text{if } t \in [\frac{1}{2}, 1] \end{cases}$$
$$f^{\leftarrow}(t) = f(1-t) \text{ for } t \in [0,1]$$

Note that, obviously, $(f^{\leftarrow})^{\leftarrow} = f$.

Second, we check compatibility with FEP-homotopy. By Proposition 1.6.8 (ii) and (v), we have

$$\forall f_1, f_2 \in \mathscr{P}(X, x_0, x_1), g_1, g_2 \in \mathscr{P}(X, x_1, x_2).\ \big(f_1 \approx f_2 \wedge g_1 \approx g_2 \ \Rightarrow\ f_1 \bullet g_1 \approx f_2 \bullet g_2\big)$$

and by Proposition 1.6.8 (iii) and (v),

$$\forall f, g \in \mathscr{P}(X, x_0, x_1).\ \big(f \approx g \ \Rightarrow\ f^{\leftarrow} \approx g^{\leftarrow}\big)$$

We see that the maps $\bullet$ and $.^{\leftarrow}$ induce operations on equivalence classes of loops.

Definition 5.1.3. Let $\langle X, \mathcal{T}\rangle$ be a topological space and $x_0 \in X$. We define operations on $\pi_1(X, x_0)$ as the unique maps with (p denotes the canonical projection)

$$\begin{tikzcd} \mathscr{L}(X,x_0) \times \mathscr{L}(X,x_0) \arrow[r, "\bullet"] \arrow[d, "p"'] & \mathscr{L}(X,x_0) \arrow[d, "p"] \\ \pi_1(X,x_0) \times \pi_1(X,x_0) \arrow[r, dashed, "\cdot"'] & \pi_1(X,x_0) \end{tikzcd} \qquad \begin{tikzcd} \mathscr{L}(X,x_0) \arrow[r, ".^{\leftarrow}"] \arrow[d, "p"'] & \mathscr{L}(X,x_0) \arrow[d, "p"] \\ \pi_1(X,x_0) \arrow[r, dashed, ".^{-1}"'] & \pi_1(X,x_0) \end{tikzcd}$$

Written explicitly, this is

$$(f/_{\approx}) \cdot (g/_{\approx}) := (f \bullet g)/_{\approx} \text{ for } f/_{\approx}, g/_{\approx} \in \pi_1(X, x_0),$$
$$(f/_{\approx})^{-1} := (f^{\leftarrow})/_{\approx} \text{ for } f/_{\approx} \in \pi_1(X, x_0).$$

Moreover, we denote by $\mathbb{1} \in \pi_1(X, x_0)$ the equivalence class of the constant loop $\mathbb{1}_{x_0} : t \mapsto x_0$.

Theorem 5.1.4. *Let $\langle X, \mathcal{T}\rangle$ be a topological space and $x_0 \in X$. Then $\pi_1(X, x_0)$ is, with the operations defined above, a group.*

Let $\langle X, \mathcal{T}\rangle$ and $\langle Y, \mathcal{V}\rangle$ be topological spaces and $x_0 \in X$, $y_0 \in Y$. For every continuous map $\phi\colon X \to Y$ with $\phi(x_0) = y_0$, the map $\pi_1(\phi)\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is a group homomorphism.

We prove validity of computation rules for paths rather than loops.

Lemma 5.1.5.

(i) $\forall f \in \mathscr{P}(X, x_0, x_1), g \in \mathscr{P}(X, x_1, x_2), h \in \mathscr{P}(X, x_2, x_3).\ (f \bullet g) \bullet h \sim_r f \bullet (g \bullet h)$

(ii) $\forall f \in \mathscr{P}(X, x_0, x_1).\ f \bullet \mathbb{1}_{x_1} \approx \mathbb{1}_{x_0} \bullet f \approx f$

(iii) $\forall f \in \mathscr{P}(X, x_0, x_1).\ f^{\leftarrow} \bullet f \approx \mathbb{1}_{x_0} \wedge f \bullet f^{\leftarrow} \approx \mathbb{1}_{x_1}$

Proof.

① *We show associativity:* Use Lemma 1.6.4 (i) and (iii), and Proposition 1.6.6 (ii), to compute

$$\begin{aligned}(f \centerdot g) \centerdot h &= \big[\big(f \cdot (g\circ\alpha)\big)\circ\beta\big] \centerdot h = \Big(\big[\big(f \cdot (g\circ\alpha)\big)\circ\beta\big] \cdot (h\circ\alpha)\Big)\circ\beta \\ &= \big[(f\circ\beta\circ\beta) \cdot (g\circ\alpha\circ\beta\circ\beta)\big] \cdot (h\circ\alpha\circ\beta) = (f\circ\beta\circ\beta) \cdot \big[(g\circ\alpha\circ\beta\circ\beta) \cdot (h\circ\alpha\circ\beta)\big] \\ &\sim_r (f\circ\beta) \cdot \big[(g\circ\beta\circ\alpha\circ\beta) \cdot (h\circ\alpha\circ\beta\circ\alpha\circ\beta)\big] = \Big(f \cdot \big[(g\circ\beta\circ\alpha) \cdot (h\circ\alpha\circ\beta\circ\alpha)\big]\Big)\circ\beta \\ &= \Big(f \cdot \big[\big(g \cdot (h\circ\alpha)\big)\circ\beta\circ\alpha\big]\Big)\circ\beta = f \centerdot \Big(\big(g \cdot (h\circ\alpha)\big)\circ\beta\Big) = f \centerdot (g \centerdot h).\end{aligned}$$

② *We show that $\mathbb{1}$ acts as unit element:* Unfolding the definition gives

$$f \centerdot \mathbb{1}_{x_1} = \big(f \cdot (\mathbb{1}_{x_0} \circ \alpha)\big) \circ \beta = \begin{cases} f(2t) & \text{if } t \in [0, \frac{1}{2}] \\ x_1 & \text{if } t \in [\frac{1}{2}, 1] \end{cases}$$

$$\mathbb{1}_{x_0} \centerdot f = \big(\mathbb{1}_{x_0} \cdot (f \circ \alpha)\big) \circ \beta = \begin{cases} x_0 & \text{if } t \in [0, \frac{1}{2}] \\ f(2t-1) & \text{if } t \in [\frac{1}{2}, 1] \end{cases}$$

The maps

$$H_1(t,s) := \begin{cases} f\big((1+s)t\big) & \text{if } t \in [0, \frac{1}{1+s}], s \in [0,1] \\ x_1 & \text{if } t \in [\frac{1}{1+s}, 1], s \in [0,1] \end{cases}$$

$$H_2(t,s) := \begin{cases} x_0 & \text{if } t \in [0, \frac{s}{2}], s \in [0,1] \\ f\big((t - \frac{s}{2})\frac{1}{1-\frac{s}{2}}\big) & \text{if } t \in [\frac{s}{2}, 1], s \in [0,1] \end{cases}$$

are well-defined and continuous (by the gluing lemma), and we see that

$$f \approx f \centerdot \mathbb{1}_{x_1} \text{ and } f \approx \mathbb{1}_{x_0} \centerdot f.$$

③ *We show that $.^{\leftarrow}$ acts as inverse element:* Unfolding the definition gives

$$(f \centerdot f^{\leftarrow})(t) = \begin{cases} f(2t) & \text{if } t \in [0, \frac{1}{2}] \\ f(2-2t) & \text{if } t \in [\frac{1}{2}, 1] \end{cases}$$

The map

$$H(t,s) := \begin{cases} f(2st) & \text{if } t \in [0, \frac{1}{2}], s \in [0,1] \\ f(s(2-2t)) & \text{if } t \in [\frac{1}{2}, 1], s \in [0,1] \end{cases}$$

is well-defined and continuous, and we see that $f \centerdot f^{\leftarrow} \approx \mathbb{1}_{x_0}$.

Applying the already proven with $f^{\leftarrow}$ instead of f, yields that $f^{\leftarrow} \centerdot f \approx \mathbb{1}_{x_1}$.

❑

Proof of Theorem 5.1.4. The fact that $\pi_1(X, x_0)$ is a group follows immediately from the lemma. For associativity remember Proposition 1.6.8 (v).

It remains to check that $\pi_1(\phi)$ is a homomorphism. To this end, let $f, g \in \mathscr{L}(X, x_0)$ be given. Lemma 1.6.4 (ii) yields

$$\phi \circ (f \centerdot g) = \phi \circ \big(f \cdot (g \circ \alpha)\big) \circ \beta = \big((\phi \circ f) \cdot (\phi \circ g \circ \alpha)\big) \circ \beta = (\phi \circ f) \centerdot (\phi \circ g),$$

and we obtain

$$\pi_1(\phi)\big(f/_{\approx} \cdot g/_{\approx}\big) = \big[\phi \circ (f \centerdot g)\big]/_{\approx} = \big[(\phi \circ f) \centerdot (\phi \circ g)\big]/_{\approx} = (\phi \circ f)/_{\approx} \cdot (\phi \circ g)/_{\approx}.$$

❑

As a corollary we obtain that $\pi_1(X, x_0)$ is a homeomorphism invariant.

Corollary 5.1.6. *Let $\langle X, \mathcal{T}\rangle$ and $\langle Y, \mathcal{V}\rangle$ be topological space, and $x_0 \in X$, $y_0 \in Y$. If $\phi\colon X \to Y$ is a homeomorphism with $\phi(x_0) = y_0$, then $\pi_1(\phi)\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism.*

Proof. We have $\phi^{-1} \circ \phi = \mathrm{id}_X$ and $\phi \circ \phi^{-1} = \mathrm{id}_Y$, and hence

$$\pi_1(\phi^{-1}) \circ \pi_1(\phi) = \pi_1(\mathrm{id}_X) = \mathrm{id}_{\pi_1(X,x_0)}, \quad \pi_1(\phi) \circ \pi_1(\phi^{-1}) = \pi_1(\mathrm{id}_Y) = \mathrm{id}_{\pi_1(Y,y_0)},$$

and hence $\pi_1(\phi)$ is an isomorphism with inverse $\pi_1(\phi^{-1})$. ❑

It is often a difficult task to compute the fundamental group of a given space, even for spaces as simple as a circle. However, one class of spaces whose fundamental group is trivial can easily be given.

Example 5.1.7. Let $\langle Z, \mathcal{T}\rangle$ be a topological vector space, X a convex subset of Z, and $x_0 \in X$. Then $\pi(X, x_0) = \{\mathbb{1}\}$.

To see this, let $f \in \mathscr{L}(X, x_0)$. The function

$$H(t, s) := (1 - s)f(t) + sx_0 \text{ for } t, s \in [0, 1]$$

is continuous and maps into X by convexity. Clearly, it is a FEP-homotopy from f to the constant path $\mathbb{1}_{x_0}$.

Remark 5.1.8. While the fundamental group is a powerful homeomorphism invariant of a topological space, it is far to weak to classify topological spaces up to homeomorphism.

We just saw that every convex subset of a topological vector space has trivial fundamental group. Thus, for example, $\pi_1([-1, 1], 0) = \pi_1(\{0\}, 0)$. However, the interval and the singleton space cannot be homeomorphic.

5.2 The fundamental group of the circle

In this section we compute the fundamental group of the unit circle in the plane (for practical reasons, we identify $\mathbb{R}^2$ with the complex numbers $\mathbb{C}$)

$$S^1 := \{z \in \mathbb{C} \mid |z| = 1\}.$$

Geometric intuition suggests that up to deformations a loop in S^1 should correspond to the number of times it winds around the origin. The proof that this intuition is indeed correct relies on the machinery of covering spaces.

Theorem 5.2.1. $\pi_1(S^1, 1) \cong \mathbb{Z}$

Proof. The map

$$f_n\colon \begin{cases} [0,1] & \to & S^1 \\ t & \mapsto & e^{2\pi i n t} \end{cases}$$

is a loop in S^1 with base point 1. We define our candidate for the required isomorphism as

$$\Phi\colon \begin{cases} \mathbb{Z} & \to & \pi_1(S^1) \\ n & \mapsto & f_n/_{\approx} \end{cases}$$

We are going to analyse this map making use of the covering exhibited in Example 4.1.10. Recall that it is given by the space $\mathbb{R}$ and the covering projection $p(t) := e^{2\pi i t}$.

The lifting of the path f_n with initial point 0 is the path

$$\tilde{f}_n\colon \begin{cases} [0,1] & \to & \mathbb{R} \\ t & \mapsto & nt \end{cases}$$

① *We show that Φ is injective:* Let $n, m \in \mathbb{Z}$, and assume that $f_n \approx f_m$. By the monodromy theorem, we have $n = \tilde{f}_n(1) = \tilde{f}_m(1) = m$.

② *We show that Φ is surjective:* Let $g \in \mathscr{L}(S^1, 1)$ be given, and let $\tilde{g}\colon [0,1] \to \mathbb{R}$ be the lifting of g with initial point 0. Then

$$e^{2\pi i \tilde{g}(1)} = (p \circ \tilde{g})(1) = g(1) = 1,$$

and hence $\tilde{g}(1) \in \mathbb{Z}$. Set $n := \tilde{g}(1)$. The paths $\tilde{g}$ and $\tilde{f}_n$ have the same initial and terminal points. The map $H(t,s) := (1-s)\tilde{g}(t) + s\tilde{f}_n(t)$ is a FEP-homotopy from $\tilde{g}$ to $\tilde{f}_n$, and thus therefore $\tilde{g} \approx \tilde{f}_n$. It follows that

$$g = p \circ \tilde{g} \approx p \circ \tilde{f}_{\tilde{g}(1)} = f_{\tilde{g}(1)}.$$

③ *We show that Φ is a group homomorphism:* Let $n, m \in \mathbb{Z}$ be given. The lifting of f_n with initial point m is given as $\tilde{f}_n^{(m)}(t) := m + nt$. The lifting of $f_m \blacksquare f_n$ with initial point 0 is thus equal to $\tilde{g} := \tilde{f}_m \blacksquare \tilde{f}_n^{(m)}$. This is a path with initial point 0 and terminal point $m+n$, and thus $\tilde{g} \approx \tilde{f}_{m+n}$. Applying the covering projection, it follows that $f_m \blacksquare f_n \approx f_{m+n}$.

❑

To demonstrate a typical way how the fundamental group can be applied, we deduce Brouwer's fixed point theorem in dimension 2.

Let B^n be the *n-ball* (here $\|.\|$ denotes the euclidean norm)

$$B^n := \{x \in \mathbb{R}^n \mid \|x\| \leqslant 1\}.$$

Corollary 5.2.2. *Every continuous map $\phi\colon B^2 \to B^2$ has a fixed point.*

We start from the usual lemma (for completeness we provide its proof).

Lemma 5.2.3. *Assume that $\phi\colon B^2 \to B^2$ is continuous and has no fixed point. Then there exists a continuous map $r\colon B^2 \to S^1$ with $r|_{S^1} = \mathrm{id}_{S^1}$.*

Proof. Let $x \in B^2$ and consider the equation

$$\|(1-\lambda)x + \lambda\phi(x)\|^2 = 1 \text{ where } \lambda \in \mathbb{R}. \tag{5.1}$$

The left side is a quadratic polynomial in λ, in fact,

$$P(\lambda) := \|(1-\lambda)x + \lambda\phi(x)\|^2 = \lambda^2\|\phi(x) - x\|^2 + \lambda\big(-2\|x\|^2 + 2\operatorname{Re}(x,\phi(x))\big) + \|x\|^2.$$

We have $P(0) = \|x\|^2 \leqslant 1$ and $\lim_{|\lambda|\to\infty} P(\lambda) = +\infty$. Thus the equation (5.1) has one solution $\lambda_x^- \in (-\infty, 0]$ and one $\lambda_x^+ \in [0,\infty)$. These two solutions coincide if and only if $\|x\| = 1$ (in which case 0 is the only solution of (5.1)). By the quadratic formula, we have

$$\lambda_x^- = \frac{1}{2\|\phi(x)-x\|^2}\left[\big(2\|x\|^2 - 2\operatorname{Re}(x,\phi(x))\big) - \sqrt{\big(2\|x\|^2 - 2\operatorname{Re}(x,\phi(x))\big)^2 - 4\|\phi(x)-x\|^2\|x\|^2}\right]$$

and we see that λ_x^- depends continuously on x.

The map defined by

$$r(x) := (1 - \lambda_x)x + \lambda_x \phi(x) \text{ for } x \in B^2$$

is thus continuous, and maps B^2 into S^1 with $r(x) = x$ if $\|x\| = 1$. ❑

Proof of Corollary 5.2.2. Assume $\phi\colon B^2 \to B^2$ is continuous and has no fixed point. Choose a continuous function $r\colon B^2 \to S^1$ with $r|_{S^1} = \mathrm{id}_{S^1}$ as in the lemma, and denote by $\iota\colon S^1 \to B^2$ be the set-theoretic inclusion map. We have

$$\begin{tikzcd} B^2 \arrow[dr, "r"] & \\ S^1 \arrow[u, "\iota"] \arrow[r, "\mathrm{id}"'] & S^1 \end{tikzcd} \quad \text{and hence} \quad \begin{tikzcd} \pi_1(B^2) \arrow[dr, "\pi_1(r)"] & \\ \pi_1(S^1) \arrow[u, "\pi_1(\iota)"] \arrow[r, "\mathrm{id}"'] & \pi_1(S^1) \end{tikzcd}$$

We see that $\pi_1(r)$ is surjective, which contradicts the fact that $\pi_1(B^2) = \{\mathbb{1}\}$ by convexity and $\pi_1(S^1) \cong \mathbb{Z}$ by Theorem 5.2.1. ❑

Remark 5.2.4. To prove Brouwer's fixed point theorem in arbitrary dimension would require machinery suitable to deal with higher dimensions: π_1 captures 1-dimensional loops.

Still, the method presented above may be thought of as one of the "intrinsically right" approaches. Another common method, working with the simplex instead of the ball and using Sperner's lemma, also exhibits an important method and is a "right" approach. However, it goes towards homology theory and thus rather belongs to algebraic topology.

The result certainly has nothing to do with differentiability. The proof given in many analysis books, using differentiable approximations and the Jacobian, completely misses the point.

5.3 Some properties of π_1

The group $\pi_1(X, x_0)$ in general depends on the base point x_0, but this dependency is easily understood.

Lemma 5.3.1. *Let $\langle X, \mathcal{T} \rangle$ be a topological space and let $h\colon [0,1] \to X$ be a path. Then*

$$\pi_1(X, h(0)) \cong \pi_1(X, h(1)).$$

Proof. We have the map

$$\phi_h\colon \begin{cases} \mathscr{L}(X, h(0)) & \to & \mathscr{L}(X, h(1)) \\ f & \mapsto & h^{\leftarrow} \centerdot f \centerdot h \end{cases}$$

Let $f, g \in \mathscr{L}(X, x_0)$. If $f \approx g$, then also $\phi_h(f) \approx \phi_h(g)$. We define Φ_h as the unique map with (p is the canoncial projection)

$$\begin{tikzcd} \mathscr{L}(X, h(0)) \arrow[r, "\phi_h"] \arrow[d, "p"'] & \mathscr{L}(X, h(1)) \arrow[d, "p"] \\ \pi_1(X, x_0) \arrow[r, dashed, "\Phi_h"'] & \pi_1(X, x_1) \end{tikzcd}$$

We have

$$(h^{\leftarrow} \bullet f \bullet h) \bullet (h^{\leftarrow} \bullet g \bullet h) \approx h^{\leftarrow} \bullet f \bullet (h \bullet h^{\leftarrow}) \bullet g \bullet h \approx h^{\leftarrow} \bullet (f \bullet g) \bullet h,$$
$$h \bullet (h^{\leftarrow} \bullet f \bullet h) \bullet h^{\leftarrow} \approx f, \quad h^{\leftarrow} \bullet (h \bullet f \bullet h^{\leftarrow}) \bullet h \approx f.$$

The first relation implies that $\Phi_h(f/_{\approx} \cdot g/_{\approx}) = \Phi_h(f/_{\approx}) \cdot \Phi_h(g/_{\approx})$, and the second that $\Phi_{h^{\leftarrow}} \circ \Phi_h = \mathrm{id}_{\pi_1(X,x_0)}$ and $\Phi_h \circ \Phi_{h^{\leftarrow}} = \mathrm{id}_{\pi_1(X,x_1)}$. ❑

In view of Lemma 5.3.1 it is justified to drop the base point and speak of the *fundamental group* $\pi_1(X)$ whenever $\langle X, \mathcal{T}\rangle$ is pathwise connected.

It is intuitively expected, and also easily proven, that one may restrict attention to pathwise connected spaces when dealing with fundamental groups.

Lemma 5.3.2. *Let $\langle X, \mathcal{T}\rangle$ be a topological space, $x_0 \in X$, and let C be the path-component of X with $x_0 \in C$. Then*

$$\pi_1(X, x_0) \cong \pi_1(C, x_0).$$

Proof. Let $\iota\colon C \to X$ be the inclusion map. Then we have the homomorphism

$$\pi_1(\iota)\colon \pi_1(C, x_0) \to \pi_1(X, x_0).$$

The image of a loop is a connected subset of X, and hence every loop in X with base point in C must lie entirely in C. Thus $\pi_1(\iota)$ is surjective. The image of a FEP-homotopy in X is a connected subset of X, and hence every FEP-homotopy in X between loops in X with base point in C must map entirely into C. Thus each two loops with a base point in C which are FEP-homotopic in X are also FEP-homotopic in C, and it follows that $\pi_1(\iota)$ is injective. ❑

Let us introduce a name for spaces with trivial fundamental group. The geometric intuition is that these are spaces which have no holes: every loop can be continuously deformed into a single point.

Definition 5.3.3. A topological space $\langle X, \mathcal{T}\rangle$ is called *simply connected*, if it is pathwise connected and $\pi_1(X) = \{\mathbb{1}\}$.

An equivalent formulation of this definition which is often practical reads as follows.

Lemma 5.3.4. *Let $\langle X, \mathcal{T}\rangle$ be a pathwise connected topological space. Then $\pi_1(X) = \{\mathbb{1}\}$, if and only if each two paths having the same initial point and the same terminal point are FEP-homotopic.*

Proof. To prove the forward implication, let $x_0, x_1 \in X$ and $f, g \in \mathcal{P}(X, x_0, x_1)$. Then $f \bullet g^{\leftarrow}$ is a loop in X with base point x_0, and hence we find a FEP-homotopy H from $f \bullet g^{\leftarrow}$ to the constant path at x_0. Explicitly, we have

$$H(t,0) = f(2t) \text{ for } t \in [0, \frac{1}{2}], \quad H(t,0) = g(2(1-t)) \text{ for } t \in [\frac{1}{2}, 1],$$
$$H(t,1) = H(0,s) = H(1,s) = x_0 \text{ for } t, s \in [0,1].$$

We define a map $K\colon [0,1] \times [0,1] \to X$ as

$$K(t,s) := \begin{cases} H(t, 2s) & \text{if } (t,s) \in [0,\frac{1}{2}] \times [0, \frac{1}{2}] \\ H(\frac{1}{2}, 4(1-t)s) & \text{if } (t,s) \in [\frac{1}{2}, 1] \times [0, \frac{1}{2}] \\ H(1-t, 2(1-s)) & \text{if } (t,s) \in [0,\frac{1}{2}] \times [\frac{1}{2}, 1] \\ H(\frac{1}{2}, 4(1-t)(1-s)) & \text{if } (t,s) \in [\frac{1}{2}, 1] \times [\frac{1}{2}, 1] \end{cases}$$

Inspecting this definition and remembering the above stated properties of H shows that K is well-defined, hence continuous by the gluing lemma, and satisfies

$$H(.,0) = f \bullet \mathbb{1}_{x_1}, \quad H(.,1) = g \bullet \mathbb{1}_{x_1}, \quad H(0,.) = x_0, \quad H(1,.) = x_1.$$

Since $f \bullet \mathbb{1}_{x_1} \approx f$ and $g \bullet \mathbb{1}_{x_1} \approx g$, it follows that $f \approx g$.

For the backward implication it is enough to note that f has the same initial and terminal point as $\mathbb{1}_{x_0}$. ❑

The argument in Example 5.1.7 which led to the fact that convex sets are simply connected deserves a more systematic treatment.

Definition 5.3.5. Let $\langle X, \mathcal{T}\rangle$ be a topological space.

(i) A subset Y of X is called a *retract* of X, if there exists a continuous map $r\colon X \to Y$ (here Y is endowed with the subspace topology $\mathcal{T}|_Y$) with $r|_Y = \mathrm{id}_Y$. Each map r with this property is called a *retraction*.

(ii) A subset Y of X is called a *deformation retract*, if there exists a continuous map $H\colon X \times [0,1] \to X$ with

$$\forall x \in X.\ H(x,0) = x, \quad H(X \times \{1\}) \subseteq Y, \quad \forall y \in Y, s \in [0,1].\ H(y,s) = y. \tag{5.2}$$

One should think of the map H in (ii) as a deformation of the identity (which is $x \mapsto H(x,0)$) to a retraction (namely, $x \mapsto H(x,1)$). In particular, if Y is a deformation retract of X, it is also a retract of X.

Proposition 5.3.6. *Let $\langle X, \mathcal{T}\rangle$ be a topological space, $Y \subseteq X$, $y_0 \in Y$, and denote by $\iota\colon Y \to X$ the inclusion map.*

(i) If Y is a retract of X and $r\colon X \to Y$ is a retraction, then $\pi_1(\iota)\colon \pi_1(Y, y_0) \to \pi_1(X, y_0)$ is injective with left-inverse $\pi_1(r)$.

(ii) If Y is a deformation retract of X, then $\pi_1(\iota)\colon \pi_1(Y, y_0) \to \pi_1(X, y_0)$ is bijective with inverse $\pi_1(r)$ (where r is a retraction given by a deformation H).

Proof. Assume that $r\colon X \to Y$ is a retraction. Then $r \circ \iota = \mathrm{id}_Y$, and $r(y_0) = y_0$, $\iota(y_0) = y_0$. Hence, $\pi_1(r) \circ \pi_1(\iota) = \mathrm{id}_{\pi_1(Y,y_0)}$.

Assume now that Y is a deformation retract of X, choose a map H as in Definition 5.3.5 (ii), and let $r\colon X \to Y$ be the retraction $r(x) := H(x,1)$. Let $f \in \mathscr{L}(X, y_0)$. Then the map

$$K\colon \begin{cases} [0,1] \times [0,1] & \to & X \\ (t,s) & \mapsto & H(f(t), s) \end{cases}$$

is a FEP-homotopy from f to $\iota \circ r \circ f$. Thus $\pi_1(\iota) \circ \pi_1(r) = \mathrm{id}_{\pi_1(X,y_0)}$. ❑

Example 5.3.7. Let $0 < r < 1 < R < \infty$, and consider the annulus

$$\mathbb{T}_{r,R} := \{z \in \mathbb{C} \mid r < |z| < R\}.$$

The unit circle S^1 is a subset of $\mathbb{T}_{r,R}$. The function

$$H(z, s) := \frac{z}{1 + s(|z| - 1)} \text{ for } z \in \mathbb{T}_{r,R}, s \in [0, 1]$$

maps $\mathbb{T}_{r,R} \times [0, 1]$ continuously into $\mathbb{T}_{r,R}$ and satisfies all properties (5.2). Hence, S^1 is a deformation retract of $\mathbb{T}_{r,R}$, and we conclude that $\pi_1(\mathbb{T}_{r,R}) = \mathbb{Z}$.

Definition 5.3.8. A topological space $\langle X, \mathcal{T}\rangle$ is called *contractible*, if there exists a point $x_0 \in X$ such that $\{x_0\}$ is a deformation retract of X.

Writing out the condition for X being contractible gives:

$$\exists x_0 \in X \;\exists H : X \times [0, 1] \to X \text{ continuous.}$$
$$H(., 0) = \mathrm{id}_X \wedge H(., 1) = x_0 \;\wedge\; H(x_0, .) = x_0 \quad (5.3)$$

Corollary 5.3.9. *If $\langle X, \mathcal{T}\rangle$ is contractible, then X is pathwise connected and $\pi_1(X) = \{\mathbb{1}\}$.*

Proof. Any point $x \in X$ can be connected with x_0 with the path $s \mapsto H(x, s)$, and by Proposition 5.3.6 (ii) we have $\pi_1(X) \cong \pi_1(\{x_0\}) = \{\mathbb{1}\}$. ❑

Example 5.3.10. Consider a slit annulus

$$\mathbb{T}_{r,R}\backslash(-\infty, 0] = \{\rho e^{i\theta} \in \mathbb{C} \,|\, r < \rho < R, \theta \in (-\pi, \pi)\}.$$

The function

$$H(\rho e^{i\theta}, s) := \frac{\rho}{1 + (1 - s)(\rho - 1)} e^{i\theta(1-s)} \text{ for } \rho \in (r, R), \theta \in (-\pi, \pi), s \in [0, 1]$$

maps $(\mathbb{T}_{r,R}\backslash(-\infty, 0]) \times [0, 1]$ continuously into $\mathbb{T}_{r,R}\backslash(-\infty, 0]$ and satisfies the properties (5.3) with the point $x_0 := 1$. Hence, $\mathbb{T}_{r,R}\backslash(-\infty, 0]$ is contractible, and we conclude that $\pi_1(\mathbb{T}_{r,R}\backslash(-\infty, 0]) = \{\mathbb{1}\}$.

5.4 Products and unions

The fundamental group of a product of spaces can be determined from the fundamental groups of the single spaces in a straightforward way.

Proposition 5.4.1. *Let $\langle X_i, \mathcal{T}_i\rangle$, $i \in I$, be a family of nonempty topological spaces, and consider the product $X := \prod_{i\in I} X_i$ endowed with the product topology. Moreover, let $z_i \in X_i$ for each $i \in I$, and set $z := (z_i)_{i\in I}$. Then*

$$\pi_1(X, z) \cong \prod_{i\in I} \pi_1(X_i, z_i).$$

Proof. Denote by $p_i : X \to X_i$ the canonical projections. Then we have the homomorphisms $\pi_1(p_i): \pi_1(X, z) \to \pi_1(X_i, z_i)$, and can consider their product map

$$\phi := \prod_{i\in I} \pi_1(p_i): \pi_1(X, z) \to \prod_{i\in I} \pi_1(X_i, z_i).$$

We are going to show that ϕ is bijective by constructing an inverse.

For loops $f_i \in \mathscr{L}(X_i, z_i)$, we can consider the product map $f := \prod_{i\in I} f_i \in \mathscr{L}(X, z)$. Assume that $g_i \in \mathscr{L}(X_i, z_i)$ are FEP-homotopic to f_i, and let $H_i\colon [0,1] \times [0,1] \to X_i$ be a FEP-homotopy from f_i to g_i. Then the product map

$$H := \prod_{i\in I} H_i\colon [0,1] \times [0,1] \to X$$

is a FEP-homotopy from f to $g := \prod_{i\in I} g_i$. Thus a map $\psi\colon \prod_{i\in I} \pi_1(X_i, z_i) \to \pi_1(X, z)$ is well-defined by setting

$$\psi\big((f_i/_{\approx})_{i\in I}\big) := \Big(\prod_{i\in I} f_i\Big)/_{\approx}. \tag{5.4}$$

We have

$$\begin{aligned}(\phi \circ \psi)\big((f_i/_{\approx})_{i\in I}\big) &= \phi\Big[\Big(\prod_{i\in I} f_i\Big)/_{\approx}\Big] = \Big(\prod_{j\in I} \pi_1(p_j)\Big)\Big[\Big(\prod_{i\in I} f_i\Big)/_{\approx}\Big] \\ &= \Big(\pi_1(p_j)\Big[\Big(\prod_{i\in I} f_i\Big)/_{\approx}\Big]\Big)_{j\in I} = \Big(\Big(p_j \circ \prod_{i\in I} f_i\Big)/_{\approx}\Big)_{j\in I} = (f_j/_{\approx})_{j\in I},\end{aligned}$$

and

$$(\psi \circ \phi)(f/_{\approx}) = \psi\Big[\Big(\prod_{i\in I} \pi_1(p_i)\Big)(f/_{\approx})\Big] = \psi\Big[\big((p_i \circ f)/_{\approx}\big)_{i\in I}\Big] = \Big(\prod_{i\in I}(p_i \circ f)\Big)/_{\approx} = f/_{\approx}.$$

❑

As an example, let us compute the fundamental group of a torus.

Example 5.4.2. The *torus* T is the product $T := S^1 \times S^1$. It can be embedded homeomorphically into $\mathbb{R}^3$: fix $0 < r < R < \infty$, and set

$$\Phi\big(e^{i\alpha}, e^{i\beta}\big) := \begin{pmatrix} (R + r\cos\alpha)\cos\beta \\ (R + r\cos\alpha)\sin\beta \\ r\sin\alpha \end{pmatrix}$$

This gives the well-known image of a doughnut.

From the above proposition, we obtain that $\pi_1(T) = \mathbb{Z} \times \mathbb{Z}$.

Also the fundamental group of a union of spaces can be determined from the fundamental groups of the single spaces (under some additional hypothesis). This is a much deeper construction, and we will return to it later. In this place, let us only observe that the group of a union cannot be too large.

Proposition 5.4.3. *Let $\langle X, \mathcal{T}\rangle$ be a topological space and let $\{U_i \mid i \in I\}$ be an open cover of X. Assume that*

(i) $\bigcap_{i\in I} U_i \neq \varnothing$

(ii) $\forall i, j \in I.\ U_i \cap U_j$ *pathwise connected*

Moreover, denote for each $i \in I$ by $\iota_i\colon U_i \to X$ the set-theoretic inclusion map. Then X is pathwise connected and the group $\pi_1(X)$ is generated by $\bigcup_{i\in I} \pi_1(\iota_i)(\pi_1(U_i))$.

Proof. The condition (ii) implies in particular that each set U_i is pathwise connected. Remembering Theorem 1.7.8 (iii), we obtain that X is pathwise connected.

Throughout the following fix a point $x_0 \in \bigcap_{i \in I} U_i$, and use this point as base point for computing fundamental groups.

Let $f \in \mathscr{L}(X, x_0)$. The family $\{f^{-1}(U_i) \mid i \in I\}$ is an open cover of $[0,1]$. Choose a Lebesgue number $\epsilon > 0$ for this cover. Next, choose $n \in \mathbb{N}$ with $\frac{1}{n} < \epsilon$, and consider the partition

$$t_k := k \cdot \frac{1}{n} \text{ for } k = 0, \ldots, n.$$

Finally, choose $i_k \in I$ for $k \in \{1, \ldots, n\}$, such that $[t_{k-1}, t_k] \subseteq f^{-1}(U_{i_k})$.

Since $U_{i_{k-1}} \cap U_{i_k}$ is pathwise connected and contains the point x_0, we find a path g_k in $U_{i_{k-1}} \cap U_{i_k}$ with initial point x_0 and terminal point $f(t_k)$. Let α_k be the affine map of $[0,1]$ onto $[t_{k-1}, t_k]$, and consider the paths $f_k \colon [0,1] \to X$ defined as

$$f_k := \begin{cases} (f|_{[0,t_1]} \circ \alpha_1) \blacksquare g_1^{\leftarrow} & \text{if } k = 1 \\ g_{k-1} \blacksquare (f|_{[t_{k-1},t_k]} \circ \alpha_k) \blacksquare g_k^{\leftarrow} & \text{if } k = 2, \ldots, n-1 \\ g_{n-1} \blacksquare (f|_{[t_{n-1},t_n]} \circ \alpha_n) & \text{if } k = n \end{cases}$$

We can represent f modulo FEP-homotopy as the product

$$\begin{aligned} f_1 \blacksquare \ldots \blacksquare f_n &= \big((f|_{[0,t_1]} \circ \alpha_1) \blacksquare g_1^{-1}\big) \blacksquare \big(g_1 \blacksquare (f|_{[t_1,t_2]} \circ \alpha_2) \blacksquare g_2^{-1}\big) \blacksquare \ldots \\ &\quad \ldots \blacksquare \big(g_{n-2} \blacksquare (f|_{[t_{n-2},t_{n-1}]} \circ \alpha_{n-1}) \blacksquare g_{n-1}^{-1}\big) \blacksquare \big(g_{n-1} \blacksquare (f|_{[t_{n-1},t_n]} \circ \alpha_n)\big) \\ &\approx (f|_{[0,t_1]} \circ \alpha_1) \blacksquare \big(g_1^{-1} \blacksquare g_1\big) \blacksquare (f|_{[t_1,t_2]} \circ \alpha_2) \blacksquare \big(g_2^{-1} \blacksquare g_2\big) \blacksquare \ldots \\ &\quad \ldots \blacksquare (f|_{[t_{n-2},t_{n-1}]} \circ \alpha_{n-1}) \blacksquare \big(g_{n-1}^{-1} \blacksquare g_{n-1}\big) \blacksquare (f|_{[t_{n-1},t_n]} \circ \alpha_n) \\ &\approx (f|_{[0,t_1]} \circ \alpha_1) \blacksquare (f|_{[t_1,t_2]} \circ \alpha_2) \blacksquare \ldots \\ &\quad \ldots \blacksquare (f|_{[t_{n-2},t_{n-1}]} \circ \alpha_{n-1}) \blacksquare (f|_{[t_{n-1},t_n]} \circ \alpha_n) \approx f \end{aligned}$$

Clearly, f_k is a loop with base point x_0, and hence we have in $\pi_1(X, x_0)$

$$f/_{\approx} = f_1/_{\approx} \cdot \ldots \cdot f_n/_{\approx}.$$

The loop f_k lies entirely in U_{i_k}, i.e., can be considered as an element of $\mathscr{L}(U_{i_k}, x_0)$. Strictly speaking, the codomain restriction of f_k to a map $\tilde{f}_k \colon [0,1] \to U_{i_k}$ belongs to $\mathscr{L}(U_{i_k}, x_0)$. Clearly, $f_k = [\pi_1(\iota_k)](\tilde{f}_k)$, and we see that $f/_{\approx}$ lies in the subgroup generated by

$$\bigcup_{k=1}^{n} \pi_1(\iota_k)\big(\pi_1(U_{i_k}, x_0)\big).$$

❑

As an example let us compute the fundamental group of a sphere with dimension at least 2.

Example 5.4.4. Let $n \geqslant 2$. The n-sphere S^n is simply connected.

Write the sphere as union of the two subsets obtained by removing the north pole and south pole, respectively. Using spherical coordinates (see, e.g., Remark 4.1.6) we can write

$$\begin{aligned} U_1 &:= S^n \backslash \{(0, \ldots, 0, 1)\} = \Phi\big([0, 2\pi]^{n-1} \times (0, \pi]\big) \\ U_2 &:= S^n \backslash \{(0, \ldots, 0, -1)\} = \Phi\big([0, 2\pi]^{n-1} \times [0, \pi)\big). \end{aligned}$$

Then

$$U_1 \cap U_2 = S^n \backslash \{(0,\dots,0,1),(0,\dots,0,-1)\} = \Phi\big([0,2\pi]^{n-1} \times (0,\pi)\big).$$

We see that U_1, U_2 and $U_1 \cap U_2$ are pathwise connected.

By means of the stereographic projection (taking as projection center the north- or south pole), U_1 and U_2 are homeomorphic to $\mathbb{R}^n$. Thus $\pi_1(U_1) = \pi_1(U_2) = \{\mathbb{1}\}$. Proposition 5.4.3 applies and yields that $\pi_1(S^n) = \{\mathbb{1}\}$.

5.5 The fundamental group of the projective space

The n-dimensional projective space $\mathbb{P}_n$ over the field of real numbers can be identified bijectively with a factor of the n-sphere. Namely, let $\sim \subseteq S^n \times S^n$ be the equivalence relation

$$x \sim y :\Leftrightarrow (x = y \ \vee \ x = -y)$$

Then $\mathbb{P}_n \cong S^n/_\sim$. Tacitly making this identification, we can topologise $\mathbb{P}_n$ with the final topology inherited from S^n via the canonical projection $p_n\colon S^n \to S^n/_\sim$. In this way, $\mathbb{P}_n$ becomes a compact space which is pathwise connected. It is also Hausdorff, since the relation $\sim$ is closed in the product topology.

Theorem 5.5.1.

$$\pi_1(\mathbb{P}_n) \cong \begin{cases} \mathbb{Z} & \text{if } n = 1 \\ \mathbb{Z}/_{2\mathbb{Z}} & \text{if } n \geqslant 2 \end{cases}$$

The proof of this theorem exhibits an important general idea. Namely, that the fundamental group is related to the fibers of a covering.

Proof. To establish the assertion for "$n = 1$", we show that $S^1/_\sim$ is homeomorphic to S^1. Thereby, we again think of S^1 as a subset of $\mathbb{C}$. Let $\psi\colon S^1 \to S^1$ be the map $\psi(z) := z^2$. Clearly, ψ is continuous and surjective. We have

$$z \sim w \ \Leftrightarrow \ \psi(z) = \psi(w)$$

Thus there is a bijection $\tilde{\psi}\colon S^1/_\sim \to S^1$ with

$$\begin{tikzcd} S^1 \arrow[r, "\psi"] \arrow[d, "p_1"'] & S^1 \\ S^1/_\sim \arrow[ur, "\tilde{\psi}"'] & \end{tikzcd}$$

Since $S_1/_\sim$ is endowed with the final topology, $\tilde{\psi}$ is continuous. Since $S^1/_\sim$ is compact and S^1 is Hausdorff, $\tilde{\psi}$ is a homeomorphism.

To treat the case "$n \geqslant 2$" we are going to analyse the fundamental group $\pi_1(S^n/_\sim)$ using the covering from Proposition 4.1.8. Recall that this covering is given by the space S^n and covering projection p_n.

To start with, let us fix notation. Set $x_0 := (1,0,\dots,0)/_\sim \in S^n/_\sim$, which will be used as base point, and set $\tilde{x}_0 := (1,0,\dots,0)$. For a loop $f \in \mathscr{L}(S^n/_\sim, x_0)$ denote by $\tilde{f}$ the lifting of f with initial point $\tilde{x}_0$.

Now we use that the covering $p_n\colon S^n \to S^n/_\sim$ is 2-fold. For every loop $f \in \mathscr{L}(S^n/_\sim, x_0)$ the terminal point of a lifting $\tilde{f}$ belongs to the two-element set $\{\tilde{x}_0, -\tilde{x}_0\}$. Now we use that S^n is simply connected. If $f, g \in \mathscr{L}(S^n/_\sim, x_0)$ with $\tilde{f}(1) = \tilde{g}(1)$, then $\tilde{f} \approx \tilde{g}$ and hence $f \approx g$. We conclude that $\pi_1(S^n/_\sim)$ has at most two elements.

Let $\mathbb{1}_{x_0}$ be the constant path at x_0. Its lifting $\tilde{\mathbb{1}}_{x_0}$ clearly is the constant path at $\tilde{x}_0$. If f is a loop with $f \approx \mathbb{1}_{x_0}$, then by the monodromy theorem $\tilde{f}(1) = \tilde{x}_0$. In order to show that $\pi_1(S^n/_\sim) \neq \{\mathbb{1}\}$, it is thus sufficient to find one loop f with $\tilde{f}(1) \neq \tilde{x}_0$.

Consider the path $\tilde{f}\colon [0,1] \to S^n$ given as

$$\tilde{f}(t) := \big(\cos(\pi t), \sin(\pi t), 0, \ldots, 0\big) \text{ for } t \in [0,1]. \tag{5.5}$$

Then $\tilde{f}(0) = \tilde{x}_0$ and $\tilde{f}(1) = -\tilde{x}_0$. Pushing $\tilde{f}$ to $S^n/_\sim$ gives a loop with base point x_0

$$f(t) := (p_n \circ \tilde{f})(t) = \big(\cos(\pi t), \sin(\pi t), 0, \ldots, 0\big)/_\sim \text{ for } t \in [0,1].$$

By definition $\tilde{f}$ is the lifting of f.

To conclude the proof, note that up to isomorphism there is only one group with two elements, namely $\mathbb{Z}/2\mathbb{Z}$. ❑

We demonstrate another typical way to apply knowledge about the fundamental group, and deduce the Borsuk-Ulam theorem (again, proving the result also in higher dimensions would require more machinery).

Corollary 5.5.2. *Let $n \geqslant 2$. Then there exists no continuous map $\phi\colon S^n \to S^1$ with $\phi(-x) = -\phi(x)$ for all $x \in S^n$.*

Proof. Let $\phi\colon S^n \to S^1$ be continuous and assume towards a contradiction that $\phi(-x) = -\phi(x)$ for all $x \in S^n$. This implies

$$\forall x, y \in S^n.\ x \sim y \ \Rightarrow\ \phi(x) \sim \phi(y)$$

Hence, there exists a map $\psi\colon S^n/_\sim \to S^1/_\sim$ with

$$\begin{tikzcd} S^n \arrow[r, "\phi"] \arrow[d, "p_n"'] & S^1 \arrow[d, "p_1"] \\ S^n/_\sim \arrow[r, "\psi"'] & S^1/_\sim \end{tikzcd}$$

Since $S^n/_\sim$ carries the final topology, ψ is continuous. We thus have the homomorphism

$$\pi_1(\psi)\colon \pi_1(S^n/_\sim) \cong \mathbb{Z}/_{2\mathbb{Z}} \ \to\ \pi_1(S^1) \cong \mathbb{Z}$$

Since $\mathbb{Z}$ contains no elements of order 2, we must have $[\pi_1(\psi)](f/_\approx) = \mathbb{1}$ for all $f/_\approx \in \pi_1(S^n/_\sim)$.

Consider the path $\tilde{f}$ from (5.5). Then

$$(\phi \circ \tilde{f})(0) = \phi(\tilde{x}_0), \quad (\phi \circ \tilde{f})(1) = \phi(-\tilde{x}_0) = -\phi(\tilde{x}_0).$$

Pushing $\phi \circ \tilde{f}$ to a path in $S^1/_\sim$ gives a loop $g := p_1 \circ (\phi \circ \tilde{f})$. By the monodromy theorem, $g \not\approx \mathbb{1}$. Pushing $\tilde{f}$ to a path in $S^n/_\sim$, gives a loop $f := p_n \circ \tilde{f}$. We have

$$[\pi_1(\psi)](f/_\approx) = [\psi \circ (p_n \circ \tilde{f})]/_\approx = [(p_1 \circ \phi) \circ \tilde{f}]/_\approx = g/_\approx \neq \mathbb{1},$$

and have reached a contradiction. ❑

5.6 The Seifert–van Kampen theorem

We saw in Proposition 5.4.3 that (under some hypothesis) the fundamental group of a union $\bigcup_{i\in I} U_i$ is generated by the union of the fundamental groups of the single spaces U_i. The *Seifert–van Kampen theorem* determines the redundancies in this generating system. In fact, it says that only the minimal relations between the elements of $\pi_1(U_i)$ hold when considered as elements of $\pi_1(X)$: Let f be a loop in some intersection $U_i \cap U_j$. Then we can consider f as a path in U_i and then as a path in X, or consider f as a path in U_j and then as a path in X. Of course, proceeding either way will result in the same element of $\pi_1(X)$.

Theorem 5.6.1. *Let $\langle X, \mathcal{T}\rangle$ be a topological space and let $\{U_i \mid i \in I\}$ be an open cover of X. Assume that*

(i) $\displaystyle\bigcap_{i\in I} U_i \neq \varnothing$

(ii) $\forall i, j, k \in I.\ U_i \cap U_j \cap U_k$ *pathwise connected*

Moreover, let $\iota_i\colon U_i \to X$ and $\iota_{ij}\colon U_i \cap U_j \to U_j$ be the set-theoretic inclusion maps. We fix a base point x_0 in $\bigcap_{i\in I} U_i$, and denote equivalence classes of loops in the respective spaces as $[f]_i \in \pi_1(U_i)$, $[f]_{ij} \in \pi_1(U_i \cap U_j)$, or $[f]_X \in \pi_1(X)$.

Let N be the smallest normal subgroup of the free product $\star_{i\in I}\pi_1(U_i)$ containing (γ_i is the embedding of $\pi_1(U_i)$ into $\star_{i\in I}\pi_1(U_i)$)

$$\Big\{ \big(\gamma_j \circ \pi_1(\iota_{ij})\big)([f]_{ij}) \cdot \big(\gamma_i \circ \pi_1(\iota_{ji})\big)([f]_{ji})^{-1} \;\big|\; i, j \in I, i \neq j, f \in \mathscr{L}(U_i \cap U_j, x_0) \Big\} \tag{5.6}$$

Then

$$\pi_1(X) \cong \mathop{\star}_{i\in I} \pi_1(U_i) \big/ N.$$

We come to the proof of this result. Throughout the following, let data be given as in the formulation of the theorem.

Let $\Phi\colon \star_{i\in I}\pi_1(U_i) \to \pi_1(X)$ be the homomorphism given by the universal property of the free product for $\langle \pi_1(X), (\pi_1(\iota_i))_{i\in I}\rangle$.

$$\begin{tikzcd}
\vdots & & \\
\pi_1(U_i) \arrow[dr, "\gamma_i"] \arrow[drr, bend left, "\pi_1(\iota_i)"] & & \\
\vdots & \mathop{\star}_{i\in I}\pi_1(U_i) \arrow[r, dashed, "\Phi"] & \pi_1(X) \\
\pi_1(U_j) \arrow[ur, "\gamma_j"'] \arrow[urr, bend right, "\pi_1(\iota_j)"'] & & \\
\vdots & &
\end{tikzcd} \tag{5.7}$$

Proposition 5.4.3 implies that Φ is surjective. Theorem 5.6.1 will be proven, if we show that the kernel of Φ is N.

The inclusion "$N \subseteq \ker\Phi$" is clear: we have $\iota_i \circ \iota_{ji} = \iota_j \circ \iota_{ij}$, since both are just the inclusion of $U_i \cap U_j$ in X, and $[f]_{ij} = [f]_{ji}$ since both are classes w.r.t. FEP-homotopy in

$U_i \cap U_j$. Thus

$$\begin{aligned}
\Phi\big((\gamma_j \circ \pi_1(\iota_{ij}))([f]_{ij})\big) &= \big(\Phi \circ \gamma_j \circ \pi_1(\iota_{ij})\big)([f]_{ij}) = \big(\pi_1(\iota_j) \circ \pi_1(\iota_{ij})\big)([f]_{ij}) \\
&= \big(\pi_1(\iota_j \circ \iota_{ij})\big)([f]_{ij}) = \big(\pi_1(\iota_i \circ \iota_{ji})\big)([f]_{ji}) \\
&= \big(\pi_1(\iota_i) \circ \pi_1(\iota_{ji})\big)([f]_{ji}) = \big(\Phi \circ \gamma_i \circ \pi_1(\iota_{ji})\big)([f]_{ji}) \\
&= \Phi\big((\gamma_i \circ \pi_1(\iota_{ji}))([f]_{ji})\big)
\end{aligned}$$

To prove the reverse inclusion, we work with the concrete form of the free product given in the proof of Theorem 1.8.1

$$\begin{tikzcd}
\pi_1(U_i) \arrow[r, "\delta_i"] \arrow[rr, bend left, "\pi_1(\iota_i)"] \arrow[dr, "\gamma_i"'] & A^* \arrow[r, "\Phi_0"] \arrow[d, "\pi"'] & \pi_1(X) \\
& A^*/_\Theta \arrow[ur, "\Phi"'] \arrow[r, dashed, "\pi'"'] & (A^*/_\Theta)/_N \arrow[u, dashed, "\Phi'"']
\end{tikzcd}$$

Set $\Gamma := \ker(\pi' \circ \pi)$. Since π is surjective, we have (Γ and $\ker \Phi_0$ are relations)

$$\ker \Phi \subseteq N \;\Leftrightarrow\; \ker \Phi_0 \subseteq \Gamma.$$

We are going to show that the inclusion on the right holds. Written out explicitly, this means that for all loops $f_i \in \mathscr{L}(U_{\lambda_i}, x_0)$, $i = 1, \ldots, n$, and $g_j \in \mathscr{L}(U_{\kappa_j}, x_0)$, $j = 1, \ldots, m$, the implication

$$f_1 \,\blacksquare \ldots \blacksquare\, f_n \approx_X g_1 \,\blacksquare \ldots \blacksquare\, g_m \;\Rightarrow\; [f_1]_{\lambda_1}[f_2]_{\lambda_2} \cdots [f_n]_{\lambda_n} \;\Gamma\; [g_1]_{\kappa_1}[g_2]_{\kappa_2} \cdots [g_m]_{\kappa_m} \tag{5.8}$$

holds.

We start with a particular case.

Lemma 5.6.2. *Let $a, b \in \mathbb{R}$, $a < b$, and $s_0, s_1 \in \mathbb{R}$, $s_0 < s_1$, and $a = t_0 < t_1 < \ldots < t_N = b$. Let $H\colon [a,b] \times [s_0, s_1] \to X$ be a continuous function with*

$$\forall s \in [s_0, s_1].\; H(a,s) = H(b,s) = x_0$$

and assume that we have $\alpha(1), \ldots, \alpha(N) \in I$ with

$$\forall k \in \{1, \ldots, N\}.\; H\big([t_{k-1}, t_k] \times [s_0, s_1]\big) \subseteq U_{\alpha(k)}$$

Further, assume that for each $(k,l) \in \{1, \ldots, N-1\} \times \{0,1\}$ we have a path $b_{k,l}$ in $U_{\alpha(k)} \cap U_{\alpha(k+1)}$ with initial point x_0 and terminal point $H(t_k, s_l)$. For $(k,l) \in \{0, N\} \times \{0,1\}$ let $b_{k,l}$ be the constant path at x_0. Denote by $c_{k,l}$ the loop

$$c_{k,l} := b_{k-1,l} \,\blacksquare\, H(.,s_l)|_{[t_{k-1},t_k]} \,\blacksquare\, b_{k,l}^{\leftarrow} \in \mathscr{L}(U_{\alpha(k)}, x_0) \text{ for } (k,l) \in \{1, \ldots, N\} \times \{0,1\}.$$

Then

$$[c_{1,0}]_{\alpha(1)}[c_{2,0}]_{\alpha(2)} \cdots [c_{N,0}]_{\alpha(N)} \;\Gamma\; [c_{1,1}]_{\alpha(1)}[c_{2,1}]_{\alpha(2)} \cdots [c_{N,1}]_{\alpha(N)}$$

Proof. For $k \in \{1, \ldots, N-1\}$ set

$$d_k := b_{k,0} \,\blacksquare\, H(t_k, .) \,\blacksquare\, b_{k,1}^{\leftarrow},$$

and let d_0 and d_N be the constant path at x_0. Note that d_k lies entirely in $U_{\alpha(k)} \cap U_{\alpha(k+1)}$. Moreover, note that $c_{k,l}$ lies in $U_{\alpha(k)}$.

The rectangle $[t_{k-1}, t_k] \times [s_0, s_1]$ is convex, and hence each two paths in this rectangle which have the same initial points and the same terminal points are FEP-homotopic. Applying the continuous function H, shows that the paths $d_{k-1} \centerdot c_{k,1}$ and $c_{k,0} \centerdot d_k$ are FEP-homotopic in $U_{\alpha(k)}$:

$$\begin{aligned} d_{k-1} \centerdot c_{k,1} &= \big(b_{k-1,0} \centerdot H(t_{k-1}, .) \centerdot \overleftarrow{b}_{k-1,1}\big) \centerdot \big(b_{k-1,1} \centerdot H(., s_1)|_{[t_{k-1},t_k]} \centerdot \overleftarrow{b}_{k,1}\big) \\ &\approx_{\alpha(k)} b_{k-1,0} \centerdot H(t_{k-1}, .) \centerdot \big(\overleftarrow{b}_{k-1,1} \centerdot b_{k-1,1}\big) \centerdot H(., s_1)|_{[t_{k-1},t_k]} \centerdot \overleftarrow{b}_{k,1} \\ &\approx_{\alpha(k)} b_{k-1,0} \centerdot \big(H(t_{k-1}, .) \centerdot H(., s_1)|_{[t_{k-1},t_k]}\big) \centerdot \overleftarrow{b}_{k,1} \\ &\approx_{\alpha(k)} b_{k-1,0} \centerdot \big(H(., s_0)|_{[t_{k-1},t_k]} \centerdot H(t_k, .)\big) \centerdot \overleftarrow{b}_{k,1} \approx_{\alpha(k)} c_{k,0} \centerdot d_k \end{aligned}$$

We obtain

$$\begin{aligned} &[c_{1,1}]_{\alpha(1)} [c_{2,1}]_{\alpha(2)} \cdots [c_{N,1}]_{\alpha(N)} \;\Theta\; [d_0]_{\alpha(1)}[c_{1,1}]_{\alpha(1)}[c_{2,1}]_{\alpha(2)} \cdots [c_{N,1}]_{\alpha(N)} \\ &\quad \Theta\; [d_0 \centerdot c_{1,1}]_{\alpha(1)}[c_{2,1}]_{\alpha(2)} \cdots [c_{N,1}]_{\alpha(N)} \;=\; [c_{1,0} \centerdot d_1]_{\alpha(1)}[c_{2,1}]_{\alpha(2)} \cdots [c_{N,1}]_{\alpha(N)} \\ &\quad \Theta\; [c_{1,0}]_{\alpha(1)}[d_1]_{\alpha(1)}[c_{2,1}]_{\alpha(2)} \cdots [c_{N,1}]_{\alpha(N)} \;\Gamma\; [c_{1,0}]_{\alpha(1)}[d_1]_{\alpha(2)}[c_{2,1}]_{\alpha(2)} \cdots [c_{N,1}]_{\alpha(N)} \\ &\quad \Theta\; [c_{1,0}]_{\alpha(1)}[d_1 \centerdot c_{2,1}]_{\alpha(2)} \cdots [c_{N,1}]_{\alpha(N)} \;=\; [c_{1,0}]_{\alpha(1)}[c_{2,0} \centerdot d_2]_{\alpha(2)} \cdots [c_{N,1}]_{\alpha(N)} \\ &\qquad \cdots \\ &\quad \Theta\; [c_{1,0}]_{\alpha(1)}[c_{2,0}]_{\alpha(2)} \cdots [c_{N,0}]_{\alpha(N)}[d_N]_{\alpha(N)} \;\Theta\; [c_{1,0}]_{\alpha(1)}[c_{2,0}]_{\alpha(2)} \cdots [c_{N,0}]_{\alpha(N)} \end{aligned}$$

❑

Now let loops $f_i \in \mathscr{L}(U_{\lambda_i}, x_0)$, $i = 1, \ldots, n$, and $g_j \in \mathscr{L}(U_{\kappa_j}, x_0)$, $j = 1, \ldots, m$, be given, and assume that

$$f_1 \centerdot \ldots \centerdot f_n \approx_X g_1 \centerdot \ldots \centerdot g_m.$$

Let $H\colon [a, b] \times [0, 1] \to X$ be a FEP-homotopy from $f_1 \centerdot \ldots \centerdot f_n$ to $g_1 \centerdot \ldots \centerdot g_m$.

① *We construct a tiling of* $[a, b] \times [0, 1]$*:* The family $\{H^{-1}(U_i) \mid i \in I\}$ is an open cover of the rectangle $[a, b] \times [0, 1]$. Let $\epsilon > 0$ be a Lebesgue number for this cover. Now choose partitions

$$a = t_0 < t_1 < \ldots < t_N = b \text{ and } 0 = s_0 < s_1 < \ldots < s_M = 1$$

such that

(i) $\max\big\{(t_k - t_{k-1})^2 + (s_l - s_{l-1})^2 \mid k = 1, \ldots, N, l = 1, \ldots, M\big\} < \epsilon^2$
(ii) The switching points between the paths f_i and g_j in their respective products appear among the t_k.

We denote indices corresponding to switching points as

$$a = \phi_0 < \phi_1 < \ldots < \phi_n = b \text{ and } a = \psi_0 < \psi_1 < \ldots < \psi_m = b,$$

so that the part of $f_1 \centerdot \ldots \centerdot f_n$ coming from f_i is $H(., 0)|_{[t_{\phi_{i-1}}, t_{\phi_i}]}$, and the part of $g_1 \centerdot \ldots \centerdot g_m$ coming from g_j is $H(., 1)|_{[t_{\psi_{j-1}}, t_{\psi_j}]}$.

Now we have the tiling given by the rectangles

$$R_{k,l} := [t_{k-1}, t_k] \times [s_{l-1}, s_l] \text{ where } k \in \{1, \ldots, N\}, l \in \{1, \ldots, M\}.$$

The nodes of the lattice formed by the boundaries of these rectangles are denoted as

$$v_{k,l} := (t_k, s_l) \text{ where } k \in \{0, \dots, N\}, l \in \{0, \dots, M\}.$$

Note that $H(v_{k,l}) = x_0$ whenever

$$k = 0 \vee k = N \vee (k,l) \in \{\phi_0, \dots, \phi_n\} \times \{0\} \vee (k,l) \in \{\psi_0, \dots, \psi_m\} \times \{M\}. \tag{5.9}$$

② *We define paths $b_{k,l}$ connecting nodes with the base point:* Since we chose the tiling sufficiently fine, we can choose for each $(k,l) \in \{0, \dots, N\} \times \{0, \dots, M\}$ an index $\alpha(k,l) \in I$ such that

(i) $\forall (k,l) \in \{1, \dots, N\} \times \{1, \dots, M\}.\ R_{k,l} \subseteq H^{-1}(U_{\alpha(k,l)})$

(ii) $\forall (k,l) \in \{1, \dots, N-1\} \times \{1, \dots, M\}.\ k \equiv l \bmod 2 \ \Rightarrow\ \alpha(k,l) = \alpha(k+1,l)$

▷ Interior nodes: Let $(k,l) \in \{1, \dots, N-1\} \times \{1, \dots, M-1\}$. Depending whether k and l have equal or unequal parity, we have $\alpha(k,l) = \alpha(k+1,l)$ or $\alpha(k,l+1) = \alpha(k+1,l+1)$. Hence the set

$$V_{k,l} := U_{\alpha(k,l)} \cap U_{\alpha(k+1,l)} \cap U_{\alpha(k,l+1)} \cap U_{\alpha(k+1,l+1)}$$

is the intersection of at most three different sets U_i. Thus, we can choose a path $b_{k,l}$ in $V_{k,l}$ with initial point x_0 and terminal point $H(v_{k,l})$. Note here that the tiles containing the node $v_{k,l}$ are exactly $R_{\alpha(k,l)}, R_{\alpha(k+1,l)}, R_{\alpha(k,l+1)}, R_{\alpha(k+1,l+1)}$.

▷ Boundary nodes (part 1): For all (k,l) as in (5.9), we let $b_{k,l}$ be the constant path at x_0.

▷ Boundary nodes (part 2): Let $k = \{1, \dots, N-1\}$ and assume that $\phi_{i-1} < t_k < \phi_i$. Then $H(t_k, 0) \in U_{\lambda_i}$. Set

$$V_{k,0} := U_{\alpha(k,1)} \cap U_{\alpha(k+1,1)} \cap U_{\lambda_i},$$

and let $b_{k,0}$ be a path in $V_{k,0}$ with initial point x_0 and terminal point $H(v_{k,0})$. The tiles containing the node $v_{k,0}$ are exactly $R_{\alpha(k,0)}, R_{\alpha(k+1,0)}$.

▷ Boundary nodes (part 3): Let $k = \{1, \dots, N-1\}$ and assume that $\psi_{j-1} < k < \psi_j$. Then $H(t_k, M) \in U_{\kappa_j}$. Set

$$V_{k,M} := U_{\alpha(k,M)} \cap U_{\alpha(k+1,M)} \cap U_{\kappa_j},$$

and let $b_{k,M}$ be a path in $V_{k,M}$ with initial point x_0 and terminal point $H(v_{k,M})$. The tiles containing the node $v_{k,M}$ are exactly $R_{\alpha(k,M)}, R_{\alpha(k+1,M)}$.

③ *Applying the lemma:* For $(k,l) \in \{1, \dots, N\} \times \{0, \dots, M\}$ let $c_{k,l}$ be the loop

$$c_{k,l} := b_{k-1,l} \bullet H(., s_l)|_{[t_{k-1}, t_k]} \bullet \overleftarrow{b_{k,l}}.$$

Our choice of the paths $b_{k,l}$ guarantees that the assumptions of the lemma are fulfilled on every level from s_{l-1} to s_l. Applying the lemma M times, we obtain that

$$[c_{1,0}]_{\alpha(1,1)}[c_{2,0}]_{\alpha(2,1)} \cdots [c_{N,0}]_{\alpha(N,1)} \ \Gamma\ [c_{1,M}]_{\alpha(1,M)}[c_{2,M}]_{\alpha(2,M)} \cdots [c_{N,M}]_{\alpha(N,M)}$$

④ *Splitting $f_1 \centerdot \ldots \centerdot f_n$ and $g_1 \centerdot \ldots \centerdot g_m$:* Our choice of paths $b_{k,0}$ and $b_{k,M}$ guarantees that

$$f_i \approx_{\lambda_i} c_{\phi_{i-1}+1,0} \centerdot \ldots \centerdot c_{\phi_i,0} \text{ and } g_j \approx_{\kappa_j} c_{\psi_{j-1}+1,M} \centerdot \ldots \centerdot c_{\psi_j,M}$$

It follows that

$$\begin{aligned}
[f_i]_{\lambda_i} &= [c_{\phi_{i-1}+1,0} \centerdot \ldots \centerdot c_{\phi_i,0}]_{\lambda_i} \;\Theta\; [c_{\phi_{i-1}+1,0}]_{\lambda_i} \cdots [c_{\phi_i,0}]_{\lambda_i} \\
&\quad \Gamma\; [c_{\phi_{i-1}+1,0}]_{\alpha(\phi_{i-1}+1,1)} \cdots [c_{\phi_i,0}]_{\alpha(\phi_i,1)}, \\
[g_j]_{\kappa_j} &= [c_{\psi_{j-1}+1,M} \centerdot \ldots \centerdot c_{\psi_j,M}]_{\kappa_j} \;\Theta\; [c_{\psi_{j-1}+1,M}]_{\kappa_j} \cdots [c_{\psi_j,M}]_{\kappa_j} \\
&\quad \Gamma\; [c_{\psi_{j-1}+1,M}]_{\alpha(\psi_{j-1}+1,M)} \cdots [c_{\psi_j,M}]_{\alpha(\psi_i,M)}.
\end{aligned}$$

Putting together, thus

$$\begin{aligned}
[f_1]_{\lambda_1} \cdots [f_n]_{\lambda_n} \;&\Gamma\; [c_{1,0}]_{\alpha(1,1)}[c_{2,0}]_{\alpha(2,1)} \cdots [c_{N,0}]_{\alpha(N,1)} \\
&\Gamma\; [c_{1,M}]_{\alpha(1,M)}[c_{2,M}]_{\alpha(2,M)} \cdots [c_{N,M}]_{\alpha(N,M)} \;\Gamma\; [g_1]_{\kappa_1} \cdots [g_m]_{\kappa_m}
\end{aligned}$$

This concludes the proof of (5.8).

The proof of the theorem is complete.

Let us explicitly state a corollary which deals with two particular situations and is often of good use.

Corollary 5.6.3. *Let $\langle X, \mathcal{T}\rangle$ be a topological space and let $\{U_i \mid i \in I\}$ be an open cover of X such that the assumptions of the Seifert–van Kampen theorem are fulfilled. Also let other notation be as in Theorem 5.6.1.*

(i) *Assume that $\pi_1(U_i \cap U_j) = \{\mathbb{1}\}$ for all $i, j \in I$, $i \neq j$. Then*

$$\pi_1(X) \cong \mathop{\text{\ding{73}}}_{i\in I} \pi_1(U_i).$$

(ii) *Assume that there exists an index $i_0 \in I$ such that $\pi_1(U_j) = \{\mathbb{1}\}$ for all $j \in I\backslash\{i_0\}$, and denote by $M \subseteq \pi_1(U_{i_0})$ the smallest normal subgroup containing*

$$\bigcup_{j\in I\backslash\{i_0\}} \pi_1(\iota_{ji_0})\big(\pi_1(U_j \cap U_{i_0})\big) \tag{5.10}$$

Further, let $\iota_{i_0}\colon U_{i_0} \to X$ be the inclusion map. Then $\pi(\iota_{i_0})\colon \pi_1(U_{i_0}) \to \pi_1(X)$ is surjective with kernel M. In particular,

$$\pi_1(X) \cong \pi_1(U_{i_0})\big/M.$$

Proof. Under the assumption of the present item (i) the set (5.6) equals $\{\mathbb{1}\}$, and hence the normal subgroup N in the Seifert–van Kampen theorem equals $\{\mathbb{1}\}$.

Assume we are in the situation of item (ii). Then we have $\gamma_j(U_j) = \{1\}$ for all $j \neq i_0$. The map γ_{i_0} is thus an isomorphism of $\pi_1(U_{i_0})$ onto $\text{\ding{73}}_{i\in I}\pi_1(U_i)$, cf. Lemma 1.8.2. We see that $\pi(\iota_{i_0})$ is surjective with kernel $\gamma_{i_0}^{-1}(N)$. The set (5.6) equals

$$\begin{aligned}
&\Big\{\big(\gamma_{i_0} \circ \pi_1(\iota_{ji_0})\big)([f]_{ji_0}) \mid j \in I, f \in \mathcal{L}(U_j \cap U_{i_0}, x_0)\Big\} \\
&\qquad\qquad \cup \Big\{\big(\gamma_{i_0} \circ \pi_1(\iota_{ji_0})\big)([f]_{ji_0})^{-1} \mid j \in I, f \in \mathcal{L}(U_j \cap U_{i_0}, x_0)\Big\},
\end{aligned}$$

and its inverse image under γ_{i_0} thus equals (5.10). It follows that

$$\pi_1(X) \cong \mathop{\text{\ding{73}}}_{i\in I} \pi_1(U_i)\big/N \cong \pi_1(U_{i_0})\big/M.$$

❑

In our formulation of the Seifert–van Kampen, we have determined the group $\pi_1(X)$ concretely: take the free product and factor out the normal subgroup generated by (5.6). Let us mention the following more structural viewpoint.

Corollary 5.6.4. *Assume we are in the situation of the Seifert–van Kampen theorem. Then $\langle \pi_1(X), \pi_1(\iota_i)\rangle$ is a colimit of the diagram*

$$\begin{tikzcd} & \pi_1(U_i) \\ \pi_1(U_i \cap U_j) \arrow[ur, dotted, "\pi_1(\iota_{ji})"] \arrow[dr, dotted, "\pi_1(\iota_{ij})"'] & \vdots \\ & \pi_1(U_j) \end{tikzcd} \tag{5.11}$$

Proof. When we proved existence of colimits in Theorem 1.9.5, we constructed a colimit as a factor of the free product. It suffices to match notation: for the diagram (5.11), the normal subgroup in the Seifert–van Kampen theorem is the same as the normal subgroup in Theorem 1.9.5. In Theorem 1.9.5 the cone maps of the colimit are obtained as projection after γ_i, which in the present situation is equal to $\pi_1(\iota_i)$ by the diagram (5.7). ❑

5.7 The bouquet of circles

Our aim in this and the following section is to prove that every group can be realised as the fundamental group of some topological space. The proof of this fact contains many interesting ideas.

Making a first step, we realise free groups. This relies on Corollary 5.6.3 (i), and a construction known in a more general setting as *wedge sum*. Intuitively, a wedge sum takes a family of topological spaces and glues them together at one point of each of them. We shall not elaborate further on the general construction.

Example 5.7.1. Let I be a nonempty set, let $\coprod_{i\in I} S^1$ be the disjoint union of $|I|$ many circles S^1, and let $\beta_i\colon S^1 \to \coprod_{i\in I} S^1$ be the inclusion maps. Further, let $\sim$ be the equivalence relation which identifies the point 1 of each of these circles. Formally, thus,

$$\coprod_{i\in I} S^1 = \bigcup_{i\in I}(S^1 \times \{i\}), \qquad \beta_i\colon \begin{cases} S^1 & \to & \bigcup_{i\in I}(S^1\times\{i\}) \\ z & \mapsto & (z,i) \end{cases}$$

$$(z,i) \sim (w,j) :\Leftrightarrow (z,i) = (w,j) \vee z = w = 1$$

Set $X := (\coprod_{i\in I} S^1)/_\sim$, denote by $p\colon \coprod_{i\in I} S^1 \to X$ the canonical projection and set $p_i := p\circ\beta_i$. We endow X with the final topology $\mathcal{T}$ induced by the family $\{p_i\colon S^1 \to X \mid i \in I\}$

$$\begin{tikzcd} S^1 \arrow[dr, dotted, "\beta_i"'] \arrow[drr, "p_i"] & & \\ \vdots & \bigcup_{i\in I}(S^1\times\{i\}) \arrow[r, dotted, "p"'] & X \\ S^1 \arrow[ur, dotted, "\beta_j"] \arrow[urr, "p_j"'] & & \end{tikzcd}$$

The space $\langle X, \mathcal{T}\rangle$ is called the *bouquet of $|I|$ circles*.

A bouquet of circles is pathwise connected, since it is the union of the pathwise connected subsets $p_i(S^1)$ which have the point $x_0 := p_i(1)$ in common.

The topology on a disjoint union is easily understood. Assume we have a family $\langle X_i, \mathcal{T}_i \rangle$ of topological spaces and endow their disjoint union with the final topology induced by the inclusion maps $\beta_i\colon X_i \to \coprod_{i\in I} X_i$. A map ϕ from $\coprod_{i\in I} X_i$ to some other topological space is continuous, if and only if all restrictions $\phi|_{\beta_i(X_i)}$ are continuous. All subsets $\beta_i(X_i)$ are open and closed in $\coprod_{i\in I} X_i$, and β_i is a homeomorphism onto its image. Given sets $B_i \subseteq X_i$ for each $i \in I$, we have

$$\bigcup_{i\in I} \beta_i(B_i) \text{ open in } \coprod_{i\in I} X_i \;\Leftrightarrow\; \forall i \in I.\ B_i \text{ open in } X_i$$

and the same with "open" replaced by "closed".

The topology of a quotient looks somewhat more complicated. Passing to a quotient topology behaves nicely when the projection is open (for example when factorising a topological group), but this is an exceptional situation.

Let us study the topology of a bouquet of circles in some more detail. We use notation as in Example 5.7.1. Moreover, given a point $z \in S^1$ and $\epsilon > 0$, we denote by $A_\epsilon(z)$ the arc

$$A_\epsilon(z) := \big\{ z \cdot e^{2\pi i t} \in S^1 \mid |t| < \epsilon \big\}.$$

Lemma 5.7.2. *Let I be a nonempty set, and $\langle X, \mathcal{T} \rangle$ the bouquet of $|I|$ circles.*

(i) *For subsets $B_i \subseteq S^1$, $i \in I$, it holds that*

$$p\Big(\bigcup_{i\in I} (B_i \times \{i\})\Big) \text{ open in } X \;\Leftrightarrow\; \Big(\forall i \in I.\ B_i \text{ open in } S^1\Big) \wedge \bigg[\Big(\forall i \in I.\ 1 \notin B_i\Big) \vee \Big(\forall i \in I.\ B_i \cup \{1\} \in \mathcal{U}(1)\Big)\bigg] \tag{5.12}$$

(ii) *Let $z \in S^1 \backslash \{1\}$ and $j \in I$. Every set of the form*

$$p_j(A_\epsilon(z)), \tag{5.13}$$

with $\epsilon > 0$ such that $1 \notin A_\epsilon(z)$, is open in X. The family of all these sets is a neighbourhood base of the point $p_j(z)$.

(iii) *Every set of the form*

$$\bigcup_{i\in I} p_i(A_{\epsilon_i}(1)) \tag{5.14}$$

with $\epsilon_i > 0$ is open in X. The family of all these sets is a neighbourhood base of the point x_0.

(iv) *The space X is Hausdorff.*

(v) *The following restrictions of p and p_j are homeomorphisms between the written sets:*

$$p\colon \bigcup_{i\in I} \big((S^1 \backslash \{1\}) \times \{i\}\big) \to X \backslash \{x_0\}, \qquad p_j\colon S^1 \to p_j(S^1) \text{ for } j \in I.$$

Each set $p_j(S^1 \backslash \{1\})$ is open in X.

Proof. Let $B_i \subseteq S^1$, $i \in I$. The equivalence (5.12) follows since for every $j \in I$

$$p_j^{-1}\Big(p\Big(\bigcup_{i\in I}(B_i \times \{i\})\Big)\Big) = \begin{cases} B_j & \text{if } \forall i \in I.\ 1 \notin B_i \\ B_j \cup \{1\} & \text{if } \exists i \in I.\ 1 \in B_i \end{cases}$$

Let $z \in S^1 \backslash \{1\}$, $\epsilon > 0$ such that $1 \notin A_\epsilon(z)$, and $j \in I$. Applying (5.12) with

$$B_i := \begin{cases} A_\epsilon(z) & \text{if } i = j \\ \varnothing & \text{if } i \neq j \end{cases}$$

shows that $p_j(A_\epsilon(z))$ is open. Since the open arcs form a basis of the topology of S^1, the family of all such sets is a neighbourhood base of $p_j(z)$.

Let $\epsilon_i > 0$, $i \in I$. Applying (5.12) with $B_i := A_{\epsilon_i}(1)$, $i \in I$, shows that $\bigcup_{i\in I} p_i(A_{\epsilon_i}(1))$ is open, and since open arcs are a basis in S^1, the family of those sets is a neighbourhood base of x_0.

Given two different points, we find open neighbourhoods of the form (5.13) or (5.14) of the respective points which are disjoint by choosing ϵ or ϵ_i sufficiently small.

We come to the proof of (v). First observe that p indeed maps $\bigcup_{i\in I} \beta_i(S^1\backslash\{1\})$ bijectively onto $X \backslash \{x_0\}$, and that both sets are open in $\coprod_{i\in I} S^1$ or X, respectively. The family of arcs

$$\Big\{\beta_i(A_\epsilon(z)) \,\big|\, i \in I, z \in S^1\backslash\{1\}, 1 \notin A_\epsilon(z)\Big\}$$

forms a base of the topology of $\bigcup_{i\in I} \beta_i(S^1\backslash\{1\})$. We have $p(\beta_i(A_\epsilon(z))) = p_j(A_\epsilon(z))$, and this set is open in X. Thus the restriction of p is an open map. It is continuous by the definition of the topology of X.

Consider now the map p_j. It is continuous by definition. It is injective, and hence induces a bijection onto its image. Each arc $A_\epsilon(z)$ with $z \in S^1\backslash\{1\}$ and $1 \notin A_\epsilon(z)$ is mapped to a set which is open in X. The image of an arc $A_\epsilon(1)$ can be written as

$$p_j(A_\epsilon(1)) = p_j(S^1) \cap \bigcup_{i\in I} p_i(A_\epsilon(1)).$$

and hence is open in the subspace topology of $p_j(S^1)$. ❑

Theorem 5.7.3. *Let I be a nonempty set, and $\langle X, \mathcal{T}\rangle$ the bouquet of $|I|$ circles. Then $\pi_1(X)$ is the free group with $|I|$ generators.*

The idea is that each copy of S^1 contributes one generator to $\pi_1(X)$, and that these loops cannot interfere since each two of them intersect only in one single point. So, ideally, we would like to apply the Seifert–van Kampen theorem with the cover $\{p_i(S^1) \,|\, i \in I\}$. Then Corollary 5.6.3 (i) would imply that $\pi_1(X)$ is isomorphic to the free product of $|I|$ copies of $\mathbb{Z}$, which is the free group with $|I|$ generators. However, the sets in the mentioned cover are not open, and therefore we have to slightly tweak this approach.

Proof of Theorem 5.7.3.

① *We define an open cover:* Consider a bunch of small arcs centered at the base point 1: e.g. take $\delta := \frac{1}{9}$ and set

$$W := \bigcup_{i\in I} p_i(A_\delta(1)).$$

Then W is an open neighbourhood of x_0. Now set

$$U_i := p_i(S^1) \cup W \text{ for } i \in I,$$

and note that $U_i = p_i(S^1 \backslash \{1\}) \cup W$.

By Lemma 5.7.2, the set U_i is an open subset of X. Moreover, clearly, $\{U_i \mid i \in I\}$ is a cover of X.

② *We check the assumptions of the Seifert–van Kampen theorem:* The point x_0 belongs to all sets U_i, and hence $\bigcap_{i \in I} U_i \neq \varnothing$.

The circle S^1 and every arc in S^1 is pathwise connected. Hence also $p_i(A_\delta(1))$ and $p_i(S^1)$ are pathwise connected, and since x_0 belongs to each of these sets it follows that W and all U_i, $i \in I$, are pathwise connected. The intersection of at least two different sets U_i is equal to W.

③ *We show that W is contractible, and hence $\pi_1(W) = \{\mathbb{1}\}$:* Let $\tilde{H}$ be the unique function with

$$\begin{tikzcd}
A_\delta(1) \times [0,1] \arrow[r, "(e^{it},s) \mapsto e^{its}"] \arrow[d, "\beta_i \times \mathrm{id}"', "(i \in I)"] & A_\delta(1) \arrow[d, "\beta_i"] \\
\left(\bigcup_{i \in I} (A_\delta(1) \times \{i\}) \right) \times [0,1] \arrow[r, dashed, "\tilde{H}"'] & \bigcup_{i \in I} (A_\delta(1) \times \{i\})
\end{tikzcd}$$

Since

$$\left(\bigcup_{i \in I} (A_\delta(1) \times \{i\}) \right) \times [0,1] = \left(\bigcup_{i \in I} (A_\delta(1) \times \{i\}) \times [0,1] \right),$$

and each set in the union on the right is open, the gluing lemma implies that $\tilde{H}$ is continuous.

Let $t, t' \in [0,1)$ and $i, i' \in I$. If $(e^{2\pi i t}, i) \sim (e^{2\pi i t'}, i')$, then either $(t = t' \wedge i = i')$ or $t = t' = 0$. In both cases, $\tilde{H}((e^{2\pi i t}, i), s) = \tilde{H}((e^{2\pi i t'}, i'), s)$ for all $s \in [0,1]$. Hence, a function $H \colon W \times [0,1] \to W$ is well-defined by

$$\begin{tikzcd}
\left(\bigcup_{i \in I} (A_\delta(1) \times \{i\}) \right) \times [0,1] \arrow[r, "\tilde{H}"] \arrow[d, "p \times \mathrm{id}"'] & \bigcup_{i \in I} (A_\delta(1) \times \{i\}) \arrow[d, "p"] \\
W \times [0,1] \arrow[r, dashed, "H"'] & W
\end{tikzcd}$$

Explicitly, thus,

$$H\big((e^{2\pi i t}, i)/_\sim, s\big) := (e^{2\pi i t s}, i)/_\sim \text{ for } i \in I, |t| < \delta.$$

Let us check that H is continuous. First, remember that the projection p is a homeomorphism between the open sets $\bigcup_{i \in I} \beta_i(A_\delta(1) \backslash \{1\})$ and $W \backslash \{x_0\}$. Thus H is continuous at every point of $(W \backslash \{x_0\}) \times [0,1]$. Second, for every neighbourhood V of x_0 of the form (5.14), we have $H(V \times [0,1]) \subseteq V$. Hence, H is also continuous at every point of $\{x_0\} \times [0,1]$.

The definition of H ensures that

$$\forall z \in A_\delta(1), i \in I.\ \ H(p_i(z), 1) = p_i(z) \ \wedge\ H(p_i(z), 0) = x_0$$
$$\forall s \in [0,1].\ \ H(x_0, s) = x_0$$

and we see that W is contractible.

④ *We show that $p_j(S^1)$ is a deformation retract of U_j, and hence $\pi_1(U_j) \cong \mathbb{Z}$:* Fix $j \in I$. The argument is almost the same as in the previous step, only that we deform only those arcs $p_i(A_\delta(1))$ with $i \neq j$.

Define a function $H\colon (p_j(S^1) \cup U_j) \times [0,1] \to U_j$

$$\begin{tikzcd}
A_\delta(1) \times [0,1] \arrow[rrr, "(e^{it},s)\mapsto e^{its}"] \arrow[dd, "\beta_i \times \mathrm{id}\ (i\in I\setminus\{j\})"'] & & & A_\delta(1) \arrow[dd, "\beta_i"] \\
& S^1 \times [0,1] \arrow[r, "(z,s)\mapsto z"] \arrow[dl, "\beta_j"'] & S^1 \arrow[dr, "\beta_j"] & \\
\Big(\beta_j(S^1) \cup \bigcup\limits_{i\in I\setminus\{j\}} \beta_i(A_\delta(1))\Big) \times [0,1] \arrow[rrr, dotted] \arrow[d, "p\times\mathrm{id}"'] & & & \beta_j(S^1) \cup \bigcup\limits_{i\in I\setminus\{j\}} \beta_i(A_\delta(1)) \arrow[d, "p"] \\
U_j \times [0,1] \arrow[rrr, dashed, "H"'] & & & U_j
\end{tikzcd}$$

Explicitly, thus,

$$H\big((e^{2\pi it}, i)/_\sim, s\big) := \begin{cases} (e^{2\pi it}, i)/_\sim & \text{if } i = j, t \in [0,1) \\ (e^{2\pi its}, i)/_\sim & \text{if } i \in I\setminus\{j\}, |t| < \delta \end{cases}$$

For the same reasons as in the previous step, H is well-defined and continuous. Moreover, the definition of H ensures that

$$\begin{aligned} &\forall z \in A_\delta(1), i \in I.\ H(p_i(z), 1) = p_i(z) \ \wedge\ H(p_i(z), 0) \in p_j(S^1) \\ &\forall z \in S^1, s \in [0,1].\ H(p_j(z), s) = p_j(z) \end{aligned}$$

and we see that indeed $p_j(S^1)$ is a deformation retract of U_j.

Since $p_j(S^1) \cong S^1$ by Lemma 5.7.2 (v), we obtain from Proposition 5.3.6 (ii) that $\pi_1(U_j) \cong \mathbb{Z}$.

❑

5.8 Spaces with prescribed fundamental group

We show how to kill selected loops by attaching disks to a space. One can image that one builds bridges over which specified loops can be deformed to a point. Again, the crucial role is played by the Seifert–van Kampen theorem (this time in the form of Corollary 5.6.3 (ii)), which ensures that not more than the specified loops become deformable to a point. The construction of the space is very much related to a general construction known as *CW-complexes*. Intuitively, a CW-complex is a family of cells of different dimensions which are glued together. Again, we do not elaborate further on the general situation.

We proceed similar as before, present a construction of a space as an example, and then establish its properties.

Example 5.8.1. Let I be a nonempty set, $\langle X, \mathcal{T}\rangle$ be the bouquet of $|I|$ circles, and $\mathcal{F}$ a family of loops in X with base point x_0. Moreover, let B^2 be the closed unit disk in $\mathbb{C}$. Consider the disjoint union

$$X \amalg \Big(\coprod_{f\in\mathcal{F}} B^2 \Big) = (X \times \{\circ\}) \cup \bigcup_{f\in\mathcal{F}} (B^2 \times \{f\}),$$

and denote by

$$\hat{\beta}_\circ \colon X \to X \amalg \coprod_{f\in\mathcal{F}} B^2, \qquad \hat{\beta}_f \colon B^2 \to X \amalg \coprod_{f\in\mathcal{F}} B^2 \text{ for } f \in \mathcal{F},$$

the inclusion maps. Further, define a relation $\sim$ on $X \amalg \coprod_{f\in\mathcal{F}} B^2$ as (Δ denotes the diagonal)

$$\begin{aligned} \sim := \Delta &\cup \Big\{ \big(\hat{\beta}_f(e^{2\pi i t}), \hat{\beta}_\circ(f(t))\big) \mid t \in [0,1], f \in \mathcal{F} \Big\} \\ &\cup \Big\{ \big(\hat{\beta}_\circ(f(t)), \hat{\beta}_f(e^{2\pi i t})\big) \mid t \in [0,1], f \in \mathcal{F} \Big\} \\ &\cup \Big\{ \big(\hat{\beta}_f(e^{2\pi i t}), \hat{\beta}_g(e^{2\pi i s})\big) \mid 0 \leqslant s, t \in [0,1], f, g \in \mathcal{F} \text{ with } f(t) = g(s) \Big\} \end{aligned}$$

This is an equivalence relation. Reflexivity and symmetry is built in the definition, and transitivity is seen by distinguishing cases.

Now set

$$\hat{X} := \Big(X \amalg \coprod_{f\in\mathcal{F}} B^2 \Big) /\sim.$$

Further, let $\hat{p} \colon X \amalg \coprod_{f\in\mathcal{F}} B^2 \to \hat{X}$ be the canonical projection, and set

$$\hat{p}_\circ := \hat{p} \circ \hat{\beta}_\circ, \qquad \hat{p}_f := \hat{p} \circ \hat{\beta}_f \text{ for } f \in \mathcal{F}.$$

We endow $\hat{X}$ with the final topology induced by the family $\{\hat{p}_\circ\} \cup \{\hat{p}_f \mid f \in \mathcal{F}\}$.

$$\begin{tikzcd}
S^1 \arrow[dr, dotted, "\beta_i"] \arrow[drr, dotted, "p_i"] & & & & B^2 \arrow[dl, dotted, "\hat{\beta}_f"'] \arrow[dddl, "\hat{p}_f"] \\
\vdots & \coprod_{i\in I} S^1 \arrow[r, dotted, "p"] & X \arrow[r, dotted, "\hat{\beta}_\circ"] \arrow[ddr, "\hat{p}_\circ"'] & X \amalg \Big(\coprod_{f\in\mathcal{F}} B^2 \Big) \arrow[dd, dotted, "\hat{p}"] & \vdots \\
S^1 \arrow[ur, dotted, "\beta_j"] \arrow[urr, dotted, "p_j"'] & & & & B^2 \arrow[ul, dotted, "\hat{\beta}_{f'}"] \arrow[dl, "\hat{p}_{f'}"] \\
 & & & \hat{X} &
\end{tikzcd}$$

The space $\hat{X}$ is pathwise connected, since it is the union of the pathwise connected subsets $(\hat{p}_\circ \circ p_i)(S^1)$, $i \in I$, and $\hat{p}_f(B^2)$, $f \in \mathcal{F}$, which have the point $\hat{x}_0 := \hat{p}_\circ(x_0)$ in common.

Given $z \in S^1$ and $\epsilon \in (0,1)$, we set

$$S_\epsilon(z) := \big\{ w \in B^2 \mid 1 - \epsilon < |w| \wedge |\arg w - \arg z| < 2\pi\epsilon, \big\}.$$

Note that $S_\epsilon(z) \cap S^1 = A_\epsilon(z)$.

Lemma 5.8.2.

(i) *The following maps, considered as maps between the written sets, are homeomorphisms:*

$$\hat{p}_0 \colon X \to \hat{p}_0(X), \qquad \hat{p}_f \colon B^2 \backslash S^1 \to \hat{p}_f(B^2 \backslash S^1).$$

(ii) *Let $x \in X$ and $O \subseteq \hat{X}$ open with $\hat{p}_\circ(x) \in O$. Then there exist $\epsilon_{f,z} \in (0,1)$, such that the set*

$$W := \hat{p}_\circ\big(\hat{p}_\circ^{-1}(O)\big) \cup \bigcup_{\substack{f \in \mathcal{F} \\ z \in \hat{p}_f^{-1}(O) \cap S^1}} \hat{p}_f\big(S_{\epsilon_{f,z}}(z)\big) \tag{5.15}$$

is open in $\hat{X}$ and contained in O.

Proof. By the definition of the relation $\sim$ the maps $\hat{p}_\circ$ and $\hat{p}_f|_{B^2 \backslash S^1}$ are injective, and by the definition of the topology on $\hat{X}$ they are continuous. We have to show that they are open as maps onto their image carrying the subspace topology. First, let $O \subseteq X$ be open. Set $W := \hat{p}_\circ(O) \cup \bigcup_{f \in \mathcal{F}} \hat{p}_f(B^2 \backslash S^1)$, then clearly $\hat{p}_\circ(O) = \hat{p}_\circ(X) \cap W$. We have

$$\hat{p}_\circ^{-1}(W) = O, \quad \hat{p}_f^{-1}(W) = \big\{ e^{2\pi i t} \mid f(t) \in O \big\} \cup (B^2 \backslash S^1),$$

and see that W is open in $\hat{X}$. Second, let $f \in \mathcal{F}$ and $O \subseteq B^2 \backslash S^1$ be open. Then

$$\hat{p}_\circ^{-1}\big(\hat{p}_f(O)\big) = \varnothing, \quad \hat{p}_f^{-1}\big(\hat{p}_f(O)\big) = O, \quad \hat{p}_g^{-1}\big(\hat{p}_f(O)\big) = \varnothing \text{ for } g \neq f,$$

and thus $\hat{p}_f(O)$ is open in $\hat{X}$.

We come to the proof of (ii). Since $\hat{p}_f^{-1}(O)$ is an open subset of B^2 we can choose for each $z \in \hat{p}_f^{-1}(O) \cap S^1$ a number $\epsilon_{f,z} \in (0,1)$ with $S_{\epsilon_{f,z}}(z) \subseteq \hat{p}_f^{-1}(O)$. Set

$$W_f := \bigcup_{z \in \hat{p}_f^{-1}(O) \cap S^1} S_{\epsilon_{f,z}}(z).$$

Then W_f is an open subset of B^2, and

$$W_f \subseteq \hat{p}_f^{-1}(O), \qquad W_f \cap S^1 = \hat{p}_f^{-1}(O) \cap S^1.$$

Now consider the set (5.15), i.e.,

$$W = \hat{p}_\circ\big(\hat{p}_\circ^{-1}(O)\big) \cup \bigcup_{g \in \mathcal{F}} \hat{p}_g(W_g).$$

The inclusion $W \subseteq O$ clearly holds, so we have to show that W is open. Again, this means that we have to check inverse images. We claim that

$$\hat{p}_\circ^{-1}(W) = \hat{p}_\circ^{-1}(O), \qquad \forall f \in \mathcal{F}.\ \hat{p}_f^{-1}(W) = W_f. \tag{5.16}$$

The first equality follows since (in the first line we use that $\hat{p}_\circ$ is injective)

$$\begin{aligned}
&\hat{p}_\circ^{-1}\big(\hat{p}_\circ\big(\hat{p}_\circ^{-1}(O)\big)\big) = \hat{p}_\circ^{-1}(O) \\
&\hat{p}_\circ^{-1}\big(\hat{p}_g(W_g)\big) = \hat{p}_\circ^{-1}\big(\hat{p}_g(W_g \cap S^1)\big) = \hat{p}_\circ^{-1}\big(\hat{p}_g(\hat{p}_g^{-1}(O) \cap S^1)\big) \subseteq \hat{p}_\circ^{-1}(O)
\end{aligned}$$

To see the second equality in (5.16), compute

$$\begin{aligned}
&\hat{p}_f^{-1}\big(\hat{p}_\circ\big(\hat{p}_\circ^{-1}(O)\big)\big) \subseteq \hat{p}_f^{-1}(O) \cap S^1 = W_f \cap S^1 \subseteq W_f,\\
&\hat{p}_f^{-1}\big(\hat{p}_f(W_f)\big) = \hat{p}_f^{-1}\big(\hat{p}_f(W_f \cap S^1)\big) \cup \hat{p}_f^{-1}\big(\hat{p}_f(W_f \backslash S^1)\big)\\
&\qquad = \hat{p}_f^{-1}\big(\hat{p}_f\big(\hat{p}_f^{-1}(O) \cap S^1\big)\big) \cup (W_f \backslash S^1) \subseteq \big(\hat{p}_f^{-1}(O) \cap S^1\big) \cup (W_f \backslash S^1) = W_f,\\
&\hat{p}_f^{-1}\big(\hat{p}_g(W_g)\big) \subseteq \hat{p}_f^{-1}(O) \cap S^1 = W_f \cap S^1 \subseteq W_f \text{ for } g \neq f.
\end{aligned}$$

Thus $\hat{p}_f^{-1}(W) \subseteq W_f$. The reverse inequality is clear. ❑

Remark 5.8.3. It follows already from Proposition 5.4.3 and the description of loops in S^1 that every loop in X is FEP-homotopic to a product of some loops

$$\psi_{j,n} \colon \left\{ \begin{array}{ccc} [0,1] & \to & X \\ t & \mapsto & p_j(e^{int}) \end{array} \right. \quad \text{where } j \in I, n \in \mathbb{Z}. \tag{5.17}$$

When working with $\pi_1(X)$, it is thus enough to consider products of loops $\psi_{j,n}$.

A technical advantage of loops of this form is that the image of such a loop is a union of whole circles $p_j(S^1)$, and that the inverse image of every point of X is finite.

Theorem 5.8.4. *Let I be a nonempty set, $\langle X, \mathcal{T} \rangle$ the bouquet of $|I|$ circles, and let $\mathcal{F}$ be a set of loops in X based at x_0 such that each $f \in \mathcal{F}$ is a product of some loops of the form (5.17). Moreover, denote by $N(\mathcal{F})$ the smallest normal subgroup of $\pi_1(X)$ containing $\mathcal{F}/_\sim$. Let $\hat{X}$ be the space constructed from $\mathcal{F}$ as in Example 5.8.1. Then*

$$\pi_1(\hat{X}) \cong \pi_1(X) \big/ N(\mathcal{F}).$$

Corollary 5.8.5. *Let G be a group. Then there exists a topological space $\langle X, \mathcal{T} \rangle$ with $\pi_1(X) \cong G$.*

Proof. Every group is isomorphic to a factor of a free group. Hence, Corollary 5.8.5 is an immediate consequence of Theorems 5.7.3 and 5.8.4. ❑

To prove Theorem 5.8.4, we would ideally like to apply the Seifert–van Kampen theorem with the cover

$$\{\hat{p}_\circ(X)\} \cup \{\hat{p}_f(B^2) \mid f \in \mathcal{F}\} \tag{5.18}$$

of $\hat{X}$. The stated assertion would then follow at once from Corollary 5.6.3 (ii). However, again there is the problem that the sets in (5.18) are not open. The way to circumvent this problem is somewhat different as in the proof of Theorem 5.7.3.

Proof of Theorem 5.8.4.

① *We define a family of sets:* For $f \in \mathcal{F}$ set

$$\begin{aligned}
O_f &:= \hat{p}_f(B^2 \backslash S^1),\\
\mathring{O}_f &:= \hat{p}_f(B^2 \backslash (S^1 \cup \{0\})),\\
U_\circ &:= \hat{p}_\circ(X) \cup \Big(\bigcup_{g \in \mathcal{F}} \mathring{O}_g\Big) = \hat{X} \backslash \{\hat{p}_g(0) \mid g \in \mathcal{F}\},\\
U_f &:= \hat{p}_\circ(X) \cup O_f \cup \Big(\bigcup_{g \in \mathcal{F} \backslash \{f\}} \mathring{O}_g\Big) = \hat{X} \backslash \{\hat{p}_g(0) \mid g \in \mathcal{F} \backslash \{f\}\}.
\end{aligned}$$

It is seen by checking inverse images that these sets are open in $\hat{X}$:

	O_f	$\mathring{O}_f$	$U_\circ$	U_f
$\hat{p}_\circ^{-1}$	$\varnothing$	$\varnothing$	X	X
$\hat{p}_f^{-1}$	$B^2 \backslash S^1$	$B^2 \backslash (S^1 \cup \{0\})$	$B^2 \backslash \{0\}$	B^2
$\hat{p}_g^{-1}$ $(g \neq f)$	$\varnothing$	$\varnothing$	$B^2 \backslash \{0\}$	$B^2 \backslash \{0\}$

Moreover, these sets are all pathwise connected. For O_f and $\mathring{O}_f$ this is obvious. For $U_\circ$ and U_f note that

$$\begin{aligned} U_\circ &= \hat{p}_\circ(X) \cup \Big(\bigcup_{g \in \mathcal{F}} \hat{p}_g(B^2 \backslash \{0\}) \Big) \\ U_f &= \hat{p}_\circ(X) \cup \hat{p}_f(B^2) \cup \Big(\bigcup_{g \in \mathcal{F} \backslash \{f\}} \hat{p}_g(B^2 \backslash \{0\}) \Big) \end{aligned}$$

and that the point $\hat{x}_0$ belongs to each set in the union.

② *We apply Seifert–van Kampen to describe $\pi_1(\hat{X})$ as colimit:* The intersection of at least two different sets U_f is equal to $U_\circ$. Hence, the open cover $\{U_f \mid f \in \mathcal{F}\}$ satisfies all assumptions of the Seifert–van Kampen theorem. Denote by $\iota_f \colon U_f \to \hat{X}$ and $\iota_{\circ f} \colon U_\circ \to U_f$ the inclusion maps. Then $\langle \pi_1(\hat{X}), \pi_1(\iota_f) \rangle$ is colimit of the diagram

$$\begin{tikzcd} & \vdots \\ & \pi_1(U_f) \\ \pi_1(U_\circ) \arrow[ur, dotted, "\pi_1(\iota_{\circ f})"] \arrow[dr, dotted, "\pi_1(\iota_{\circ g})"'] & \vdots \\ & \pi_1(U_g) \\ & \vdots \end{tikzcd} \tag{5.19}$$

③ *We apply Seifert–van Kampen to analyse the maps $\pi_1(\iota_{\circ f})$:* We have

$$U_f = U_\circ \cup O_f \text{ and } U_\circ \cap O_f = \mathring{O}_f.$$

Hence, $\{U_\circ, O_f\}$ is an open cover of U_f and satisfies the assumptions of the Seifert–van Kampen theorem. Since O_f is a continuous image of a convex set, we have $\pi_1(O_f) = \{\mathbb{1}\}$.

Corollary 5.6.3 (ii) yields that $\pi_1(\iota_{\circ f}) \colon \pi_1(U_\circ) \to \pi_1(U_f)$ is surjective and that $\ker \iota_{\circ f}$ is the smallest normal subgroup of $\pi_1(U_\circ)$ containing $\pi_1(\mathring{\iota}_{f\circ})(\pi_1(\mathring{O}_f))$. Here we denote by $\mathring{\iota}_{f\circ} \colon \pi_1(\mathring{O}_f) \to \pi_1(U_\circ)$ the inclusion map.

The circle $\frac{1}{2} \cdot S^1 = \{z \in \mathbb{C} \mid |z| = \frac{1}{2}\}$, which is homeomorphic to S^1, is a deformation retract of $B^2 \backslash (S^1 \cup \{0\})$. Therefore, $\pi_1(\mathring{O}_f) \cong \mathbb{Z}$, and the equivalence class of the loop

$$h_f(t) := \hat{p}_f\Big(\frac{1}{2} e^{2\pi i t}\Big)$$

is a generator of $\pi_1(\mathring{O}_f)$. The normal subgroup of $\pi_1(U_\circ)$ generated by $\pi_1(\mathring{\iota}_{f\circ})(\pi_1(\mathring{O}_f))$ is thus equal to the normal subgroup generated by the singleton set $\{(\mathring{\iota}_{f\circ} \circ h_f)/_{\approx}\}$.

④ *We show that $\hat{p}_\circ(X)$ is a deformation retract of $U_\circ$:* Define a map $H\colon U_\circ \times [0,1] \to \hat{p}_\circ(X)$ as

$$\begin{cases} H(\hat{p}_\circ(x), s) := \hat{p}_\circ(x) & \text{for } x \in X, s \in [0,1] \\ H(\hat{p}_f(z), s) := \hat{p}_f\big(\frac{z}{(1-s)+s|z|}\big) & \text{for } z \in B^2\backslash\{0\}, s \in [0,1] \end{cases} \tag{5.20}$$

Since $\hat{p}_f$ is a homeomorphism of $B^2\backslash S^1$ onto its image, and this image is an open subset of $\hat{X}$, the function H is continuous at every point of $\bigcup_{f\in\mathcal{F}} \mathring{O}_f$. Let $x \in X$ and $O \subseteq \hat{X}$ be an open set with $\hat{p}_\circ(x) \in O$. According to Lemma 5.8.2, we find an open set W of the form (5.15) which is contained in O. Clearly, it contains the point $\hat{p}_\circ(x)$. For every point $z \in \hat{p}_f^{-1}(O) \cap S^1$ we have

$$H\big(\hat{p}_f(S_{\epsilon_{f,z}}(z)\backslash S^1) \times [0,1]\big) \subseteq \hat{p}_f(S_{\epsilon_{f,z}}(z))$$

It follows that $H(W \times [0,1]) \subseteq W$. We conclude that H is continuous also at every point of $\hat{p}_\circ(X) \times [0,1]$.

⑤ *We produce an isomorphic diagram:* Let H be as in (5.20), and let $r\colon U_\circ \to \hat{p}_\circ(X)$ be the retraction $r := H(.,1)$. Moreover, let $\iota_\circ\colon \hat{p}_\circ(X) \to U_\circ$ be the inclusion map.

$$\begin{tikzcd} X \arrow[r, bend left, "\hat{p}_\circ"] & \hat{p}_\circ(X) \arrow[l, bend left, "\hat{p}_\circ^{-1}"] \arrow[r, dotted, bend left, "\iota_\circ"] & U_\circ \arrow[l, dotted, bend left, "r"] \end{tikzcd}$$

Then we have the isomorphisms

$$\begin{tikzcd} \pi_1(X) \arrow[r, bend left, "\pi_1(\hat{p}_\circ)"] & \pi_1(\hat{p}_\circ(X)) \arrow[l, bend left, "\pi_1(\hat{p}_\circ^{-1})"] \arrow[r, bend left, "\pi_1(\iota_\circ)"] & \pi_1(U_\circ) \arrow[l, bend left, "\pi_1(r)"] \end{tikzcd}$$

The isomorphism $\pi_1(\hat{p}_\circ^{-1}) \circ \pi_1(r)$ maps $(\mathring{\iota}_{f\circ} \circ h_f)/_{\approx}$ to the equivalence class of the loop

$$\big(\hat{p}_\circ^{-1} \circ r \circ \mathring{\iota}_{f\circ} \circ h_f\big)(t) = \hat{p}_\circ^{-1}\big(\hat{p}_f(e^{2\pi i t})\big) = \hat{p}_\circ^{-1}\big(\hat{p}_\circ(f(t))\big) = f(t).$$

Hence, we get an isomorphism with (p_f is the canonical projection)

$$\begin{tikzcd} \pi_1(U_\circ) \arrow[r, "\pi_1(\iota_{\circ f})"] \arrow[d, "\pi_1(\hat{p}_\circ^{-1})\circ\pi_1(r)"'] & \pi_1(U_f) \arrow[d, dashed, "\cong"] \\ \pi_1(X) \arrow[r, "p_f"] & \pi_1(X)/_{N(f)} \end{tikzcd}$$

Since the isomorphism on the left is independent of f, the diagram (5.19) is isomorphic (in the sense of Theorem 1.9.5 (ii)) to

$$\begin{tikzcd} & \pi_1(X)/_{N(f)} \\ \pi_1(X) \arrow[ur, dotted, "p_f"] \arrow[dr, dotted, "p_g"'] & \vdots \\ & \pi_1(X)/_{N(g)} \end{tikzcd} \tag{5.21}$$

By uniqueness of colimits in Theorem 1.9.5 and the computation of the colimit in Example 1.9.4, the group $\pi_1(\hat{X})$ is isomorphic to $\pi_1(X)/_{N(\mathcal{F})}$. ❑

5.9 Free homotopy and homotopy equivalence

We define a general variant of homotopy.

Definition 5.9.1. Let $\langle X, \mathcal{T}\rangle$ and $\langle Y, \mathcal{V}\rangle$ be topological spaces, and let $\phi, \psi \colon X \to Y$ be continuous functions.

(i) We say that ϕ and ψ are *homotopic* (more specifically, *freely homotopic*) if there exists a continuous function $H \colon X \times [0,1] \to Y$ with

$$\forall x \in X.\ H(x,0) = \phi(x) \ \wedge\ H(x,1) = \psi(x)$$

Every function H with this property is called a *homotopy* (more specifically, a *free homotopy*) from ϕ to ψ.

(ii) Let $A \subseteq X$. We say that ϕ and ψ are *homotopic relative to* A, if there exists a continuous function $H \colon X \times [0,1] \to Y$ with

$$\begin{aligned}&\forall x \in X.\ H(x,0) = \phi(x) \ \wedge\ H(x,1) = \psi(x)\\ &\forall a \in A\ \forall s, s' \in [0,1].\ H(a,s) = H(a,s')\end{aligned}$$

If ϕ and ψ are homotopic relative to A we write $\phi \approx_A \psi$, and every function H as above is called a *relative homotopy* from ϕ to ψ.

Note that free homotopy is nothing but homotopy relative to $\varnothing$, and FEP-homotopy of paths defined on some interval $[a,b]$ is homotopy relative to $\{a,b\}$.

Proposition 5.9.2. *Let $\langle X, \mathcal{T}\rangle$ and $\langle Y, \mathcal{V}\rangle$ be topological spaces and $\phi, \psi \colon X \to Y$ continuous functions. Assume that ϕ and ψ are freely homotopic, and let H be a free homotopy from ϕ to ψ. Further, let $x_0 \in X$, denote by $h \colon [0,1] \to Y$ the path $h(s) := H(x_0, s)$, and let Φ_h be the isomorphism*

$$\Phi_h \colon \begin{cases} \pi_1(Y, \psi(x_0)) & \to \quad \pi_1(Y, \phi(x_0)) \\ \qquad f/_{\approx} & \mapsto \quad (h \centerdot f \centerdot h^{\leftarrow})/_{\approx} \end{cases}$$

with inverse $\Phi_{h^{-1}}$, cf. Lemma 5.3.1.

Then $\pi_1(\phi) = \Phi_h \circ \pi_1(\psi)$

$$\begin{tikzcd}
& \pi_1(X,x_0) \arrow[dl, "\pi_1(\psi)"'] \arrow[dr, "\pi_1(\phi)"] & \\
\pi_1(X,\psi(x_0)) \arrow[rr, bend left, "\Phi_h"] & & \pi_1(X,\phi(x_0)) \arrow[ll, bend left, "\Phi_{h^{-1}}"]
\end{tikzcd}$$

Proof. Let $f \in \mathscr{L}(X, x_0)$ be given. We have to produce a FEP-homotopy in Y from $\phi \circ f$ to $h \centerdot (\psi \circ f) \centerdot h^{\leftarrow}$. For $t, s \in [0,1]$ define

$$K(t,s) := \begin{cases} h(3t) & \text{if } t \leqslant \frac{s}{3} \\ H\Big(f\big((t - \frac{s}{3})(1 - \frac{2s}{3})^{-1}\big), s\Big) & \text{if } \frac{s}{3} \leqslant t \leqslant 1 - \frac{s}{3} \\ h(3(1-t)) & \text{if } 1 - \frac{s}{3} \leqslant t \end{cases}$$

Inspecting these formulae shows first of all that K is well-defined, hence by the gluing lemma continuous, and that

$$K(0,s) = K(1,s) = h(0) = \phi(x_0), \quad K(t,0) = H(f(t),0) = (\phi \circ f)(t).$$

Moreover, we see that

$$K(t,1) = \begin{cases} h(3t) & \text{if } t \leqslant \frac{1}{3} \\ (\psi \circ f)\big((t-\frac{1}{3})(1-\frac{2}{3})^{-1}\big) & \text{if } \frac{1}{3} \leqslant t \leqslant \frac{2}{3} \\ h(3(1-t)) & \text{if } 1-\frac{1}{3} \leqslant t \end{cases}$$

which is a reparameterisation of $h \centerdot (\psi \circ f) \centerdot h^{\leftarrow}$. ❑

Corollary 5.9.3. *Let $\langle X, \mathcal{T}\rangle$ and $\langle Y, \mathcal{V}\rangle$ be topological spaces and $\phi, \psi \colon X \to Y$ continuous functions which are freely homotopic. Let $x_0 \in X$ and assume that $\phi(x_0) = \psi(x_0)$ (denote this point as y_0).*

(i) *There exists an inner automorphism Φ of the group $\pi_1(Y, y_0)$ such that $\pi_1(\phi) = \Phi \circ \pi_1(\psi)$.*

(ii) *If $\pi_1(Y, y_0)$ is commutative or ϕ and ψ are homotopic relative to $\{x_0\}$, then $\pi_1(\phi) = \pi_1(\psi)$.*

Proof. Since $\phi(x_0) = \psi(x_0)$, the path $h(s) := H(x_0, s)$ is a loop in Y with base point y_0. Thus we can write

$$\Phi_h(f) = (h \centerdot f \centerdot h^{\leftarrow})/_{\approx} = h/_{\approx} \cdot f/_{\approx} \cdot (h^{\leftarrow}/_{\approx}) = h/_{\approx} \cdot f/_{\approx} \cdot (h/_{\approx})^{-1} \text{ for } f \in \mathscr{L}(Y, y_0).$$

If $\pi_1(Y, y_0)$ is commutative, the identity map is the only inner automorphism. If $\phi \approx_{\{x_0\}} \psi$, we can choose H such that h is the constant path at x_0. ❑

Definition 5.9.4. Let $\langle X, \mathcal{T}\rangle$ and $\langle Y, \mathcal{V}\rangle$ be topological spaces. A continuous map $\phi \colon X \to Y$ is called a *homotopy equivalence*, if

$$\exists \psi \colon Y \to X \text{ continuous. } \psi \circ \phi \approx_{\varnothing} \mathrm{id}_X \wedge \phi \circ \psi \approx_{\varnothing} \mathrm{id}_Y$$

The spaces $\langle X, \mathcal{T}\rangle$ and $\langle Y, \mathcal{V}\rangle$ are said to have the *same homotopy type*, if there exists a homotopy equivalence between them.

The relation to have the same homotopy type clearly is an equivalence relation. If X and Y are homeomorphic, or if Y is a deformation retract of X, then X and Y have the same homotopy type. In the first case we have $\phi^{-1} \circ \phi = \mathrm{id}_X, \phi \circ \phi^{-1} = \mathrm{id}_Y$ for a homeomorphism $\phi \colon X \to Y$, and in the second case we have $\iota \circ r \approx_{\varnothing} \mathrm{id}_X, r \circ \iota = \mathrm{id}_Y$ when $\iota \colon Y \to X$ is the inclusion and $r = H(.,1)$ with H as in (5.2).

It is a consequence of Proposition 5.9.2 that homotopy equivalent spaces have the same fundamental group.

Corollary 5.9.5. *Let $\langle X, \mathcal{T}\rangle$ and $\langle Y, \mathcal{V}\rangle$ be topological spaces, $x_0 \in X$, and $\phi \colon X \to Y$ a homotopy equivalence. Then $\pi_1(\phi) \colon \pi_1(X, x_0) \to \pi_1(Y, \phi(x_0))$ is an isomorphism.*

Proof. Let $\psi \colon Y \to X$ be such that $\psi \circ \phi \approx_{\varnothing} \mathrm{id}_X$ and $\phi \circ \psi \approx_{\varnothing} \mathrm{id}_Y$. Moreover, denote $y_0 := \phi(x_0)$.

Proposition 5.9.2 provides us with the respective isomorphisms in the diagrams

$$\begin{tikzcd}
\pi_1(X,x_0) \arrow[r, "\mathrm{id}"] \arrow[d, "\pi_1(\phi)"'] \arrow[dr, dotted, "\pi_1(\psi\circ\phi)"] & \pi_1(X,x_0) \arrow[d, "\cong"] \\
\pi_1(Y,y_0) \arrow[r, "\pi_1(\psi)"'] & \pi_1(X,\psi(y_0))
\end{tikzcd}
\qquad
\begin{tikzcd}
\pi_1(Y,y_0) \arrow[r, "\mathrm{id}"] \arrow[d, "\pi_1(\psi)"'] \arrow[dr, dotted, "\pi_1(\phi\circ\psi)"] & \pi_1(Y,y_0) \arrow[d, "\cong"] \\
\pi_1(X,\psi(y_0)) \arrow[r, "\pi_1(\phi)"'] & \pi_1(X,\phi(\psi(y_0)))
\end{tikzcd}$$

The left diagram shows that the map $\pi_1(\phi)\colon \pi_1(X,x_0) \to \pi_1(Y,\phi(x_0))$ is injective. The right one shows that $\pi_1(\phi)\colon \pi_1(X,\psi(y_0)) \to \pi_1(Y,\phi(\psi(y_0)))$ is surjective.

Let h be the path in X with initial point x_0 and terminal point $(\psi\circ\phi)(x_0) = \psi(y_0)$ obtained from some free homotopy H from id_X to $\psi\circ\phi$ (as $h(s) := H(x_0,s)$). Then the path $\phi\circ h$ has initial point y_0 and terminal point $\phi(\psi(y_0))$. By their definition, the isomorphisms from Lemma 5.3.1 satisfy

$$\begin{tikzcd}
\pi_1(X,x_0) \arrow[r, "\pi_1(\phi)"] \arrow[d, dashed, "\cong"] & \pi_1(Y,y_0) \arrow[d, dashed, "\cong"] \\
\pi_1(X,\psi(y_0)) \arrow[r, "\pi_1(\phi)"'] & \pi_1(X,\phi(\psi(y_0)))
\end{tikzcd}$$

and we see that also $\pi_1(\phi)\colon \pi_1(X,x_0) \to \pi_1(Y,\phi(x_0))$ is surjective. ❏

We can now prove an interesting structural property of topological groups. The formulation in the below Theorem 5.9.7 is general.

Definition 5.9.6. Let $\langle X,\mathcal{T}\rangle$ be a topological space and $x_0 \in X$. Then $\langle X,\mathcal{T},x_0\rangle$ is called a *H-space* (the letter H stands to recognise the influence of H.Hopf) , if there exists a continuous function $m\colon X\times X \to X$ with

$$m(x_0,x_0) = x_0, \qquad m(x_0,.) \approx_{\{x_0\}} \mathrm{id}_X, \quad m(.,x_0) \approx_{\{x_0\}} \mathrm{id}_X\,.$$

We think of m as a deformation of a multiplication. For example every topological group is a H-space with the group multiplication for m and the unit element for x_0. Then it even holds that $m(x_0,.) = m(.,x_0) = \mathrm{id}_X$ and multiplication is associative. When passing to a deformation, of course algebraic properties will get lost. But in view of corollary 5.9.5 one might expect that topological properties are retained.

Theorem 5.9.7. *Let $\langle X,\mathcal{T},x_0\rangle$ be a H-space. Then $\pi_1(X,x_0)$ is commutative.*

Proof. Denote by $k\colon X \to X$ the constant map $k(x) := x_0$, and by $\mathbb{1}_{x_0}$ the constant loop based at x_0. Then we can write

$$m(x_0,.) = m\circ(k\times\mathrm{id}_X), \qquad m(.,x_0) = m\circ(\mathrm{id}_X \times k).$$

Corollary 5.9.3 (ii) implies that

$$\pi_1\big(m\circ(k\times\mathrm{id}_X)\big) = \mathrm{id}_{\pi_1(X,x_0)}, \qquad \pi_1\big(m\circ(k\times\mathrm{id}_X)\big) = \mathrm{id}_{\pi_1(X,x_0)}\,.$$

Let $\psi\colon \pi_1(X,x_0)\times\pi_1(X,x_0)\to\pi_1(X\times X,(x_0,x_0))$ be the isomorphism (5.4). We obtain that for each $f\in\mathscr{L}(X,x_0)$

$$\begin{aligned}
f/_{\approx} &= \pi_1\big(m\circ(k\times\mathrm{id}_X)\big)(f/_{\approx}) = \big(\pi_1(m)\circ\pi_1(k\times\mathrm{id}_X)\big)(f/_{\approx})\\
&= \pi_1(m)\big(([k\times\mathrm{id}_X]\circ f)/_{\approx}\big) = \pi_1(m)\big(\big(\underbrace{[k\circ f]}_{=\mathbb{1}_{x_0}}\times\underbrace{[\mathrm{id}_X\circ f]}_{=f}\big)/_{\approx}\big)\\
&= \pi_1(m)\big(\psi(\mathbb{1}_{x_0}/_{\approx},f/_{\approx})\big) = \big(\pi_1(m)\circ\psi\big)(\mathbb{1}_{x_0}/_{\approx},f/_{\approx})
\end{aligned}$$

and analogously

$$f/_{\approx} = \big(\pi_1(m)\circ\psi\big)(f/_{\approx},\mathbb{1}_{x_0}/_{\approx}).$$

In the direct product of $\pi_1(X,x_0)\times\pi_1(X,x_0)$ each two elements $(\mathbb{1}_{x_0}/_{\approx},f/_{\approx})$ and $(g/_{\approx},\mathbb{1}_{x_0}/_{\approx})$ commute. It follows that

$$\begin{aligned}
(g/_{\approx})\cdot(f/_{\approx}) &= \big(\pi_1(m)\circ\psi\big)(g/_{\approx},\mathbb{1}_{x_0}/_{\approx})\cdot\big(\pi_1(m)\circ\psi\big)(\mathbb{1}_{x_0}/_{\approx},f/_{\approx})\\
&= \big(\pi_1(m)\circ\psi\big)\big((g/_{\approx},\mathbb{1}_{x_0}/_{\approx})\cdot(\mathbb{1}_{x_0}/_{\approx},f/_{\approx})\big)\\
&= \big(\pi_1(m)\circ\psi\big)\big((\mathbb{1}_{x_0}/_{\approx},f/_{\approx})\cdot(g/_{\approx},\mathbb{1}_{x_0}/_{\approx})\big)\\
&= \big(\pi_1(m)\circ\psi\big)(\mathbb{1}_{x_0}/_{\approx},f/_{\approx})\cdot\big(\pi_1(m)\circ\psi\big)(g/_{\approx},\mathbb{1}_{x_0}/_{\approx}) = (f/_{\approx})\cdot(g/_{\approx})
\end{aligned}$$

❏

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