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Contents

Part I

Geometry

Chapter 1

Geometry of inner product spaces

1.1 Inner product spaces

We start with recalling some vocabulary from linear algebra. Throughout our exposition, with exception of the below example dealing with Minkowski space-time, linear spaces will be over the scalar field $\mathbb{C}$ of complex numbers.

DEA1 **1.1.1 Definition.** Let $\mathcal{L}$ be a linear space. An *inner product* on $\mathcal{L}$ is a map

$$[.,.] : \mathcal{L} \times \mathcal{L} \to \mathbb{C}$$

such that

(IP1) $[x+y, z] = [x,z] + [y,z], \quad x,y,z \in \mathcal{L}.$

(IP2) $[\alpha x, y] = \alpha [x,y], \quad x,y \in \mathcal{L},\ \alpha \in \mathbb{C}.$

(IP3) $[x,y] = \overline{[y,x]}, \quad x,y \in \mathcal{L}.$

If $[.,.]$ is an inner product on $\mathcal{L}$, we will speak of $\langle \mathcal{L}, [.,.] \rangle$ as an *inner product space*[1]. //

Note that we do not require any definiteness properties, like e.g. $[x,x] \geq 0$, $x \in \mathcal{L}$.

DEA2 **1.1.2 Definition.** Let $\langle \mathcal{L}, [.,.] \rangle$ be an inner product space.

(*i*) An element $x \in \mathcal{L}$ is called

$$\begin{aligned} \textit{positive} &:\Leftrightarrow [x,x] > 0 & \quad \textit{nonnegative} &:\Leftrightarrow [x,x] \geq 0 \\ \textit{negative} &:\Leftrightarrow [x,x] < 0 & \quad \textit{nonpositive} &:\Leftrightarrow [x,x] \leq 0 \\ \textit{neutral} &:\Leftrightarrow [x,x] = 0 & & \end{aligned}$$

(*ii*) A linear subspace $\mathcal{M}$ of $\mathcal{L}$ is called

[1] Often we will drop explicit notation of the inner product, and speak of an 'inner product space $\mathcal{L}$', implicitly understanding that on $\mathcal{L}$ an inner product $[.,.]$ is given.

$$\begin{aligned} \textit{positive definite} &:\Leftrightarrow [x,x] > 0, x \in \mathcal{M}\setminus\{0\} \ ^{2} \\ \textit{negative definite} &:\Leftrightarrow [x,x] < 0, x \in \mathcal{M}\setminus\{0\} \\ \textit{neutral} &:\Leftrightarrow [x,x] = 0, x \in \mathcal{M} \end{aligned}$$

$$\begin{aligned} \textit{positive semidefinite} &:\Leftrightarrow [x,x] \geq 0, x \in \mathcal{M} \ ^{3} \\ \textit{negative semidefinite} &:\Leftrightarrow [x,x] \leq 0, x \in \mathcal{M} \end{aligned}$$

The set of all linear subspaces of $\mathcal{L}$ will be denoted by $\operatorname{Sub}\mathcal{L}$. The set of all positive definite subspaces by $\operatorname{Sub}_{>0}\mathcal{L}$, and the notations $\operatorname{Sub}_{\geq 0}\mathcal{L}$, $\operatorname{Sub}_{<0}\mathcal{L}$, $\operatorname{Sub}_{\leq 0}\mathcal{L}$, $\operatorname{Sub}_{=0}\mathcal{L}$, have their obvious meaning.

(*iii*) A linear subspace $\mathcal{M}$ of $\mathcal{L}$ is called

$$\begin{aligned} \textit{definite} &:\Leftrightarrow \mathcal{M} \in \operatorname{Sub}_{>0}\mathcal{L} \vee \mathcal{M} \in \operatorname{Sub}_{<0}\mathcal{L} \\ \textit{semidefinite} &:\Leftrightarrow \mathcal{M} \in \operatorname{Sub}_{\geq 0}\mathcal{L} \vee \mathcal{M} \in \operatorname{Sub}_{\leq 0}\mathcal{L} \end{aligned}$$

$$\textit{indefinite} :\Leftrightarrow \mathcal{M} \notin \operatorname{Sub}_{\geq 0}\mathcal{L} \wedge \mathcal{M} \notin \operatorname{Sub}_{\leq 0}\mathcal{L}$$

(*iv*) An inner product $[.,.]$ on $\mathcal{L}$, or the inner product space $\langle \mathcal{L}, [.,.]\rangle$, is called *positive definite*, *negative definite*, etc., if the subspace $\mathcal{L}$ of $\langle \mathcal{L}, [.,.]\rangle$ has the corresponding property.

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Also, we do not require nondegeneracy.

DEA57 **1.1.3 Definition.** Let $\langle \mathcal{L}, [.,.]\rangle$ be an inner product space.

(*i*) An element x of $\mathcal{L}$ is called *isotropic*, if $[x,y] = 0$ for all $y \in \mathcal{L}$. The set of all isotropic elements of $\mathcal{L}$ is called the *isotropic part* of $\mathcal{L}$, and will be denoted by $\langle \mathcal{L}, [.,.]\rangle^{\circ}$ [4].

(*ii*) An inner product $[.,.]$ on $\mathcal{L}$, or the inner product space $\langle \mathcal{L}, [.,.]\rangle$, is called *degenerated*, if $\mathcal{L}^{\circ} \neq \{0\}$. If $\mathcal{L}^{[\circ]} = \{0\}$, it is called *nondegenerated*.

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examples: minkowski space-time, dirichlet space

EXA3 *1.1.4 Example.* In order to visualize geometric notions it is more practical to use a linear space over the field $\mathbb{R}$. Consider for example the linear space $\mathbb{R}^2$ endowed with the inner product $[.,.]$ defined as

$$\left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right] := x_1 y_1 - x_2 y_2, \ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^2 .$$

Then $\langle \mathbb{R}^2, [.,.]\rangle$ is an indefinite inner product space.

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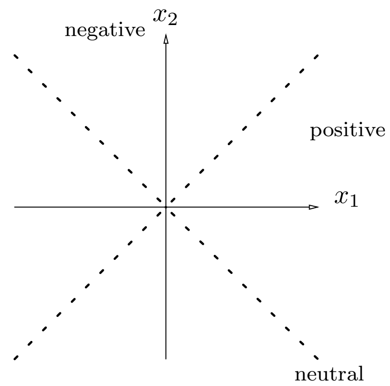
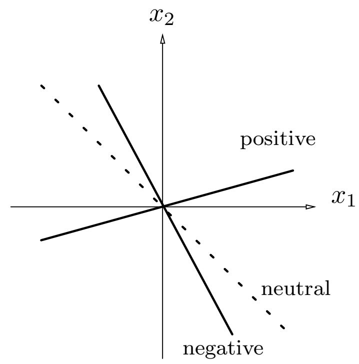
[2] Sometimes instead of 'positive definite subspace' or 'negative definite subspace' one also uses the shorter terms *positive subspace* or *negative subspace*, respectively.

[3] Sometimes positive semidefinite subspaces are also called *nonnegative*, and negative semidefinite ones *nonpositive*.

[4] Often we use the shorthand notation $\mathcal{L}^{\circ}$ or, a little more specific, $\mathcal{L}^{[\circ]}$.



positive/negative/neutral
elements



positve/negative/neutral
subspaces

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It is a basic fact that in semidefinite inner product spaces the *Schwarz inequality* holds.

LEA58 **1.1.5 Lemma.** *Let $\langle \mathcal{L}, [.,.]\rangle$ be a semidefinite inner product space. Then*

$$|[x,y]| \leq |[x,x]|^{\frac{1}{2}} \cdot |[y,y]|^{\frac{1}{2}}, \quad x,y \in \mathcal{L}\,.$$

Proof. Consider the case that $\mathcal{L}$ is positive semidefinite, the case that $\mathcal{L}$ is negative semidefinite is settled with the same argument.

Set $A := [x,x]$, $B := |[x,y]|$, and $C := [y,y]$, and let $\alpha \in \mathbb{C}$, $|\alpha| = 1$ be such that $\alpha[y,x] = B$. We have

$$0 \leq [x - t\alpha y, x - t\alpha y] = [x,x] - t\alpha[y,x] - t\overline{\alpha}[x,y] + t^2[y,y], \quad t \in \mathbb{R}\,,$$

i.e. $A - 2tB + t^2C \geq 0$ for all $t \in \mathbb{R}$. If $C = 0$, thus also $B = 0$. If $C \neq 0$, we choose $t = \frac{B}{C}$ to obtain $AC - B^2 \geq 0$. ❑

COA9 **1.1.6 Corollary.** *Let $\langle \mathcal{L}, [.,.]\rangle$ be a semidefinite inner product space. Then each neutral element is isotropic.*

Proof. Assume that $x \in \mathcal{L}$ and $[x,x] = 0$. Then

$$|[x,y]| \leq [x,x]^{\frac{1}{2}} \cdot [y,y]^{\frac{1}{2}} = 0, \quad y \in \mathcal{L}\,.$$

❑

Structure preserving maps deserve to be named.

DEA4 **1.1.7 Definition.** A map $\phi : \mathcal{L}_1 \to \mathcal{L}_2$ between two inner product spaces $\langle \mathcal{L}_1, [.,.]_1\rangle$ and $\langle \mathcal{L}_2, [.,.]_2\rangle$ is called *isometric* (or an *isometry*), if it is linear and satisfies

$$[\phi x, \phi y]_2 = [x,y]_1, \quad x,y \in \mathcal{L}_1\,.$$

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Note that, clearly, the composition of two isometric maps is again isometric. Also, the identity map of one inner product space onto itself is isometric.

EXA58 *1.1.8 Example.* Let $\langle \mathcal{L}, [.,.]\rangle$ be an inner product space.

(*i*) Let $\mathcal{M}$ be a linear subspace of $\mathcal{L}$. Then $\mathcal{M}$ becomes an inner product space if endowed with the inner product inherited from $\mathcal{L}$, i.e. if we set

$$[x,y]_{\mathcal{M}} := [x,y], \quad x,y \in \mathcal{M}\,.$$

Then the set-theoretic inclusion $\iota : \mathcal{M} \to \mathcal{L}$ is an isometry.

(*ii*) Let $\mathcal{M}$ be a linear subspace of $\mathcal{L}^\circ$, and denote by $\pi : \mathcal{L} \to \mathcal{L}/_{\mathcal{M}}$ the canonical projection. Then an inner product is well-defined on the factor space $\mathcal{L}/_{\mathcal{M}}$ by

$$[\pi x, \pi y]_{\mathcal{L}/_{\mathcal{M}}} := [x,y], \quad x,y \in \mathcal{L}\,.$$

The canonical projection π is an isometry of $\mathcal{L}$ onto $\mathcal{L}/_{\mathcal{M}}$.

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We will occasionally use the following homomorphy theorem.

LEA5 **1.1.9 Lemma.** *Let $\langle \mathcal{L}_1, [.,.]_1\rangle$ and $\langle \mathcal{L}_2, [.,.]_2\rangle$ be inner product spaces, and let $\phi : \mathcal{L}_1 \to \mathcal{L}_2$ be isometric. Then*

$$\phi^{-1}\big([\operatorname{ran}\phi]^\circ\big) = \mathcal{L}_1^\circ\,. \tag{1.1.1}$$ A59

There exists a bijective isometry $\tilde{\phi}$ with

$$\begin{tikzcd} \mathcal{L}_1 \arrow[r, "\phi"] \arrow[d] & \operatorname{ran}\phi \arrow[d] \\ \mathcal{L}_1/_{\mathcal{L}_1^\circ} \arrow[r, "\tilde{\phi}"'] & \operatorname{ran}\phi/_{[\operatorname{ran}\phi]^\circ} \end{tikzcd} \tag{1.1.2}$$ A60

where the downwards arrows are the respective canonical projections.

Proof. Let $x \in \mathcal{L}_1^\circ$ and $y \in \operatorname{ran}\phi$. Then we can write $y = \phi z$ with some $z \in \mathcal{L}_1$, and hence obtain

$$[\phi x, y]_2 = [\phi x, \phi z]_2 = [x,z]_1 = 0\,.$$

This shows that $\phi(\mathcal{L}_1^\circ) \subseteq [\operatorname{ran}\phi]^\circ$. Conversely, let $x \in \mathcal{L}_1$, and assume that $\phi x \in [\operatorname{ran}\phi]^\circ$. Then

$$[x,y]_1 = [\phi x, \phi y]_2 = 0,\ y \in \mathcal{L}_1\,,$$

and hence $x \in \mathcal{L}_1^\circ$. This shows (1.1.1).

By (1.1.1), there exits a linear and bijective map $\tilde{\phi}$ making the diagram (1.1.2) commute. Since the canonical projections are isometric, the map $\tilde{\phi}$ also has this property. ❑

1.2 Orthogonality

DEA6 **1.2.1 Definition.** Let $\langle \mathcal{L}, [.,.]\rangle$ be an inner product space.

(*i*) Two elements $x, y \in \mathcal{L}$ are called *orthogonal*, if $[x, y] = 0$. In this case we write $x[\perp]y$. Two subsets $A, B \subseteq \mathcal{L}$ are called *orthogonal*, if $[x, y] = 0$ for all $x \in A$ and $y \in B$, and in this case we write $A[\perp]B$. [5]

(*ii*) Let $A \subseteq \mathcal{L}$. The set

$$\mathcal{L}[-]A := \{x \in \mathcal{L} : x[\perp]y, y \in A\}$$

is called the *orthogonal companion* of A. [6]

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Let $\mathcal{M}$ be a linear subspace of an inner product space $\langle \mathcal{L}, [.,.] \rangle$. The orthogonal companion $\mathcal{L}[-]\mathcal{M}$ need not be a complement of $\mathcal{M}$ in the sense that $\mathcal{M} \dot{+} \mathcal{M}^\perp = \mathcal{L}$. It may happen that $\mathcal{M} \cap \mathcal{M}^\perp \neq \{0\}$ or $\mathcal{M} + \mathcal{M}^\perp \neq \mathcal{L}$, or both.

Let us note that the isotropic part of an inner product space $\mathcal{L}$ is nothing else but $\mathcal{L}^\perp$. In fact, we should be more careful with abuse of language and rather write $\mathcal{L}^\circ = \mathcal{L}[-]\mathcal{L}$, namely for the following reason: If $\mathcal{M}$ is a linear subspace of $\mathcal{L}$, it is itself an inner product space with the inner product inherited from $\mathcal{L}$. Then $\mathcal{M}^\circ = \mathcal{M}[-]\mathcal{M}$. The symbol $\mathcal{M}^\perp$ however may have two essentially different meanings ($\mathcal{M}^\circ$ or $\mathcal{L}[-]\mathcal{M}$).

DEA7 **1.2.2 Definition.** Let $\langle \mathcal{L}, [.,.] \rangle$ be an inner product space, and let $y \in \mathcal{L}$. Then we denote by $[., y]$ the linear functional

$$[., y] : \left\{ \begin{array}{ccc} \mathcal{L} & \to & \mathbb{C} \\ x & \mapsto & [x, y] \end{array} \right.$$

//

If $A \subseteq \mathcal{L}$, then clearly

$$A^\perp = \bigcap_{y \in A} \ker \big([., y]\big) . \tag{1.2.1}$$ A8

Thus $A^\perp$ is a linear subspace of $\mathcal{L}$. In particular, the isotropic part $\mathcal{L}^\circ$ of $\mathcal{L}$ is a linear subspace of $\mathcal{L}$.

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EXA10 *1.2.3 Example.* Consider the inner product space (over the scalar field $\mathbb{R}$) $\langle \mathbb{R}^3, [.,.] \rangle$ where the inner product $[.,.]$ is defined as

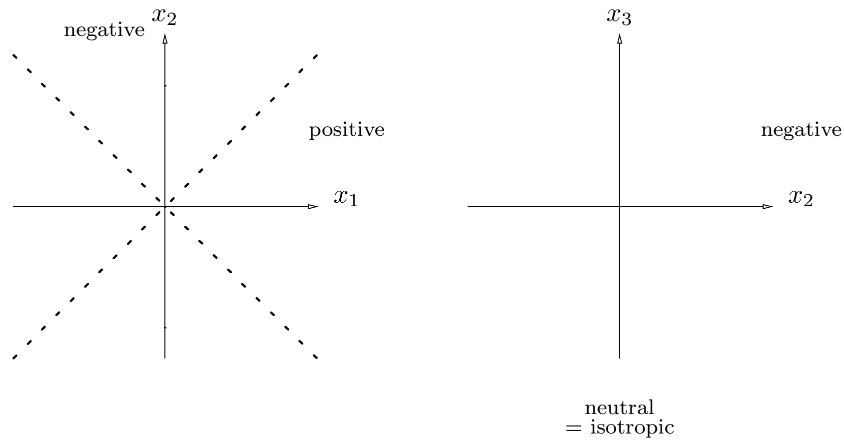
$$\left[\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \right] := x_1 y_1 - x_2 y_2, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in \mathbb{R}^3 .$$

Then $\langle \mathbb{R}^3, [.,.] \rangle$ is a degenerated indefinite inner product space.

The subspace $\mathcal{M}_1 := \{(x_1, x_2, x_3)^T : x_3 = 0\}$ is nothing else but the non-degenerated indefinite inner product space considered in Example 1.1.4. The subspace $\mathcal{M}_2 := \{(x_1, x_2, x_3)^T : x_1 = 0\}$ is a negative semidefinite degenerated space.

[5] If explicit notation of the inner product is not needed, we will write $x \perp y$ or $A \perp B$.

[6] If it is not necessary to emphasize the space $\mathcal{L}$ within which the orthogonal companion is taken, we will write $A^{[\perp]}$ or, even less specific, $A^\perp$.



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If $\mathcal{L}$ is an inner product space and $\mathcal{M} \in \operatorname{Sub}\mathcal{L}$, we may have $\mathcal{M} + \mathcal{M}^\perp \neq \mathcal{L}$. For example, consider the subspace $\mathcal{M} := \operatorname{span}\{(1,1)^T\}$ in Example 1.1.4. Then $\mathcal{M} = \mathcal{M}^\perp$, and hence certainly $\mathcal{M} + \mathcal{M}^\perp \neq \mathcal{L}$.

DEA11 **1.2.4 Definition.** Let $\langle \mathcal{L}, [.,.] \rangle$ be an inner product space, and let $\mathcal{M} \in \operatorname{Sub}\mathcal{L}$. Then $\mathcal{M}$ is called *orthocomplemented*, if $\mathcal{M} + \mathcal{M}^\perp = \mathcal{L}$. //

Let us note that, if $\mathcal{L} = \{0\}$ and $\mathcal{M}$ is finite-dimensional, then

$$\mathcal{M} + \mathcal{M}^\perp \neq \mathcal{L} \iff \mathcal{M} \cap \mathcal{M}^\perp \neq \{0\}\,.$$

In general these two conditions need not coincide.

Orthocomplemented subspaces are of interest, since they allow for orthogonal projections. Thereby a projection $P : \mathcal{L} \to \mathcal{L}$ is called *orthogonal*, if $\operatorname{ran} P \perp \ker P$. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be linear subspaces of $\mathcal{L}$. Then $\mathcal{L}_1 + \mathcal{L}_2$ denotes the sum of them. If $\mathcal{L}_1 \cap \mathcal{L}_2 = \{0\}$, and we wish to put emphasize on this fact, we will use the notation $\mathcal{L}_1 \dot{+} \mathcal{L}_2$ and speak of a direct sum. Similarly, if $\mathcal{L}_1 \perp \mathcal{L}_2$, we will sometimes write $\mathcal{L}_1[+]\mathcal{L}_2$ and speak of an orthogonal sum. The combined symbol $\mathcal{L}_1[\dot{+}]\mathcal{L}_2$ will have the obvious meaning.

PRA12 **1.2.5 Proposition.** *Let $\langle \mathcal{L}, [.,.] \rangle$ be an inner product space, and let $\mathcal{M} \in \operatorname{Sub}\mathcal{L}$. Then $\mathcal{M}$ is orthocomplemented if and only if there exists an orthogonal projection P with $\operatorname{ran} P = \mathcal{M}$. In this case the projection P is unique if and only if $\mathcal{M}$ is nondegenerated.*

Proof. A projection P with $\operatorname{ran} P = \mathcal{M}$ is uniquely determined by the subspace $\ker P$. This subspace has the property that $\mathcal{M} \dot{+} \ker P = \mathcal{L}$. Moreover, the projection P is orthogonal if and only if $\ker P \subseteq \mathcal{M}^\perp$. We see that the set of all orthogonal projections whose range is $\mathcal{M}$ corresponds bijectively to the set of all linear subspaces $\mathcal{M}'$ of $\mathcal{M}^\perp$ with $\mathcal{M} \dot{+} \mathcal{M}' = \mathcal{L}$. This set, however, is nonempty if and only if $\mathcal{M} + \mathcal{M}^\perp = \mathcal{L}$. Moerover, it contains exactly one element if and only if $\mathcal{M} \dot{+} \mathcal{M}^\perp = \mathcal{L}$. ❑

COA13 **1.2.6 Corollary.** *Let $\langle \mathcal{L}, [.,.] \rangle$ be an inner product space, and let $\mathcal{M}_1, \mathcal{M}_2 \in \operatorname{Sub}\mathcal{L}$ be such that*

$$\mathcal{M}_1 \perp \mathcal{M}_2,\ \mathcal{M}_1 \cap \mathcal{M}_2 = \{0\}\,.$$

If $\mathcal{M}_1$ and $\mathcal{M}_2$ are both orthocomplemented, so is $\mathcal{M}_1[\dot{+}]\mathcal{M}_2$.

Proof. By the above proposition, there exist orthogonal projections of $\mathcal{L}$ onto $\mathcal{M}_1$ and $\mathcal{M}_2$. Let us denote them by P_1 and P_2, respectively. Since $\mathcal{M}_1 \perp \mathcal{M}_2$, we have $P_1P_2 = P_2P_1 = 0$. Since $\mathcal{M}_1 \cap \mathcal{M}_2 = \{0\}$, we have $\ker(P_1 + P_2) = \ker P_1 \cap \ker P_2$. We see that $(P_1+P_2)^2 = P_1+P_2$, and $\ker(P_1+P_2) \perp \operatorname{ran}(P_1+P_2)$. Thus $P_1 + P_2$ is an orthogonal projection. Clearly, $\operatorname{ran}(P_1 + P_2) = \mathcal{M}_1[\dot{+}]\mathcal{M}_2$. Therefore, the space $\mathcal{M}_1[\dot{+}]\mathcal{M}_2$ is orthocomplemented. ❑

Let us collect some simple properties of orthocomplemented subspaces.

LEA14 **1.2.7 Lemma.** *Let $\mathcal{M}$ be an orthocomplemented subspace of $\langle \mathcal{L}, [.,.] \rangle$. Then the following hold:*

(*i*) $\mathcal{M}^\circ \subseteq \mathcal{L}^\circ$.

(*ii*) $\mathcal{M}^\perp$ *is orthocomplemented.*

(*iii*) $(\mathcal{M}^\perp)^\circ = \mathcal{L}^\circ$.

(*iv*) $\mathcal{M}^{\perp\perp} = \mathcal{M} + \mathcal{L}^\circ$.

Proof. The inclusion (*i*) holds since

$$\mathcal{M}^\circ = \mathcal{M} \cap \mathcal{M}^\perp \subseteq (\mathcal{M}^\perp + \mathcal{M})^\perp = \mathcal{L}^\perp = \mathcal{L}^\circ\,.$$

The assertion (*ii*) follows since $\mathcal{M}^{\perp\perp} \supseteq \mathcal{M}$, and hence $\mathcal{M}^\perp + \mathcal{M}^{\perp\perp} \supseteq \mathcal{M}^\perp + \mathcal{M} = \mathcal{L}$. To see (*iii*), note that

$$\mathcal{L}^\circ \subseteq \underbrace{\mathcal{M}^{\perp\perp} \cap \mathcal{M}^\perp}_{=(\mathcal{M}^\perp)^\circ} = (\mathcal{M}^\perp + \mathcal{M})^\perp = \mathcal{L}^\perp = \mathcal{L}^\circ\,.$$

Finally, for the proof of (*iv*), let $x \in \mathcal{M}^{\perp\perp}$ be given. Write $x = y + z$ with $y \in \mathcal{M}$, $z \in \mathcal{M}^\perp$. Then, by the already proved item (*iii*) and the fact that $\mathcal{M} \subseteq \mathcal{M}^{\perp\perp}$, we have

$$z = x - y \in \mathcal{M}^{\perp\perp} \cap \mathcal{M}^\perp = \mathcal{L}^\circ\,.$$

Hence $x \in \mathcal{M} + \mathcal{L}^\circ$. The converse inclusion $\mathcal{M} + \mathcal{L}^\circ \subseteq \mathcal{M}^{\perp\perp}$ in (*iv*) is is trivial. ❑

In general it is hard to decide whether or not a given subspace $\mathcal{M}$ is orthocomplemented. A simple, but important, example of orthocomplemented subspaces is given by the following proposition.

PRA15 **1.2.8 Proposition.** *Let $\mathcal{M}$ be a finite-dimensional subspace of $\mathcal{L}$ with $\mathcal{M}^\circ \subseteq \mathcal{L}^\circ$. Then $\mathcal{M}$ is orthocomplemented.*

Proof. Since $m := \dim \mathcal{M} < \infty$, we can find a basis $\{b_1, \ldots, b_n, b_{n+1}, \ldots, b_m\}$ of $\mathcal{M}$, such that

$$[b_i, b_j] = \begin{cases} \pm 1\,, & i = j \in \{1, \ldots, n\} \\ 0 \quad, & \text{otherwise} \end{cases}$$

Thereby we have $\mathcal{M}^\circ = \operatorname{span}\{b_{n+1}, \ldots, b_m\}$. Put $\mathcal{M}_1 := \operatorname{span}\{b_1, \ldots, b_n\}$, and define $P : \mathcal{L} \to \mathcal{M}_1$ as

$$Px := \sum_{i=1}^{n} \frac{[x, b_i]}{[b_i, b_i]} b_i \,.$$

Then P is linear and $P^2 = P$. Moreover, $\operatorname{ran} P = \mathcal{M}_1$ and $\ker P = \operatorname{ran}(I - P) \perp \mathcal{M}_1$. It follows that $\mathcal{M}_1 + \mathcal{M}_1^\perp = \mathcal{L}$. Since $\operatorname{span}\{b_{n+1}, \ldots, b_m\} \subseteq \mathcal{L}^\circ$, we have $\mathcal{M}^\perp = \mathcal{M}_1^\perp$. Thus also $\mathcal{M} + \mathcal{M}^\perp = \mathcal{L}$. ❑

1.3 Orthogonal decompositions and angular operators

DEA16 **1.3.1 Definition.** Let $\langle \mathcal{L}, [.,.] \rangle$ be an inner product space. A pair $\mathfrak{j} := (\mathcal{L}_1, \mathcal{L}_2)$ of linear subspaces of $\mathcal{L}$ is called an *orthogonal decomposition* of $\mathcal{L}$, if

$$\mathcal{L} = \mathcal{L}_1 [\dot{+}] \mathcal{L}_2 \,.$$

In this case, we will denote by $P_\mathfrak{j}^1$ and $P_\mathfrak{j}^2$ the orthogonal projections with

$$\operatorname{ran} P_\mathfrak{j}^1 = \mathcal{L}_1,\ \ker P_\mathfrak{j}^1 = \mathcal{L}_2, \qquad \operatorname{ran} P_\mathfrak{j}^2 = \mathcal{L}_2,\ \ker P_\mathfrak{j}^2 = \mathcal{L}_1 \,.$$

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Let $\mathfrak{j}$ be an orthogonal decomposition of $\mathcal{L}$. Then it is clear that the following relations hold:

$$\begin{gathered} P_\mathfrak{j}^1 + P_\mathfrak{j}^2 = I,\ P_\mathfrak{j}^1 P_\mathfrak{j}^2 = P_\mathfrak{j}^2 P_\mathfrak{j}^1 = 0\,, \\ [P_\mathfrak{j}^1 x, P_\mathfrak{j}^1 y] = [P_\mathfrak{j}^1 x, y] = [x, P_\mathfrak{j}^1 y]\,, \\ [P_\mathfrak{j}^2 x, P_\mathfrak{j}^2 y] = [P_\mathfrak{j}^2 x, y] = [x, P_\mathfrak{j}^2 y]\,, \\ [x, y] = [P_\mathfrak{j}^1 x, P_\mathfrak{j}^1 y] + [P_\mathfrak{j}^2 x, P_\mathfrak{j}^2 y]\,. \end{gathered}$$

DEA17 **1.3.2 Definition.** Let $\langle \mathcal{L}, [.,.] \rangle$ be an inner product space. A pair $\mathfrak{J} := (\mathcal{L}_+, \mathcal{L}_-)$ of linear subspaces of $\mathcal{L}$ is called an *fundamental decomposition* of $\mathcal{L}$, if

(*i*) $\mathcal{L}_+ \in \operatorname{Sub}_{>0} \mathcal{L}$ and $\mathcal{L}_- \in \operatorname{Sub}_{<0} \mathcal{L}$.

(*ii*) $\mathcal{L} = \mathcal{L}_+ [\dot{+}] \mathcal{L}_- [\dot{+}] \mathcal{L}^\circ$.

In this case, we will denote by $P_\mathfrak{J}^+$ and $P_\mathfrak{J}^-$ the orthogonal projections with

$$\operatorname{ran} P_\mathfrak{J}^+ = \mathcal{L}_+, \ker P_\mathfrak{J}^+ = \mathcal{L}_- + \mathcal{L}^\circ, \quad \operatorname{ran} P_\mathfrak{J}^- = \mathcal{L}_-, \ker P_\mathfrak{J}^- = \mathcal{L}_+ + \mathcal{L}^\circ \,.$$

Moreover, we put

$$J_\mathfrak{J} := P_\mathfrak{J}^+ - P_\mathfrak{J}^-,\ (x, y)_\mathfrak{J} := [Jx, y] \,.$$

The projections $P_\mathfrak{J}^\pm$ are called the *fundamental projections*, the map $J_\mathfrak{J}$ the *fundamental symmetry* associated with $\mathfrak{J}$. //

Again it is clear that the following relations hold:

$$P_{\mathfrak{J}}^{+}P_{\mathfrak{J}}^{-}=P_{\mathfrak{J}}^{-}P_{\mathfrak{J}}^{+}=0,\ P_{\mathfrak{J}}^{+}J_{\mathfrak{J}}=J_{\mathfrak{J}}P_{\mathfrak{J}}^{+}=P_{\mathfrak{J}}^{+},\ P_{\mathfrak{J}}^{-}J_{\mathfrak{J}}=J_{\mathfrak{J}}P_{\mathfrak{J}}^{-}=-P_{\mathfrak{J}}^{-}\,,$$
$$J_{\mathfrak{J}}|_{\mathcal{L}_+}=I_{\mathcal{L}_+},\ J_{\mathfrak{J}}|_{\mathcal{L}_-}=-I_{\mathcal{L}_-},\ \ker J_{\mathfrak{J}}=\mathcal{L}^\circ\,,$$
$$J_{\mathfrak{J}}^2=P_{\mathfrak{J}}^{+}+P_{\mathfrak{J}}^{-},\ \left(J_{\mathfrak{J}}|_{\mathcal{L}_++\mathcal{L}_-}\right)^{-1}=J_{\mathfrak{J}}|_{\mathcal{L}_++\mathcal{L}_-}\,,$$
$$[P_{\mathfrak{J}}^{\pm}x,P_{\mathfrak{J}}^{\pm}y]=[P_{\mathfrak{J}}^{\pm}x,y]=[x,P_{\mathfrak{J}}^{\pm}y],\ [x,y]=[P_{\mathfrak{J}}^{+}x,P_{\mathfrak{J}}^{+}y]+[P_{\mathfrak{J}}^{-}x,P_{\mathfrak{J}}^{-}y]\,,$$
$$[J_{\mathfrak{J}}x,y]=[x,J_{\mathfrak{J}}y],\ [J_{\mathfrak{J}}x,J_{\mathfrak{J}}y]=[x,y],\ [J_{\mathfrak{J}}x,y]=[P_{\mathfrak{J}}^{+}x,P_{\mathfrak{J}}^{+}y]-[P_{\mathfrak{J}}^{-}x,P_{\mathfrak{J}}^{-}y]\,.$$

Next we collect some basic properties of $(.,.)_{\mathfrak{J}}$.

LEA18 **1.3.3 Lemma.** *Let $\mathfrak{J}=(\mathcal{L}_+,\mathcal{L}_-)$ be a fundamental decomposition of the inner product space $\langle\mathcal{L},[.,.]\rangle$. Then $(.,.)_{\mathfrak{J}}$ is a positive semidefinite inner product on $\mathcal{L}$. We have*

$$\langle\mathcal{L},(.,.)_{\mathfrak{J}}\rangle^\circ=\langle\mathcal{L},[.,.]\rangle^\circ,\ \mathcal{L}=\mathcal{L}_+(\dot{+})_{\mathfrak{J}}\mathcal{L}_-(\dot{+})_{\mathfrak{J}}\mathcal{L}^\circ\,,$$
$$(J_{\mathfrak{J}}x,y)_{\mathfrak{J}}=(x,J_{\mathfrak{J}}y)_{\mathfrak{J}},\ (J_{\mathfrak{J}}x,J_{\mathfrak{J}}y)_{\mathfrak{J}}=(x,y)_{\mathfrak{J}}\,.$$

Put $p_{\mathfrak{J}}(x):=(x,x)_{\mathfrak{J}}^{\frac12}$, then $p_{\mathfrak{J}}$ is a seminorm, and we have

$$|[x,y]|\leq p_{\mathfrak{J}}(x)\cdot p_{\mathfrak{J}}(y),\ x,y\in\mathcal{L}\,. \tag{1.3.1}$$ A19

We have $p_{\mathfrak{J}}^{-1}(\{0\})=\mathcal{L}^\circ$. Thus $p_{\mathfrak{J}}$ is a norm if and only if $\mathcal{L}^\circ$ is nondegenerated. In this case, we will also use the notation $\|.\|_{\mathfrak{J}}$ instead of $p_{\mathfrak{J}}$.

Proof. Since $[J_{\mathfrak{J}}x,y]=[x,J_{\mathfrak{J}}y]$, $x,y\in\mathcal{L}$, the map $(.,.)_{\mathfrak{J}}:\mathcal{L}\times\mathcal{L}\to\mathbb{C}$ is an inner product. Moreover, since $\mathcal{L}_+$ is positive and $\mathcal{L}_-$ is negative,

$$(x,x)_{\mathfrak{J}}=[P_{\mathfrak{J}}^{+}x,P_{\mathfrak{J}}^{+}x]-[P_{\mathfrak{J}}^{-}x,P_{\mathfrak{J}}^{-}x]\geq 0,\ x\in\mathcal{L}\,.$$

Clearly, $\mathcal{L}^{[\circ]}\subseteq\mathcal{L}^{(\circ)_{\mathfrak{J}}}$. Conversely, assume that $x\in\mathcal{L}^{(\circ)_{\mathfrak{J}}}$. Then $x[\perp]\operatorname{ran}J_{\mathfrak{J}}=\mathcal{L}_++\mathcal{L}_-$. Since in any case $x[\perp]\mathcal{L}^{[\circ]}$, it follows that $x[\perp]\mathcal{L}$, i.e. $x\in\mathcal{L}^{[\circ]}$.

Since $\mathcal{L}_+$ and $\mathcal{L}_-$ are $J_{\mathfrak{J}}$-invariant and $[.,.]$-orthogonal, it follows that also $\mathcal{L}_+(\perp)_{\mathfrak{J}}\mathcal{L}_-$. Together with what we already saw, $\mathcal{L}=\mathcal{L}_+(\dot{+})_{\mathfrak{J}}\mathcal{L}_-(\dot{+})_{\mathfrak{J}}\mathcal{L}^\circ$. Next, we compute

$$(J_{\mathfrak{J}}x,y)_{\mathfrak{J}}=[J_{\mathfrak{J}}^2x,y]=[P_{\mathfrak{J}}^{+}x,y]+[P_{\mathfrak{J}}^{-}x,y]=[P_{\mathfrak{J}}^{+}x,P_{\mathfrak{J}}^{+}y]+[P_{\mathfrak{J}}^{-}x,P_{\mathfrak{J}}^{-}y]=$$
$$=[P_{\mathfrak{J}}^{+}x-P_{\mathfrak{J}}^{-}x,P_{\mathfrak{J}}^{+}y-P_{\mathfrak{J}}^{-}y]=[J_{\mathfrak{J}}x,J_{\mathfrak{J}}y]=(x,J_{\mathfrak{J}}y)_{\mathfrak{J}}$$

and

$$(J_{\mathfrak{J}}x,J_{\mathfrak{J}}y)_{\mathfrak{J}}=[J_{\mathfrak{J}}^2x,J_{\mathfrak{J}}y]=[J_{\mathfrak{J}}(P_{\mathfrak{J}}^{+}+P_{\mathfrak{J}}^{-})x,y]=$$
$$=[(P_{\mathfrak{J}}^{+}-P_{\mathfrak{J}}^{-})x,y]=[J_{\mathfrak{J}}x,y]=(x,y)_{\mathfrak{J}}\,.$$

To show (1.3.1), let $x,y\in\mathcal{L}$ be given and put $x_\pm:=P_{\mathfrak{J}}^{\pm}$, $y_\pm:=P_{\mathfrak{J}}^{\pm}$. Then, by the Schwartz inequality in $\langle\mathcal{L}_+,[.,.]|_{\mathcal{L}_+\times\mathcal{L}_+}\rangle$, $\langle\mathcal{L}_-,-[.,.]|_{\mathcal{L}_-\times\mathcal{L}_-}\rangle$, and $\mathbb{R}^2$,

$$\big|[x,y]\big|\leq\big|[x_+,y_+]\big|+\big|[x_-,y_-]\big|\leq$$
$$\leq[x_+,x_+]^{\frac12}[y_+,y_+]^{\frac12}+(-[x_-,x_-])^{\frac12}(-[y_-,y_-])^{\frac12}\leq$$
$$\leq\left([x_+,x_+]^{\frac12\cdot2}+(-[x_-,x_-])^{\frac12\cdot2}\right)^{\frac12}\cdot\left([y_+,y_+]^{\frac12\cdot2}+(-[y_-,y_-])^{\frac12\cdot2}\right)^{\frac12}=$$
$$=(x,x)_{\mathfrak{J}}\cdot(y,y)_{\mathfrak{J}}\,.$$

❑

DEA20 **1.3.4 Definition.** An inner product space $\langle\mathcal{L},[.,.]\rangle$ is called *decomposable*, if there exists a fundamental decomposition of $\mathcal{L}$. //

Let us in this place only remark that not every inner product space is decomposable. Actually, being decomposable is a strong and important property for an inner product space. An explicit example for non-decomposability will be given later, cf. Example 2.3.4.

Orthogonal decompositions with semidefinite summands are closely related to fundamental decompositions.

LEA21 **1.3.5 Lemma.** *Let $\langle\mathcal{L},[.,.]\rangle$ be an inner product space.*

(*i*) *Let $\mathfrak{J}=(\mathcal{L}_+,\mathcal{L}_-)$ be a fundamental decomposition of $\mathcal{L}$, and let $\mathcal{L}_0^+,\mathcal{L}_0^-\in$ $\operatorname{Sub}\mathcal{L}$ with $\mathcal{L}_0^+\dot{+}\mathcal{L}_0^-=\mathcal{L}^\circ$. Put*

$$\mathcal{L}_1:=\mathcal{L}_++\mathcal{L}_0^+,\ \mathcal{L}_2:=\mathcal{L}_-+\mathcal{L}_0^-\ ,$$

then $\mathcal{L}_1$ is positive semidefinite, $\mathcal{L}_2$ is negative semidefinite, and $\mathfrak{j}:=(\mathcal{L}_1,\mathcal{L}_2)$ is an orthogonal decomposition of $\mathcal{L}$.

(*ii*) *Let $\mathfrak{j}=(\mathcal{L}_1,\mathcal{L}_2)$ be an orthogonal decomposition of $\mathcal{L}$, and assume that $\mathcal{L}_1$ is positive semidefinite and $\mathcal{L}_2$ is negative semidefinite. Choose linear subspaces $\mathcal{L}_+$ and $\mathcal{L}_-$ such that $\mathcal{L}_1=\mathcal{L}_+\dot{+}\mathcal{L}_1^\circ$ and $\mathcal{L}_2=\mathcal{L}_-\dot{+}\mathcal{L}_2^\circ$. Then $\mathfrak{J}:=(\mathcal{L}_+,\mathcal{L}_-)$ is a fundamental decomposition of $\mathcal{L}$.*

Proof. The assertion (*i*) is clear. For (*ii*) note first that, because of the Schwartz inequality, $\mathcal{L}_+$ is positive definite and $\mathcal{L}_-$ is negative definite, cf. Lemma **??**. Moreover, since $\mathcal{L}_1[\perp\mathcal{L}_2$ and $\mathcal{L}_1+\mathcal{L}_2=\mathcal{L}$, we have $\mathcal{L}^\circ=\mathcal{L}_1^\circ+\mathcal{L}_2^\circ$. ❑

DEA22 **1.3.6 Definition.** Let $\mathfrak{j}=(\mathcal{L}_1,\mathcal{L}_2)$ be an orthogonal decomposition of the inner product space $\langle\mathcal{L},[.,.]\rangle$. Then we denote

$$\operatorname{Sub}_{\mathfrak{j}}:=\big\{\mathcal{M}\in\operatorname{Sub}\mathcal{L}:\ \mathcal{M}\cap\mathcal{L}_2=\{0\}\big\}\,.$$

If $\mathcal{M}\in\operatorname{Sub}_{\mathfrak{j}}$, we define the *angular operator* of $\mathcal{M}$ with respect to $\mathfrak{j}$ as

$$\mathfrak{a}_{\mathfrak{j}}(\mathcal{M}):\begin{cases} P_{\mathfrak{j}}^1\mathcal{M} & \to\quad \mathcal{L}_2\\ \qquad x & \mapsto\quad P_{\mathfrak{j}}^2\circ(P_{\mathfrak{j}}^1|_{\mathcal{M}})^{-1}x\end{cases}$$

//

Note here that the requirement $\mathcal{M}\in\operatorname{Sub}_{\mathfrak{j}}$ just means that $P_{\mathfrak{j}}^1|_{\mathcal{M}}$ is injective. Thus $\mathfrak{a}_{\mathfrak{j}}(\mathcal{M})$ is well-defined.

For a given orthogonal decomposition $\mathfrak{j}=(\mathcal{L}_1,\mathcal{L}_2)$ of $\langle\mathcal{L},[.,.]\rangle$, let us denote by $\mathfrak{A}_{\mathfrak{j}}$ the set

$$\mathfrak{A}_{\mathfrak{j}}:=\big\{(D,K):\ D\in\operatorname{Sub}\mathcal{L}_1, K:D\to\mathcal{L}_2\text{ linear}\big\}\,.$$

The set $\mathfrak{A}_{\mathfrak{j}}$ is partially ordered in a natural way, namely with the relation '$\preceq$' defined as

$$(D_1,K_1)\preceq(D_2,K_2)\ :\Leftrightarrow\ D_1\subseteq D_2, K_2|_{D_1}=K_1$$

Note that, clearly, the set $\operatorname{Sub}_{\mathfrak{j}}$ is partially ordered with respect to set-theoretic inclusion.

PRA23 **1.3.7 Proposition.** *Let $\mathfrak{j}$ be an orthogonal decomposition of the inner product space $\langle \mathcal{L}, [.,.]\rangle$. The assignment*

$$\mathfrak{a} : \mathcal{M} \mapsto \big(P_{\mathfrak{j}}^1 \mathcal{M}, \mathfrak{a}_{\mathfrak{j}}(\mathcal{M})\big)$$

is an order-preserving bijection of $\operatorname{Sub}_{\mathfrak{j}}$ *onto* $\mathfrak{A}_{\mathfrak{j}}$. *Its inverse is given as*

$$(D, K) \mapsto \{x + Kx : x \in D\}. \tag{1.3.2}$$ A24

Proof. The fact that $\mathfrak{a}$ maps $\operatorname{Sub}_{\mathfrak{j}}$ into $\mathfrak{A}_{\mathfrak{j}}$, and that it preserves the respective orders, is obvious. Denote the map defined by (1.3.2) by $\mathfrak{b}$. Assume that $(D, K) \in \mathfrak{A}_{\mathfrak{j}}$. Clearly, $\mathfrak{M} := \mathfrak{b}(D, K)$ is a linear subspace of $\mathcal{L}$. Moreover, if $y = x+Kx \in \mathcal{M}\cap\mathcal{L}_2$ for some $x \in D$, then $x = y-Kx \in D\cap\mathcal{L}_2 \subseteq \mathcal{L}_1\cap\mathcal{L}_2 = \{0\}$. Thus $x = 0$ and hence also $y = 0$. We see that $\mathcal{M} \in \operatorname{Sub}_{\mathfrak{j}}$, and thus that $\mathfrak{b}$ maps $\mathfrak{A}_{\mathfrak{j}}$ into $\operatorname{Sub}_{\mathfrak{j}}$.

We shall show that $\mathfrak{a} \circ \mathfrak{b} = \mathrm{id}_{\mathfrak{A}_{\mathfrak{j}}}$. Let $(D, K) \in \mathfrak{A}_{\mathfrak{j}}$ be given, and put $\mathcal{M} :$ $\mathfrak{b}(D, K)$. We have $z \in P_{\mathfrak{j}}^1\mathcal{M}$ if and only if $z = P_{\mathfrak{j}}^1(x+Kx)$ with some $x \in D$. It follows that $z = x$ and $P_{\mathfrak{j}}^2(x+Kx) = Kx$. Thus $P_{\mathfrak{j}}^1\mathcal{M} = D$ and $\mathfrak{a}_{\mathfrak{j}}(\mathcal{M})z = Kz$, $z \in P_{\mathfrak{j}}^1\mathcal{M}$, i.e. $\mathfrak{a}(\mathcal{M}) = (D, K)$.

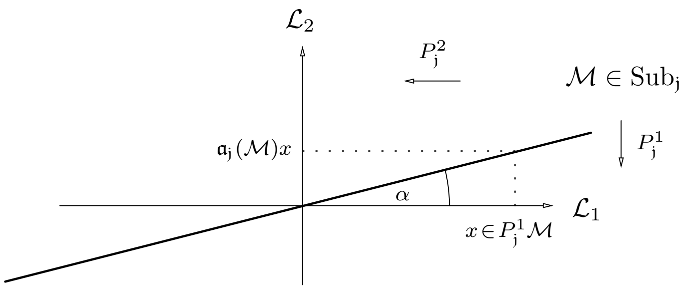
It remains to establish $\mathfrak{b} \circ \mathfrak{a} = \mathrm{id}_{\operatorname{Sub}_{\mathfrak{j}}}$. Let $\mathcal{M} \in \operatorname{Sub}_{\mathfrak{j}}$ be given. If $y \in \mathcal{M}$, then $y = P_{\mathfrak{j}}^1 y + P_{\mathfrak{j}}^2 y$. However, if we set $x := P_{\mathfrak{j}}^1 y$, then $x \in P_{\mathfrak{j}}^1\mathcal{M}$ and $P_{\mathfrak{j}}^2 y = \mathfrak{a}_{\mathfrak{j}}(\mathcal{M})x$. Hence $y = x + \mathfrak{a}_{\mathfrak{j}}(\mathcal{M})x$. We conclude that $\mathcal{M} \subseteq (\mathfrak{b}\circ\mathfrak{a})\mathcal{M}$. To see the other inclusion, let $z = x + \mathfrak{a}_{\mathfrak{j}}(\mathcal{M})x$ with some $x \in P_{\mathfrak{j}}^1\mathcal{M}$ be given. Write $x = P_{\mathfrak{j}}^1 y$ with some $y \in \mathcal{M}$, then $\mathfrak{a}_{\mathfrak{j}}(\mathcal{M})x = P_{\mathfrak{j}}^2 y$, and we conclude that $z = P_{\mathfrak{j}}^1 y + P_{\mathfrak{j}}^2 y = y \in \mathcal{M}$. It follows that $\mathcal{M} = (\mathfrak{b}\circ\mathfrak{a})\mathcal{M}$. ❑

EXA25 *1.3.8 Example.* Consider the (x, y)-plane $\mathbb{R}^2$ endowed with the euclidean inner product, and let

$$\mathcal{L}_1 := \{\begin{pmatrix} x \\ y \end{pmatrix} : y = 0\},\ \mathcal{L}_2 := \{\begin{pmatrix} x \\ y \end{pmatrix} : x = 0\}.$$

Then $\mathfrak{j} := (\mathcal{L}_1, \mathcal{L}_2)$ is an orthogonal decomposition of $\mathbb{R}^2$. The projections $P_{\mathfrak{j}}^1$ and $P_{\mathfrak{j}}^2$ are the orthogonal projections onto the x-axis and onto the y-axis, respectively.

A linear subspace $\mathcal{M}$ of $\mathbb{R}^2$ belongs to $\operatorname{Sub}_{\mathfrak{j}}$ if and only if either $\mathcal{M} = \{0\}$ or $\mathcal{M}$ is a line through the origin different from the y-axis. Let $\mathcal{M} \in \operatorname{Sub}_{\mathfrak{j}}$, $\mathcal{M} \neq \{0\}$. Then $P_{\mathfrak{j}}^1\mathcal{M} = \mathcal{L}_1$. If $\alpha \in (-\pi, \pi)$ denotes the angle between $\mathcal{M}$ and the x-axis, then the action of the angular operator $\mathfrak{a}_{\mathfrak{j}}(\mathcal{M})$ is multiplication by $\tan\alpha$.



$\mathfrak{a}_{\mathfrak{j}}(\mathcal{M})x = x \cdot \tan\alpha$

//

Sometimes it is useful to know that orthogonality can be characterized via angular operators.

LEA26 **1.3.9 Lemma.** *Let $\langle\mathcal{L},[.,.]\rangle$ be an inner product space, and let $\mathfrak{j}=(\mathcal{L}_1,\mathcal{L}_2)$ be an orthogonal decomposition of $\mathcal{L}$. Denote by $\bar{\mathfrak{j}}$ the orthogonal decomposition $\bar{\mathfrak{j}}:=(\mathcal{L}_2,\mathcal{L}_1)$ of $\mathcal{L}$. Let $\mathcal{M}_1\in\mathrm{Sub}_{\mathfrak{j}}$ and $\mathcal{M}_2\in\mathrm{Sub}_{\bar{\mathfrak{j}}}$. Then $\mathcal{M}_1\perp\mathcal{M}_2$ if and only if*

$$-\big[\mathfrak{a}_{\mathfrak{j}}(\mathcal{M}_1)x,y\big]=\big[x,\mathfrak{a}_{\bar{\mathfrak{j}}}(\mathcal{M}_2)y\big],\quad x\in P_{\mathfrak{j}}^1\mathcal{M}_1,\ y\in P_{\mathfrak{j}}^2(\mathcal{M}_2)\,.$$

Proof. We have

$$\mathcal{M}_1=\big\{x+\mathfrak{a}_{\mathfrak{j}}(\mathcal{M}_1)x:\,x\in P_{\mathfrak{j}}^1\mathcal{M}_1\big\},\ \mathcal{M}_2=\big\{x+\mathfrak{a}_{\bar{\mathfrak{j}}}(\mathcal{M}_2)x:\,x\in P_{\mathfrak{j}}^2\mathcal{M}_2\big\}\,.$$

Hence, $\mathcal{M}_1\perp\mathcal{M}_2$ if and only if

$$0=\big[x+\mathfrak{a}_{\mathfrak{j}}(\mathcal{M}_1)x,y+\mathfrak{a}_{\bar{\mathfrak{j}}}(\mathcal{M}_2)y\big],\quad x\in P_{\mathfrak{j}}^1\mathcal{M}_1,\ y\in P_{\mathfrak{j}}^2(\mathcal{M}_2)\,.$$

❑

1.4 Semidefinite subspaces

For certain orthogonal decompositions, in particular for such arising from fundamental decompositions, the sets of definite/semindefinite/neutral subspaces can be described with help of angular operators.

PRA27 **1.4.1 Proposition.** *Let $\mathfrak{j}=(\mathcal{L}_1,\mathcal{L}_2)$ be an orthogonal decomposition of the inner product space $\langle\mathcal{L},[.,.]\rangle$, and let $\mathcal{M}\in\mathrm{Sub}\,\mathcal{L}$.*

(i) Assume that $\mathcal{L}_2$ is negative semidefinite. Then

$$\mathcal{M}\in\mathrm{Sub}_{>0}\,\mathcal{L}\iff\\ \mathcal{M}\in\mathrm{Sub}_{\mathfrak{j}}\ \textit{and}\ -[\mathfrak{a}_{\mathfrak{j}}(\mathcal{M})x,\mathfrak{a}_{\mathfrak{j}}(\mathcal{M})x]<[x,x],\ x\in P_{\mathfrak{j}}^1\mathcal{M}\setminus\{0\}$$

(ii) Assume that $\mathcal{L}_2$ is negative definite. Then

$$\mathcal{M}\in\mathrm{Sub}_{\geq 0}\,\mathcal{L}\iff\\ \mathcal{M}\in\mathrm{Sub}_{\mathfrak{j}}\ \textit{and}\ -[\mathfrak{a}_{\mathfrak{j}}(\mathcal{M})x,\mathfrak{a}_{\mathfrak{j}}(\mathcal{M})x]\leq[x,x],x\in P_{\mathfrak{j}}^1\mathcal{M}$$

$$\mathcal{M}\in\mathrm{Sub}_{=0}\,\mathcal{L}\iff\\ \mathcal{M}\in\mathrm{Sub}_{\mathfrak{j}}\ \textit{and}\ -[\mathfrak{a}_{\mathfrak{j}}(\mathcal{M})x,\mathfrak{a}_{\mathfrak{j}}(\mathcal{M})x]=[x,x],x\in P_{\mathfrak{j}}^1\mathcal{M}$$

(iii) Assume that $\mathcal{L}_2$ is positive semidefinite. Then

$$\mathcal{M}\in\mathrm{Sub}_{<0}\,\mathcal{L}\iff\\ \mathcal{M}\in\mathrm{Sub}_{\mathfrak{j}}\ \textit{and}\ [\mathfrak{a}_{\mathfrak{j}}(\mathcal{M})x,\mathfrak{a}_{\mathfrak{j}}(\mathcal{M})x]<-[x,x],x\in P_{\mathfrak{j}}^1\mathcal{M}\setminus\{0\}$$

(*iv*) *Assume that* $\mathcal{L}_2$ *is positive definite. Then*

$$\mathcal{M} \in \mathrm{Sub}_{\geq 0}\,\mathcal{L} \iff \mathcal{M} \in \mathrm{Sub}_{\mathfrak{j}} \text{ and } [\mathfrak{a}_{\mathfrak{j}}(\mathcal{M})x, \mathfrak{a}_{\mathfrak{j}}(\mathcal{M})x] \leq -[x,x], x \in P_{\mathfrak{j}}^1\mathcal{M}$$

$$\mathcal{M} \in \mathrm{Sub}_{=0}\,\mathcal{L} \iff \mathcal{M} \in \mathrm{Sub}_{\mathfrak{j}} \text{ and } [\mathfrak{a}_{\mathfrak{j}}(\mathcal{M})x, \mathfrak{a}_{\mathfrak{j}}(\mathcal{M})x] = -[x,x], x \in P_{\mathfrak{j}}^1\mathcal{M}$$

Proof. We shall restrict explicit proof to the item (i), the other items are seen in the same way.

Assume first that $\mathcal{M}$ is positive definite. Since $\mathcal{L}_2$ is negative semidefinite, we have $\mathcal{M} \cap \mathcal{L}_2 = \{0\}$. Hence $\mathcal{M} = \{x + \mathfrak{a}_{\mathfrak{j}}(\mathcal{M})x : x \in P_{\mathfrak{j}}^1\mathcal{M}\}$. Let $x \in P_{\mathfrak{j}}^1\mathcal{M}$, $x \neq 0$, then

$$0 < [x + \mathfrak{a}_{\mathfrak{j}}(\mathcal{M})x, x + \mathfrak{a}_{\mathfrak{j}}(\mathcal{M})x] = [x,x] + [\mathfrak{a}_{\mathfrak{j}}(\mathcal{M})x, \mathfrak{a}_{\mathfrak{j}}(\mathcal{M})x]\,.$$

Conversely, if $\mathcal{M} \in \mathrm{Sub}_{\mathfrak{j}}$ satisfies the condition on the right side, we just read the above inequality backwards to obtain that $x + \mathfrak{a}_{\mathfrak{j}}(\mathcal{M})x$ is positive for each $x \in P_{\mathfrak{j}}^1\mathcal{M}$. ❑

This proposition has an immediate, but noteworthy, corollary.

COA28 **1.4.2 Corollary.** *Let* $\langle \mathcal{L}, [.,.]\rangle$ *be a nondegenerated inner product space, and let* $\mathfrak{J} = (\mathcal{L}_+, \mathcal{L}_-)$ *be a fundamental decomposition of* $\mathcal{L}$*, so that* $\mathfrak{J}$ *is also an orthogonal decomposition of* $\mathcal{L}$*. Moreover, let* $\mathcal{M} \in \mathrm{Sub}\,\mathcal{L}$*. Then* $\mathcal{M}$ *is positive semidefinite if and only if its angular operator* $\mathfrak{a}_{\mathfrak{J}}(\mathcal{M})$ *is well-defined and is a* $p_{\mathfrak{J}}|_{\mathcal{P}_{\mathfrak{J}}^+\mathcal{M}}$*-to-*$p_{\mathfrak{J}}|_{\mathcal{P}_{\mathfrak{J}}^-\mathcal{M}}$*–contraction.* ❑

Each of the sets $\mathrm{Sub}_{\mathsf{index}}$, where index is one of $\mathfrak{j}, >0, \geq 0, =0, \leq 0, <0$, is ordered by set-theoretic inclusion. Maximal elements will be of interest. First let us note the following consequence of Proposition 1.3.7.

COA29 **1.4.3 Corollary.** *Let* $\mathfrak{j} = (\mathcal{L}_1, \mathcal{L}_2)$ *be an orthogonal decomposition of* $\langle \mathcal{L}, [.,.]\rangle$*, and let* $\mathcal{M} \in \mathrm{Sub}_{\mathfrak{j}}$*. Then* $\mathcal{M}$ *is a maximal element of* $\mathrm{Sub}_{\mathfrak{j}}$ *if and only if* $P_{\mathfrak{j}}^1\mathcal{M} = \mathcal{L}_1$*.*

Proof. Since the map $\mathfrak{a}$ is an order-preserving bijection of $\mathrm{Sub}_{\mathfrak{j}}$ onto $\mathfrak{A}_{\mathfrak{j}}$, maximal elements of $\mathrm{Sub}_{\mathfrak{j}}$ correspond to maximal elements of $\mathfrak{A}_{\mathfrak{j}}$. However, clearly, an element $(D,K) \in \mathfrak{A}_{\mathfrak{j}}$ is maximal if and only if $D = \mathcal{L}_1$. ❑

COA30 **1.4.4 Corollary.** *For each element* $\mathcal{M} \in \mathrm{Sub}_{\mathfrak{j}}$*, there exists a maximal element* $\mathcal{M}'$ *of* $\mathrm{Sub}_{\mathfrak{j}}$ *with* $\mathcal{M} \subseteq \mathcal{M}'$*.*

Proof. Let $\mathcal{M} \in \mathrm{Sub}_{\mathfrak{j}}$ be given. Choose a projection P of $\mathcal{L}_1$ onto $P_{\mathfrak{j}}^1\mathcal{M}$, and define $K := \mathfrak{a}_{\mathfrak{j}}(\mathcal{M})P$. Then $(\mathcal{L}_1, K) \in \mathfrak{A}_{\mathfrak{j}}$, and clearly $(P_{\mathfrak{j}}^1\mathcal{M}, \mathfrak{a}_{\mathfrak{j}}(\mathcal{M})) \preceq (\mathcal{L}_1, K)$. Thus the subspace $\mathcal{M}' := \mathfrak{a}^{-1}(\mathcal{L}_1, K)$ is maximal in $\mathfrak{A}_{\mathfrak{j}}$ and contains $\mathcal{M}$. ❑

The obstacle when trying to construct maximal elements of, say, $\mathrm{Sub}_{\geq 0}\,\mathfrak{L}$ in the same way, is that we not only have to extend $\mathfrak{a}_{\mathfrak{j}}(\mathcal{M})$, but also to retain the condition $-[Kx, Kx] \leq [x,x]$. In general this is not possible, however, existence of maximal elements is ensured by Zorn's Lemma.

LEA31 **1.4.5 Lemma.** *Let $\mathcal{M} \in \mathrm{Sub}_{>0}\,\mathcal{L}$. Then there exists a maximal element $\mathcal{M}'$ of $\mathrm{Sub}_{>0}\,\mathcal{L}$ such that $\mathcal{M} \subseteq \mathcal{M}'$. The same assertion holds for $\mathrm{Sub}_{\textsf{index}}$, where* $\textsf{index}$ *is one of $\geq 0, = 0, \leq 0, < 0$.*

Proof. The union of an ascending chain of positive subspaces of $\mathcal{L}$ is again a positive subspace. Hence we may apply Zorn's Lemma. The assertion for the other classes of subspaces is seen in the same way. ❑

Of course, Zorn's Lemma would also apply to $\mathrm{Sub}_{\mathrm{j}}$, but the argument used above is more enlightening.

Since $\mathrm{Sub}_{>0}\,\mathcal{L} \subseteq \mathrm{Sub}_{\geq 0}\,\mathcal{L}$, an element of $\mathrm{Sub}_{>0}\,\mathcal{L}$ which is maximal in $\mathrm{Sub}_{\geq 0}\,\mathcal{L}$ is clearly also maximal in $\mathrm{Sub}_{>0}\,\mathcal{L}$. The converse, however, need not hold true. The same remark is valid for each pair of classes $\mathrm{Sub}_{=0}\,\mathcal{L} \subseteq \mathrm{Sub}_{\geq 0}\,\mathcal{L}$, $\mathrm{Sub}_{<0}\,\mathcal{L} \subseteq \mathrm{Sub}_{\leq 0}\,\mathcal{L}$, $\mathrm{Sub}_{=0}\,\mathcal{L} \subseteq \mathrm{Sub}_{\leq 0}\,\mathcal{L}$. In this context maximal neutral subspaces have an interesting property.

PRA32 **1.4.6 Proposition.** *Let $\mathcal{M}$ be a neutral subspace of $\mathcal{L}$. Then the following are equivalent:*

(i) *$\mathcal{M}$ is maximal in $\mathrm{Sub}_{=0}\,\mathcal{L}$.*

(ii) *$\mathcal{M}^\perp$ is semidefinite and $\mathcal{M}^{\perp\perp} = \mathcal{M}$.*

(iii) *$\mathcal{M}$ is maximal in $\mathrm{Sub}_{\geq 0}\,\mathcal{L}$ or maximal in $\mathrm{Sub}_{\leq 0}\,\mathcal{L}$.*

Proof.
Step 1, (i) $\Rightarrow$ (ii): Assume that $\mathcal{M}$ is maximal neutral. Since $\mathcal{M}$ is neutral, we have $\mathcal{M} \subseteq \mathcal{M}^\perp$, and hence $\mathcal{M}^{\perp\perp} \subseteq (\mathcal{M}^{\perp\perp})^\perp$. Thus $\mathcal{M}^{\perp\perp}$ is also neutral. However, $\mathcal{M}^{\perp\perp} \supseteq \mathcal{M}$, and we conclude from maximality of $\mathcal{M}$ that $\mathcal{M}^{\perp\perp} = \mathcal{M}$.

Next assume that $\mathcal{M}^\perp$ is indefinite, and choose $x_+, x_- \in \mathcal{M}^\perp$ with $[x_+, x_+] = 1$ and $[x_-, x_-] = -1$. Let $\phi \in \mathbb{R}$ be such that $e^{i\phi}[x_-, x_+] \in i\mathbb{R}$, and put $x_0 := x_+ + e^{i\phi}x_-$. Then $x_0 \perp \mathcal{M}$ and

$$[x_0, x_0] = [x_+ + e^{i\phi}x_-, x_+ + e^{i\phi}x_-] =$$
$$= [x_+, x_+] + e^{i\phi}[x_-, x_+] + e^{-i\phi}[x_+, x_-] + [x_-, x_-] = 0\,.$$

Thus $\mathcal{M}_0 := \mathrm{span}(\mathcal{M} \cup \{x_0\})$ is neutral. By maximality of $\mathcal{M}$, it follows that $x_0 \in \mathcal{M}$. Thus $x_- \in \mathrm{span}(\mathcal{M} \cup \{x_+\})$. The subspace $\mathrm{span}(\mathcal{M} \cup \{x_+\})$, however, is positive semidefinite. We have reached a contradiction.

Step 2, (ii) $\Rightarrow$ (iii): Let us consider the case that $\mathcal{M}^\perp$ is positive semidefinite. Let $\mathcal{M}_-$ be a negative semidefinite subspace which contains $\mathcal{M}$. By the Schwartz inequality, we have $\mathcal{M} \perp \mathcal{M}_-$, i.e. $\mathcal{M}_- \subseteq \mathcal{M}^\perp$. Since $\mathcal{M}^\perp$ is positive semidefinite, this implies that $\mathcal{M}_-$ is neutral. Hence, again by the Schwartz inequality, $\mathcal{M}_- \perp \mathcal{M}^\perp$. It follows that $\mathcal{M}_- \subseteq \mathcal{M}^{\perp\perp} = \mathcal{M}$. We have shown that $\mathcal{M}$ is maximal in $\mathrm{Sub}_{\leq 0}\,\mathcal{L}$.

The case that $\mathcal{M}^\perp$ is negative semidefinite is treated in the same way.

Step 3, (iii) $\Rightarrow$ (i): This is clear, as we have already noted before the present proposition. ❑

REA33 *1.4.7 Remark.* In the proof of *(ii)* $\Rightarrow$ *(iii)* above we have shown that, for a maximal neutral subspace $\mathcal{M}$, $\mathcal{M}^\perp$ being positive semidefinite implies that $\mathcal{M}$ is maximal nonpositive. Analogously, $\mathcal{M}^\perp$ being negative semidefinite implies that $\mathcal{M}$ is maximal nonnegative. //

The above result motivates the following definition.

DEA34 **1.4.8 Definition.** Let $\mathcal{M}$ be a subspace of $\langle\mathcal{L},[.,.]\rangle$. Then $\mathcal{M}$ is called *hypermaximal neutral*, if $\mathcal{M}$ is neutral and maximal in both, $\operatorname{Sub}_{\leq 0}\mathcal{L}$ and $\operatorname{Sub}_{\geq 0}\mathcal{L}$. //

COA35 **1.4.9 Corollary.** *Let $\mathcal{M}$ be a subspace of $\langle\mathcal{L},[.,.]\rangle$. Then $\mathcal{M}$ is hypermaximal neutral if and only if $\mathcal{M}^\perp=\mathcal{M}$.*

Proof. Assume that $\mathcal{M}^\perp=\mathcal{M}$. First of all this implies that $\mathcal{M}$ is neutral. Also it follows that $\mathcal{M}^\perp=\mathcal{M}^{\perp\perp}$, and hence $\mathcal{M}=\mathcal{M}^{\perp\perp}$. Moreover, $\mathcal{M}^\perp$ is neutral, and hence by Remark 1.4.7 $\mathcal{M}$ is maximal nonnegative and maximal nonpositive.

Conversely, assume that $\mathcal{M}$ is maximal nonnegative and maximal nonpositive. Then $\mathcal{M}$ is maximal neutral, and hence $\mathcal{M}^\perp$ is semidefinite. Since $\mathcal{M}\subseteq\mathcal{M}^\perp$, maximality of $\mathcal{M}$ implies that $\mathcal{M}=\mathcal{M}^\perp$. ❑

EXA36 *1.4.10 Example.* Let $\langle\mathcal{H},[.,.]\rangle$ be a Hilbert space, and consider the linear space $\mathcal{H}\times\mathcal{H}$ endowed with the inner product

$$[\![(x,y),(a,b)]\!]:=i\big([x,b]-[y,a]\big),\ (x,y),(a,b)\in\mathcal{H}\times\mathcal{H}\,.$$

Let T be a densely defined closed operator in $\mathcal{H}$. Then the Hilbert space adjoint T^* of T is defined as the operator with domain

$$\operatorname{dom}T^*:=\Big\{x\in\mathcal{H}:\ \exists y\in\mathcal{H}\text{ s.t. }[Ta,x]=[a,y],a\in\operatorname{dom}T\Big\}$$

which assigns to an element $x\in\operatorname{dom}T^*$ the element $T^*x:=y$ which exists uniquely by the definition of $\operatorname{dom}T^*$. Equivalently, we could define T^* via its graph $\operatorname{graph}T^*$ as

$$\operatorname{graph}T^*=\big\{(x,y):\ [b,x]=[a,y]\,,(a,b)\in\operatorname{graph}T\big\}$$

Now observe that this just says that

$$\operatorname{graph}T^*=(\operatorname{graph}T)^{[\![\perp]\!]}\,.$$

Hence, an operator T is symmetric if and only if it is neutral, and it is selfadjoint if and only if it is hypermaximal neutral. //

Finally, let us mention a result which relates maximal semidefiniteness of $\mathcal{M}$ and $\mathcal{M}^\perp$.

PRA37 **1.4.11 Proposition.** *Let $\langle\mathcal{L},[.,.]\rangle$ be an inner product space, and let $\mathcal{M}\in\operatorname{Sub}\mathcal{L}$. Then the following hold:*

(*i*) *If $\mathcal{M}$ is maximal in $\operatorname{Sub}_{\geq 0}\mathcal{L}$ or maximal in $\operatorname{Sub}_{>0}\mathcal{L}$, then $\mathcal{M}^\perp\in\operatorname{Sub}_{\leq 0}\mathcal{L}$.*

(*ii*) *If $\mathcal{M}\in\operatorname{Sub}_{>0}\mathcal{L}$ and $\mathcal{M}$ is maximal in $\operatorname{Sub}_{\geq 0}\mathcal{L}$, then $\mathcal{M}^\perp\in\operatorname{Sub}_{<0}\mathcal{L}$.*

(*iii*) *If $\mathcal{M}$ is maximal in $\operatorname{Sub}_{>0}\mathcal{L}$ and orthocomplemented, then $\mathcal{M}^\perp$ is maximal in $\operatorname{Sub}_{\leq 0}\mathcal{L}$.*

Proof. To see (i), assume that $x \in \mathcal{M}^\perp$ and $[x,x] > 0$. Then the linear space $\mathcal{M}_1 := \operatorname{span}(\mathcal{M} \cup \{x\})$ belongs to $\operatorname{Sub}_{\geq 0}\mathcal{L}$ or $\operatorname{Sub}_{>0}\mathcal{L}$, respectively. By maximality of $\mathcal{M}$, it follows that $\mathcal{M}_1 = \mathcal{M}$, i.e. $x \in \mathcal{M}$. Thus $x \in \mathcal{M} \cap \mathcal{M}^\perp$, and hence $[x,x] = 0$, which contradicts our choice of x.

For (ii) we argue in the same way. Assume that $x \in \mathcal{M}^\perp \setminus \{0\}$ and $[x,x] \geq 0$. Then $\mathcal{M}_1 := \operatorname{span}(\mathcal{M} \cup \{x\}) \in \operatorname{Sub}_{\geq 0}\mathcal{L}$ and, by maximality, $\mathcal{M}_1 = \mathcal{M}$. Again it follows that $[x,x] = 0$ which now contradicts the fact that $\mathcal{M}$ is positive.

It remains to show (iii). Under the hypothesis of (iii), by the already proved item (i), the subspace $\mathcal{M}^\perp$ is nonpositive. Let $\mathcal{M}_1 \in \operatorname{Sub}_{\leq 0}\mathcal{L}$ be such that $\mathcal{M}^\perp \subseteq \mathcal{M}_1$, and let $x \in \mathcal{M}_1$. Since $\mathcal{M}$ is orthocomplemented, we can write $x = y + z$ with $y \in \mathcal{M}$ and $z \in \mathcal{M}^\perp$. It follows that $y = x - z \in \mathcal{M} \cap \mathcal{M}_1$. However, since $\mathcal{M}$ is positive and $\mathcal{M}_1$ is nonpositive, we have $\mathcal{M} \cap \mathcal{M}_1 = \{0\}$. Thus $x = z \in \mathcal{M}^\perp$. We have shown that $\mathcal{M}_1 = \mathcal{M}^\perp$. ❑

1.5 Inner product spaces with finite negative index

DEA38 **1.5.1 Definition.** For an inner product space $\langle \mathcal{L}, [.,.] \rangle$, we define its *negative index* as the cardinal number

$$\operatorname{ind}_-\langle \mathcal{L}, [.,.] \rangle := \sup \big\{ \dim \mathcal{M} : \mathcal{M} \in \operatorname{Sub}_{<0}\mathcal{L} \big\} .$$

Completely parallel, we define its *positive index* as

$$\operatorname{ind}_+\langle \mathcal{L}, [.,.] \rangle := \sup \big\{ \dim \mathcal{M} : \mathcal{M} \in \operatorname{Sub}_{>0}\mathcal{L} \big\} .$$

Moreover, we will use the notation

$$\operatorname{ind}_0\langle \mathcal{L}, [.,.] \rangle := \dim \mathcal{L}^\circ ,$$

and speak of the *degree of degeneracy* of $\mathcal{L}$. //

PRA39 **1.5.2 Proposition.** *Let $\langle \mathcal{L}, [.,.] \rangle$ be an inner product space, and assume that there exists a subspace $\mathcal{M}_0$ which is finite-dimensional and maximal in $\operatorname{Sub}_{<0}\mathcal{L}$. Set $\kappa := \dim \mathcal{M}_0$. Then the following hold:*

(i) *We have*

$$\dim \mathcal{M} = \begin{cases} \kappa & , \quad \mathcal{M} \text{ maximal in } \operatorname{Sub}_{<0}\mathcal{L} \\ \kappa + \operatorname{ind}_0 \mathcal{L}, & \quad \mathcal{M} \text{ maximal in } \operatorname{Sub}_{\leq 0}\mathcal{L} \end{cases}$$

In particular, $\operatorname{ind}_- \mathcal{L} = \kappa$.

(ii) *A negative definite subspace $\mathcal{M}$ of $\mathcal{L}$ is maximal in $\operatorname{Sub}_{<0}\mathcal{L}$, if and only if $\dim \mathcal{M} = \kappa$. In case that $\operatorname{ind}_0 \mathcal{L} < \infty$, a negative semidefinite subspace $\mathcal{M}$ of $\mathcal{L}$ is maximal in $\operatorname{Sub}_{\leq 0}\mathcal{L}$, if and only if $\dim \mathcal{M} = \kappa + \operatorname{ind}_0 \mathcal{L}$.*

(iii) *The space $\mathcal{L}$ is decomposable. More precisely, for each maximal element $\mathcal{M}$ of $\operatorname{Sub}_{<0}\mathcal{L}$, the subspace $\mathcal{M}^\perp$ is maximal in $\operatorname{Sub}_{\geq 0}\mathcal{L}$ and $(\mathcal{M}, \mathcal{M}^\perp)$ is an orthogonal decomposition of $\mathcal{L}$.*

Proof. Since the space $\mathcal{M}_0$ is finite-dimensional and nondegenerated, it is orthocomplemented. Since $\mathcal{M}_0$ is maximal in $\operatorname{Sub}_{<0}\mathcal{L}$, its orthogonal companion is nonnegative. Moreover, we have $(\mathcal{M}_0^{\perp})^{\circ} = \mathcal{L}^{\circ}$. In the following let $\mathfrak{j}$ denote the orthogonal decomposition $\mathfrak{j} := (\mathcal{M}_0, \mathcal{M}_0^{\perp})$ of $\mathcal{L}$, and let $\mathcal{M} \in \operatorname{Sub}_{<0}\mathcal{L}$ be given.

We have $\operatorname{Sub}_{<0}\mathcal{L} \subseteq \operatorname{Sub}_{\mathfrak{j}}$, hence the projection $P_{\mathfrak{j}}^1$ maps $\mathcal{M}$ injectively into $\mathcal{M}_0$. in particular, $\dim\mathcal{M} \leq \kappa$. We already see that $\operatorname{ind}_-\mathcal{L} = \kappa$, and hence that each negative subspace with dimension equal to κ must be maximal negative.

Since $\mathcal{M}_0$ is finite-dimensional and definite, there exists an $[.,.]$-orthogonal projection P of $\mathcal{M}_0$ onto $P_{\mathfrak{j}}^1\mathcal{M}$. Let $\tilde{\mathcal{M}} \in \operatorname{Sub}_{\mathfrak{j}}$ be the unique subspace with

$$\mathfrak{a}_{\mathfrak{j}}(\tilde{\mathcal{M}}) = \mathfrak{a}_{\mathfrak{j}}\mathcal{M}P\,.$$

Then $\tilde{\mathcal{M}}$ is maximal in $\operatorname{Sub}_{\mathfrak{j}}$ and extends $\mathcal{M}$. Moreover, we have

$$\begin{aligned}\big[\mathfrak{a}_{\mathfrak{j}}(\tilde{\mathcal{M}})x, \mathfrak{a}_{\mathfrak{j}}(\tilde{\mathcal{M}})x\big] = \big[\mathfrak{a}_{\mathfrak{j}}\mathcal{M}Px, \mathfrak{a}_{\mathfrak{j}}\mathcal{M}Px\big] \leq \\ \leq -[Px,Px] \leq -[x,x], \quad x \in \mathcal{M}_0\,.\end{aligned} \tag{1.5.1}$$ A40

However, if $x \notin \ker P$ then the first inequality is strict, and if $x \in \ker P \setminus \{0\}$ the second one is strict. Hence, $\tilde{\mathcal{M}} \in \operatorname{Sub}_{<0}\mathcal{L}$. We conclude that each maximal element $\mathcal{M}$ of $\operatorname{Sub}_{<0}\mathcal{L}$ is mapped by bijectively onto $\mathcal{M}_0$ by $P_{\mathfrak{j}}^1$. Thus $\dim\mathcal{M} = \dim\mathcal{M}_0 = \kappa$. This finishes the proof of the first formula in (i) and of the first half of (ii).

For the proof of the second formula in (i), we first reduce to the case that $\mathcal{L}$ is nondegenerated. Let $\mathcal{M}$ be a maximal element of $\operatorname{Sub}_{\leq 0}\mathcal{L}$. Since with $\mathcal{M}$ also $\mathcal{M}+\mathcal{L}^{\circ}$ is nonpositive, we conclude that $\mathcal{L}^{\circ} \subseteq \mathcal{M}$. Denote by $\pi : \mathcal{L} \to \mathcal{L}/\mathcal{L}^{\circ}$ the canonical projection, then $\pi(\mathcal{M}) \in \operatorname{Sub}_{\leq 0}(\mathcal{L}/\mathcal{L}^{\circ})$. If $\mathcal{N} \in \operatorname{Sub}_{\leq 0}(\mathcal{L}/\mathcal{L}^{\circ})$ and $\mathcal{N} \supseteq \pi(\mathcal{M})$, then $\pi^{-1}(\mathcal{N}) \in \operatorname{Sub}_{\leq 0}\mathcal{L}$ and $\pi^{-1}(\mathcal{N}) \supseteq \pi^{-1}(\pi(\mathcal{M})) = \mathcal{M}$. By maximality, $\pi^{-1}(\mathcal{N}) = \mathcal{M}$, and hence $\mathcal{N} = \pi(\mathcal{M})$. Thus $\pi(\mathcal{M})$ is maximal in $\operatorname{Sub}_{\leq 0}(\mathcal{L}/\mathcal{L}^{\circ})$. If the desired assertion had already been proved for $\mathcal{L}/\mathcal{L}^{\circ}$, we could conclude that

$$\dim\mathcal{M} = \dim\pi(\mathcal{M}) + \dim\mathcal{L}^{\circ} = \kappa + \operatorname{ind}_0\mathcal{L}\,.$$

Assume that $\mathcal{L}$ is nondegenerated. Then $\mathcal{M}_0^{\perp}$ is positive definite, and hence $\operatorname{Sub}_{\leq 0}\mathcal{L} \subseteq \operatorname{Sub}_{\mathfrak{j}}$. If $\mathcal{M} \in \operatorname{Sub}_{\leq 0}\mathcal{L}$, thus $\dim\mathcal{M} \leq \kappa$. Moreover, we can in the same way as above extend $\mathcal{M}$ to the maximal element $\tilde{\mathcal{M}}$ of $\operatorname{Sub}_{\mathfrak{j}}$. Due to (1.5.1), we have $\tilde{\mathcal{M}} \in \operatorname{Sub}_{\leq 0}\mathcal{L}$. Again it follows that $P_{\mathfrak{j}}^1$ maps maximal elements of$\operatorname{Sub}_{\leq 0}\mathcal{L}$ bijectively onto $\mathcal{M}_0$. This finishes the proof of the second formula in (i).

For the proof of the remaining implication in (ii) assume that $\operatorname{ind}_0\mathcal{L} < \infty$, and let $\mathcal{M} \in \operatorname{Sub}_{\leq 0}\mathcal{L}$ with $\dim\mathcal{M} = \kappa + \operatorname{ind}_0\mathcal{L}$ be given. If $\mathcal{M}_1 \in \operatorname{Sub}_{\leq 0}\mathcal{L}$ and $\mathcal{M} \subseteq \mathcal{M}_1$, then

$$\kappa + \operatorname{ind}_0\mathcal{L} = \dim\mathcal{M} \leq \dim\mathcal{M}_1 \leq \kappa + \operatorname{ind}_0\mathcal{L}\,.$$

Thus $\mathcal{M} = \mathcal{M}_1$, and we conclude that $\mathcal{M}$ is maximal in $\operatorname{Sub}_{\leq 0}\mathcal{L}$.

We have by now shown that each maximal negative subspace has dimension κ, in particular, it is finite-dimensional. Hence, it is orthocomplemented and $\mathcal{M}^{\perp} \in \operatorname{Sub}_{\geq 0}\mathcal{L}$. Thus $(\mathcal{M}^{\perp}, \mathcal{M})$ is an orthogonal decomposition of $\mathcal{L}$ with semidefinite components, and thus $\mathcal{L}$ is decomposable, cf. Lemma 1.3.5. This is (iii). ❑

Of course, with the obvious modifications, the analogous statements hold for an inner product space which contains a finite-dimensional maximal positive subspace.

1.6 Dual pairs

DEA41 **1.6.1 Definition.** Let $\langle\mathcal{L},[.,.]\rangle$ be an inner product space, and let $\mathcal{M},\mathcal{N} \in \operatorname{Sub}\mathcal{L}$. Then $\mathcal{M},\mathcal{N}$ is called a *dual pair*, if

$$\mathcal{M}\cap\mathcal{N}^\perp = \mathcal{M}^\perp\cap\mathcal{N} = \{0\}\,.$$

We will use the notation $\mathcal{M}\#\mathcal{N}$ to indicate that $\mathcal{M},\mathcal{N}$ form a dual pair.

If $\mathcal{M}$ and $\mathcal{N}$ are neutral, one also says that $\mathcal{M}$ and $\mathcal{N}$ are *skewly linked* in order to express that $\mathcal{M}\#\mathcal{N}$. //

Existence of dual pairs follows with help of a variant of the Gram-Schmidt orthogonalization proceedure.

LEA42 **1.6.2 Lemma.** *Let $\langle\mathcal{L},[.,.]\rangle$ be an inner product space, let $\mathcal{M} \in \operatorname{Sub}\mathcal{L}$, $\dim\mathcal{M}<\infty$, and let $\mathcal{L}_0\in\operatorname{Sub}\mathcal{L}$ be such that $\mathcal{M}\cap\mathcal{L}_0^\perp=\{0\}$.*

(i) There exists a subspace $\mathcal{N}\subseteq\mathcal{L}_0$, such that $\mathcal{M}\#\mathcal{N}$.

(ii) If $\mathcal{M}$ is neutral and $\mathcal{M}\subseteq\mathcal{L}_0$, then the space $\mathcal{N}$ in (i) can be chosen to be neutral.

Proof. We will use induction on $n:=\dim\mathcal{M}$ to construct elements $e_1,\dots,e_n$, $f_1,\dots,f_n$, such that

$$\begin{gathered}\{e_1,\dots,e_n\}\text{ is basis of }\mathcal{M},\quad f_j\in\mathcal{L}_0,\ j=1,\dots,n\\ [e_i,f_j]=\delta_{ij},\ i,j=1,\dots,n\end{gathered} \tag{1.6.1}$$ A43

Once this has been done, put $\mathcal{N}:=\operatorname{span}\{f_1,\dots,f_n\}$. Then, clearly, $\mathcal{N}\subseteq\mathcal{L}_0$ and $\mathcal{M}\#\mathcal{N}$.

Consider the case $n=1$. Choose $e_1\in\mathcal{M}\setminus\{0\}$, and let $f_1\in\mathcal{L}_0$ be such that $[e_1,f_1]=1$. This choice is possible, since $e_1\notin\mathcal{L}_0^\perp$. Obviously, the elements e_1,f_1 satisfy (1.6.1).

Let a subspace $\mathcal{M}$ with $\dim\mathcal{M}=n+1$ be given. Choose $\mathcal{M}'\subseteq\mathcal{M}$ with $\dim\mathcal{M}'=n$, and let e_j,f_j, $j=1,\dots,n$, be elements satisfying (1.6.1) for $\mathcal{M}'$. Choose $g\in\mathcal{M}\setminus\mathcal{M}'$, and put

$$e_{n+1}:=g-\sum_{j=1}^n[g,f_j]e_j\,.$$

Then $\{e_1,\dots,e_{n+1}\}$ is a basis for $\mathcal{M}$, and

$$[e_{n+1},f_k]=0,\ k=1,\dots,n\,.$$

Since $e_{n+1}\notin\mathcal{L}_0^\perp$, we can choose $h\in\mathcal{L}_0$ with $[e_{n+1},h]=1$. Put

$$f_{n+1}:=h-\sum_{j=1}^n[h,e_j]f_j\,,$$

then $f_{n+1} \in \mathcal{L}_0$ and
$$[e_k, f_{n+1}] = 0,\ k = 1, \ldots, n\,.$$
It remains to compute
$$[e_{n+1}, f_{n+1}] = [e_{n+1}, h] - \sum_{j=1}^{n} [h, e_j]\,\underbrace{[f_j, e_{n+1}]}_{=0} = 1\,.$$

For the proof of (ii), assume that $\mathcal{M}$ is neutral. Let elements e_j, f_j, $j = 1, \ldots, n$, satisfy (1.6.1). Choose a matrix $A = (a_{ij})_{i,j=1}^{n} \in \mathbb{C}^{n\times n}$, such that
$$A + A^* = -\big([f_i, f_j]\big)_{i,j=1}^{n}\,.$$
For example, we may put
$$a_{ij} := \begin{cases} [f_i, f_j]\quad, & i < j \\ \frac{1}{2}[f_i, f_j]\,, & i = j \\ 0 \qquad\quad\ , & i > j \end{cases}$$
Define elements
$$f_k' := f_k + \sum_{j=1}^{n} a_{kj} e_j,\ k = 1, \ldots, n\,.$$
Then, since $\mathcal{M}$ is neutral,
$$[e_i, f_j'] = [e_i, f_j] = \delta_{ij},\ i, j = 1, \ldots, n\,,$$
and hence $\mathcal{N}' := \operatorname{span}\{f_1', \ldots, f_n'\}$ satisfies $\mathcal{M}\#\mathcal{N}'$. Moreover,
$$[f_k', f_l'] = [f_k, f_l] + \sum_{j=1}^{n} a_{kj}[e_j, f_l'] + \sum_{j=1}^{n} \overline{a_{lj}}[f_k', e_j] =$$
$$= [f_k, f_l] + a_{kl} + \overline{a_{lk}} = 0,\ k, l = 1, \ldots, n\,,$$
and hence $\mathcal{N}'$ is neutral. ❑

Let us collect some simple properties of dual pairs.
$\boxed{\downarrow\downarrow\downarrow \text{ ordentlich !}}$

LEA44 **1.6.3 Lemma.** *Let $\langle \mathcal{L}, [.,.]\rangle$ be an inner product space, let $\mathcal{M}, \mathcal{N} \in \operatorname{Sub}\mathcal{L}$, $\mathcal{M}, \mathcal{N} \neq \{0\}$, and assume that $\mathcal{M}\#\mathcal{N}$. Then*

(i) $\mathcal{M} \cap \mathcal{L}^\circ = \mathcal{N} \cap \mathcal{L}^\circ = \{0\}$.

(ii) If $\mathcal{M}$ is neutral, then $\mathcal{M} \cap \mathcal{N} = \{0\}$ and $\mathcal{M} \dot{+} \mathcal{N}$ is nondegenerated.

Assume additionally that $\dim \mathcal{M} < \infty$. *Then*

(iii) We have $\dim \mathcal{N} = \dim \mathcal{M}$.

(iv) If $\mathcal{M}$ is neutral, there exists a neutral subspace $\mathcal{N}'$ with
$$\mathcal{M}\#\mathcal{N}', \quad \mathcal{M} + \mathcal{N}' = \mathcal{M} + \mathcal{N}\,.$$

(*v*) *If* $\{e_1,\dots,e_n\}$ *is a basis of* $\mathcal{M}$, *then there exists a basis* $\{f_1,\dots,f_n\}$ *of* $\mathcal{N}$, *such that*
$$[e_i,f_j]=\delta_{ij},\ i,j=1,\dots,n\,. \tag{1.6.2}$$ A45

(*vi*) *We have* $\mathcal{M}\dot{+}\mathcal{N}^\perp=\mathcal{M}^\perp\dot{+}\mathcal{N}=\mathcal{L}$.

Proof. The first item follows since

$$\mathcal{M}\cap\mathcal{L}^\circ\subseteq\mathcal{M}\cap\mathcal{N}^\perp=\{0\},\ \mathcal{N}\cap\mathcal{L}^\circ\subseteq\mathcal{N}\cap\mathcal{M}^\perp=\{0\}\,.$$

Assume that $\mathcal{M}$ is neutral, i.e. $\mathcal{M}\subseteq\mathcal{M}^\perp$. Then $\mathcal{M}\cap\mathcal{N}\subseteq\mathcal{M}^\perp\cap\mathcal{N}=\{0\}$. Let $z\in(\mathcal{M}+\mathcal{N})^\perp$, and write $z=x+y$ with $x\in\mathcal{M}$, $y\in\mathcal{N}$. Then

$$y=z-x\in(\mathcal{M}+\mathcal{N})^\circ+\mathcal{M}\subseteq\mathcal{M}^\perp.$$

However, $y\in\mathcal{N}$, and it follows that $y=0$. Thus $x=z\in(\mathcal{M}+\mathcal{N})^\circ\subseteq\mathcal{N}^\perp$. Since $x\in\mathcal{M}$, we obtain that also $x=0$. Thus $z=0$, and we conclude that $(\mathcal{M}+\mathcal{N})^\circ=\{0\}$.

From now on assume that $\mathcal{M}$ is finite-dimensional. For the proof of (*iii*) choose a basis $\{e_1,\dots,e_n\}$ of $\mathcal{M}$. Then $\mathcal{M}^\perp=\bigcap_{j=1}^n\ker[.,e_j]$, and hence $\operatorname{codim}\mathcal{M}^\perp\leq n$. However, since $\mathcal{N}\cap\mathcal{M}^\perp=\{0\}$, we have $\dim\mathcal{N}\leq\operatorname{codim}\mathcal{M}^\perp$. Thus $\dim\mathcal{N}\leq\dim\mathcal{M}$ and in particular is finite. Exchanging the roles of $\mathcal{M}$ and $\mathcal{N}$ yields $\dim\mathcal{N}=\dim\mathcal{M}$.

The assertion in (*iv*) follows from applying Lemma 1.6.2, (*ii*), with $\mathcal{L}_0:=\mathcal{M}+\mathcal{N}$. For the proof of (*v*), choose a basis $\{e_1,\dots,e_n\}$ of $\mathcal{M}$, and put

$$\mathcal{M}_k:=\operatorname{span}\big(\{e_1,\dots,e_n\}\setminus\{e_k\}\big),\ k=1,\dots,n\,.$$

Then $\operatorname{codim}\mathcal{M}_k^\perp\leq\dim\mathcal{M}_k=n-1$. Thus $\mathcal{N}\cap\mathcal{M}_k^\perp\neq\{0\}$. If $x\in\mathcal{N}\cap\mathcal{M}_k^\perp$, $x\neq0$, then $[x,e_k]\neq0$ since x cannot be orthogonal to all of $\mathcal{M}$. Thus we may choose

$$f_k\in\mathcal{N}\cap\mathcal{M}_k^\perp,\ [e_k,f_k]=1,\quad k=1,\dots,n\,,$$

i.e. $f_k\in\mathcal{N}$ and (1.6.2) holds. These relations imply that $\{f_1,\dots,f_n\}$ are linearly independent, and hence is a basis for $\mathcal{N}$.

Finally, we come to the proof of (*vi*). According to the already proved item (*v*), we may choose bases $\{e_1,\dots,e_n\}$ and $\{f_1,\dots,f_n\}$ of $\mathcal{M}$ and $\mathcal{N}$, respectively, which are connected by (1.6.2). For $x\in\mathcal{L}$, put

$$x_{\mathcal{M}}:=\sum_{j=1}^n[x,f_j]e_j\,.$$

$\boxed{=\text{Projektion}}$

Then $x_{\mathcal{M}}\in\mathcal{M}$, and $[x_{\mathcal{M}},f_k]=[x,f_k]$, $k=1,\dots,n$. This shows that $x-x_{\mathcal{M}}\perp\mathcal{N}$, and it follows that $\mathcal{M}+\mathcal{N}^\perp=\mathcal{L}$. The relation $\mathcal{M}^\perp+\mathcal{N}=\mathcal{L}$ is seen in the same way. ❑

1.7 Orthogonal coupling

Let us explicitly state the following simple geometric facts.

REA46 *1.7.1 Remark.* Let $\langle\mathcal{L},[.,.]\rangle$ be an inner product space, and let $\mathcal{M}$ be a linear subspace of $\mathcal{L}$. Then $\langle\mathcal{M},[.,.]|_{\mathcal{M}\times\mathcal{M}}$ is an inner product space. We have

$$\operatorname{ind}_-\mathcal{M}\leq\operatorname{ind}_-\mathcal{L},\quad \operatorname{ind}_0\mathcal{M}\geq\dim(\mathcal{M}\cap\mathcal{L}^\circ)\,.$$

The inclusion map $\iota:\mathcal{M}\to\mathcal{L}$ is isometric. Let $\langle\mathcal{N},[.,.]_\mathcal{N}\rangle$ be an inner product space, and let $\phi:\mathcal{N}\to\mathcal{M}$. Then ϕ is isometric if and only if $\iota\circ\phi$ is. //

REA47 *1.7.2 Remark.* Let $\langle\mathcal{L}_i,[.,.]_i\rangle$, $i=1,\dots,n$, be inner product spaces, and define

$$\mathcal{L}:=\prod_{i=1}^n\mathcal{L}_i,\ [x,y]:=\sum_{i=1}^n[\pi_i x,\pi_i y]_i\,,$$

where π_i denotes the canonical projection of $\mathcal{L}$ onto $\mathcal{L}_i$. Then $\langle\mathcal{L},[.,.]\rangle$ is an inner product space. We have $\operatorname{ind}_0\mathcal{L}=\sum_{i=1}^n\operatorname{ind}_0\mathcal{L}_i$, in fact

$$\mathcal{L}^\circ=\prod_{i=1}^n\mathcal{L}_I^\circ\,.$$

Moreover,

$$\operatorname{ind}_-\mathcal{L}=\sum_{i=1}^n\operatorname{ind}_-\mathcal{L}_i\,.$$

Let $\iota_i:\mathcal{L}_i\to\mathcal{L}$, $i=1,\dots,n$, be the canonical embedding

$$\iota_i(x):=(0,\dots,\underset{\substack{\uparrow\\ i\text{-th place}}}{x},\dots,0)\,.$$

Then ι_i is isometric. //

REA48 *1.7.3 Remark.* Let $\langle\mathcal{L},[.,.]\rangle$ be an inner product space, and let $\mathcal{M}$ be a linear subspace of $\mathcal{L}$ with $\mathcal{M}\subseteq\mathcal{L}^\circ$. Moreover, denote by $\pi:\mathcal{L}\to\mathcal{L}/\mathcal{M}$ the canonical projection. Then an inner product $[.,.]_\sim$ on $\mathcal{L}/\mathcal{M}$ is well-defined by

$$[\pi x,\pi y]_\sim:=[x,y],\ x,y\in\mathcal{L}\,.$$

The canonical projection π is isometric. We have

$$\operatorname{ind}_-\mathcal{L}/\mathcal{M}=\operatorname{ind}_-\mathcal{L},\quad \operatorname{ind}_0\mathcal{L}/\mathcal{M}=\operatorname{ind}_0\mathcal{L}-\dim\mathcal{M}\,.$$

Let $\langle\mathcal{N},[.,.]_\mathcal{N}\rangle$ be an inner product space, and let $\phi:\mathcal{L}/\mathcal{M}\to\mathcal{N}$. Then ϕ is isometric if and only if $\phi\circ\pi$ is. //

We will in this section study a geometric construction which is a combination of product and factorization. Our starting point is the following observation.

REA49 *1.7.4 Remark.* If $\langle\mathcal{L}_1,[.,.]_1\rangle$ and $\langle\mathcal{L}_2,[.,.]_2\rangle$ are nondegenerated inner product spaces, then the direct and orthogonal sum $\mathcal{L}_1[\dot{+}]\mathcal{L}_2$ is (up to isomorphisms) the unique inner product space containing $\mathcal{L}_1$ and $\mathcal{L}_2$ isometrically as orthogonal subspaces which together span the whole space. //

If we move from the nondegenerated to the degenerated situation, then a space with this property will not be unique anymore.

DEA50 **1.7.5 Definition.** Let $\langle \mathcal{L}_1, [.,.]_1\rangle$ and $\langle \mathcal{L}_2, [.,.]_2\rangle$ be inner product spaces, and let α be a linear subspace of $\mathcal{L}_1^\circ \times \mathcal{L}_2^\circ$. Then the *orthogonal coupling* $\mathcal{L}_1 \boxplus_\alpha \mathcal{L}_2$ of $\mathcal{L}_1$ and $\mathcal{L}_2$ along α is defined as

$$\mathcal{L}_1 \boxplus_\alpha \mathcal{L}_2 := \big(\mathcal{L}_1[\dotplus]\mathcal{L}_2\big)/_\alpha .$$

Moreover, let ι_j be the canonical embedding of $\mathcal{L}_j$ into $\mathcal{L}_1[\dotplus]\mathcal{L}_2$, π_α the canonical projection of $\mathcal{L}_1[\dotplus]\mathcal{L}_2$ onto $(\mathcal{L}_1[\dotplus]\mathcal{L}_2)/_\alpha$, and define $\iota_1^\alpha := \pi_\alpha \circ \iota_1$, $\iota_2^\alpha := \pi_\alpha \circ \iota_2$, that is

$$\begin{tikzcd} & \mathcal{L}_1[\dotplus]\mathcal{L}_2 \arrow[d, "\pi_\alpha"] \\ \mathcal{L}_j \arrow[ur, "\iota_j"] \arrow[r, "\iota_j^\alpha"'] & \mathcal{L}_1 \boxplus_\alpha \mathcal{L}_2 \end{tikzcd}$$

//

Let us note the following facts:

REA51 *1.7.6 Remark.* Let $\langle \mathcal{L}_1, [.,.]_1\rangle$ and $\langle \mathcal{L}_2, [.,.]_2\rangle$ be inner product spaces, and let α be a linear subspace of $\mathcal{L}_1^\circ \times \mathcal{L}_2^\circ$.

(*i*) We have

$$\operatorname{ind}_- \mathcal{L}_1 \boxplus_\alpha \mathcal{L}_2 = \operatorname{ind}_- \mathcal{L}_1 + \operatorname{ind}_- \mathcal{L}_2$$
$$\operatorname{ind}_0 \mathcal{L}_1 \boxplus_\alpha \mathcal{L}_2 = \operatorname{ind}_0 \mathcal{L}_1 + \operatorname{ind}_0 \mathcal{L}_2 - \dim \alpha$$

This follows from the formulas for negative index and degree of degeneracy given in Remark 1.7.2 and Remark 1.7.3.

(*ii*) Since $\mathcal{L}_1^\circ \times \mathcal{L}_2^\circ = (\mathcal{L}_1[\dotplus]\mathcal{L}_2)^\circ$, the mappings $\iota_1^\alpha : \mathcal{L}_1 \to \mathcal{L}_1 \boxplus_\alpha \mathcal{L}_2$ and $\iota_2^\alpha : \mathcal{L}_2 \to \mathcal{L}_1 \boxplus_\alpha \mathcal{L}_2$ are both isometric. Moreover,

$$\iota_1^\alpha(\mathcal{L}_1) \perp \iota_2^\alpha(\mathcal{L}_2) \quad\text{and}\quad \mathcal{L}_1 \boxplus_\alpha \mathcal{L}_2 = \operatorname{ran} \iota_1^\alpha + \operatorname{ran} \iota_2^\alpha .$$

This is obvious from the definition and the fact that $\iota_1(\mathcal{L}_1) \perp \iota_2(\mathcal{L}_2)$.

(*iii*) The mappings ι_1^α and ι_2^α are both injective if and only if the linear subspace α is the graph of a bijective map $\alpha : \operatorname{dom}\alpha \to \operatorname{ran}\alpha$ between some linear subspaces $\operatorname{dom}\alpha \subseteq \mathcal{L}_1^\circ$ and $\operatorname{ran}\alpha \subseteq \mathcal{L}_2^\circ$. To see this, note that

$$(0, x_2) \in \alpha \iff \iota_2^\alpha(x_2) = 0, \quad (x_1, 0) \in \alpha \iff \iota_1^\alpha(x_1) = 0 .$$

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PRA52 **1.7.7 Proposition.** *Let $\langle \mathcal{L}_1, [.,.]_1\rangle$ and $\langle \mathcal{L}_2, [.,.]_2\rangle$ be inner product spaces, and let $\langle \mathcal{L}, [.,.]\rangle$ be an inner product space together with isometric maps $\iota_j' : \mathcal{L}_j \to \mathcal{L}$, $j = 1, 2$, such that $\iota_1'(\mathcal{L}_1) \perp \iota_2'(\mathcal{L}_2)$. Then there exists a unique linear subspace $\alpha \subseteq \mathcal{L}_1^\circ \times \mathcal{L}_2^\circ$, such that*

$$\begin{tikzcd} \mathcal{L}_1 \arrow[r, "\iota_1^\alpha"] \arrow[dr, "\iota_1'"'] & \mathcal{L}_1 \boxplus_\alpha \mathcal{L}_2 \arrow[d, dotted, "\psi"] & \mathcal{L}_2 \arrow[l, "\iota_2^\alpha"'] \arrow[dl, "\iota_2'"] \\ & \mathcal{L} & \end{tikzcd} \tag{1.7.1}$$ A53

with some injective and isometric linear map ψ. Also the map ψ is uniquely determined by the diagram (1.7.1)*. In fact, α and ψ are given by*

$$\alpha = \big\{(x_1,x_2)\in\mathcal{L}_1\times\mathcal{L}_2 : \iota_1'(x_1) = -\iota_2'(x_2)\big\},\quad \psi\big((x_1,x_2)/_\alpha\big) = \iota_1'(x_1)+\iota_2'(x_2)\,.$$

The map ι_j^α is injective if and only if ι_j' has this property, $j=1,2$. Moreover, $\operatorname{ran}\psi = \operatorname{ran}\iota_1' + \operatorname{ran}\iota_2'$*, in particular ψ is bijective if and only if* $\operatorname{ran}\iota_1' + \operatorname{ran}\iota_2' = \mathcal{L}$*.*

Proof. First we show existence of α and ϕ. Since $\iota_1'(\mathcal{L}_1)\perp\iota_2'(\mathcal{L}_2)$, the map $\phi(x) := \iota_1'(x_1)+\iota_2'(x_2)$, $x=(x_1,x_2)$, $x_1\in\mathcal{L}_1$, $x_2\in\mathcal{L}_2$, is an isometry of $\mathcal{L}_1[\dot+]\mathcal{L}_2$ into $\mathcal{L}$. It satisfies

$$\begin{tikzcd} \mathcal{L}_1 \arrow[r, "\iota_1"] \arrow[dr, "\iota_1'"'] & \mathcal{L}_1[\dot+]\mathcal{L}_2 \arrow[d, "\phi"] & \mathcal{L}_2 \arrow[l, "\iota_2"'] \arrow[dl, "\iota_2'"] \\ & \mathcal{L} & \end{tikzcd} \tag{1.7.2}$$ A54

and we have

$$\ker\phi = \big\{(x_1,x_2)\in\mathcal{L}_1[\dot+]\mathcal{L}_2 :\ \iota_1'(x_1) = -\iota_2'(x_2)\big\}\,.$$

Let $(x_1,x_2)\in\ker\phi$ be given. If $y_1\in\mathcal{L}_1$, then

$$\big[(x_1,x_2),(y_1,0)\big]_{\mathcal{L}_1[\dot+]\mathcal{L}_2} = [x_1,y_1]_1 = \big[\iota_1'(x_1),\iota_1'(y_1)\big] = \big[-\iota_2'(x_2),\iota_1'(y_1)\big] = 0\,.$$

Similarly, $[(x_1,x_2),(0,y_2)]=0$ for all $y_2\in\mathcal{L}_2$. Hence $\ker\phi\subseteq(\mathcal{L}_1[\dot+]\mathcal{L}_2)^\circ$, i.e. $\alpha:=\ker\phi$ qualifies as a subspace being used in the definition of $\mathcal{L}_1\boxplus_\alpha\mathcal{L}_2$. Let ψ be the injective isometry with

$$\begin{tikzcd} \mathcal{L}_1[\dot+]\mathcal{L}_2 \arrow[r, "\phi"] \arrow[d, "\pi_\alpha"'] & \mathcal{L} \\ (\mathcal{L}_1[\dot+]\mathcal{L}_2)/\alpha \arrow[ur, dotted, "\psi"'] & \end{tikzcd}$$

Then we have

$$\begin{tikzcd} & \mathcal{L}_1[\dot+]\mathcal{L}_2 \arrow[d, "\pi_{\alpha'}"] \arrow[dd, bend left=60, "\phi"] \arrow[d, phantom, shift right=20, "\#"] & \\ \mathcal{L}_1 \arrow[ur, "\iota_1"] \arrow[r, "\iota_1^{\alpha'}"] \arrow[dr, "\iota_1'"'] & \mathcal{L}_1\boxplus_{\alpha'}\mathcal{L}_2 \arrow[d, "\psi'"] \arrow[r, phantom, "\#"] & {} \\ & \mathcal{L} & \end{tikzcd} \tag{1.7.3}$$ A55

and hence (1.7.1) commutes. Moreover, clearly, $\operatorname{ran}\psi = \operatorname{ran}\phi = \operatorname{ran}\iota_1' + \operatorname{ran}\iota_2'$. Since ψ is injective, it follows that ι_j^α is injective if and only if ι_j' is.

In order to show uniqueness, assume that $\alpha'\subseteq\mathcal{L}_1^\circ\times\mathcal{L}_2^\circ$ and $\psi':\mathcal{L}_1\boxplus_{\alpha'}\mathcal{L}_2\to\mathcal{L}$ also have the stated properties. If $x_1\in\mathcal{L}_1$, then (1.7.1) gives

$$\psi'\big((x_1,0)/_{\alpha'}\big) = (\psi'\circ\iota_1^{\alpha'})(x_1) = \iota_1'(x_1)\,.$$

Similarly, $\psi'((0, x_2)/_{\alpha'}) = \iota_2'(x_2)$, $x_2 \in \mathcal{L}_2$, and hence by linearity $\psi'((x_1, x_2)/_{\alpha'}) = \iota_1'(x_1) + \iota_2'(x_2)$. Since ψ' is injective, it follows that $\alpha \subseteq \ker \psi' = \alpha'$. Conversely, since ψ' is a function, $(x_1, x_2) \in \alpha'$ implies that $\iota_1'(x_1) = -\iota_2'(x_2)$. Altogether, we see that $\alpha' = \alpha$ and $\psi' = \psi$. ❑

Combining Proposition 1.7.7 with Remark 1.7.6, (*iii*), we obtain the following corollary.

COA56 **1.7.8 Corollary.** *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be inner product spaces. An inner product space contains isomorphic copies of $\mathcal{L}_1$ and $\mathcal{L}_2$ as orthogonal subspaces which span the whole space, if and only if it is isomorphic to $\mathcal{L}_1 \boxplus_\alpha \mathcal{L}_2$ with some bijective map α between subspaces of $\mathcal{L}_1^\circ$ and $\mathcal{L}_2^\circ$.* ❑

Chapter 2

Topological inner product spaces

2.1 Definition of TIPS

Let us recall the definition of vector topologies.

DEB1 **2.1.1 Definition** (Vector topologies)**.** Let $\mathcal{L}$ be a linear space. A topology $\mathcal{T}$ on $\mathcal{L}$ is called a *vector topology*, if

(VT) The maps $+ : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ and $\cdot : \mathbb{C} \times \mathcal{L} \to \mathcal{L}$ are continuous, when $\mathcal{L}\times\mathcal{L}$ and $\mathbb{C}\times\mathcal{L}$ are endowed with the respective product topologies.

If $\mathcal{T}$ is a vector topology on $\mathcal{L}$, we will speak of $\langle \mathcal{L}, \mathcal{T} \rangle$ as a *topological vector space*.

A vector topology $\mathcal{T}$ on $\mathcal{L}$ is called *locally convex*, if

(LC) There exists a neighbourhood base of 0 which consists of convex sets.

If $\mathcal{T}$ is a locally convex vector topology on $\mathcal{L}$, we will speak of $\langle \mathcal{L}, \mathcal{T} \rangle$ as a *locally convex space*. //

Note that we do not require any seperation properties, like e.g. that each singleton set is closed. Still, the usual relation between locally convex vector topologies and families of seminorms, as elaborated e.g. in [**?**, Theorem 1.36–Remark 1.38], is present. Thereby Hausdorff topologies correspond to seperating families of seminorms.

A combination of the notions of 'inner product space' and 'locally convex space', together with the natural compatibility requirement, leads to the notion of 'topological inner product space'.

DEB2 **2.1.2 Definition.** A triple $\langle \mathcal{L}, [.,.], \mathcal{T} \rangle$ is called a *topological inner product space*, if

(TIPS1) $\mathcal{L}$ is a linear space.

(TIPS2) $[.,.]$ is an inner product on $\mathcal{L}$.

(TIPS3) $\mathcal{T}$ is a locally convex vector topology on $\mathcal{L}$ and the map $[.,.] : \mathcal{L} \times \mathcal{L} \to \mathbb{C}$ is continuous, when $\mathcal{L} \times \mathcal{L}$ is endowed with the product topology.

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2.2 Compatible topologies

We will often take the viewpoint that an inner product space $\langle \mathcal{L}, [.,.] \rangle$ is given, and ask for topologies $\mathcal{T}$ such that $\langle \mathcal{L}, [.,.], \mathcal{T} \rangle$ is a topological inner product space.

DEB4 **2.2.1 Definition.** Let $\langle \mathcal{L}, [.,.] \rangle$ be an inner product space. A locally convex vector topology $\mathcal{T}$ on $\mathcal{L}$ is called a *compatible topology*, if $\langle \mathcal{L}, [.,.], \mathcal{T} \rangle$ is a topological inner product space.

The set of all compatible topologies on $\langle \mathcal{L}, [.,.] \rangle$ will be denoted by $\operatorname{Top}\langle \mathcal{L}, [.,.] \rangle$. As usual, we will sometimes write $\operatorname{Top}\mathcal{L}$ if explicit mentioning the pregiven inner product on $\mathcal{L}$ is not necessary. //

The fact whether or not a vector topology is compatible, can be formulated in terms of seminorms.

PRB5 **2.2.2 Proposition.** *Let $\langle \mathcal{L}, [.,.] \rangle$ be an inner product space, and let $\mathcal{T}$ be a locally convex vector topology on $\mathcal{L}$. Then the following are equivalent:*

(*i*) $\mathcal{T} \in \operatorname{Top}\langle \mathcal{L}, [.,.] \rangle$.

(*ii*) *There exists a $\mathcal{T}$-continuous seminorm $p : \mathcal{L} \to [0, \infty)$ and a constant $\alpha > 0$, such that*

$$|[x, y]| \leq \alpha p(x) p(y), \quad x, y \in \mathcal{L}\,. \tag{2.2.1}$$ B6

(*iii*) *There exists a $\mathcal{T}$-continuous seminorm $p : \mathcal{L} \to [0, \infty)$ and a constant $\beta > 0$, such that*

$$|[x, x]|^{\frac{1}{2}} \leq \beta p(x), \quad x \in \mathcal{L}\,. \tag{2.2.2}$$ B7

Proof. Let $\{p_i : i \in I\}$ be a family of $\mathcal{T}$-continuous seminorms such that the set of all finite intersections of balls

$$U(p_i, \epsilon) = \{x \in \mathcal{L} : p_i(x) \leq \epsilon\}, \quad i \in I, \epsilon > 0$$

forms a $\mathcal{T}$-neighbourhood base at 0.

Step 1; (i) $\Rightarrow$ (ii): Assume that $[.,.]$ is $\mathcal{T} \times \mathcal{T}$-continuous. In particular, the inner product $[.,.]$ is continuous at the point $(0,0)$. Hence, there exists $\epsilon > 0$ and a finite subset I_0 of I, such that

$$|[x, y]| \leq 1 \text{ whenever } p_i(x), p_i(y) \leq \epsilon,\ i \in I_0\,.$$

Set $p := \max_{i \in I_0} p_i$, then p is a $\mathcal{T}$-continuous seminorm. Moreover,

$$|[x, y]| \leq 1 \text{ whenever } p(x), p(y) \leq \epsilon\,.$$

Assume that $p(x), p(y) \neq 0$. Then we obtain

$$|[x, y]| = \frac{p(x)p(y)}{\epsilon^2} \Big| \Big[\frac{\epsilon}{p(x)} x, \frac{\epsilon}{p(y)} y \Big] \Big| \leq \frac{1}{\epsilon^2} p(x) p(y)\,.$$

Assume that $p(x)\neq 0$ but $p(y)=0$. Then we have, for each $\lambda>0$,

$$|[x,y]|=\frac{p(x)}{\epsilon\lambda}\Big|\big[\frac{\epsilon}{p(x)}x,\lambda y\big]\Big|\leq\frac{p(x)}{\epsilon}\frac{1}{\lambda}\,.$$

Thus $[x,y]=0$. In the same way we find that the inner product $[x,y]$ vanishes if $p(x)=0,p(y)\neq 0$ or if $p(x)=p(y)=0$. Thus (2.2.1) holds with $\alpha:=\epsilon^{-2}$.

Step 2; (ii) $\Rightarrow$ (i): Let $\alpha>0$ be such that (2.2.1) holds. Let $(x_0,y_0)\in\mathcal{L}\times\mathcal{L}$ and $\epsilon>0$ be given, and choose a neighbourhood U of 0 such that $p(x)\leq\epsilon$, $x\in U$. We have

$$|[x,y]-[x_0,y_0]|\leq|[x-x_0,y]|+|[x_0,y-y_0]|\leq$$
$$\leq\alpha p(x-x_0)p(y)+\alpha p(x_0)p(y-y_0)\,.$$

Hence, if $x\in x_0+U$, $y\in y_0+U$,

$$|[x,y]-[x_0,y_0]|\leq\alpha\epsilon\big((p(y_0)+\epsilon)+p(x_0)\big)\,.$$

Thus $[.,.]$ is continuous at (x_0,y_0).

Step 3; (ii) $\iff$ (iii): The implication $(ii)\Rightarrow(iii)$ is trivial, in fact we can take the same seminorm p and the constant $\beta:=\sqrt{\alpha}$. In order to show the converse implication, assume that the seminorm p and the constant $\beta>0$ satisfy (2.2.2) in (iii). We will show that (2.2.1) holds with the same seminorm p and the constant $\alpha:=4\beta^2$.

The first step is to prove that

$$|[x,y]|\leq\beta^2\big(p(x)+p(y)\big)^2,\quad x,y\in\mathcal{L}\,. \tag{2.2.3}$$ B8

To this end let $x,y\in\mathcal{L}$ be given, and choose $\gamma\in\mathbb{C}$ with $|\gamma|=1$ such that $\gamma[x,y]\geq 0$. Then

$$2|[x,y]|=|[\gamma x,y]+[y,\gamma x]|=\Big|\frac{1}{2}[\gamma x+y,\gamma x+y]-\frac{1}{2}[\gamma x-y,\gamma x-y]\Big|\leq$$
$$\leq\frac{1}{2}\beta^2p(\gamma x+y)^2+\frac{1}{2}\beta^2p(\gamma x-y)^2\leq\beta^2\big(p(x)+p(y)\big)^2\,,$$

and this is (2.2.3).

Assume that $p(x),p(y)\neq 0$, then (2.2.3) gives

$$|[x,y]|=p(x)p(y)\Big|\big[\frac{x}{p(x)},\frac{y}{p(y)}\big]\Big|\leq p(x)p(y)\beta^2 2^2=4\beta^2p(x)p(y)\,.$$

If $p(x)=0$, we obtain from (2.2.3) that for each $\lambda>0$

$$|[x,y]|=\frac{1}{\lambda}|[\lambda x,y]|\leq\frac{1}{\lambda}\beta^2p(y)^2\,.$$

Thus $[x,y]$ vanishes. If $p(y)=0$, it follows in the same way that $[x,y]=0$. Altogether, we see that (2.2.1) holds with $\alpha:=4\beta^2$. ❏

Let p be a seminorm on $\mathcal{L}$. Then we will denote the topology induced on $\mathcal{L}$ by the one-element family $\{p\}$ as $\mathcal{T}_p$.

COB9 **2.2.3 Corollary.** *Let $\langle \mathcal{L}, [.,.] \rangle$ be an inner product space, and let p be a seminorm on $\mathcal{L}$. Then the following are equivalent:*

(*i*) $\mathcal{T}_p \in \operatorname{Top}\langle \mathcal{L}, [.,.] \rangle$.

(*ii*) *The seminorm p satifies* (2.2.1) *with some constant $\alpha > 0$.*

(*iii*) *The seminorm p satifies* (2.2.2) *with some constant $\beta > 0$.*

Proof. The seminorm p is $\mathcal{T}_p$-continuous. Hence (*ii*) or (*iii*) imply (*i*). Conversely, assume that $\mathcal{T}_p \in \operatorname{Top}\mathcal{L}$. Then, for each $\mathcal{T}_p$-continuous seminorm p_1 there exists a constant $\gamma > 0$ with $p_1(x) \leq \gamma p(x)$, $x \in \mathcal{L}$. Hence (*i*) implies (*ii*) and (*iii*). ❑

It is an immediate consequence that decomposable inner product spaces can be made into topological inner product spaces.

EXB10 *2.2.4 Example.* Let $\langle \mathcal{L}, [.,.] \rangle$ be an inner product space, and let $\mathfrak{J}$ be a fundamental decomposition of $\mathcal{L}$. By Lemma 1.3.3, the seminorm $p_{\mathfrak{J}}$ satisfies an inequality of the form (2.2.1), actually with $\alpha = 1$. Thus $\mathcal{T}_{p_{\mathfrak{J}}} \in \operatorname{Top}\mathcal{L}$. //

Let us collect some elementary, but useful, facts about topological inner product spaces.

LEB11 **2.2.5 Lemma.** *Let $\langle \mathcal{L}, [.,.], \mathcal{T} \rangle$ be a topological inner product space. Then the following hold:*

(*i*) *Each of the functionals $[., y]$, $y \in \mathcal{L}$, is continuous.*

(*ii*) *For each subset $A \subseteq \mathcal{L}$, its orthogonal companion $A^\perp$ is $\mathcal{T}$-closed. Moreover, $(\overline{A})^\perp = A^\perp$.*

(*iii*) $\overline{\{0\}} \subseteq \mathcal{L}^\circ$.

Proof. Clearly, $\mathcal{T} \times \mathcal{T}$-continuity of $[.,.]$ implies $\mathcal{T}$-continuity of $x \mapsto [x, y]$ whenever $y \in \mathcal{L}$ is fixed. This is (*i*).

Since $A^\perp = \bigcap_{y \in A} \ker[., y]$, this set is closed, and this is the first part of (*ii*). The inclusion $(\overline{A})^\perp \subseteq A^\perp$ is trivial. To show the converse inclusion, let $x \in A^\perp$. The function $y \mapsto [x, y]$, $y \in \mathcal{L}$, is continuous and vanishes on A. Thus it also vanishes on $\overline{A}$, i.e. $x \in (\overline{A})^\perp$.

To show (*iii*), let $x \in \overline{\{0\}}$ be given. Then, for each continuous linear functional ϕ on $\mathcal{L}$ we have $\phi(x) = 0$. Since $[., y]$ is continuous, it follows that $x \in \mathcal{L}^\circ$. ❑

2.3 Existence of compatible topolgies

Let $\langle \mathcal{L}, [.,.] \rangle$ be an inner product space. The question whether $\operatorname{Top}\langle \mathcal{L}, [.,.] \rangle$ is nonempty, is a nontrivial matter.

EXB12 *2.3.1 Example. We give an example of an inner product space with no compatible topologies.*

Set

$$\mathbb{C}^{\mathbb{Z}}_{\mathrm{lf}} := \left\{ (\xi_j)_{j \in \mathbb{Z}} : \ \xi_j \in \mathbb{C}, \ \exists N \in \mathbb{Z} : \xi_j = 0, j < N \right\},$$

and define an inner product $[.,.]$ on $\mathbb{C}^{\mathbb{Z}}_{\mathrm{lf}}$ by

$$\big[(\xi_j)_{j\in\mathbb{Z}},(\eta_j)_{j\in\mathbb{Z}}\big] := \sum_{j\in\mathbb{Z}} \xi_j\overline{\eta_{-j-1}}, \quad (\xi_j)_{j\in\mathbb{Z}},(\eta_j)_{j\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}_{\mathrm{lf}}\,. \tag{2.3.1}$$ B13

Note here that in the sum on the right hand side of this relation contains only finitely many nonzero summands.

Assume that p is a seminorm on $\mathbb{C}^{\mathbb{Z}}_{\mathrm{lf}}$ and that $\alpha > 0$ is a constant, such that (2.2.1) holds. Let $e_k := (\delta_{kj})_{j\in\mathbb{Z}}$, and consider the sequence $x := (\xi_j)_{j\in\mathbb{Z}}$ where

$$\xi_j := \begin{cases} (j+1)\max\{p(e_{-j-1}),1\}\,, & j\geq 0\\ 0 & ,\quad j<0\end{cases}$$

Then, for each $k\in\mathbb{N}_0$,

$$(k+1)\max\{p(e_{-k-1}),1\} = \xi_k = [x,e_{-k-1}] \leq \alpha p(x)p(e_{-k-1}) \leq$$

$$\leq \alpha p(x)\max\{p(e_{-k-1}),1\}\,.$$

It follows that $k+1\leq \alpha p(x)$, $k\in\mathbb{N}_0$, and we have reached a contradiction. Thus $\mathrm{Top}\langle\mathbb{C}^{\mathbb{Z}}_{\mathrm{lf}},[.,.]\rangle = \emptyset$. //

As we have observed in Example 2.2.4, fundamental decompositions of an inner product space $\langle\mathcal{L},[.,.]\rangle$ are a source for compatible vector topologies. These topologies are constructed intrinsically from the inner product, and hence may be regarded as the most natural elements of $\mathrm{Top}\langle\mathcal{L},[.,.]\rangle$.

DEB14 **2.3.2 Definition.** Let $\langle\mathcal{L},[.,.]\rangle$ be an inner product space. If $\mathfrak{J}$ is a fundamental decomposition of $\mathcal{L}$, then the topology induced by the seminorm $p_{\mathfrak{J}}$ will be denoted by $\mathcal{T}_{\mathfrak{J}}$.

An element $\mathcal{T}\in\mathrm{Top}\langle\mathcal{L},[.,.]\rangle$ is called a *decomposition topology*, if there exists a fundamental decomposition $\mathfrak{J}$ of $\langle\mathcal{L},[.,.]\rangle$, such that $\mathcal{T}=\mathcal{T}_{\mathfrak{J}}$. The set of all decomposition topologies of $\langle\mathcal{L},[.,.]\rangle$ will be denoted by $\mathrm{Top}_{\mathrm{dec}}\langle\mathcal{L},[.,.]\rangle$. //

The question whether an inner product space $\langle\mathcal{L},[.,.]\rangle$ possesses fundamental decompositions, in other words whether $\mathrm{Top}_{\mathrm{dec}}\langle\mathcal{L},[.,.]\rangle$ is nonempty, is again a nontrivial matter. Let us show one result which says that existence of well-behaved compatible topologies implies decomposability.

THB15 **2.3.3 Theorem.** *Let $\langle\mathcal{L},[.,.]\rangle$ be an inner product space, and assume that there exists an inner product $(.,.)$ on $\mathcal{L}$, such that $\langle\mathcal{L},(.,.)\rangle$ is a Hilbert space and the topology induced by $(.,.)$ on $\mathcal{L}$ is compatible. Then $\mathcal{L}$ is decomposable, and there exists a fundamental decomposition $\mathfrak{J}=(\mathcal{L}_+,\mathcal{L}_-)$ of $\mathcal{L}$, such that each of*

$$\mathcal{L}_+,\quad \mathcal{L}_-,\quad \mathcal{L}_++\mathcal{L}_-,\quad \mathcal{L}_++\mathcal{L}^\circ,\quad \mathcal{L}_-+\mathcal{L}^\circ$$

is $(.,.)$-closed.

Proof. Since $[.,.]$ is a $(.,.)$-continuous sesquilinearform on $\mathcal{L}$, there exists a bounded linear operator G on the Hilbert space $\langle\mathcal{L},(.,.)\rangle$, such that

$$[x,y]=(Gx,y),\quad x,y\in\mathcal{L}\,. \tag{2.3.2}$$ B16

Since $[.,.]$ actually is an inner product, G is selfadjoint. Let E denote the spectral measure of G, and set

$$\mathcal{L}_+ := \operatorname{ran} E\big((0,\infty)\big), \quad \mathcal{L}_- := \operatorname{ran} E\big((-\infty,0)\big)\,.$$

Then $\mathcal{L}_+$ and $\mathcal{L}_-$ are $(.,.)$-closed, $(.,.)$-orthogonal, and G-invariant subspaces of $\mathcal{L}$. We have $\mathcal{L}^\circ = \ker G$, and hence

$$\mathcal{L}_+ + \mathcal{L}^\circ = \operatorname{ran} E\big([0,\infty)\big), \quad \mathcal{L}_- + \mathcal{L}^\circ = \operatorname{ran} E\big((-\infty,0]\big)\,.$$

Clearly, also $\mathcal{L}_+ + \mathcal{L}_- = \operatorname{ran} E(\mathbb{R}\setminus\{0\})$. We see that each of these spaces is $(.,.)$-closed, and that

$$\mathcal{L} = \mathcal{L}_+ (\dot{+}) \mathcal{L}_- (\dot{+}) \mathcal{L}^\circ\,. \tag{2.3.3}$$ B17

Since $\mathcal{L}_+$ is G-invariant, and $(.,.)$-orthogonal to $\mathcal{L}_-$, we have

$$[x,y] = (Gx,y) = 0, \quad x \in \mathcal{L}_+, y \in \mathcal{L}_-\,,$$

i.e. $\mathcal{L}_+[\perp]\mathcal{L}_-$. It follows together with (2.3.3) that $\mathcal{L}_+^{[\circ]}[\perp]\mathcal{L}$ and $\mathcal{L}_-^{[\circ]}[\perp]\mathcal{L}$, and hence that $\mathcal{L}_+$ and $\mathcal{L}_-$ are both nondegenerated.

For $x \in \mathcal{L}$ denote by $E_{x,x}$ the positive Borel measure $\Delta \mapsto (E(\Delta)x,x)$. Then, by (2.3.2),

$$[x,x] = \int_{\mathbb{R}} t\, dE_{x,x}\,.$$

If $x \in \mathcal{L}_+$ and $\Delta \subseteq (-\infty,0]$, then

$$E(\Delta)x = E(\Delta)E((0,\infty))x = 0\,.$$

It follows that

$$[x,x] = \int_{(0,\infty)} t\, dE_{x,x} \geq 0, \quad x \in \mathcal{L}_+\,.$$

This shows that $\mathcal{L}_+$ is positive semidefinite. Since $\mathcal{L}_+$ is nondegenerated, it follows that it is actually positive definite. Similarly, we have

$$[x,x] = \int_{(-\infty,0)} t\, dE_{x,x} \leq 0, \quad x \in \mathcal{L}_-\,,$$

and conclude that $\mathcal{L}_-$ is negative definite.

Altogether, we have shown that $\mathfrak{J} := (\mathcal{L}_+, \mathcal{L}_-)$ is a fundamental decomposition of $\mathcal{L}$ which possesses the required additional properties. ❑

The following two examples show that the assumption in this theorem, that the compatible topology is induced by a Hilbert space inner product, cannot be weakened.

EXB18 2.3.4 *Example. We give an example of an inner product space which has a compatible topology induced by a positive definite inner product, but is not decomposable.*

Consider the linear space $\mathcal{L} := \mathbb{C}^{\mathbb{Z}}_{\mathrm{lf}} \cap \ell^2(\mathbb{Z})$ endowed with the inner product $[.,.]$ defined by (2.3.1).

If $(.,.)$ denotes the usual $\ell^2(\mathbb{Z})$-inner product, then

$$|[x,y]| \leq (x,x)^{\frac12}(y,y)^{\frac12}, \quad x,y \in \mathcal{L}\,.$$

Hence the topology induced by $(.,.)$ on $\mathcal{L}$ is a compatible topology.

Consider the map $\phi : \mathcal{L} \to \mathcal{L}$ defined as

$$\phi\big((\xi_j)_{j\in\mathbb{Z}}\big) := \big(\xi_j \chi_{-\mathbb{N}}(j)\big)_{j\in\mathbb{Z}},$$

where $\chi_{-\mathbb{N}}$ denotes the characteristic function of the set $-\mathbb{N}$, i.e.

$$\chi_{-\mathbb{N}}(j) := \begin{cases} 1, & j \in -\mathbb{N} \\ 0, & \text{otherwise} \end{cases}$$

Then

$$\operatorname{ran}\phi = \big\{(\xi_j)_{j\in\mathbb{Z}} : \xi_j \in \mathbb{C},\ \exists N \in \mathbb{Z} : \xi_j = 0, j < N, j \geq 0\big\} =$$

$$= \operatorname{span}\{e_k : k \in -\mathbb{N}\}.$$

$$\ker\phi = \big\{(\xi_j)_{j\in\mathbb{Z}} : \xi_j \in \mathbb{C},\ \xi_j = 0, j < 0\big\} \cong \ell^2(\mathbb{N}) \qquad (2.3.4)$$ B19

We see that $\ker\phi$ is a neutral subspace of $\mathcal{L}$.

Let $\mathcal{M}$ be a definite subspace of $\mathcal{L}$. Then $\mathcal{M} \cap \ker\phi = \{0\}$, and hence $\phi|_{\mathcal{M}}$ is injective. Thus $\operatorname{ran}\phi$ contains an isomorphic copy of $\mathcal{M}$, in particular, the dimension of $\mathcal{M}$ is at most countable.

Assume that $\mathfrak{J} = (\mathcal{L}_+, \mathcal{L}_-)$ is a fundamental decomposition of $\mathcal{L}$. Then the dimensions of $\mathcal{L}_+$ and $\mathcal{L}_-$ are at most countable. Since, for each $(\xi_j)_{j\in\mathbb{Z}} \in \mathcal{L}$ and $k \in \mathbb{Z}$ we have $[(\xi_j)_{j\in\mathbb{Z}}, e_{-k-1}] = \xi_k$, the space $\mathcal{L}$ is nondegenerated. Thus $\mathcal{L} = \mathcal{L}_+ \dot{+} \mathcal{L}_-$, and we conclude that also the dimension of $\mathcal{L}$ is at most countable. This contradicts the fact that $\mathcal{L}$ contains an isomorphic copy of the Banach space $\ell^2(\mathbb{N})$, cf. (2.3.4). //

EXB20 *2.3.5 Example. We give an example of an inner product space which admits a compatible topology induced by a Banach space norm, but is not decomposable.*

Let $X := \ell^p$ with $p \in (1,\infty)\setminus\{2\}$. Then X is a reflexive Banach space whose norm is not equivalent to any norm induced by an inner product. This follows since X contains noncomplemented closed subspaces.

Let conjugate linear and isometric mappings $.^\#$ on X and X', respectively, be defined as

$$(x_n)^\#_{n\in\mathbb{N}} := (\overline{x_n})_{n\in\mathbb{N}}, \quad (x_n)_{n\in\mathbb{N}} \in X,$$

$$\phi^\#(x) := \overline{\phi(x^\#)}, \quad \phi \in X'.$$

Note that both of these maps are involutions.

Consider the vector space $\mathcal{L} := X \times X'$, and set

$$\|(x,\phi)\| := \|x\|_X + \|\phi\|_{X'}, \qquad \big[(x,\phi),(y,\psi)\big] := \phi(y^\#) + \psi^\#(x).$$

Then $\langle \mathcal{L}, \|.\|\rangle$ is a Banach space and $[.,.]$ is an inner product on $\mathcal{L}$. Moreover, note that $[.,.]$ is nondegenerated. We have

$$\big|\big[(x,\phi),(y,\psi)\big]\big| \leq |\phi(y^\#)| + |\psi^\#(x)| \leq \|\phi\|_{X'}\|y\|_X + \|\psi\|_{X'}\|x\|_X \leq$$

$$\leq \|(x,\phi)\|\|(y,\psi)\|,$$

and hence the topology induced on $\mathcal{L}$ by $\|.\|$ is compatible. In particular, each functional $[.,(y,\psi)]$ is continuous. We will show that every continuous linear functional is of this form. To this end, let $\Phi \in \langle \mathcal{L}, \|.\|\rangle'$. Then the map

$$\alpha : x \mapsto \Phi(x,0), \quad x \in X,$$

belongs to X'. The map

$$\eta : \phi \mapsto \Phi(0,\phi), \quad \phi \in X',$$

belongs to X''. Hence, by reflexivity, there exists an element $y \in X$ with $\eta(\phi) = \phi(y)$, $\phi \in X'$. It follows that

$$\Phi(x,\phi) = \Phi(x,0)+\Phi(0,\phi) = \alpha(x)+\phi(y) = \big[(x,\phi),(\alpha^{\#},y^{\#})\big], \quad (x,\phi) \in X\times X'.$$

Assume that $\mathcal{L}$ is decomposable and let $\mathfrak{J}$ be a fundamental decomposition of $\mathcal{L}$. Then, borrowing from the later Proposition 2.4.3, (iii), we have $\mathcal{T}_{\mathfrak{J}} \subseteq \mathcal{T}_{\|.\|}$. It follows that

$$\big\{[.,(y,\psi)] : (y,\psi) \in \mathcal{L}\big\} \subseteq \langle \mathcal{L}, \mathcal{T}_{\mathfrak{J}}\rangle' \subseteq \langle \mathcal{L}, \mathcal{T}_{\|.\|}\rangle' = \big\{[.,(y,\psi)] : (y,\psi) \in \mathcal{L}\big\},$$

and hence that

$$\langle \mathcal{L}, \mathcal{T}_{\mathfrak{J}}\rangle' = \langle \mathcal{L}, \mathcal{T}_{\|.\|}\rangle'.$$

By the Hahn-Banach Theorem, this implies that the norms $\|.\|_{\mathfrak{J}}$ and $\|.\|$ are equivalent. Thus also the norm $\|.\|_{\mathfrak{J}}|_{X\times\{0\}}$ is equivalent to $\|.\|_X = \|.\||_{X\times\{0\}}$. We have reached a contradiction, since $\|.\|_{\mathfrak{J}}$ is induced by an inner product. //

2.4 Subclasses of $\mathrm{Top}\langle \mathcal{L}, [.,.]\rangle$

We have already seen in Theorem 2.3.3 that existence of compatible topologies with specific properties may allow for specific conclusions. In this section we will investigate some subclasses of $\mathrm{Top}\langle \mathcal{L}, [.,.]\rangle$ more systematically.

DEB21 **2.4.1 Definition.** Let $\langle \mathcal{L}, [.,.]\rangle$ be an inner product space, and let $\mathcal{T} \in \mathrm{Top}\langle \mathcal{L}, [.,.]\rangle$. Then we write $\mathcal{T} \in \mathrm{Top}_{\mathsf{index}}\langle \mathcal{L}, [.,.]\rangle$, where '$\mathsf{index}$' may be one of

$$\mathrm{sn}\ /\ \mathrm{n}\ /\ \mathrm{ip}\ /\ \mathrm{ip}^+\ /\ \mathrm{Bs}\ /\ \mathrm{Hs},$$

if $\mathcal{T}$ is induced by:

sn $\leftrightsquigarrow$	a single seminorm	n $\leftrightsquigarrow$	a norm
ip $\leftrightsquigarrow$	a positive semidefinite inner product	ip^+ $\leftrightsquigarrow$	a positive definite inner product
Bs $\leftrightsquigarrow$	a norm turning $\mathcal{L}$ into a Banach space	Hs $\leftrightsquigarrow$	an inner product turning $\mathcal{L}$ into a Hilbert space

//

The relation between these subclasses of $\mathrm{Top}\langle \mathcal{L}, [.,.]\rangle$ can be pictured as follows:

$$
\begin{tikzcd}
 & \mathrm{Top} \arrow[d, dash] & \\
 & \mathrm{Top}_{\mathrm{sn}} \arrow[dl, dash] \arrow[d, dash] & \\
\mathrm{Top}_{\mathrm{n}} \arrow[d, dash] \arrow[dr, dash] & \mathrm{Top}_{\mathrm{ip}} \arrow[d, dash] \arrow[dr, dash] & \\
\mathrm{Top}_{\mathrm{Bs}} \arrow[dr, dash] & \mathrm{Top}_{\mathrm{ip}^+} \arrow[d, dash] & \mathrm{Top}_{\mathrm{dec}} \\
 & \mathrm{Top}_{\mathrm{Hs}} &
\end{tikzcd}
$$

Let us collect some results concerning these classes.

PRB22 **2.4.2 Proposition.** *Let $\langle\mathcal{L},[.,.]\rangle$ be an inner product space.*

(i) We have $\mathrm{Top}\,\mathcal{L}\neq\emptyset$ if and only if $\mathrm{Top}_{\mathrm{sn}}\,\mathcal{L}\neq\emptyset$. More exactly: if $\mathcal{T}\in\mathrm{Top}\,\mathcal{L}$, then there exists a seminorm p such that $\mathcal{T}_p\in\mathrm{Top}\,\mathcal{L}$ and $\mathcal{T}_p\subseteq\mathcal{T}$.

(ii) We have $\mathrm{Top}_{\mathrm{sn}}\,\mathcal{L}\neq\emptyset$ if and only if $\mathrm{Top}_{\mathrm{n}}\,\mathcal{L}\neq\emptyset$. More exactly: if $\mathcal{T}\in\mathrm{Top}_{\mathrm{sn}}\,\mathcal{L}$, then there exists a norm $\|.\|$ such that $\mathcal{T}_{\|.\|}\in\mathrm{Top}\,\mathcal{L}$ and $\mathcal{T}_{\|.\|}\supseteq\mathcal{T}$.

(iii) We have $\mathrm{Top}_{\mathrm{ip}}\,\mathcal{L}\neq\emptyset$ if and only if $\mathrm{Top}_{\mathrm{ip}^+}\,\mathcal{L}\neq\emptyset$. More exactly: if $\mathcal{T}\in\mathrm{Top}_{\mathrm{sn}}\,\mathcal{L}$, then there exists a positive definite inner product $(.,.)$ such that $\mathcal{T}_{(.,.)}\in\mathrm{Top}\,\mathcal{L}$ and $\mathcal{T}_{(.,.)}\supseteq\mathcal{T}$.

(iv) Assume that $\mathcal{T}\in\mathrm{Top}_{\mathrm{Bs}}\,\mathcal{L}$, and that $\mathfrak{J}=(\mathcal{L}_+,\mathcal{L}_-)$ is a fundamental decomposition of $\mathcal{L}$. If $\mathcal{L}_+$ and $\mathcal{L}_-$ are $\mathcal{T}$-closed, then $P^+_{\mathfrak{J}}$ and $P^-_{\mathfrak{J}}$ are $\mathcal{T}$-to-$\mathcal{T}|_{\mathcal{L}_+}$-continuous ($\mathcal{T}$-to-$\mathcal{T}|_{\mathcal{L}_+}$-continuous, respectively). Moreover, we have $\mathcal{T}_{\mathfrak{J}}\subseteq\mathcal{T}$.

Proof.
Item (i): Let $\mathcal{T}\in\mathrm{Top}\,\mathcal{L}$. By Proposition 2.2.2 there exists a $\mathcal{T}$-continuous seminorm p which satisfies (2.2.1) with some $\alpha>0$. As we have already noted in Corollary 2.2.3, the topology $\mathcal{T}_p$ induced by this seminorm is compatible. However, $\mathcal{T}$-continuity of p implies that $\mathcal{T}_p\subseteq\mathcal{T}$.

Item (ii): Let p be a seminorm with $\mathcal{T}_p\in\mathrm{Top}\,\mathcal{L}$ be given. By Corollary 2.2.3, p satisfies (2.2.2) with some $\beta>0$. Put $\mathcal{L}_0:=p^{-1}(\{0\})$, and choose a positive semidefinite inner product $(.,.)_0$ on $\mathcal{L}_0$. This can be done e.g. by choosing a basis of $\mathcal{L}_0$ and defining the inner product so that this basis becomes an orthonormal basis. Moreover, choose a linear subspace $\mathcal{L}_1$ such that $\mathcal{L}=\mathcal{L}_1\dot{+}\mathcal{L}_0$, denote by P_0 the projection of $\mathcal{L}$ onto $\mathcal{L}_0$ with kernel $\mathcal{L}_1$, and let p_0 be the seminorm $p_0(x):=(P_0x,P_0x)_0^{\frac{1}{2}}$. Define

$$\|x\|:=\sqrt{p(x)^2+(P_0x,P_0x)_0}=\left\|\begin{pmatrix}p(x)\\p_0(x)\end{pmatrix}\right\|_{\mathbb{R}^2},\quad x\in\mathcal{L}\,.$$

Using the triangular inequality of the euclidean norm in $\mathbb{R}^2$, we obtain that $\|.\|$ is a seminorm. However, $\|x\|=0$ implies that $p(x)=0$ and $(P_0x,P_0x)_0=0$

which is $P_0 x = 0$. For $x \in p^{-1}(\{0\})$ we have $P_0 x = x$, and thus it follows that $x = 0$. Therefore $\|.\|$ is actually a norm. Moreover,

$$\|x\| \geq p(x), \quad x \in \mathcal{L}\,,$$

and thus $\mathcal{T}_{\|.\|} \supseteq \mathcal{T}_p$. In particular, the inner product $[.,.]$ is $\mathcal{T}_{\|.\|} \times \mathcal{T}_{\|.\|}$-continuous.

Item (iii): Assume that the seminorm p is induced by an inner product $(.,.)$, i.e.

$$p(x) = (x,x)^{\frac{1}{2}}, \quad x \in \mathcal{L}\,.$$

Then the norm $\|.\|$ defined in the above given proof of (ii) is induced by the inner product

$$(x,y)_1 := (x,y) + (P_0 x, P_0 y)_0, \quad x,y \in \mathcal{L}\,.$$

This inner product, however, is positive definite since $\|.\|$ is a norm.

Item (iv): Let $\mathcal{T} \in \operatorname{Top}_{\mathrm{Bs}} \mathcal{L}$, and let $\mathfrak{J} = (\mathcal{L}_+, \mathcal{L}_-)$ be a fundamental decomposition whose components are $\mathcal{T}$-closed. Consider the fundamental projection $P_{\mathfrak{J}}^+ : \mathcal{L} \to \mathcal{L}_+$. Let $x_n \in \mathcal{L}$, $n \in \mathbb{N}$, and assume that

$$x_n \xrightarrow{\mathcal{T}} x,\ P_{\mathfrak{J}}^+ x_n \xrightarrow{\mathcal{T}|_{\mathcal{L}_+}} y_+\,.$$

If $z \in \mathcal{L}_+$, then

$$[P_{\mathfrak{J}}^+ x, z] = [x,z] = \lim_{n\to\infty} [x_n, z] = \lim_{n\to\infty} [P_{\mathfrak{J}}^+ x_n, z] = [y_+, z]\,,$$

and we conclude that $P_{\mathfrak{J}}^+ x = y_+$. Thus the graph of $P_{\mathfrak{J}}^+$ is closed. Since $\mathcal{L}_+$ is $\mathcal{T}$-closed, it is itself a Banach space. Hence, by the Closed Graph Theorem, $P_{\mathfrak{J}}^+$ is $\mathcal{T}$-to-$\mathcal{T}|_{\mathcal{L}_+}$-continuous. In the same way, we see that $P_{\mathfrak{J}}^-$ is $\mathcal{T}$-to-$\mathcal{T}|_{\mathcal{L}_-}$-continuous.

Choose a norm $\|.\|$ which induces $\mathcal{T}$, and let $\alpha > 0$ be such that (2.2.1) holds for $\|.\|$. Then we have

$$\|x\|_{\mathfrak{J}}^2 = (x,x)_{\mathfrak{J}} = [Jx, x] = [P_{\mathfrak{J}}^+ x, x] - [P_{\mathfrak{J}}^- x, x] \leq$$

$$\leq \alpha \|P_{\mathfrak{J}}^+ x\| \cdot \|x\| + \alpha \|P_{\mathfrak{J}}^- x\| \cdot \|x\| \leq \alpha\big(\|P_{\mathfrak{J}}^+\| + \|P_{\mathfrak{J}}^-\|\big)\|x\|^2\,,$$

and it follows that $\mathcal{T}_{\mathfrak{J}} \subseteq \mathcal{T}$. ❑

PRB3 **2.4.3 Proposition.** *Let $\langle \mathcal{L}, [.,.] \rangle$ be a nondegenerated inner product space.*

(i) We have $\operatorname{Top}_{\mathrm{sn}} \mathcal{L} = \operatorname{Top}_{\mathrm{n}} \mathcal{L}$ and $\operatorname{Top}_{\mathrm{ip}} \mathcal{L} = \operatorname{Top}_{\mathrm{ip}^+} \mathcal{L}$. More exactly: each seminorm p with $\mathcal{T}_p \in \operatorname{Top} \mathcal{L}$ is a norm, and each positive semidefinite inner product $(.,.)$ with $\mathcal{T}_{(.,.)} \in \operatorname{Top} \mathcal{L}$ is positive definite.

(ii) $|\operatorname{Top}_{\mathrm{Bs}} \mathcal{L}| \leq 1$ contains at most one element.

(iii) If $\mathcal{T}_0 \in \operatorname{Top}_{\mathrm{dec}} \mathcal{L}$ and $\mathcal{T} \in \operatorname{Top}_{\mathrm{Bs}} \mathcal{L}$, then $\mathcal{T}_0 \subseteq \mathcal{T}$.

Proof.
Item (i): To prove this item, it is enough to remember that for each seminorm p with $\mathcal{T}_p \in \operatorname{Top} \mathcal{L}$ we have $p^{-1}(\{0\}) \subseteq \mathcal{L}^{\circ}$, cf. Lemma 2.2.5, (ii).

Item (ii): Let $\mathcal{T}_1,\mathcal{T}_2 \in \mathrm{Top}\,\mathcal{L}$, and consider the identity map $\mathrm{id} : \langle\mathcal{L},\mathcal{T}_1\rangle \to \langle\mathcal{L},\mathcal{T}_2\rangle$. Let $(x_i)_{i\in I}$ be a net in $\mathcal{L}$, and assume that

$$x_i \xrightarrow{\mathcal{T}_1} x,\ x_i \xrightarrow{\mathcal{T}_2} x\,.$$

Then, for each $\phi \in \langle\mathcal{L},\mathcal{T}_1\rangle' \cap \langle\mathcal{L},\mathcal{T}_2\rangle'$, we have

$$\phi(x) = \lim_{i\in I}\phi(x_i) = \phi(y)\,.$$

In particular, this applies with each functional $[.,y]$, $y \in \mathcal{L}$, and we conclude that $x - y \in \mathcal{L}^\circ$.

If $[.,.]$ is nondegenerated, the above argument tells us that the graph of the identity map $\mathrm{id} : \langle\mathcal{L},\mathcal{T}_1\rangle \to \langle\mathcal{L},\mathcal{T}_2\rangle$ is closed. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are both induced by Banach space norms, the Closed Graph Theorem yields that it is continuous. Since id is bijective, it is a homeomorphism, and we conclude that $\mathcal{T}_1 = \mathcal{T}_2$.

Item (iii): Let $\mathcal{T} \in \mathrm{Top}\,\mathcal{L}$ and let $\mathfrak{J} = (\mathcal{L}_+,\mathcal{L}_-)$ be a fundamental decomposition. Since $\mathcal{L}$ is nondegenerated, we have $\mathcal{L}_+ = \mathcal{L}_-^\perp$ and $\mathcal{L}_- = \mathcal{L}_+^\perp$. Hence $\mathcal{L}_+$ and $\mathcal{L}_-$ are both closed, and we may apply the already proved item (v). ❑

2.5 Minimal elements of $\mathrm{Top}\langle\mathcal{L},[.,.]\rangle$

The set $\mathrm{Top}\,\mathcal{L}$ is ordered by set-theoretic inclusion. Clearly, if $\mathcal{T}_1$ and $\mathcal{T}_2$ are vector topologies on $\mathcal{L}$ with $\mathcal{T}_1 \subseteq \mathcal{T}_2$ and $\mathcal{T}_1$ is compatible, then also $\mathcal{T}_2$ is compatible. Hence, asking for large elements of $\mathrm{Top}\,\mathcal{L}$ will lead to the same questions and answers as for the set of all vector topologies. When asking for small elements of $\mathrm{Top}\,\mathcal{L}$, the situation is more specific and depends on the additional structure provided by the inner product. In particular, the investigation of minimal elements of $\mathrm{Top}\,\mathcal{L}$ is interesting in various respects.

DEB23 **2.5.1 Definition.** Let $\langle\mathcal{L},[.,.]\rangle$ be an inner product space. We denote the set of all minimal elements of $\mathrm{Top}\langle\mathcal{L},[.,.]\rangle$ by $\mathrm{Top}_{\min}\,\mathcal{L}$. That is, we write $\mathcal{T} \in \mathrm{Top}_{\min}\langle\mathcal{L},[.,.]\rangle$ if $\mathcal{T} \in \mathrm{Top}\langle\mathcal{L},[.,.]\rangle$ and

$$\mathcal{T}' \in \mathrm{Top}\langle\mathcal{L},[.,.]\rangle, \mathcal{T}' \subseteq \mathcal{T} \implies \mathcal{T}' = \mathcal{T}\,.$$

//

In the study of $\mathrm{Top}_{\min}\,\mathcal{L}$ the notion of the polar of a seminorm is useful.

DEB24 **2.5.2 Definition.** Let $\langle\mathcal{L},[.,.]\rangle$ be an inner product space, and let p be a seminorm on $\mathcal{L}$ with $\mathcal{T}_p \in \mathrm{Top}\,\mathcal{L}$. Then the map $p' : \mathcal{L} \to [0,\infty)$ which is defined as

$$p'(x) := \sup_{p(y)\le 1} |[x,y]|, \quad x \in \mathcal{L}\,,$$

is called the *polar of* p. //

Note that the supremum in the defining relation for p' is finite, since $\mathcal{T}_p \in \mathrm{Top}\,\mathcal{L}$ means that there exists some $\alpha > 0$ with $|[x,y]| \le \alpha p(x)p(y)$, $x,y \in \mathcal{L}$. Thus,

$$p'(x) = \sup_{p(y)\le 1} |[x,y]| \le \alpha p(x), \quad x \in \mathcal{L}\,. \tag{2.5.1}$$ B25

Actually, for given $x \in \mathcal{L}$, the number $p'(x)$ is the smallest constant γ such that the inequality $|[x,x]| \leq \gamma p(x)$ holds for all $x \in \mathcal{L}$.

Let us collect some simple properties of polars.

LEB26 **2.5.3 Lemma.** *Let $\langle \mathcal{L}, [.,.] \rangle$ be an inner product space, and let p, p_1, p_2 be seminorms on $\mathcal{L}$ with $\mathcal{T}_p, \mathcal{T}_{p_1}, \mathcal{T}_{p_2} \in \operatorname{Top} \mathcal{L}$. Then the following hold:*

(i) p' is a seminorm.

(ii) If $p_1 \leq p_2$, then $p_1' \geq p_2'$. For each $\lambda > 0$ we have $(\lambda p)' = \frac{1}{\lambda} p'$.

(iii) If $\mathcal{T}_{p_1} \subseteq \mathcal{T}_{p_2}$, then $\mathcal{T}_{p_1'} \supseteq \mathcal{T}_{p_2'}$.

(iv) We have $\mathcal{T}_{p'} \subseteq \mathcal{T}_p$.

Proof. To see (i), we compute

$$p'(\lambda x) = \sup_{p(y) \leq 1} |[\lambda x, y]| = |\lambda| \sup_{p(y) \leq 1} |[x,y]| \,,$$

$$p'(x_1 + x_2) = \sup_{p(y) \leq 1} |[x_1 + x_2, y]| \leq \sup_{p(y) \leq 1} |[x_1, y]| + \sup_{p(y) \leq 1} |[x_2, y]| \,.$$

For the first part of (ii) note that $p_1 \leq p_2$ implies that $\{y \in \mathcal{L} : p_1(y) \leq 1\} \supseteq \{y \in \mathcal{L} : p_2(y) \leq 1\}$. Hence,

$$p_1'(x) = \sup_{p_1(y) \leq 1} |[x,y]| \geq \sup_{p_1(y) \leq 1} |[x,y]| = p_2'(x) \,.$$

Next, for each $\lambda > 0$ we have

$$(\lambda p)'(x) = \sup_{(\lambda p)(y) \leq 1} |[x,y]| = \sup_{p(\lambda y) \leq 1} |[x,y]| = \sup_{p(z) \leq 1} |[x, \frac{z}{\lambda}]| = \frac{1}{\lambda} p'(x) \,.$$

In order to show (iii), assume that $\mathcal{T}_{p_1} \subseteq \mathcal{T}_{p_2}$. Then p_1 is $\mathcal{T}_2$-continuous, and hence there exists a constant $\gamma > 0$ with $p_1(x) \leq \gamma p_2(x)$, $x \in \mathcal{L}$. It follows that

$$p_1' \geq (\gamma p_2)' = \frac{1}{\gamma} p_2' \,,$$

and hence that $\mathcal{T}_{p_1'} \supseteq \mathcal{T}_{p_2'}$.

Finally, item (iv) is immediate from (2.5.1). ❑

Item (iii) of the above lemma implies that the following notion is well-defined.

DEB27 **2.5.4 Definition.** Let $\langle \mathcal{L}, [.,.] \rangle$ be an inner product space, and let $\mathcal{T} \in \operatorname{Top}_{\mathrm{sn}} \mathcal{L}$. Choose a seminorm p with $\mathcal{T} = \mathcal{T}_p$, and define $\mathcal{T}' := \mathcal{T}_{p'}$. The vector topology $\mathcal{T}'$ is called the *polar of $\mathcal{T}$*. //

From items (iii) and (iv) of Lemma 2.5.3 we immediately obtain the following corollary.

COB28 **2.5.5 Corollary.** *Let $\langle \mathcal{L}, [.,.] \rangle$ be an inner product space. If $\mathcal{T} \in \operatorname{Top}_{\mathrm{sn}} \langle \mathcal{L}, [.,.] \rangle$, then $\mathcal{T}' \subseteq \mathcal{T}$. If $\mathcal{T}_1, \mathcal{T}_2 \in \operatorname{Top}_{\mathrm{sn}} \langle \mathcal{L}, [.,.] \rangle$ with $\mathcal{T}_1 \subseteq \mathcal{T}_2$, then $\mathcal{T}_1' \supseteq \mathcal{T}_2'$.* ❑

We can now give a characterization of minimal compatible topologies.

THB29 **2.5.6 Theorem.** *Let $\langle\mathcal{L},[.,.]\rangle$ be an inner product space. Then*

$$\mathrm{Top}_{\min}\langle\mathcal{L},[.,.]\rangle = \big\{\mathcal{T}\in\mathrm{Top}_{\mathrm{sn}}\,\mathcal{L}:\ \mathcal{T}=\mathcal{T}'\big\}\,. \tag{2.5.2}$$ B30

Proof. The inclusion '$\supseteq$' in (2.5.2) is easy to see. Assume that $\mathcal{T}\in\mathrm{Top}_{\mathrm{sn}}\,\mathcal{L}$ and satisfies $\mathcal{T}'=\mathcal{T}$. Let $\mathcal{T}_1\in\mathrm{Top}\,\mathcal{L}$ with $\mathcal{T}_1\subseteq\mathcal{T}$ be given. By Proposition 2.4.2, (i), there exists $\mathcal{T}_2\in\mathrm{Top}_{\mathrm{sn}}\,\mathcal{L}$ with $\mathcal{T}_2\subseteq\mathcal{T}_1$. Since $\mathcal{T}_2\subseteq\mathcal{T}$, it follows from Corollary 2.5.5 that

$$\mathcal{T}_2\supseteq\mathcal{T}_2'\supseteq\mathcal{T}'=\mathcal{T}\,,$$

i.e. $\mathcal{T}_2=\mathcal{T}$. Thus $\mathcal{T}_1=\mathcal{T}$, and we conclude that $\mathcal{T}$ is a minimal element of $\mathrm{Top}\,\mathcal{L}$.

In order to establish the reverse inclusion, we prove the following

Claim: If $\mathcal{T}\in\mathrm{Top}_{\mathrm{sn}}\,\mathcal{L}$, there exists $\mathcal{T}_\infty\in\mathrm{Top}_{\mathrm{sn}}\,\mathcal{L}$ with $\mathcal{T}_\infty\subseteq\mathcal{T}$ and $\mathcal{T}_\infty=\mathcal{T}_\infty'$.

Since, by Proposition 2.4.2, (i), $\mathrm{Top}_{\min}\,\mathcal{L}\subseteq\mathrm{Top}_{\mathrm{sn}}\,\mathcal{L}$, the inequality '$\subseteq$' in (2.5.2) will follow immediately from this claim.

We come to the proof of the above claim. Let p be a seminorm with $\mathcal{T}=\mathcal{T}_p$. We inductively define maps $p_n:\mathcal{L}\to\mathbb{R}$, $n\in\mathbb{N}$, as follows. Choose a constant $\alpha>0$ such that p satisfies (2.2.1) with this constant, and set

$$p_1(x):=\sqrt{\alpha}\,p(x),\quad x\in\mathcal{L}\,.$$

If p_n has already been defined, set

$$p_{n+1}(x):=\sqrt{\frac{1}{2}\big(p_n(x)^2+p_n'(x)^2\big)}=\Big\|\frac{1}{2}\binom{p_n(x)}{p_n'(x)}\Big\|_{\mathbb{R}^2},\quad x\in\mathcal{L}\,.$$

Clearly, each map p_n is a seminorm. Next we verify by induction that

$$|[x,y]|\le p_n(x)p_n(y),\ p_{n+1}(x)\le p_n(x),\quad x,y\in\mathcal{L},n\in\mathbb{N}\,. \tag{2.5.3}$$ B31

Let $n=1$. Then, by the definition of p_1 and (2.2.1), we have

$$|[x,y]|\le\alpha p(x)p(y)=p_1(x)p_1(y),\quad x,y\in\mathcal{L}\,.$$

The estimate (2.5.1) yields that

$$p_1'=\frac{1}{\sqrt{\alpha}}\,p'\le\sqrt{\alpha}p=p_1\,,$$

and we conclude that $p_2\le p_1$. Assume that (2.5.3) has already been proved for some $n\in\mathbb{N}$. From the definition of the polar p_n' we deduce that

$$|[x,y]|\le p_n'(x)p_n(y),\ |[x,y]|\le p_n(x)p_n'(y),\quad x,y\in\mathcal{L}\,. \tag{2.5.4}$$ B32

Thus

$$|[x,y]|\le\frac{1}{2}\big(p_n'(x)p_n(y)+p_n(x)p_n'(y)\big)\le$$

$$\le\frac{1}{2}\big(p_n(x)^2+p_n'(x)^2\big)^{\frac{1}{2}}\cdot\big(p_n(y)^2+p_n'(y)^2\big)^{\frac{1}{2}}=p_{n+1}(x)p_{n+1}(y)\,.$$

Using (2.5.1), we obtain that $p_{n+1}'\le p_{n+1}$ and hence that $p_{n+2}\le p_{n+1}$.

Due to monotonicity the limit

$$p_\infty(x) := \lim_{n\to\infty} p_n(x), \quad x \in \mathcal{L},$$

exists. Clearly, p_∞ is a seminorm. Moreover, we have $p_\infty \leq p_n$, $n \in \mathbb{N}$, and hence in particular $p_\infty \leq \sqrt{\alpha}\, p$. Passing to the limit in (2.5.3) gives

$$|[x,y]| \leq p_\infty(x) p_\infty(y), \quad x,y \in \mathcal{L}.$$

Next we will show that

$$p'_\infty(x) = \lim_{n\to\infty} p'_n(x), \quad x \in \mathcal{L}. \tag{2.5.5}$$ B33

We have $p_\infty \leq p_{n+1} \leq p_n$, and hence $p'_n \leq p'_{n+1} \leq p'_\infty$. Hence the limit on the right hand side of (2.5.5) exists and the inequality '$\geq$' in (2.5.5) holds. Conversely, passing to the limit in (2.5.4) gives

$$|[x,y]| \leq \big(\lim_{n\to\infty} p'_n(x)\big) \cdot p_\infty(y), \quad y \in \mathcal{L},$$

and we conclude that $p'_\infty(x) \leq \lim_{n\to\infty} p'_n(x)$. This establishes (2.5.5).

Finally, passing to the limit in the definition of p_{n+1} and using (2.5.5) gives

$$p_\infty(x) = \sqrt{\frac{1}{2}\big(p_\infty(x)^2 + p'_\infty(x)^2\big)},$$

and hence $p_\infty(x) = p'_\infty(x)$.

Setting $\mathcal{T}_\infty := \mathcal{T}_{p_\infty}$, we have constructed an element of $\mathrm{Top}_{\mathrm{sn}}\,\mathcal{L}$ with the required properties. This finishes the proof of our claim, and hence the proof of the theorem. ❑

COB34 **2.5.7 Corollary.** *Let $\langle \mathcal{L}, [.,.]\rangle$ be an inner product space. Then the following hold:*

(*i*) *For each $\mathcal{T} \in \mathrm{Top}\,\mathcal{L}$, there exists $\mathcal{T}_0 \in \mathrm{Top}_{\min}\,\mathcal{L}$ with $\mathcal{T}_0 \subseteq \mathcal{T}$.*

(*ii*) *We have $\mathrm{Top}_{\mathrm{dec}}\,\mathcal{L} \subseteq \mathrm{Top}_{\min}\,\mathcal{L}$.*

Proof. The claim explicitly stated in the proof of Theorem 2.5.6, together with (2.5.2) and Proposition 2.4.2, (*i*), says that we can find $\mathcal{T}_0$ as required in (*i*).

For the proof of (*ii*) we have, in view of Theorem 2.5.6, to show that for each $\mathcal{T} \in \mathrm{Top}_{\mathrm{dec}}\,\mathcal{L}$ the equality $\mathcal{T} = \mathcal{T}'$ holds. Let a fundamental decomposition $\mathfrak{J} = (\mathcal{L}_+, \mathcal{L}_-)$ be given, and let $\langle \mathcal{H}, (.,.)_\mathcal{H}\rangle$ be the Hilbert space completion of the positive definite inner product space $\langle \mathcal{L}/_{\mathcal{L}^\circ}, (.,.)_\mathfrak{J}/_{\mathcal{L}^\circ}\rangle$. Moreover, denote by ι the canonical map $\iota : \mathcal{L} \to \mathcal{H}$, i.e. projection followed by embedding. Then ι is isometric, and has dense range. We compute

$$p_\mathfrak{J}(x) = (\iota x, \iota x)_\mathcal{H}^{\frac{1}{2}} = \sup_{\substack{z\in\mathcal{H}\\ \|z\|_\mathcal{H}\leq 1}} |(\iota x, z)_\mathcal{H}| = \sup_{\substack{u\in\mathcal{L}\\ p_\mathfrak{J}(u)\leq 1}} |(\iota x, \iota u)_\mathcal{H}| =$$

$$= \sup_{\substack{u\in\mathcal{L}\\ p_\mathfrak{J}(u)\leq 1}} |(x,u)_\mathfrak{J}| = \sup_{\substack{u\in\mathcal{L}\\ p_\mathfrak{J}(u)\leq 1}} |[x, Ju]| = \sup_{\substack{y\in\mathcal{L}\\ p_\mathfrak{J}(y)\leq 1}} |[x,y]| = p'_\mathfrak{J}(x).$$

❑

There exist situations when $\mathrm{Top}\,\mathcal{L}$ has a unique minimal element, cf. the below Theorem 2.5.10. Because of Corollary 2.5.7, (i), this element will then be the smallest element of $\mathrm{Top}\,\mathcal{L}$.

DEB35 **2.5.8 Definition.** Let $\langle\mathcal{L},[.,.]\rangle$ be an inner product space. A positive definite subspace $\mathcal{M}$ of $\mathcal{L}$ is called *intrinsically complete*, if $\langle\mathcal{M},[.,.]|_{\mathcal{M}\times\mathcal{M}}\rangle$ is a Hilbert space. Similarly, a negative definite subspace $\mathcal{M}$ is called intrinsically complete, if $\langle\mathcal{M},-[.,.]\rangle$ is a Hilbert space. //

The following property of an inner product space appears frequently.

DEB55 **2.5.9 Definition.** Let $\langle\mathcal{L},[.,.]\rangle$ be an inner product space. Then $\mathcal{L}$ is called *semicompletely decomposable*, if there exists a fundamental decomposition $\mathfrak{J}=(\mathcal{L}_+,\mathcal{L}_-)$ of $\mathcal{L}$ such that (at least) one of $\mathcal{L}_+$ and $\mathcal{L}_-$ is intrinsically complete. //

THB36 **2.5.10 Theorem.** *Let $\langle\mathcal{L},[.,.]\rangle$ be a semicompletely decomposable inner product space. Then $\mathrm{Top}\,\mathcal{L}$ contains a smallest element. This element is the decomposition topology $\mathcal{T}_{\mathfrak{J}}$ whenever $\mathfrak{J}$ is a fundamental decomposition with (at least) one intrinsically complete component.*

Proof. Let $\mathfrak{J}=(\mathcal{L}_+,\mathcal{L}_-)$ be a fundamental decomposition with (at least) one intrinsically complete component. For definiteness, let us assume that $\mathcal{L}_+$ is intrinsically complete. The case that $\mathcal{L}_-$ satisfies this hypothesis is treated in the same way.

We shall prove that $\mathcal{T}_{\mathfrak{J}}$ is the unique minimal, and thus smallest, element of $\mathrm{Top}\,\mathcal{L}$. To this end let $\mathcal{T}\in\mathrm{Top}_{\min}\mathcal{L}$ be given.

Step 1: According to Theorem 2.5.6, there exists a seminorm p and a constant $\gamma>0$ such that

$$\mathcal{T}=\mathcal{T}_p,\quad p(x)\leq\gamma p'(x),\quad x\in\mathcal{L}\,. \tag{2.5.6}$$ B37

Moreover choose $\alpha>0$ according to (2.2.1), i.e. such that

$$|[x,y]|\leq\alpha p(x)p(y),\quad x,y\in\mathcal{L}\,. \tag{2.5.7}$$ B38

Consider the Hilbert space $\langle\mathcal{L}_+,[.,.]\rangle$. The norm induced by $[.,.]$ on $\mathcal{L}_+$ is nothing else but $p_{\mathfrak{J}}|_{\mathcal{L}_+}$. Hence, for each $y\in\mathcal{L}$, the functional $[.,y]|_{\mathcal{L}_+}$ is belongs to $\langle\mathcal{L}_+,[.,.]\rangle'$. Consider the family of functionals

$$\{[.,y]:\,p(y)\leq 1\}\subseteq\langle\mathcal{L}_+,[.,.]\rangle'\,.$$

By (2.5.7) this family is pointwise bounded and hence, by the Principle of Uniform Boundedness, uniformly bounded. This means that there exists a constant $C>0$ such that

$$\|[.,y]\|\leq C,\quad p(y)\leq 1\,, \tag{2.5.8}$$ B39

where $\|.\|$ denotes the norm in $\langle\mathcal{L}_+,[.,.]\rangle'$. Putting together (2.5.6) and (2.5.8) yields

$$p(x)\leq\gamma p'(x)=\gamma\sup_{p(y)\leq 1}|[x,y]|\leq\gamma C p_{\mathfrak{J}}(x),\quad x\in\mathcal{L}_+\,. \tag{2.5.9}$$ B40

Step 2: Let $x\in\mathcal{L}$. Then, according to (2.5.6) and (2.5.9),

$$p(P_{\mathfrak{J}}^+x)^2\leq(\gamma C)^2p_{\mathfrak{J}}(P_{\mathfrak{J}}^+x)^2=(\gamma C)^2[P_{\mathfrak{J}}^+x,P_{\mathfrak{J}}^+x]=(\gamma C)^2[P_{\mathfrak{J}}^+x,x]\leq$$

$$\leq (\gamma C)^2 \alpha p(P_{\mathfrak{J}}^+ x) p(x) .$$

Hence,

$$p(P_{\mathfrak{J}}^+ x) \leq \alpha(\gamma C)^2 p(x), \quad x \in \mathcal{L} .$$

Since $x + J\mathfrak{J}x - 2P_{\mathfrak{J}}^+ x \in \mathcal{L}^\circ$, it follows that

$$p_{\mathfrak{J}}(x)^2 = [J\mathfrak{J}x, x] = [2P_{\mathfrak{J}}^+ x - x, x] \leq \alpha p(2P_{\mathfrak{J}}^+ x - x) \cdot p(x) \leq$$

$$\leq \alpha(2\alpha(\gamma C)^2 + 1) p(x) \cdot p(x) .$$

We conclude that $\mathcal{T}_{\mathfrak{J}} \subseteq \mathcal{T}_p = \mathcal{T}$, and hence, by minimality of $\mathcal{T}$, that $\mathcal{T}_{\mathfrak{J}} = \mathcal{T}$. ❏

2.6 Uniqueness of decomposition topologies

As we already noticed, decomposition topologies are of particular interest since they arise intrinsically from the inner product. It is thus a most well-behaved situation if there exists a unique decomposition topology. We know that the question of existence of decomposition topologies, i.e. whether or not the space under consideration is decomposable, is a nontrivial matter. The following example shows that also uniqueness is not always present.

EXB41 *2.6.1 Example. We give an example of an inner product space with two different decomposition topologies.*

Consider the linear space

$$\mathcal{L} := \mathbb{C}_f^{\mathbb{Z}} := \big\{ (\xi_j)_{j\in\mathbb{Z}} : \exists N \in \mathbb{N} \text{ s.t. } \xi_j = 0, |j| > N \big\} ,$$

endowed with the inner product

$$\big[(\xi_j)_{j\in\mathbb{Z}}, (\eta_j)_{j\in\mathbb{Z}} \big] := \sum_{j\geq 0} \xi_j \overline{\eta_j} - \sum_{j<0} \xi_j \overline{\eta_j} .$$

Denote $e_n := (\delta_{nj})_{j\in\mathbb{Z}}$, $n \in \mathbb{Z}$, and put

$$\mathcal{L}_+^1 := \operatorname{span}\{e_n : n \geq 0\}, \quad \mathcal{L}_-^1 := \operatorname{span}\{e_n : n < 0\} .$$

Then, clearly, $\mathfrak{J}_1 := (\mathcal{L}_+^1, \mathcal{L}_-^1)$ is a fundamental decomposition of $\mathcal{L}$.

Define elements $f_n \in \mathcal{L}$, $n \in \mathbb{Z}$, as

$$f_n := e_n + \frac{|n|}{|n|+1} e_{-n}, \quad n \in \mathbb{Z} ,$$

and put

$$\mathcal{L}_+^2 := \operatorname{span}\{f_n : n \geq 0\}, \quad \mathcal{L}_-^2 := \operatorname{span}\{f_n : n < 0\} .$$

We have

$$f_n - \frac{|n|}{|n|+1} f_{-n} = \Big(e_n + \frac{|n|}{|n|+1} e_{-n} \Big) - \frac{|n|}{|n|+1} \Big(e_{-n} + \frac{|n|}{|n|+1} e_n \Big) =$$

$$= e_n \Big(1 - \frac{|n|^2}{(|n|+1)^2} \Big) = \frac{2|n|+1}{(|n|+1)^2} e_n, \quad n \in \mathbb{Z} ,$$

and hence $\mathcal{L}_+^2 + \mathcal{L}_-^2 = \mathcal{L}$. Let us compute inner products. By the definition of $[.,.]$, we have

$$[f_n, f_m] = 0, \quad |n| \neq |m|,$$

$$[f_n, f_n] = [e_n, e_n] + \frac{|n|^2}{(|n|+1)^2}[e_{-n}, e_{-n}] = \begin{cases} \frac{2|n|+1}{(|n|+1)^2}, & n \geq 0 \\ -\frac{2|n|+1}{(|n|+1)^2}, & n < 0 \end{cases}$$

$$[f_n, f_{-n}] = \big[e_n + \frac{|n|}{|n|+1}e_{-n}, e_{-n} + \frac{|n|}{|n|+1}e_n\big] =$$

$$= \frac{|n|}{|n|+1}[e_n, e_n] + \frac{|n|}{|n|+1}[e_{-n}, e_{-n}] = 0, \quad n \in \mathbb{N}.$$

It follows that $\mathcal{L}_+^2 \in \operatorname{Sub}_{>0}\mathcal{L}$, $\mathcal{L}_-^2 \in \operatorname{Sub}_{<0}\mathcal{L}$, and that $\mathcal{L}_+^2 \perp \mathcal{L}_-^2$. Thus $\mathfrak{J}_2 := (\mathcal{L}_+^2, \mathcal{L}_-^2)$ is a fundamental decomposition of $\mathcal{L}$.

Consider the sequence $(f_n)_{n\in\mathbb{N}}$. Then we have

$$\|f_n\|_{\mathfrak{J}_1}^2 = \|e_n + \frac{n}{n+1}e_{-n}\|_{\mathfrak{J}_1}^2 = [e_n, e_n] + \big[\frac{n}{n+1}e_{-n}, \frac{n}{n+1}e_{-n}\big] =$$

$$= 1 + \frac{n^2}{(n+1)^2},$$

$$\|f_n\|_{\mathfrak{J}_2}^2 = [f_n, f_n] = \frac{2n+1}{(n+1)^2}.$$

We see that $\|f_n\|_{\mathfrak{J}_2} \to 0$ whereas $\|f_n\|_{\mathfrak{J}_1} \geq 1$.

Consider the sequence $(n^{-\frac{1}{2}}e_n)_{n\in\mathbb{N}}$. Then

$$\|\frac{e_n}{\sqrt{n}}\|_{\mathfrak{J}_1}^2 = [\frac{e_n}{\sqrt{n}}, \frac{e_n}{\sqrt{n}}] = \frac{1}{n},$$

$$\|\frac{e_n}{\sqrt{n}}\|_{\mathfrak{J}_2}^2 = \Big\|\frac{(n+1)^2}{(2n+1)\sqrt{n}}\big(f_n - \frac{n}{n+1}f_{-n}\big)\Big\|_{\mathfrak{J}_2}^2 =$$

$$= \Big(\frac{(n+1)^2}{(2n+1)\sqrt{n}}\Big)^2\Big([f_n, f_n] - \big[\frac{n}{n+1}f_{-n}, \frac{n}{n+1}f_{-n}\big]\Big) =$$

$$= \Big(\frac{(n+1)^2}{(2n+1)\sqrt{n}}\Big)^2\Big(\frac{2n+1}{(n+1)^2} + \frac{n^2}{(n+1)^2}\frac{2n+1}{(n+1)^2}\Big) =$$

$$= \frac{(n+1)^4}{(2n+1)^2 n}\frac{2n+1}{(n+1)^2}\Big(1 + \frac{n^2}{(n+1)^2}\Big)$$

We see that $\|n^{-\frac{1}{2}}e_n\|_{\mathfrak{J}_1} \to 0$ whereas $\|n^{-\frac{1}{2}}e_n\|_{\mathfrak{J}_2} \to 1$. //

The following statement gives two useful conditions under which at most one decomposition topology exists.

THB42

2.6.2 Theorem. *Let $\langle\mathcal{L}, [.,.]\rangle$ be an inner product space, and assume that (at least) one of the following hypothesis holds true:*

(*i*) *$\mathcal{L}$ is semicompletely decomposable.*

(*ii*) *The inner product $[.,.]$ is nondegenerated and $\operatorname{Top}_{\mathrm{Bs}}\mathcal{L} \neq \emptyset$.*

Then there exists at most one decomposition topology on $\mathcal{L}$.

Proof. Under the assumption (i), the present assertion is an immediate consequence of our previous results. Namely, by Corollary 2.5.7, (ii), each decomposition topology is minimal, and by Theorem 2.5.10 there exists only one minimal element.

For the proof under the hypothesis (ii), assume that $\mathcal{L}$ is nondegenerated, let $\mathcal{T} \in \operatorname{Top}_{\mathrm{Bs}} \mathcal{L}$, and choose a norm $\|.\|$ which induces $\mathcal{T}$. Let two fundamental decompositions $\mathfrak{J}_1 = (\mathcal{L}_+^1, \mathcal{L}_-^1)$ and $\mathfrak{J}_2 = (\mathcal{L}_+^2, \mathcal{L}_-^2)$ of $\mathcal{L}$ be given. Since $\mathcal{L}$ is nondegenerated, each of $\mathcal{L}_\pm^j$, $j = 1, 2$, is $\mathcal{T}$-closed. By Proposition 2.4.2, (iv), the fundamental symmetries J_1, J_2 corresponding to $\mathfrak{J}_1$ and $\mathfrak{J}_2$ are $\mathcal{T}$-to-$\mathcal{T}$-continuous, and $\mathcal{T}_{\mathfrak{J}_1}, \mathcal{T}_{\mathfrak{J}_2} \subseteq \mathcal{T}$. Put $T := J_1 J_2$, then $J_1 T = J_2$ and hence

$$(Tx, y)_{\mathfrak{J}_1} = [J_2 x, y] = [x, J_2 y] = (x, Ty)_{\mathfrak{J}_1}, \quad x, y \in \mathcal{L} .$$

For each $n \geq 0$ we have

$$p_{\mathfrak{J}_1}\big(T^{2^n} x\big)^2 = \big(T^{2^n} x, T^{2^n} x\big)_{\mathfrak{J}_1} = \big(T^{2^{n+1}} x, x\big)_{\mathfrak{J}_1} \leq p_{\mathfrak{J}_1}\big(T^{2^{n+1}} x\big) p_{\mathfrak{J}_1}(x) . \quad (2.6.1)$$ B43

We will show by induction that

$$p_{\mathfrak{J}_1}(Tx) \leq p_{\mathfrak{J}_1}\big(T^{2^n} x\big)^{2^{-n}} \cdot p_{\mathfrak{J}_1}(x)^{1-2^{-n}}, \quad n \geq 0 . \quad (2.6.2)$$ B44

If $n = 0$, this is just $p_{\mathfrak{J}_1}(Tx) \leq p_{\mathfrak{J}_1}(Tx) \cdot 1$, and hence trivially true. Assume that (2.6.2) holds for some $n \in \mathbb{N}_0$. Then, using (2.6.1), it follows that

$$\begin{aligned} p_{\mathfrak{J}_1}(Tx) &\leq p_{\mathfrak{J}_1}\big(T^{2^n} x\big)^{2^{-n}} \cdot p_{\mathfrak{J}_1}(x)^{1-2^{-n}} \leq \\ &\leq \big(p_{\mathfrak{J}_1}\big(T^{2^{n+1}} x\big) p_{\mathfrak{J}_1}(x)\big)^{\frac{1}{2} \cdot 2^{-n}} \cdot p_{\mathfrak{J}_1}(x)^{1-2^{-n}} = \\ &= p_{\mathfrak{J}_1}\big(T^{2^{n+1}} x\big)^{2^{-(n+1)}} \cdot p_{\mathfrak{J}_1}(x)^{1-2^{-n}+2^{-(n+1)}} = \\ &= p_{\mathfrak{J}_1}\big(T^{2^{n+1}} x\big)^{2^{-(n+1)}} \cdot p_{\mathfrak{J}_1}(x)^{1-2^{-(n+1)}} . \end{aligned}$$

This finishes the proof of (2.6.2).

Since $\mathcal{T}_{\mathfrak{J}_1} \subseteq \mathcal{T}$, there exists a constant $\gamma > 0$ such that $p_{\mathfrak{J}_1}(x) \leq \gamma \|x\|$, $x \in \mathcal{L}$. Moreover, denote by $\|T\|$ the $\|.\|$-to-$\|.\|$-operator norm of T. Then we obtain, with help of (2.6.2),

$$p_{\mathfrak{J}_1}(Tx) \leq \gamma^{2^{-n}} \|T^{2^n} x\|^{2^{-n}} \cdot p_{\mathfrak{J}_1}(x)^{1-2^{-n}} \leq \gamma^{2^{-n}} \|T\| \cdot \|x\|^{2^{-n}} \cdot p_{\mathfrak{J}_1}(x)^{1-2^{-n}} .$$

Passing to the limit $n \to \infty$ gives $p_{\mathfrak{J}_1}(Tx) \leq \|T\| p_{\mathfrak{J}_1}(x)$. From this we obtain

$$\begin{aligned} p_{\mathfrak{J}_2}(x)^2 &= [J_2 x, x] = [J_1 T x, x] = (Tx, x)_{\mathfrak{J}_1} \leq \\ &\leq p_{\mathfrak{J}_1}(Tx) p_{\mathfrak{J}_1}(x) \leq \|T\| p_{\mathfrak{J}_1}(x)^2, \quad x \in \mathcal{L} , \end{aligned}$$

i.e. $\mathcal{T}_{\mathfrak{J}_2} \subseteq \mathcal{T}_{\mathfrak{J}_1}$. Since decomposition topologies are minimal, cf. Corollary 2.5.7, (ii), this implies that $\mathcal{T}_{\mathfrak{J}_1} = \mathcal{T}_{\mathfrak{J}_2}$. ❑

Let us note explicitly that (ii) does not imply (i), even if we assume the existence of a fundamental decomposition. This comes for the following reason: If, under the hypothesis (ii), $\mathfrak{J} = (\mathcal{L}_+, \mathcal{L}_-)$ is a fundamental decomposition of a nondegenerated space, then $\mathcal{L}_+$ and $\mathcal{L}_-$ are $\mathcal{T}$-closed, i.e. complete with respect to the norm $\|.\|$. But this does not necessarily imply that $\mathcal{L}_+$or $\mathcal{L}_-$ is intrinsically complete.

example ??

COB45 **2.6.3 Corollary.** *Let $\langle\mathcal{L},[.,.]\rangle$ be an inner product space, and assume that* $\operatorname{ind}_-\mathcal{L}<\infty$ *or* $\operatorname{ind}_+\mathcal{L}<\infty$*. Then* $|\operatorname{Top}_{\mathrm{dec}}\mathcal{L}|=1$*. In other words, for each two fundamental decompositions $\mathfrak{J}_1,\mathfrak{J}_2$ of $\mathcal{L}$, there exist constants $\gamma_1,\gamma_2>0$ such that*

$$\gamma_1 p_{\mathfrak{J}_1}(x)\leq p_{\mathfrak{J}_2}(x)\leq \gamma_2 p_{\mathfrak{J}_1}(x),\quad x\in\mathcal{L}\,.$$

Proof. Assume e.g. that $\operatorname{ind}_-\mathcal{L}<\infty$. Then, by Proposition 1.5.2, (*iii*), the space $\mathcal{L}$ is decomposable. The negative definite component in a fundamental decomposition is finite dimensional, and hence intrinsically complete. The assertion follows. ❑

It is a noteworthy fact that, for nondegenerated spaces $\mathcal{L}$, completeness of a component in a fundamental decomposition does not depend on the particular choice of the fundamental decomposition. In particular, if $\mathcal{L}$ is semicompletely decomposable, then either for every fundamental decomposition the positive definite component is intrinsically complete or for every fundamental decomposition the negative definite component is intrinsically complete (or both).

PRB46 **2.6.4 Proposition.** *Let $\langle\mathcal{L},[.,.]\rangle$ be nondegenerated, and let $\mathfrak{J}=(\mathcal{L}_+,\mathcal{L}_-)$ and $\mathfrak{J}'=(\mathcal{L}_+',\mathcal{L}_-')$ be two fundamental decompositions of $\mathcal{L}$. If $\mathcal{L}_+'$ is intrinsically complete, so is $\mathcal{L}_+$. The same holds for $\mathcal{L}_-'$ and $\mathcal{L}_-$.*

The proof of this fact is based on the following observation, which will also be useful later on.

LEB47 **2.6.5 Lemma.** *Let $\mathfrak{J}=(\mathcal{L}_+,\mathcal{L}_-)$ be a fundamental decomposition of the inner product space $\langle\mathcal{L},[.,.]\rangle$, and let $\mathfrak{j}$ be the orthogonal decomposition $\mathfrak{j}:=(\mathcal{L}_+,\mathcal{L}_-+\mathcal{L}^\circ)$. If $\mathcal{M}\in\operatorname{Sub}_{\geq 0}\mathcal{L}\cap\operatorname{Sub}_{\mathfrak{j}}$, then the fundamental projection $P_{\mathfrak{J}}^+$ maps $\mathcal{M}$ $p_{\mathfrak{J}}$-bicontinuously onto $P_{\mathfrak{J}}^+\mathcal{M}$.*

Proof. Since $\mathcal{M}\in\operatorname{Sub}_{\mathfrak{j}}$ and, with the notation of Definition 1.3.1, $P_{\mathfrak{J}}^+=P_{\mathfrak{j}}^1$, the map $P_{\mathfrak{J}}^+$ is a bijection of $\mathcal{M}$ onto $P_{\mathfrak{J}}^+\mathcal{M}$. Clearly, $p_{\mathfrak{J}}(P_{\mathfrak{J}}^+x)\leq p_{\mathfrak{J}}(x)$, $x\in\mathcal{L}$, and hence $P_{\mathfrak{J}}^+$ is $p_{\mathfrak{J}}$-continuous.

To see boundedness of $(P_{\mathfrak{J}}^+|_{\mathcal{M}})^{-1}$, let $x\in\mathcal{M}$ be given. Since $\mathcal{M}\in\operatorname{Sub}_{\geq 0}\mathcal{L}$, we obtain the estimate

$$p_{\mathfrak{J}}(x)^2=[P_{\mathfrak{J}}^+x,P_{\mathfrak{J}}^+x]-[P_{\mathfrak{J}}^-x,P_{\mathfrak{J}}^-x]=2[P_{\mathfrak{J}}^+x,P_{\mathfrak{J}}^+x]-\underbrace{[x,x]}_{\geq 0}\leq 2p_{\mathfrak{J}}(P_{\mathfrak{J}}^+(x))^2\,.$$

❑

Note that always $\operatorname{Sub}_{>0}\mathcal{L}\subseteq\operatorname{Sub}_{\geq 0}\mathcal{L}\cap\operatorname{Sub}_{\mathfrak{j}}$. For nondegenerated spaces $\mathcal{L}$, we even have $\operatorname{Sub}_{\geq 0}\mathcal{L}\subseteq\operatorname{Sub}_{\mathfrak{j}}$. Let us moreover point out the following fact.

REB48 *2.6.6 Remark.* If $\langle\mathcal{L},[.,.]\rangle$ is a positive definite inner product space, and $\mathcal{M}\in\operatorname{Sub}_{>0}\mathcal{L}$ is intrinsically complete, then $\mathcal{M}$ is orthocomplemented. This follows, since the usual proof of existence of orthogonal projections in a Hilbert space uses completeness of the subspace but not completeness of the whole space. //

Proof (of Proposition 2.6.4). Assume that $\mathcal{L}_+'$ is intrinsically complete, i.e. complete with respect to $p_{\mathfrak{J}'}$. First of all, Theorem 2.6.2 implies that $\mathcal{T}_{\mathfrak{J}}=\mathcal{T}_{\mathfrak{J}'}$.

Hence, $\mathcal{L}'_+$ is also $p_{\mathfrak{J}}$-complete. By the above lemma, $P_{\mathfrak{J}}^+\mathcal{L}'_+$ is a $p_{\mathfrak{J}}$-complete subspace of the positive definite inner product space $\mathcal{L}_+$.

If $P_{\mathfrak{J}}^+\mathcal{L}'_+ = \mathcal{L}_+$, we are done. Assume that $P_{\mathfrak{J}}^+\mathcal{L}'_+ \neq \mathcal{L}_+$. Then, by the above remark, there exists an element $x_0 \in \mathcal{L}_+$ with

$$x_0 \perp P_{\mathfrak{J}}^+\mathcal{L}'_+, \quad x_0 \notin P_{\mathfrak{J}}^+\mathcal{L}'_+ .$$

The subspace $\mathcal{M}' := \operatorname{span}(\mathcal{L}'_+ \cup \{x_0\})$ is thus a proper and positive definite extension of $\mathcal{L}'_+$. This implies $\mathcal{M}' \cap \mathcal{L}'_- \neq \{0\}$, and we have reached a contradiction since $\mathcal{M}'$ is positive and $\mathcal{L}'_-$ is negative. ❏

If there exists a unique decomposition topology on an inner product space, this topology is the most natural element of $\operatorname{Top}\mathcal{L}$, and appears in several contexts. Hence, it deserves to be named.

DEB49 **2.6.7 Definition.** Let $\langle \mathcal{L}, [.,.] \rangle$ be an inner product space, and assume that $|\operatorname{Top}_{\mathrm{dec}}\mathcal{L}| = 1$. Then we denote the unique decomposition topology of $\mathcal{L}$ by $\mathcal{T}^{\curlywedge}$. Moreover, we let $\mathcal{L}^{\curlywedge} := \langle \mathcal{L}, \mathcal{T}^{\curlywedge} \rangle'$ be the topological dual space of $\mathcal{L}$ with respect to the topology $\mathcal{T}^{\curlywedge}$. //

REB50 *2.6.8 Remark.* Assume that $|\operatorname{Top}_{\mathrm{dec}}\mathcal{L}| = 1$, so that $\mathcal{T}^{\curlywedge}$ is well-defined.

(*i*) We have $\mathcal{T}^{\curlywedge} \in \operatorname{Top}_{\mathrm{sn}}\mathcal{L}$, and it is a Hausdorff topology if and only if $\mathcal{L}$ is nondegenerated.

(*ii*) In the situation that $\mathcal{L}$ is nondegenerated, so that $\mathcal{T}^{\curlywedge}$ is induced by some norm, we will freely speak of 'completeness with respect to $\mathcal{T}^{\curlywedge}$', 'Cauchy sequence with respect to $\mathcal{T}^{\curlywedge}$', etc. meaning 'complete with respect to some norm inducing $\mathcal{T}^{\curlywedge}$', 'Cauchy sequence with respect to some norm inducing $\mathcal{T}^{\curlywedge}$', etc.

//

2.7 Subspaces, products, factors

The constructs mentioned in the title of this section are defined in a natural way, and give rise to topological inner product spaces.

PRB51 **2.7.1 Proposition.** *Let $\langle \mathcal{L}, [.,.], \mathcal{T} \rangle$ be a topological inner product space, and let $\mathcal{M}$ be a linear subspace of $\mathcal{L}$. Then $\langle \mathcal{M}, [.,.]|_{\mathcal{M}\times\mathcal{M}}, \mathcal{T}|_{\mathcal{M}} \rangle$ is a topological inner product space.*

The inclusion map $\iota : \mathcal{M} \to \mathcal{L}$ is a morphism. Let $\langle \mathcal{N}, [.,.]_{\mathcal{N}}, \mathcal{T}_{\mathcal{N}} \rangle$ be a topological inner product space, and let $\phi : \mathcal{N} \to \mathcal{M}$. Then ϕ is a morphism if and only if $\iota \circ \phi$ is such.

Proof. First of all note that the restriction $\mathcal{T}|_{\mathcal{M}}$ is a locally convex vector topology on $\mathcal{M}$. Clearly, $\mathcal{T}$-continuity of $[.,.]$ implies that $[.,.]|_{\mathcal{M}\times\mathcal{M}}$ is continuous with respect to $\mathcal{T}|_{\mathcal{M}}$. Thus $\langle \mathcal{M}, [.,.]|_{\mathcal{M}\times\mathcal{M}}, \mathcal{T}|_{\mathcal{M}} \rangle$ is a topological inner product space.

The fact that the inclusion map is linear, isometric, and continuous, is also clear. Let $\langle \mathcal{N}, [.,.]_{\mathcal{N}}, \mathcal{T}_{\mathcal{N}} \rangle$ be another topological inner product space, and let $\phi : \mathcal{N} \to \mathcal{M}$. If ϕ is a morphism, then $\iota\circ\phi$ is, as a composition of two morphisms, itself such. Conversely, $\mathcal{T}_{\mathcal{N}}$-to-$\mathcal{T}$-continuity of $\iota\circ\phi$ implies $\mathcal{T}_{\mathcal{N}}$-to-$\mathcal{T}|_{\mathcal{M}}$-continuity of ϕ, since $\mathcal{T}_{\mathcal{M}}$ is the inital topology with respect to $\{\iota\}$. Moreover, isometry of $\iota\circ\phi$ implies

$$[\phi x, \phi y]_{\mathcal{M}\times\mathcal{M}} = [\iota\phi x, \iota\phi y] = [x,y]_{\mathcal{N}}, \quad x,y\in\mathcal{N}\,.$$

Finally, since ι is injective, linearity of $\iota\circ\phi$ implies linearity of ϕ. ❑

PRB52 **2.7.2 Proposition.** *Let $\langle\mathcal{L}_i, [.,.]_i, \mathcal{T}_i\rangle$, $i=1,\dots,n$, be topological inner product spaces, and define*

$$\mathcal{L} := \prod_{i=1}^n \mathcal{L}_i,\ \ [x,y] := \sum_{i=1}^n [\pi_i x, \pi_i y]_i,\ \ \mathcal{T} := \prod_{i=1}^n \mathcal{T}_i\,,$$

where π_i denotes the canonical projection of $\mathcal{L}$ onto $\mathcal{L}_i$. Then $\langle\mathcal{L}, [.,.], \mathcal{T}\rangle$ is a topological inner product space.

Let $\iota_i : \mathcal{L}_i \to \mathcal{L}$, $i=1,\dots,n$, be the canonical embedding

$$\iota_i(x) := (0,\dots,\underset{\substack{\uparrow\\ i\text{-th place}}}{x},\dots,0)\,.$$

Then ι_i is a morphism. Let $\langle\mathcal{N}, [.,.]_{\mathcal{N}}, \mathcal{T}_{\mathcal{N}}\rangle$ be a topological inner product space, and let $\phi:\mathcal{N}\to\mathcal{L}$. Then ϕ is a morphism if and only if ϕ is isometric and $\pi_i\circ\phi$, $i=1,\dots,n$, are all continuous.

Proof. The product topology is a locally convex topology on $\mathcal{L}$. Clearly, $[.,.]$ is, as a sum of continuous functions, itself continuous.

Isometry and continuity of ι_i is immediate. The fact that continuity of $\pi_i\circ\phi$ implies continuity of ϕ is the universal property of initial topologies. ❑

Note that, in the situation of Proposition 2.7.2, the maps $\pi_i\circ\phi$ need not be isometric.

PRB53 **2.7.3 Proposition.** *Let $\langle\mathcal{L}, [.,.], \mathcal{T}\rangle$ be a toplogical inner product space, and let $\mathcal{M}$ be a linear subspace of $\mathcal{L}$ with $\mathcal{M}\subseteq\mathcal{L}^\circ$. Then an inner product $[.,.]_\sim$ on $\mathcal{L}/\mathcal{M}$ is well-defined by*

$$[\pi x, \pi y]_\sim := [x,y], \quad x,y\in\mathcal{L}\,,$$

where π denotes the canonical projection. The triple $\langle\mathcal{L}/\mathcal{M}, [.,.]_\sim, \mathcal{T}/\mathcal{M}\rangle$, where $\mathcal{T}/\mathcal{M}$ denotes the quotient topology, is a topological inner product space.

The canonical projection $\pi:\mathcal{L}\to\mathcal{L}/\mathcal{M}$ is a morphism. Let $\langle\mathcal{N}, [.,.]_{\mathcal{N}}, \mathcal{T}_{\mathcal{N}}\rangle$ be a topological inner product space, and let $\phi:\mathcal{L}/\mathcal{M}\to\mathcal{N}$. Then ϕ is a morphism if and only if $\phi\circ\pi$ is such.

Proof. The quotient topology is a locally convex vector topology on the factor space. The fact that $[.,.]_\sim$ is well-defined, follows since $\mathcal{M}\subseteq\mathcal{L}^\circ$. Since π maps open sets to open sets, $[.,.]_\sim$ is $\mathcal{T}/\mathcal{M}$-continuous. Clearly, π is a morphism and continuity of $\phi\circ\pi$ implies continuity of ϕ. Moreover,

$$[\pi x, \pi y]_\sim = [x,y] = [\phi\pi x, \phi\pi y]_{\mathcal{N}}, \quad x,y\in\mathcal{L}\,.$$

❑

Sometimes it is practical to have available a weak version of the 1st Homomorphism Theorem. This is an immediate consequence of the above statements.

COB54 **2.7.4 Corollary.** *Let $\langle \mathcal{L}_1, [.,.]_1, \mathcal{T}_1 \rangle$ and $\langle \mathcal{L}_2, [.,.]_2, \mathcal{T}_2 \rangle$ be topological inner product spaces, and let $\phi : \mathcal{L}_1 \to \mathcal{L}_2$ be a morphism. Then there exists a unique morphism $\hat{\phi}$ such that*

$$\begin{tikzcd} \langle \mathcal{L}_1, [.,.]_1, \mathcal{T}_1 \rangle \arrow[r, "\phi"] \arrow[d, "\pi"'] & \langle \mathcal{L}_2, [.,.]_2, \mathcal{T}_2 \rangle \\ \langle \mathcal{L}_1 / \ker \phi, [.,.]_{1,\sim}, \mathcal{T} / \ker \phi \rangle \arrow[r, dotted, "\hat{\phi}"'] & \langle \operatorname{ran} \phi, [.,.]_2|_{\operatorname{ran}\phi \times \operatorname{ran}\phi}, \mathcal{T}_2|_{\operatorname{ran}\phi} \rangle \arrow[u, "\iota"'] \end{tikzcd}$$

This map $\hat{\phi}$ is bijective.

Proof. Existence of a linear and bijective map $\hat{\phi}$ with the above diagram is standard. Isometry of $\hat{\phi}$ is clear, and continuity follows from the universal property of initial and final topologies. ❑

Note that, although ϕ is bijective, we do not know in general that ϕ is an isomorphism.

Chapter 3

Classes of complete TIPS. I. Krein spaces

3.1 Definition of Krein spaces

Let $\langle \mathcal{L}, [.,.] \rangle$ be a nondegenerated and decomposable inner product space. By Proposition 2.6.4 the positive subspaces appearing in fundamental decompositions of $\mathcal{L}$ are either all intrinsically complete or all not intrinsically complete. The same holds for the negative components in fundamental decompositions of $\mathcal{L}$.

DEC14 **3.1.1 Definition.** An inner product space $\langle \mathcal{K}, [.,.] \rangle$ is called a *Krein space*, if

(KS1) $\mathcal{K}$ is nondegenerated.

(KS2) There exists a fundamental decomposition $(\mathcal{K}_+, \mathcal{K}_-)$ of $\mathcal{K}$ whose components $\mathcal{K}_+$ and $\mathcal{K}_-$ are both intrinsically complete.

//

Let us start with some immediate reformulations of this definition.

REC15 *3.1.2 Remark.* Let $\langle \mathcal{K}, [.,.] \rangle$ be an inner product space. Then the following are equivalent:

(i) $\langle \mathcal{K}, [.,.] \rangle$ is a Krein space.

(ii) There exists a Hilbert space $\mathcal{H}_1$ and a anti-Hilbert space $\mathcal{H}_2$, such that

$$\langle \mathcal{K}, [.,.] \rangle = \mathcal{H}_1 [\dot{+}] \mathcal{H}_2 \,.$$

$\boxed{[\dot{+}] \text{ to be defined earlier (§1)}}$

(iii) $\mathcal{K}$ is nondegenerated, $|\operatorname{Top}_{\mathrm{dec}} \mathcal{K}| = 1$, and $\mathcal{K}$ is complete with respect to $\mathcal{T}^{\curlywedge}$.

//

REC16 *3.1.3 Remark.* Let $\langle \mathcal{K}, [.,.] \rangle$ be a Krein space.

(*i*) It follows from our previous discussions, more precisely from Proposition 2.4.3, (*ii*), Theorem 2.5.10, and Theorem 2.6.2, that

$$\mathrm{Top}_{\mathrm{Hs}}\,\mathcal{K} = \mathrm{Top}_{\mathrm{Bs}}\,\mathcal{K} = \mathrm{Top}_{\mathrm{dec}}\,\mathcal{K} = \mathrm{Top}_{\min}\,\mathcal{K} = \{\mathcal{T}^{\curlywedge}\}\,.$$

Hence, Krein spaces may be considered in a canonical way as a particular kind of topological inner product spaces. Namely if we additionally endow $\langle \mathcal{K}, [.,.]\rangle$ with the topology $\mathcal{T}^{\curlywedge}$, and we will refer to $\mathcal{T}^{\curlywedge}$ as the *Krein space topology* of $\mathcal{K}$.

Unless the contrary is stated explicitly, a Krein space $\mathcal{K}$ will always be understood as the topological inner product space $\langle \mathcal{K}, [.,.], \mathcal{T}^{\curlywedge}\rangle$.

(*ii*) Let $\mathfrak{J}$ be a fundamental decomposition of $\mathcal{K}$. Then $\langle \mathcal{K}, (.,.)_{\mathfrak{J}}\rangle$ is a Hilbert space. Hence, the map $y \mapsto (.,y)_{\mathfrak{J}}$ is a conjugate linear bijection of $\mathcal{K}$ onto the topological dual space of $\mathcal{K}$. However, the fundamental symmetry J is a linear bijection of $\mathcal{K}$ onto itself, and we have $(.,y)_{\mathfrak{J}} = [.,Jy]$. Hence, also the map $y \mapsto [.,y]$ is a conjugate linear bijection of $\mathcal{K}$ onto its dual.

//

↓↓↓↓↓↓↓↓↓↓↓↓

remove notion of KS-morphism... everywhere. maybe somewhere footnote

On first sight it might seem natural to define a morphism of a Krein space $\langle \mathcal{K}_1, [.,.]\rangle$ to another Krein space $\langle \mathcal{K}_2, [.,.]_2\rangle$ as a linear and isometric map of $\mathcal{K}_1$ into $\mathcal{K}_2$. Interestingly, this notion would be too weak in many respects; it is necessary to include continuity into the definition.

DEC17 **3.1.4 Definition.** Let $\langle \mathcal{K}_1, [.,.]_1\rangle$ and $\langle \mathcal{K}_2, [.,.]_2\rangle$ be topological inner product spaces. Then ϕ is called a (KS–) *morphism of* $\mathcal{K}_1$ *to* $\mathcal{K}_2$, if ϕ is a linear map of $\mathcal{K}_1$ into $\mathcal{K}_2$ which is isometric and $\mathcal{T}_1^{\curlywedge}$-to-$\mathcal{T}_2^{\curlywedge}$-continuous. //

Formulated in an abstract way, one could say that we consider Krein spaces, which are by definition a particular kind of inner product spaces and by Remark 3.1.3, (*i*), a particular kind of topological inner product spaces, rather as subcategory of TIPS than of inner product spaces.

↑↑↑↑↑↑↑↑↑↑↑↑↑↑↑↑

Let us turn to some alternative definitions of Krein spaces. First we proceed via Gram operators. Revisiting the proof of Theorem 2.3.3, we can deduce the next statement.

THC18 **3.1.5 Theorem.** *Let $\langle \mathcal{K}, [.,.]\rangle$ be an inner product space. Then $\langle \mathcal{K}, [.,.]\rangle$ is a Krein space, if and only if there exists an inner product $(.,.)$ on $\mathcal{K}$ which turns $\mathcal{K}$ into a Hilbert space, induces a compatible topology, and has the property that the Gram operator of $[.,.]$ with respect to $(.,.)$ is boundedly invertible as an operator on the Hilbert space $\langle \mathcal{K}, (.,.)\rangle$.*

In this case, for each Hilbert space inner product on $\mathcal{K}$ which induces a compatible topology, the corresponding Gram operator is boundedly invertible.

Proof. If $\langle\mathcal{K},[.,.]\rangle$ is a Krein space, then $\mathcal{K}$ becomes a Hilbert space if endowed with the inner product $(.,.)_{\mathfrak{J}}$, where $\mathfrak{J}$ is any fundamental decomposition of $\mathcal{K}$. Moreover, the Gram-operator of $[.,.]$ with respect to $(.,.)_{\mathfrak{J}}$ is just the fundamental symmetry $J_{\mathfrak{J}}$. The fundamental symmetry, however, is $(.,.)_{\mathfrak{J}}$-unitary, and in particular boundedly invertible.

Conversely, assume that $(.,.)$ is an inner product on $\mathcal{K}$ which turns $\mathcal{K}$ into a Hilbert space and satisfies the stated conditions. Denote the Gram-operator of $[.,.]$ with respect to $(.,.)$ by G. Then G is a bounded and selfadjoint operator in the Hilbert space $\langle\mathcal{K},(.,.)\rangle$, and $0\in\rho(G)$. We employ the same construction as already used in the proof of Theorem 2.3.3. Let E be the spectral measure of G, and put

$$\mathcal{K}_+ := \operatorname{ran} E\big((0,\infty)\big),\ \mathcal{K}_- := \operatorname{ran} E\big((-\infty,0)\big)\,.$$

Then we know that $(\mathcal{K}_+,\mathcal{K}_-)$ is a fundamental decomposition of $\mathcal{K}$, and that $\mathcal{K}_+$ and $\mathcal{K}_-$ are both $(.,.)$-closed.

We have to show that $\mathcal{K}$ is nondegenerated and that $\mathcal{K}_\pm$ are intrinsically complete. This will follow from the fact that $0\in\rho(G)$. First, clearly,

$$\mathcal{K}^{[\circ]} = \ker G = \{0\}\,.$$

Next, choose $c>0$ such that $(-c,c)\subseteq\rho(G)$. Then $E((0,\infty)) = E([c,\|G\|])$, i.e. $\mathcal{K}_+ = \operatorname{ran} E([c,\|G\|])$. Hence, for each $x\in\mathcal{K}_+$,

$$(x,x) = \int_{[c,\|G\|]} 1\,dE_{x,x},\quad [x,x] = (Gx,x) = \int_{[c,\|G\|]} t\,dE_{x,x}\,.$$

It follows that

$$c(x,x)\le[x,x]\le\|G\|(x,x),\ x\in\mathcal{K}_+\,, \tag{3.1.1}$$ C19

i.e. the norms induced by $(.,.)$ and $[.,.]$ are equivalent. Since $\mathcal{K}_+$ is $(.,.)$-closed, it is $(.,.)$-complete, and we conclude that $\mathcal{K}_+$ is also $[.,.]$-complete. The fact that $\mathcal{K}_-$ is intrinsically complete is seen in exactly the same way.

Finally, assume that $\langle\mathcal{K},[.,.]\rangle$ is a Krein space, and let $(.,.)$ be any Hilbert space inner product on $\mathcal{K}$ which induces a compatible topology. Denote the corresponding Gram-operator of $[.,.]$ by G, and fix a fundamental decomposition $\mathfrak{J}$ of $\mathcal{K}$. The norms $\|.\|$ and $\|.\|_{\mathfrak{J}}$ induced by $(.,.)$ and $(.,.)_{\mathfrak{J}}$, respectively, both turn $\mathcal{K}$ into a Hilbert space, and are hence equivalent, cf. Proposition 2.4.3, (*ii*). In particular, there exist ($\|.\|$- or $\|.\|_{\mathfrak{J}}$-) bounded operators G',G'', with

$$(x,y) = (G'x,y)_{\mathfrak{J}},\ (x,y)_{\mathfrak{J}} = (G''x,y),\quad x,y\in\mathcal{K}\,.$$

Since

$$(G'G''x,y)_{\mathfrak{J}} = (G''x,y) = (x,y)_{\mathfrak{J}},\quad (G''G'x,y) = (G'x,y)_{\mathfrak{J}} = (x,y)\,,$$

we have $G'G'' = G''G' = I$. Hence $0\in\rho(G')$, $0\in\rho(G'')$, and $(G')^{-1} = G''$. However,

$$(G'Gx,y)_{\mathfrak{J}} = (Gx,y) = [x,y] = (J_{\mathfrak{J}}x,y)_{\mathfrak{J}}\,,$$

and hence $G'G = J_{\mathfrak{J}}$. This gives $G = G''J_{\mathfrak{J}}$, and it follows that G is ($\|.\|$- or $\|.\|_{\mathfrak{J}}$-) boundedly invertible. ❑

REC20 *3.1.6 Remark.* For later reference, let us explicitly point out the following fact: If $(.,.)$ is a Hilbert space inner product on $\mathcal{K}$ with $\mathcal{T}_{(.,.)} \in \operatorname{Top}\mathcal{K}$, then a fundamental decomposition of $\mathcal{K}$ is given by

$$\mathfrak{J} := \big(\operatorname{ran} E(-\infty, 0), \operatorname{ran} E(0, \infty)\big)\,,$$

where E is the spectral measure of the Gram operator of $[.,.]$ with respect to $(.,.)$. //

EXC21 *3.1.7 Example.* Often Krein spaces occur reading Theorem 3.1.5 backwards, i.e. as follows: Let $\langle \mathcal{H}, (.,.)\rangle$ be a Hilbert space, and let G be a bounded and boundedly invertible operator on $\mathcal{H}$. Define

$$[x,y] := (Gx, y),\ \ x,y \in \mathcal{H}\,.$$

Then $\langle \mathcal{H}, [.,.]\rangle$ is a Krein space. //

REC22 *3.1.8 Remark.* Let $\langle \mathcal{K}_1, [.,.]_1\rangle$ and $\langle \mathcal{K}_2, [.,.]_2\rangle$ be Krein spaces, and consider their direct and orthogonal sum $\mathcal{K} := \mathcal{K}_1[\dot{+}]\mathcal{K}_2$. Then $\langle \mathcal{K}, [.,.]_+\rangle$ is a Krein space.

In order to see this, remember that the inner product $[.,.]_+$ on a direct and orthogonal sum is defined as

$$[x_1 + x_2, y_1 + y_2]_+ = [x_1, y_1]_1 + [x_2, y_2]_2, \quad x_1, y_1 \in \mathcal{K}_1, x_2, y_2 \in \mathcal{K}_2\,,$$

and that $\mathcal{K}_1[\dot{+}]\mathcal{K}_2$ contains $\mathcal{K}_1$ and $\mathcal{K}_2$ as orthogonal subspaces.

Choose fundamental decompositions $\mathfrak{J}_1 = (\mathcal{K}_{1,+}, \mathcal{K}_{1,-})$ and $\mathfrak{J}_2 = (\mathcal{K}_{2,+}, \mathcal{K}_{2,-})$ of $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively. Then the pair

$$\mathfrak{J} := \big(\mathcal{K}_{1,+}[\dot{+}]\mathcal{K}_{2,+}, \mathcal{K}_{1,-}[\dot{+}]\mathcal{K}_{2,-}\big)$$

is a fundamental decomposition of $\mathcal{K}$. Clearly, its components are intrinsically complete. The Krein space topology of $\mathcal{K}_1[\dot{+}]\mathcal{K}_2$ is equal to the product topology of the Krein space topologies of $\mathcal{K}_1$ and $\mathcal{K}_2$. //

3.2 Fundamental decompositions

In a Krein space those subspaces which are components of a fundamental decomposition can be described.

DEC23 **3.2.1 Definition.** Let $\langle \mathcal{K}, [.,.]\rangle$ be a Krein space, let $\mathfrak{J}$ be a fundamental decomposition of $\mathcal{K}$, and let $\mathcal{M} \in \operatorname{Sub}\mathcal{K}$. Then $\mathcal{M}$ is called *uniformly positive*, if there exists a constant $\gamma > 0$ such that

$$[x,x] \geq \gamma \|x\|_{\mathfrak{J}}^2, \quad x \in \mathcal{M}\,.$$

The subspace $\mathcal{M}$ is called *uniformly negative*, if there exists a constant $\gamma > 0$ such that

$$-[x,x] \geq \gamma \|x\|_{\mathfrak{J}}^2, \quad x \in \mathcal{M}\,.$$

The set of all uniformly positive subspaces of $\mathcal{K}$ will be denoted by $\operatorname{Sub}_{\gg 0}\mathcal{K}$, the set of all uniformly negative ones by $\operatorname{Sub}_{\ll 0}\mathcal{K}$. //

Note that, since the norms induced on a Krein space by each two fundamental decompositions are equivalent, the definition of 'uniformly positive' and 'uniformly negative' does not depend on the particular choice of the fundamental decomposition $\mathfrak{J}$ in Definition 3.2.1.

THC24 **3.2.2 Theorem.** *Let $\langle \mathcal{K}, [.,.] \rangle$ be a Krein space, and let $\mathcal{L}_+, \mathcal{L}_- \in \operatorname{Sub} \mathcal{K}$. Then there exists a fundamental decomposition $\mathfrak{J} = (\mathcal{K}_+, \mathcal{K}_-)$ of $\mathcal{K}$ with*

$$\mathcal{L}_+ \subseteq \mathcal{K}_+ \quad \text{and} \quad \mathcal{L}_- \subseteq \mathcal{K}_-$$

if and only if

$$\mathcal{L}_+ \in \operatorname{Sub}_{\gg 0} \mathcal{K}, \ \mathcal{L}_- \in \operatorname{Sub}_{\ll 0} \mathcal{K}, \quad \mathcal{L}_+ \perp \mathcal{L}_- . \tag{3.2.1}$$ C25

Thereby $\mathfrak{J}$ can be chosen such that $\mathcal{L}_+ = \mathcal{K}_+$ if and only if, in addition to (3.2.1), *$\mathcal{L}_+$ is maximal in $\operatorname{Sub}_{\gg 0} \mathcal{K}$. In this case, $\mathcal{L}_+$ is even maximal in $\operatorname{Sub}_{\geq 0} \mathcal{K}$. The analogous statement holds for $\mathcal{L}_-$.*

The proof of this result depends on the following lemmata; the crucial one giving an extension property for operators between Hilbert spaces.

LEC26 **3.2.3 Lemma.** *Let $\langle \mathcal{H}_1, (.,.)_1 \rangle$ and $\langle \mathcal{H}_2, (.,.)_2 \rangle$ be Hilbert spaces, and let*

$$T_1 : \operatorname{dom} T_1 \subseteq \mathcal{H}_1 \to \mathcal{H}_2, \quad T_2 : \operatorname{dom} T_2 \subseteq \mathcal{H}_2 \to \mathcal{H}_1 ,$$

be bounded linear operators. Assume that

$$\big(T_1 x, y\big)_2 = \big(x, T_2 y\big)_1, \quad x \in \operatorname{dom} T_1, \ y \in \operatorname{dom} T_2 . \tag{3.2.2}$$ C27

Then there exist linear operators

$$\tilde{T}_1 : \mathcal{H}_1 \to \mathcal{H}_2, \quad \tilde{T}_2 : \mathcal{H}_2 \to \mathcal{H}_1 ,$$

with

$$\tilde{T}_j|_{\operatorname{dom} T_j} = T_j, \ \|\tilde{T}_j\| \leq \max\{\|T_1\|, \|T_2\|\}, \ j = 1,2, \quad \tilde{T}_1^* = \tilde{T}_2 .$$

Proof.
Step 1: First we extend T_1 and T_2 by continuity to operators

$$\bar{T}_1 : \overline{\operatorname{dom} T_1} \subseteq \mathcal{H}_1 \to \mathcal{H}_2, \quad \bar{T}_2 : \overline{\operatorname{dom} T_2} \subseteq \mathcal{H}_2 \to \mathcal{H}_1 .$$

Then $\|\bar{T}_j\| = \|T_j\|$, $j = 1,2$, and

$$\big(\bar{T}_1 x, y\big)_2 = \big(x, \bar{T}_2 y\big)_1, \quad x \in \overline{\operatorname{dom} T_1}, \ y \in \overline{\operatorname{dom} T_2} ,$$

i.e. the pair of operators $\bar{T}_1$ and $\bar{T}_2$ satisfies the same hypothesis as T_1 and T_2 do. Hence, for the rest of the proof we may assume that $\operatorname{dom} T_1$ and $\operatorname{dom} T_2$ are closed subspaces of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.

Step 2: Denote by P_2 the orthogonal projection of $\mathcal{H}_2$ onto $\operatorname{dom} T_2$, put $\gamma := \max\{\|T_1\|, \|T_2\|\}$, and define

$$[x, y] := \gamma^2 (x, y)_1 - \big((T_2 P_2)^* x, (T_2 P_2)^* y\big)_2, \ x, y \in \mathcal{H}_1 .$$

Thereby, the adjoint $(T_2 P_2)^*$ is understood as the adjoint of the bounded operator $T_2 P_2$ acting between the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. Then $[.,.]$ is an inner product on $\mathcal{H}_1$. Since

$$\|(T_2 P_2)^*\| = \|T_2 P_2\| \leq \|T_2\| \leq \gamma ,$$

we have

$$[x,x] = \gamma^2\|x\|_1^2 - \|(T_2P_2)^*x\|_2^2 \geq (\gamma^2 - \|(T_2P_2)^*\|^2)\|x\|_1 \geq 0,\ x \in \mathcal{H}_1\,,$$

i.e. $[.,.]$ is positive semidefinite. If $x \in \operatorname{dom} T_1$, then we have by our assumption (3.2.2)

$$(P_2T_1x, y)_2 = (T_1x, P_2y)_2 = (x, T_2P_2y)_1,\ y \in \mathcal{H}_2\,.$$

Hence,

$$(T_2P_2)^*x = P_2T_1x,\ x \in \operatorname{dom} T_1\,. \tag{3.2.3}$$ C28

It follows that

$$\begin{aligned}\|(I-P_2)T_1x\|_2^2 = \|T_1x\|_2^2 - \|P_2T_1x\|_2^2 = \|T_1x\|_2^2 - \|(T_2P_2)^*x\|_2^2 \leq \\ \leq \gamma^2\|x\|_1^2 - \|(T_2P_2)^*x\|_2^2 = [x,x],\ x \in \operatorname{dom} T_1\,.\end{aligned} \tag{3.2.4}$$ C29

Let $\hat{\mathcal{H}}_1$ be the Hilbert space completion of the positive definite inner product space $\langle \mathcal{H}_1/\mathcal{H}_1^{[\circ]}, [.,.]\rangle$, and denote by $\iota : \mathcal{H}_1 \to \hat{\mathcal{H}}_1$ the canonical map, i.e. projection followed by embedding. Since $\operatorname{ran}(I-P_2)T_1 \subseteq \operatorname{ran}(I-P_2)$ and, by (3.2.4), $\operatorname{dom} T_1 \cap \ker\iota = \operatorname{dom} T_1 \cap \mathcal{H}_1^{[\circ]} \subseteq \ker(I-P_2)T_1$, there exists a linear operator $V_0 : \iota(\operatorname{dom} T_1) \subseteq \hat{\mathcal{H}}_1 \to \operatorname{ran}(I-P_2)$ with $V_0 \circ \iota = (I-P_2)T_1$. Once more by (3.2.4), we have $\|V_0\| \leq 1$. Let $V_1 : \hat{\mathcal{H}}_1 \to \operatorname{ran}(I-P_2)$ be an extension of V_0 with $\|V_1\| = \|V_0\|$. For example, V_1 can be taken as $\bar{V}_0P$, where $\bar{V}_0$ is the extension by continuity of V_0 to $\overline{\iota(\operatorname{dom} T_1)}$ and P is the orthogonal projection of $\hat{\mathcal{H}}_1$ onto $\overline{\iota(\operatorname{dom} T_1)}$. We are in the situation

$$\begin{tikzcd}
\operatorname{dom} T_1 \arrow[rr, "(I-P_2)T_1"] \arrow[d, "\iota"'] & & \operatorname{ran}(I-P_2) \subseteq \mathcal{H}_2 \\
\iota(\operatorname{dom} T_1) \arrow[urr, "V_0"] \quad \subseteq & \hat{\mathcal{H}}_1 \arrow[ur, "V_1"'] &
\end{tikzcd} \tag{3.2.5}$$ C30

Define

$$W := V_1 \circ \iota : \mathcal{H}_1 \to \operatorname{ran}(I-P_2) \subseteq \mathcal{H}_2\,.$$

Clearly, then

$$\|Wx\|_2^2 = \|V_1(\iota x)\|_2^2 \leq \|\iota x\|_{\hat{\mathcal{H}}_1}^2 = [x,x],\ x \in \mathcal{H}_1\,.$$

Define

$$\tilde{T}_1 := (T_2P_2)^* + W : \mathcal{H}_1 \to \mathcal{H}_2\,.$$

Then, for each $x \in \operatorname{dom} T_1$, we obtain from (3.2.3) and (3.2.5) that

$$\tilde{T}_1x = (T_2P_2)^*x + Wx = P_2T_1x + (I-P_2)T_1x = T_1x\,,$$

i.e. $\tilde{T}_1|_{\operatorname{dom} T_1} = T_1$. Moreover, since $\operatorname{ran}(T_2P_2)^* = (\ker T_2P_2)^\perp \subseteq (\ker P_2)^\perp = \operatorname{ran} P_2$, we have

$$\|\tilde{T}_1x\|_2^2 = \|(T_2P_2)^*x + Wx\|_2^2 = \|(T_2P_2)^*x\|_2^2 + \|Wx\|_2^2 \leq$$

$$\leq \|(T_2P_2)^*x\|_2^2 + [x,x] = \gamma^2\|x\|_1,\ x \in \mathcal{H}_1\,,$$

i.e. $\|\tilde{T}_1\| \leq \gamma$. Finally, for $x \in \mathcal{H}_1$ and $y \in \operatorname{dom} T_2 = \operatorname{ran} P_2$, we have

$$(\tilde{T}_1 x, y)_2 = \big((T_2P_2)^* x + Wx, y\big)_2 = \big((T_2P_2)^* x, y\big)_2 =$$

$$= (x, T_2P_2 y)_1 = (x, T_2 y)_1 \,.$$

Altogether, we see that the pair of operators $\tilde{T}_1$ and T_2 satisfies the same hypothesis as T_1 and T_2 do, and that $\max\{\|\tilde{T}_1\|, \|T_2\|\} \leq \gamma$.

Step 3: Applying what we showed in Step 2 with the pair of operators T_2 and $\tilde{T}_1$ in place of T_1 and T_2, gives an operator $\tilde{T}_2 : \mathcal{H}_2 \to \mathcal{H}_1$ with $\tilde{T}_2|_{\operatorname{dom} T_2} = T_2$, $\|\tilde{T}_2\| \leq \gamma$, and

$$(\tilde{T}_2 y, x)_1 = (y, \tilde{T}_1)_2, \;\; y \in \mathcal{H}_2, x \in \mathcal{H}_1 \,,$$

i.e. $\tilde{T}_2 = \tilde{T}_1^*$. ❑

REC31 *3.2.4 Remark.* It is worth to have a little closer look at the particular case that $\operatorname{dom} T_2 = \{0\}$ in Lemma 3.2.3. The hypothesis (3.2.2), as well as the conclusion that $\tilde{T}_2$ extends T_2, becomes in this case of course void.

(*i*) We have $T_2P_2 = 0$, and hence $[.,.] = \|T_1\|^2 (.,.)_1$. Hence, $\hat{\mathcal{H}}_1 = \mathcal{H}_1$, and

$$\tilde{T}_1 = \bar{T}_1 P \tag{3.2.6}$$ C32

where $\bar{T}_1$ is the extension by continuity of T_1 to $\overline{\operatorname{dom} T_1}$, and P is the orthogonal projection of $\mathcal{H}_1$ onto $\overline{\operatorname{dom} T_1}$. Clearly, we have $\|\tilde{T}_1\| = \|T_1\|$.

Of course, this particular case could have been treated much simpler by using (3.2.6) as the definition of $\tilde{T}_1$ and setting $\tilde{T}_2 := \tilde{T}_1^*$.

(*ii*) Assume that $\operatorname{dom} T_1$ is closed, and that T_1 satisfies

$$\|T_1 x\|_2 < \alpha \cdot \|x\|_1, \; x \in \operatorname{dom} T_1 \setminus \{0\} \,,$$

with some $\alpha > 0$. Since $\tilde{T}_1$ is given by (3.2.6), we obtain

$$\|\tilde{T}_1 x\|_2 = \|T_1(Px)\|_2 < \alpha \cdot \|Px\|_1 \leq \alpha \cdot \|x\|_1, \; x \notin \ker P \,,$$

and

$$\|\tilde{T}_1 x\|_2 = 0 < \alpha \cdot \|x\|_1, \; x \in \ker P \setminus \{0\} \,.$$

Together, it follows that $\|\tilde{T}_1 x\| < \alpha \cdot \|x\|_1$, $x \in \mathcal{H}_1 \setminus \{0\}$.

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Next we translate uniform definiteness and orthogonal complements into the language of angular operators.

LEC33 **3.2.5 Lemma.** *Let $\langle \mathcal{K}, [.,.] \rangle$ be a Krein space, and let $\mathcal{M} \in \operatorname{Sub} \mathcal{K}$. Then the following are equivalent:*

(*i*) $\mathcal{M} \in \operatorname{Sub}_{\gg 0} \mathcal{K}$.

(*ii*) *For each fundamental decomposition $\mathfrak{J} = (\mathcal{K}_+, \mathcal{K}_-)$ of $\mathcal{K}$, we have $\mathcal{M} \in \operatorname{Sub}_{\mathfrak{J}}$ and $\|\mathfrak{a}_{\mathfrak{J}}(\mathcal{M})\| < 1$. Here $\|.\|$ denotes the operator norm between the Hilbert spaces $\langle \mathcal{K}_+, [.,.] \rangle$ and $\langle \mathcal{K}_-, -[.,.] \rangle$.*

(*iii*) *There exists a fundamental decomposition* $\mathfrak{J}$ *of* $\mathcal{K}$*, such that* $\mathcal{M} \in \mathrm{Sub}_{\mathfrak{J}}$ *and* $\|\mathfrak{a}_{\mathfrak{J}}(\mathcal{M})\| < 1$.

With the obvious modifications, the analogous statement holds for 'uniformly negative' instead of 'uniformly positive'.

Proof. We first show that (*i*) implies (*ii*). To this end let a fundamental decomposition $\mathfrak{J} = (\mathcal{K}_+, \mathcal{K}_-)$ of $\mathcal{K}$ be given. Since $\mathcal{M}$ is in particular nonnegative, we have $\mathcal{M} \in \mathrm{Sub}_{\mathfrak{J}}$. Let $\gamma > 0$ be such that $[x,x] \geq \gamma \|x\|_{\mathfrak{J}}$, $x \in \mathcal{M}$. This inequality, however, is nothing else but

$$\|P_{\mathfrak{J}}^+ x\|_{\mathfrak{J}}^2 - \|P_{\mathfrak{J}}^- x\|_{\mathfrak{J}}^2 \geq \gamma\big(\|P_{\mathfrak{J}}^+ x\|_{\mathfrak{J}}^2 + \|P_{\mathfrak{J}}^- x\|_{\mathfrak{J}}^2\big),\ x \in \mathcal{M}\,,$$

and hence we have

$$\|P_{\mathfrak{J}}^- x\|_{\mathfrak{J}}^2 \leq \frac{1-\gamma}{1+\gamma}\|P_{\mathfrak{J}}^+ x\|_{\mathfrak{J}}^2,\ x \in \mathcal{M}\,,$$

i.e.

$$\|\mathfrak{a}_{\mathfrak{J}}(\mathcal{M})y\|_{\mathfrak{J}}^2 \leq \frac{1-\gamma}{1+\gamma}\|y\|_{\mathfrak{J}}^2,\ y \in P_{\mathfrak{J}}^+\mathcal{M}\,.$$

Thus

$$\|\mathfrak{a}_{\mathfrak{J}}(\mathcal{M})\| \leq \frac{1-\gamma}{1+\gamma} < 1\,.$$

The implication $(ii) \Rightarrow (iii)$ is trivial.

Assume that $\mathfrak{J}$ is a fundamental decomposition of $\mathcal{K}$, that $\mathcal{M} \in \mathrm{Sub}_{\mathfrak{J}}$, and that $\|\mathfrak{a}_{\mathfrak{J}}(\mathcal{M})\| < 1$. Then, for each $x \in P_{\mathfrak{J}}^+\mathcal{M}$, we have

$$\big[x + \mathfrak{a}_{\mathfrak{J}}(\mathcal{M})x, x + \mathfrak{a}_{\mathfrak{J}}(\mathcal{M})x\big] = [x,x] + \big[\mathfrak{a}_{\mathfrak{J}}(\mathcal{M})x, \mathfrak{a}_{\mathfrak{J}}(\mathcal{M})x\big] =$$

$$= \|x\|_{\mathfrak{J}}^2 - \|\mathfrak{a}_{\mathfrak{J}}(\mathcal{M})x\|_{\mathfrak{J}}^2 \geq \|x\|_{\mathfrak{J}}^2 \underbrace{\big(1 - \|\mathfrak{a}_{\mathfrak{J}}(\mathcal{M})\|^2\big)}_{>0} \geq$$

$$\geq \frac{1 - \|\mathfrak{a}_{\mathfrak{J}}(\mathcal{M})\|^2}{1 + \|\mathfrak{a}_{\mathfrak{J}}(\mathcal{M})\|^2} \cdot \|x + \mathfrak{a}_{\mathfrak{J}}(\mathcal{M})x\|_{\mathfrak{J}}^2\,.$$

Hence $\mathcal{M}$ is uniformly positive.

The case of uniformly negative subspaces is treated in the same way. ❑

LEC34 **3.2.6 Lemma.** *Let* $\mathfrak{J} = (\mathcal{K}_+, \mathcal{K}_-)$ *be a fundamental decomposition of the Krein space* $\langle \mathcal{K}, [.,.]\rangle$ *Moreover, let* K *be a bounded operator of the Hilbert space* $\langle \mathcal{K}_+, [.,.]\rangle$ *into the Hilbert space* $\langle \mathcal{K}_-, -[.,.]\rangle$*, and put*

$$\mathcal{M} := \big\{x + Kx : \, x \in \mathcal{K}_+\big\}\,,$$

so that $\mathcal{M} \in \mathrm{Sub}_{\mathfrak{J}}$ *and* $\mathfrak{a}_{\mathfrak{J}}(\mathcal{M}) = K$*. Then* $\mathcal{M}^\perp \in \mathrm{Sub}_{\bar{\mathfrak{J}}}$*, where* $\bar{\mathfrak{J}}$ *is the orthogonal decomposition* $\bar{\mathfrak{J}} := (\mathcal{K}_-, \mathcal{K}_+)$ *of* $\mathcal{K}$*, and*

$$\mathfrak{a}_{\bar{\mathfrak{J}}}(\mathcal{M}^\perp) = K^*\,,$$

where K^* *denotes the Hilbert space adjoint of* K*.*

Proof. Assume that $z \in \mathcal{M}^\perp \cap \mathcal{K}_+$. Then

$$0 = [z, x + Kx] = [z, x],\ x \in \mathcal{K}_+ ,$$

and hence $z = 0$. Thus $\mathcal{M}^\perp \cap \mathcal{K}_+ = \{0\}$, i.e. $\mathcal{M}^\perp \in \mathrm{Sub}_{\bar{\mathfrak{J}}}$.

If $y \in \mathcal{K}_-$, then for each $x \in \mathcal{K}_+$ we have

$$[x + Kx, y + K^*y] = [x, K^*y] + [Kx, y] = [x, K^*y] - (-[Kx, y]) = 0 .$$

This shows that

$$\{y + K^*y : y \in \mathcal{K}_-\} \subseteq \mathcal{M}^\perp . \tag{3.2.7}$$ $\boxed{\text{C35}}$

Since K^* is defined on all of $\mathcal{K}_-$, the space on the left side is maximal in $\mathrm{Sub}_{\bar{\mathfrak{J}}}$, cf. Corollary 1.4.3. It follows that in (3.2.7) already equality holds, and thus also $\mathfrak{a}_{\bar{\mathfrak{J}}}(\mathcal{M}^\perp) = K^*$. ❑

Proof (of Theorem 3.2.2).
Step 1: Assume first that $\mathfrak{J} = (\mathcal{K}_+, \mathcal{K}_-)$ is a fundamental decomposition with $\mathcal{L}_+ \subseteq \mathcal{K}_+$ and $\mathcal{L}_- \subseteq \mathcal{K}_-$. Then, clearly, $\mathcal{L}_+ \perp \mathcal{L}_-$ and

$$[x, x] = (x, x)_{\mathfrak{J}},\ x \in \mathcal{L}_+, \quad -[x, x] = (x, x)_{\mathfrak{J}},\ x \in \mathcal{L}_- .$$

Thus $\mathcal{L}_+ \in \mathrm{Sub}_{\gg 0}\,\mathcal{K}$ and $\mathcal{L}_- \in \mathrm{Sub}_{\ll 0}\,\mathcal{K}$.

Assume that even $\mathcal{L}_+ = \mathcal{K}_+$, and let $\mathcal{M} \in \mathrm{Sub}\,K$ with $\mathcal{M} \supsetneq \mathcal{L}_+$. Choose $x \in \mathcal{M} \setminus \mathcal{L}_+$, then

$$y := x - P_{\mathfrak{J}}^+ x \in \big(\mathcal{M} \cap \mathcal{K}_-\big) \setminus \{0\} .$$

Thus $[y, y] < 0$, and we conclude that $\mathcal{M} \notin \mathrm{Sub}_{\geq 0}\,\mathcal{K}$. Therefore, $\mathcal{L}_+$ is maximal in $\mathrm{Sub}_{\geq 0}\,\mathcal{K}$ and hence in particular maximal in $\mathrm{Sub}_{\gg 0}\,\mathcal{K}$. The case of $\mathcal{L}_-$ is treated in the same way.

Step 2: Assume that $\mathcal{L}_+$ and $\mathcal{L}_-$ satisfiy (3.2.1). Let $\mathfrak{J} := (\mathcal{K}_+, \mathcal{K}_-)$ be a fundamental decomposition of $\mathcal{K}$, and consider the operators

$$\mathfrak{a}_{\mathfrak{J}}(\mathcal{L}_+) : \mathcal{K}_+ \to \mathcal{K}_-, \quad \mathfrak{a}_{\bar{\mathfrak{J}}}(\mathcal{L}_-) : \mathcal{K}_- \to \mathcal{K}_+ ,$$

where $\bar{\mathfrak{J}}$ is the orthogonal decomposition $\bar{\mathfrak{J}} := (\mathcal{K}_-, \mathcal{K}_+)$ of $\mathcal{K}$. Note that, since $\mathcal{L}_+$ is positive and $\mathcal{L}_-$ is negative, we have $\mathcal{L}_+ \in \mathrm{Sub}_{\mathfrak{J}}$ and $\mathcal{L}_- \in \mathrm{Sub}_{\bar{\mathfrak{J}}}$. Since $\mathcal{L}_+$ and $\mathcal{L}_-$ are in fact uniformly definite, Lemma 3.2.5 gives $\|\mathfrak{a}_{\mathfrak{J}}(\mathcal{L}_+)\|, \|\mathfrak{a}_{\bar{\mathfrak{J}}}(\mathcal{L}_-)\| < 1$. By Lemma 1.3.9, the fact that $\mathcal{L}_+ \perp \mathcal{L}_-$ implies that

$$-[\mathfrak{a}_{\mathfrak{J}}(\mathcal{L}_+)x, y] = [x, \mathfrak{a}_{\bar{\mathfrak{J}}}(\mathcal{L}_-)y],\ x \in P_{\mathfrak{J}}^+\mathcal{L}_+, y \in P_{\mathfrak{J}}^-\mathcal{L}_- .$$

Lemma 3.2.3, applied with the Hilbert spaces $\langle \mathcal{K}_+, [.,.] \rangle$, $\langle \mathcal{K}_-, -[.,.] \rangle$, and the pair of operators

$$\mathfrak{a}_{\mathfrak{J}}(\mathcal{L}_+) : P_{\mathfrak{J}}^+\mathcal{L}_+ \subseteq \mathcal{K}_+ \to \mathcal{K}_-, \quad \mathfrak{a}_{\bar{\mathfrak{J}}}(\mathcal{L}_-) : P_{\mathfrak{J}}^-(\mathcal{L}_-) \subseteq \mathcal{K}_- \to \mathcal{K}_+ ,$$

furnishes us with operators

$$\tilde{T}_+ : \mathcal{K}_+ \to \mathcal{K}_-, \quad \tilde{T}_- : \mathcal{K}_- \to \mathcal{K}_+ ,$$

with

$$\tilde{T}_+|_{P_{\mathfrak{J}}^+\mathcal{L}_+} = \mathfrak{a}_{\mathfrak{J}}(\mathcal{L}_+), \quad \tilde{T}_-|_{P_{\mathfrak{J}}^-(\mathcal{L}_-)} = \mathfrak{a}_{\bar{\mathfrak{J}}}(\mathcal{L}_-) ,$$

$$\|\tilde{T}_\pm\| \leq \max\big\{\mathfrak{a}_{\mathfrak{J}}(\mathcal{L}_+), \mathfrak{a}_{\tilde{\mathfrak{J}}}(\mathcal{L}_-)\big\} < 1\,, \tag{3.2.8}$$ C36

and $\tilde{T}_- = \tilde{T}_+^*$. Let $\mathcal{K}'_+ \in \operatorname{Sub}_{\mathfrak{J}}$ and $\mathcal{K}'_- \in \operatorname{Sub}_{\tilde{\mathfrak{J}}}$ be those subspaces with

$$\mathfrak{a}_{\mathfrak{J}}(\mathcal{K}'_+) = \tilde{T}_+, \quad \mathfrak{a}_{\tilde{\mathfrak{J}}}(\mathcal{K}'_-) = \tilde{T}_-\,.$$

Then, by Lemma 3.2.6, we have $\mathcal{K}_+ = \mathcal{K}_-^\perp$. By (3.2.8), we have $\mathcal{K}'_+ \in \operatorname{Sub}_{\gg 0}\mathcal{K}$ and $\mathcal{K}'_- \in \operatorname{Sub}_{\ll 0}\mathcal{K}$. Moreover, clearly, $\mathcal{K}'_+ \supseteq \mathcal{L}_+$ and $\mathcal{K}'_- \supseteq \mathcal{L}_-$.

Step 3: The next task is to show that $\mathcal{K}'_+ + \mathcal{K}'_- = \mathcal{K}$. Let $z \in \mathcal{K}$ be given, and define elements $x \in \mathcal{K}_+$ and $y \in \mathcal{K}_-$ as

$$x := (I_{\mathcal{K}_+} - \tilde{T}_-\tilde{T}_+)^{-1}(P_{\mathfrak{J}}^+ - \tilde{T}_-P_{\mathfrak{J}}^-)z,\ y := (I_{\mathcal{K}_-} - \tilde{T}_+\tilde{T}_-)^{-1}(P_{\mathfrak{J}}^- - \tilde{T}_+P_{\mathfrak{J}}^+)z\,.$$

Note here that $\|\tilde{T}_-\tilde{T}_+\|, \|\tilde{T}_+\tilde{T}_-\| < 1$. Moreover, set

$$x' := x + \tilde{T}_+x \in \mathcal{K}'_+,\ y' := y + \tilde{T}_-y \in \mathcal{K}'_-, \quad z' := x' + y' \in \mathcal{K}'_+ + \mathcal{K}'_-\,.$$

Then we have

$$P_{\mathfrak{J}}^+ z' = x + \tilde{T}_-y,\ P_{\mathfrak{J}}^- z' = y + \tilde{T}_+x\,,$$

and hence

$$(P_{\mathfrak{J}}^- - \tilde{T}_+P_{\mathfrak{J}}^+)z' = (y + \tilde{T}_+x) - \tilde{T}_+(x + \tilde{T}_-y) = (I_{\mathcal{K}_+} - \tilde{T}_+\tilde{T}_-)y = (P_{\mathfrak{J}}^- - \tilde{T}_+P_{\mathfrak{J}}^+)z\,,$$

$$(P_{\mathfrak{J}}^+ - \tilde{T}_-P_{\mathfrak{J}}^-)z' = (x + \tilde{T}_-y) - \tilde{T}_-(y + \tilde{T}_+x) = (I_{\mathcal{K}_-} - \tilde{T}_-\tilde{T}_+)x = (P_{\mathfrak{J}}^+ - \tilde{T}_-P_{\mathfrak{J}}^-)z\,.$$

Thus

$$z - z' \in \ker(P_{\mathfrak{J}}^- - \tilde{T}_+P_{\mathfrak{J}}^+) \cap \ker(P_{\mathfrak{J}}^+ - \tilde{T}_-P_{\mathfrak{J}}^-)\,.$$

However, let $w \in \ker(P_{\mathfrak{J}}^- - \tilde{T}_+P_{\mathfrak{J}}^+) \cap \ker(P_{\mathfrak{J}}^+ - \tilde{T}_-P_{\mathfrak{J}}^-)$, then

$$(I_{\mathcal{K}_-} - \tilde{T}_-\tilde{T}_+)P_{\mathfrak{J}}^+ w = P_{\mathfrak{J}}^+ w - \tilde{T}_-\tilde{T}_+P_{\mathfrak{J}}^+ w = P_{\mathfrak{J}}^+ w - \tilde{T}_-P_{\mathfrak{J}}^- w = 0\,,$$

$$(I_{\mathcal{K}_+} - \tilde{T}_+\tilde{T}_-)P_{\mathfrak{J}}^- w = P_{\mathfrak{J}}^- w - \tilde{T}_+\tilde{T}_-P_{\mathfrak{J}}^- w = P_{\mathfrak{J}}^- w - \tilde{T}_+P_{\mathfrak{J}}^+ w = 0\,.$$

It follows that $P_{\mathfrak{J}}^- w = P_{\mathfrak{J}}^+ w = 0$, and hence that $w = 0$.

We conclude that $z = z'$, i.e. $z \in \mathcal{K}'_+ + \mathcal{K}'_-$. Altogether, we have constructed a fundamental decomposition with the required properties, namely $\mathfrak{J}' := (\mathcal{K}'_+, \mathcal{K}'_-)$.

Step 4: Assume that, besides (3.2.1), $\mathcal{L}_+$ is maximal in $\operatorname{Sub}_{\gg 0}\mathcal{K}$. By what we have proved in Step 2, there exists a fundamental decomposition $\mathfrak{J} = (\mathcal{K}_+, \mathcal{K}_-)$ with $\mathcal{K}_+ \supseteq \mathcal{L}_+$. As we saw in Step 1, $\mathcal{K}_+ \in \operatorname{Sub}_{\gg 0}\mathcal{K}$, and maximality of $\mathcal{L}_+$ implies that $\mathcal{K}_+ = \mathcal{L} - +$. The case of $\mathcal{L}_-$ is treated in the same way. ❑

Let us explicitly state the following immediate corollary of Theorem 3.2.2.

COC37 **3.2.7 Corollary.** *Let $\langle\mathcal{K}, [.,.]\rangle$ be a Krein space, and let $\mathcal{M} \in \operatorname{Sub}\mathcal{K}$. Then the following are equivalent:*

- *(i) $\mathcal{M}$ is a maximal element of $\operatorname{Sub}_{\gg 0}\mathcal{K}$.*
- *(ii) There exists a fundamental decomposition $\mathfrak{J} = (\mathcal{K}_+, \mathcal{K}_-)$ of $\mathcal{K}$ with $\mathcal{K}_+ = \mathcal{M}$.*

(iii) $(\mathcal{M}, \mathcal{M}^\perp)$ *is a fundamental decomposition of* $\mathcal{K}$.

The analogous set of equivalences holds for maximal uniformly negative subspaces.

Proof. Assume that $\mathcal{M}$ is maximal uniformly positive. Applying Theorem 3.2.2 with the pair of subspaces $\mathcal{M}$ and $\{0\}$, gives a fundamental decomposition $\mathfrak{J} = (\mathcal{K}_+, \mathcal{K}_-)$ with $\mathcal{K}_+ = \mathcal{M}$. Clearly, then we have $\mathcal{K}_- = \mathcal{K}_+^\perp = \mathcal{M}^\perp$. Hence *(i)* implies *(ii)* and *(iii)*. Conversely, if $\mathcal{M}$ is the component of some fundamental decomposition, then Theorem 3.2.2 implies that $\mathcal{M} \in \operatorname{Sub}_{\gg 0} \mathcal{K}$. ❑

Let us further exploit the method which led to Theorem 3.2.2 in order to obtain some information on maximal semidefinite subspaces of a Krein space. Note that, if $\mathfrak{J} = (\mathcal{K}_+, \mathcal{K}_-)$ is a fundamental decomposition of a Krein space $\mathcal{K}$, then

$$\operatorname{Sub}_{\gg 0} \mathcal{K} \subseteq \operatorname{Sub}_{>0} \mathcal{K} \subseteq \operatorname{Sub}_{\geq 0} \mathcal{K} \subseteq \operatorname{Sub}_{\mathfrak{J}} .$$

PRC38 **3.2.8 Proposition.** *Let* $\langle \mathcal{K}, [.,.] \rangle$ *be a Krein space, let* $\mathfrak{J} = (\mathcal{K}_+, \mathcal{K}_-)$ *be a fundamental decomposition of* $\mathcal{K}$, *and let* $\mathcal{M} \in \operatorname{Sub} \mathcal{K}$. *Assume that* $\mathcal{M}$ *has one of the following properties:*

(i) $\mathcal{M}$ *is a maximal element of* $\operatorname{Sub}_{\geq 0} \mathcal{K}$.

(ii) $\mathcal{M}$ *is a maximal element of* $\operatorname{Sub}_{\gg 0} \mathcal{K}$.

(iii) $\mathcal{M}$ *is closed and a maximal element of* $\operatorname{Sub}_{>0} \mathcal{K}$.

Then $\mathcal{M}$ *is already a maximal element of* $\operatorname{Sub}_{\mathfrak{J}}$.

Proof. Let $\mathcal{M} \in \operatorname{Sub}_{\geq 0} \mathcal{K}$ be given, and assume that $\mathcal{M}$ is not maximal in $\operatorname{Sub}_{\mathfrak{J}}$. By virtue of Corollary 1.4.3, this just means that $P_{\mathfrak{J}}^+ \mathcal{M} \subsetneq \mathcal{K}_+$. Consider the angular operator

$$\mathfrak{a}_{\mathfrak{J}}(\mathcal{M}) : P_{\mathfrak{J}}^+ \mathcal{M} \subsetneq \mathcal{K}_+ \to \mathcal{K}_- .$$

Then $\mathfrak{a}_{\mathfrak{J}}(\mathcal{M})$ is bounded, actually $\|\mathfrak{a}_{\mathfrak{J}}(\mathcal{M})\| \leq 1$. Let $\tilde{T} : \mathcal{K}_+ \to \mathcal{K}_-$ be the extension of $\mathfrak{a}_{\mathfrak{J}}(\mathcal{M})$ to all of $\mathcal{K}_+$ discussed in Remark 3.2.4. Then

$$\|\tilde{T}\| = \|\mathfrak{a}_{\mathfrak{J}}(\mathcal{M})\| \leq 1 . \tag{3.2.9}$$ C39

Let $\tilde{\mathcal{M}} \in \operatorname{Sub}_{\mathfrak{J}}$ be the subspace with $\mathfrak{a}_{\mathfrak{J}}(\tilde{\mathcal{M}}) = \tilde{T}$. Then $\tilde{\mathcal{M}}$ is maximal in $\operatorname{Sub}_{\mathfrak{J}}$, and $\tilde{\mathcal{M}} \supseteq \mathcal{M}$.

Assume that $\mathcal{M}$ is maximal in $\operatorname{Sub}_{\geq 0} \mathcal{K}$. By (3.2.9) we have $\tilde{\mathcal{M}} \in \operatorname{Sub}_{\geq 0} \mathcal{K}$, and it follows that $\mathcal{M} = \tilde{\mathcal{M}}$. Assume that $\mathcal{M}$ is maximal in $\operatorname{Sub}_{\gg 0} \mathcal{K}$. Then in (3.2.9) actually '< 1' holds, and we conclude that $\tilde{\mathcal{M}} \in \operatorname{Sub}_{\gg 0} \mathcal{K}$. It follows again that $\mathcal{M} = \tilde{\mathcal{M}}$.

Finally, assume that $\mathcal{M}$ is closed and maximal in $\operatorname{Sub}_{>0} \mathcal{K}$. Closedness implies that $\mathcal{M}$ is complete in the norm $\|.\|_{\mathfrak{J}}$. By Lemma 2.6.5, also $P_{\mathfrak{J}}^+ \mathcal{M}$ is complete in the norm $\|.\|_{\mathfrak{J}}$, and hence closed in $\mathcal{K}_+$. Moreover, since $\mathcal{M} \in \operatorname{Sub}_{>0} \mathcal{K}$, we have

$$\|\mathfrak{a}_{\mathfrak{J}}(\mathcal{M}) x\|_{\mathfrak{J}} < \|x\|_{\mathfrak{J}}, \ x \in P_{\mathfrak{J}}^+(\mathcal{M}) \setminus \{0\} .$$

Remark 3.2.4, *(ii)*, implies that also

$$\|\tilde{T} x\|_{\mathfrak{J}} < \|x\|_{\mathfrak{J}}, \ x \in \mathcal{K}_+ \setminus \{0\} ,$$

and hence that $\tilde{\mathcal{M}} \in \operatorname{Sub}_{>0} \mathcal{K}$. Maximality of $\mathcal{M}$ in $\operatorname{Sub}_{<0} \mathcal{K}$ yields $\mathcal{M} = \tilde{\mathcal{M}}$. ❑

Note here that the obstacle mentioned in the discussion preceeding Lemma 1.4.5 vanishes due to completeness, which enters the discussion in the form of Remark 3.2.4.

COC40 **3.2.9 Corollary.** *Let $\langle\mathcal{K},[.,.]\rangle$ be a Krein space, and let $\mathcal{M}_1,\mathcal{M}_2\in\operatorname{Sub}\mathcal{K}$. Assume that $\mathcal{M}_1$ has one of the properties (i), (ii), or (iii), stated in Proposition 3.2.8, and that $\mathcal{M}_2$ also has one of the properties (i), (ii), or (iii), stated in Proposition 3.2.8. Then there exists a linear, bijective, and bicontinuous map of $\mathcal{M}_1$ onto $\mathcal{M}_2$.*

Proof. Choose a fundamental decomposition $\mathfrak{J}=(\mathcal{K}_+,\mathcal{K}_-)$ of $\mathcal{K}$. By Corollary 1.4.3, $P_{\mathfrak{J}}^+$ maps maximal elements of $\operatorname{Sub}_{\mathfrak{J}}$ bijectively onto $\mathcal{K}_+$. In particular, if $\mathcal{M}$ satisfies (i), (ii), or (iii) of Proposition 3.2.8, this will be the case. By Lemma 2.6.5, $P_{\mathfrak{J}}^+$ is bicontinuous. ❑

REC41 *3.2.10 Remark.* The statements analogous to Proposition 3.2.8 and Corollary 3.2.9 for negative semidefinite subspaces hold by the same proofs. Thereby, we should replace '≥ 0, >0, $\gg 0$' by '≤ 0, <0, $\ll 0$', and the fundamental decomposition $\mathfrak{J}$ by the orthogonal decomposition $\tilde{\mathfrak{J}}:=(\mathcal{K}_-,\mathcal{K}_+)$. For example, then we have

$$\operatorname{Sub}_{\ll 0}\mathcal{K}\subseteq\operatorname{Sub}_{<0}\mathcal{K}\subseteq\operatorname{Sub}_{\leq 0}\mathcal{K}\subseteq\operatorname{Sub}_{\tilde{\mathfrak{J}}}\ .$$

//

COC42 **3.2.11 Corollary.** *Let $\langle\mathcal{K},[.,.]\rangle$ be a Krein space, and let $\mathfrak{J}=(\mathcal{K}_+,\mathcal{K}_-)$ be a fundamental decomposition of $\mathcal{K}$. Then*

$$\operatorname{ind}_+\mathcal{K}:=\dim\mathcal{K}_+,\quad \operatorname{ind}_-\mathcal{K}:=\dim\mathcal{K}_-\ .$$

Proof. Let $\mathcal{M}\in\operatorname{Sub}_{>0}\mathcal{K}$. Then there exists a maximal element $\tilde{\mathcal{M}}$ of $\operatorname{Sub}_{\geq 0}\mathcal{K}$ with $\tilde{\mathcal{M}}\supseteq\mathcal{M}$. By Corollary 3.2.9, $\dim\tilde{\mathcal{M}}=\dim\mathcal{K}_+$. Hence, $\operatorname{ind}_-\mathcal{K}\leq\dim\mathcal{K}_+$. However, $\mathcal{K}_+$ itself is a positive subspace of $\mathcal{K}$, and thus the converse inequality is trivial. The equality $\operatorname{ind}_-\mathcal{K}=\dim\mathcal{K}_-$ is seen in the same way. ❑

The next statement gives an improvement of Proposition 1.4.11.

COC43 **3.2.12 Corollary.** *Let $\langle\mathcal{K},[.,.]\rangle$ be a Krein space. If $\mathcal{M}$ is maximal in $\operatorname{Sub}_{\geq 0}\mathcal{K}$, then $\mathcal{M}^\perp$ is maximal in $\operatorname{Sub}_{\leq 0}\mathcal{K}$.*

Proof. The angular operator $\mathfrak{a}_{\mathfrak{J}}(\mathcal{M})$ is bounded and defined on all of $\mathcal{K}_+$. Lemma 3.2.6 gives $\mathcal{M}^\perp\in\operatorname{Sub}_{\tilde{\mathfrak{J}}}$ and $\mathfrak{a}_{\tilde{\mathfrak{J}}}(\mathcal{M}^\perp)=\mathfrak{a}_{\mathfrak{J}}(\mathcal{M})^*$. In particular, the domain of this angular operator is all of $\mathcal{K}_-$, and hence $\mathcal{M}^\perp$ is maximal in $\operatorname{Sub}_{\tilde{\mathfrak{J}}}$. Moreover, as we already know from Proposition 1.4.11, $\mathcal{M}^\perp\in\operatorname{Sub}_{\leq 0}\mathcal{K}$. Thus, $\mathcal{M}^\perp$ is maximal in $\operatorname{Sub}_{\leq 0}\mathcal{K}$. ❑

3.3 Orthocomplemented subspaces

Let us start with some corollaries of Theorem 2.3.3.

PRC44 **3.3.1 Proposition.** *Let $\langle\mathcal{K},[.,.]\rangle$ be a Krein space, and let $\mathcal{L}$ be a closed subspace of $\mathcal{K}$. Then the following hold:*

(i) $\mathcal{L}$ possesses a fundamental decomposition $\mathfrak{J}_{\mathcal{L}}=(\mathcal{L}_+,\mathcal{L}_-)$ such that each of $\mathcal{L}_\pm$, $\mathcal{L}_\pm+\mathcal{L}^\circ$, and $\mathcal{L}_++\mathcal{L}_-$ are closed in $\mathcal{K}$.

(*ii*) *We have* $\mathcal{L}^{\perp\perp} = \mathcal{L}$ *and* $(\mathcal{L}^{\perp})^{\circ} = \mathcal{L}^{\circ}$.

Proof. Let $\mathfrak{J} = (\mathcal{K}_+, \mathcal{K}_-)$ be a fundamental decomposition of $\mathcal{K}$, then $\langle \mathcal{K}, (.,.)_{\mathfrak{J}} \rangle$ is a Hilbert space. Since $\mathcal{L}$ is closed in $\mathcal{K}$, also $\langle \mathcal{L}, (.,.)_{\mathfrak{J}} \rangle$ is a Hilbert space. By Theorem 2.3.3 there exists a fundamental decomposition $\mathfrak{J}_{\mathcal{L}}$ such that each of the subspaces listed in (i) is closed in $\langle \mathcal{L}, (.,.)_{\mathfrak{J}} \rangle$ and thus also in $\mathcal{K}$.

We come to the proof of (ii). We have

$$x[\perp]\mathcal{L} \iff \forall y \in \mathcal{L} : [x, y] = 0 \iff \forall y \in \mathcal{L} : (x, J_{\mathfrak{J}} y)_{\mathfrak{J}} = 0\,,$$

i.e. $\mathcal{L}^{[\perp]} = (J_{\mathfrak{J}}\mathcal{L})^{(\perp)_{\mathfrak{J}}} = J_{\mathfrak{J}} \cdot \mathcal{L}^{(\perp)_{\mathfrak{J}}}$. It follows that

$$\mathcal{L}^{[\perp][\perp]} = \big(J_{\mathfrak{J}}\mathcal{L}^{(\perp)_{\mathfrak{J}}}\big)^{[\perp]} = \big(J_{\mathfrak{J}}J_{\mathfrak{J}}\mathcal{L}^{(\perp)_{\mathfrak{J}}}\big)^{(\perp)_{\mathfrak{J}}} = \mathcal{L}^{(\perp)_{\mathfrak{J}}(\perp)_{\mathfrak{J}}} = \mathcal{L}\,. \tag{3.3.1}$$ C45

This also implies that

$$(\mathcal{L}^{\perp})^{\circ} = \mathcal{L}^{\perp} \cap \mathcal{L}^{\perp\perp} = \mathcal{L}^{\perp} \cap \mathcal{L} = \mathcal{L}^{\circ}\,.$$

❑

COC46 **3.3.2 Corollary.** *Let* $\langle \mathcal{K}, [.,.] \rangle$ *be a Krein space, and let* $\mathcal{L} \in \operatorname{Sub} \mathcal{K}$.

(*i*) *We have* $\overline{\mathcal{L}} = \mathcal{L}^{\perp\perp}$.

(*ii*) $\mathcal{L}$ *is dense in* $\mathcal{K}$ *if and only if* $\mathcal{L}^{\perp} = \{0\}$.

Proof. We can do the computation (3.3.1) stopping before the last equality sign. However, since $\langle \mathcal{K}, (.,.)_{\mathfrak{J}} \rangle$ is a Hilbert space, $\mathcal{L}^{(\perp)_{\mathfrak{J}}(\perp)_{\mathfrak{J}}} = \overline{\mathcal{L}}$.

The space $J_{\mathfrak{J}}\mathcal{L}$ is dense in $\mathcal{K}$ if and only if $(J_{\mathfrak{J}}\mathcal{L})^{(\perp)_{\mathfrak{J}}} = \{0\}$, i.e. if and only if $\mathcal{L}^{[\perp]} = \{0\}$. Since $J_{\mathfrak{J}}$ is a homeomorphism, $\mathcal{L}$ is dense if and only if $J_{\mathfrak{J}}\mathcal{L}$ is. ❑

Let $\mathcal{L}$ be a closed and nondegenerated subspace of a Krein space. Although, by Proposition 3.3.1, there exists a fundamental decomposition $(\mathcal{L}_+, \mathcal{L}_-)$ of $\mathcal{L}$ whose components are closed (and hence complete) in the norm of $\mathcal{K}$, this does not mean that $\mathcal{L}_{\pm}$ are intrinsically complete, cf. Example 3.3.5.

THC47 **3.3.3 Theorem.** *Let* $\langle \mathcal{K}, [.,.] \rangle$ *be a Krein space, and let* $\mathcal{L} \in \operatorname{Sub} \mathcal{K}$. *Then the following are equivalent:*

(*i*) $\mathcal{L}$ *is orthocomplemented.*

(*ii*) $\mathcal{L}$ *is closed in* $\mathcal{K}$, *nondegenerated, and for each fundamental decomposition* $\mathfrak{J}_{\mathcal{L}} = (\mathcal{L}_+, \mathcal{L}_-)$ *of* $\mathcal{L}$ *there exists a fundamental decomposition* $\mathfrak{J}_{\mathcal{K}} = (\mathcal{K}_+, \mathcal{K}_-)$ *of* $\mathcal{K}$ *with*

$$\mathcal{L}_+ \subseteq \mathcal{K}_+ \quad \textit{and} \quad \mathcal{L}_- \subseteq \mathcal{K}_-\,. \tag{3.3.2}$$ C48

(*ii'*) $\mathcal{L}$ *is closed in* $\mathcal{K}$, *nondegenerated, and there exist fundamental decompositions* $\mathfrak{J}_{\mathcal{L}} = (\mathcal{L}_+, \mathcal{L}_-)$ *and* $\mathfrak{J}_{\mathcal{K}} = (\mathcal{K}_+, \mathcal{K}_-)$ *of* $\mathcal{L}$ *and* $\mathcal{K}$, *respectively, such that* (3.3.2) *holds.*

(*iii*) $\mathcal{L}$ *is decomposable, nondegenerated, and for each fundamental decomposition* $\mathfrak{J}_{\mathcal{L}} = (\mathcal{L}_+, \mathcal{L}_-)$ *of* $\mathcal{L}$ *we have*

$$\mathcal{L}_+ \in \operatorname{Sub}_{\gg 0} \mathcal{K},\ \mathcal{L}_- \in \operatorname{Sub}_{\ll 0} \mathcal{K}\,, \tag{3.3.3}$$ C49

and $\mathcal{L}_+$ *and* $\mathcal{L}_-$ *are closed in* $\mathcal{K}$.

(*iii'*) *We have* $\mathcal{L} = \mathcal{L}_+[\dot{+}]\mathcal{L}_-$ *with some subspaces* $\mathcal{L}_\pm$ *satisfying* (3.3.3) *and being closed in* $\mathcal{K}$.

(*iv*) $\mathcal{L}$ *is closed in* $\mathcal{K}$ *and* $\langle\mathcal{L}, [.,.]\rangle$ *is a Krein space.*

Proof. The proof will proceed as follows:

$$\begin{array}{ccccccccc} (i) & \Rightarrow & (ii) & \Rightarrow & (ii') & & & & \\ & & \Downarrow & & \Downarrow & & & & \\ & & (iii) & \Rightarrow & (iii') & \Rightarrow & (iv) & \Rightarrow & (i) \end{array}$$

Thereby, the implication $(iii) \Rightarrow (iii')$ is trivial.

$(i) \Rightarrow (ii)$*:* Since $\mathcal{L}$ is orthocomplemented, we have $\mathcal{L}^\circ \subseteq \mathcal{K}^\circ = \{0\}$ and $\mathcal{L}^{\perp\perp} = \mathcal{L}$, cf. Lemma 1.2.7. The first relation says that $\mathcal{L}$ is nondegenerated, the second one implies that $\mathcal{L}$ is closed. Since with $\mathcal{L}$ also $\mathcal{L}^\perp$ is orthocomplemented, also the space $\mathcal{L}^\perp$ is nondegenerated and closed. By Proposition 3.3.1, $\mathcal{L}$ and $\mathcal{L}^\perp$ are decomposable.

Let $\mathfrak{J} = (\mathcal{L}_+, \mathcal{L}_-)$ and $\mathfrak{J}' = (\mathcal{L}'_+, \mathcal{L}'_-)$ be fundamental decompositions of $\mathcal{L}$ and $\mathcal{L}^\perp$, respectively, and put

$$\mathcal{K}_+ := \mathcal{L}_+ + \mathcal{L}'_+, \quad \mathcal{K}_- := \mathcal{L}_- + \mathcal{L}'_- \,.$$

Clearly, then $\mathcal{K}_+ \perp \mathcal{K}_-$ and $\mathcal{K}_+ + \mathcal{K}_- = \mathcal{L} + \mathcal{L}^\perp = \mathcal{K}$. Since $\mathcal{L}_+, \mathcal{L}'_+ \in \operatorname{Sub}_{>0}\mathcal{K}$ and $\mathcal{L}_+ \perp \mathcal{L}'_+$, we also have $\mathcal{K}_+ \in \operatorname{Sub}_{>0}\mathcal{K}$. Similarly, $\mathcal{K}_- \in \operatorname{Sub}_{<0}\mathcal{K}$, and we see that $\mathfrak{J}_\mathcal{K} := (\mathcal{K}_+, \mathcal{K}_-)$ is a fundamental decomposition of $\mathcal{K}$. Obviously, (3.3.2) holds.

$(ii) \Rightarrow (ii'), (iii)$*:* $\mathcal{L}$ being closed, implies that it is decomposable. Thus, under the hypothesis (ii), the assertion (ii') follows immediately. For (iii), note that each subspace of a component of some fundamental decomposition is uniformly definite. Moreover, we can write $\mathcal{L}_+ = \mathcal{L} \cap \mathcal{L}_-^\perp$ and $\mathcal{L}_- = \mathcal{L} \cap \mathcal{L}_+^\perp$, where the orthogonal complement is understood in $\mathcal{K}$. Hence $\mathcal{L}_+$ and $\mathcal{L}_-$ are closed in $\mathcal{K}$.

This argument also shows that the implication $(ii') \Rightarrow (iii')$ holds.

$(iv) \Rightarrow (i)$*:* Let $\mathfrak{J}$ be a fundamental decomposition of $\mathcal{K}$, and let G denote the Gram operator of $[.,.]$ with respect to $(.,.) := (.,.)_\mathfrak{J}$. Moreover, let P denote the $(.,.)$-orthogonal projection of $\mathcal{K}$ onto $\mathcal{L}$. Then, for $x, y \in \mathcal{L}$, we have

$$[x, y] = (Gx, y) = (Gx, Py) = (PGx, y)\,. \tag{3.3.4}$$ C50

The inner product $(.,.)|_{\mathcal{L}\times\mathcal{L}}$ turns $\mathcal{L}$ into a Hilbert space, and the inner product $[.,.]|_{\mathcal{L}\times\mathcal{L}}$ is continuous with respect to it. By (3.3.4) the Gram operator of $[.,.]|_{\mathcal{L}\times\mathcal{L}}$ with respect to $(.,.)|_{\mathcal{L}\times\mathcal{L}}$ is $PG|_\mathcal{L}$. Theorem 3.1.5 implies that $PG|_\mathcal{L}$ is boundedly invertible as a operator on $\langle\mathcal{L}, (.,.)|_{\mathcal{L}\times\mathcal{L}}\rangle$. In particular, $\operatorname{ran}(PG|_\mathcal{L}) = \mathcal{L}$.

To show that $\mathcal{K} = \mathcal{L} + \mathcal{L}^\perp$, let $x \in \mathcal{K}$ be given. Since $PGx \in \mathcal{L}$, there exists an element $x_0 \in \mathcal{L}$ such that $PGx_0 = PGx$. It follows that, for each $y \in \mathcal{L}$,

$$[x - x_0, y] = (G(x - x_0), y) = (G(x - x_0), Py) = (PG(x - x_0), y) = 0\,.$$

Thus $x - x_0 \in \mathcal{L}^\perp$, and we have shown that $x = x_0 + (x - x_0) \in \mathcal{L} + \mathcal{L}^\perp$.

(iii′) ⇒ (iv): Let $\mathcal{L}_+$ and $\mathcal{L}_-$ be as in *(iii′)*, and let $\mathfrak{J}$ denote a fundamental decomposition of $\mathcal{K}$. Since $\mathcal{L}_+$ is uniformly positive, the norms

$$\|x\| := [x,x]^{\frac{1}{2}},\ x \in \mathcal{L},$$

and $\|.\|_{\mathfrak{J}}|_{\mathcal{L}}$ are equivalent. Since $\mathcal{L}_+$ is $\|.\|_{\mathfrak{J}}$-closed, it is $\|.\|_{\mathfrak{J}}$-complete. By equivalence of norms, $\mathcal{L}_+$ is also complete with respect to $\|.\|$, i.e. intrinsically complete. The same argument shows that $\mathcal{L}_-$ is intrinsically complete, and therefore $\mathcal{L} = \mathcal{L}_+[\dot{+}]\mathcal{L}_-$ is a Krein space.

The subspace $\mathcal{L}_+$ is closed in $\mathcal{K}$ and, by what we saw above, $\langle \mathcal{L}_+, [.,.]\rangle$ is a Hilbert space. Thus, by the already proved implication $(iv) \Rightarrow (i)$, $\mathcal{L}_+$ is orthocomplemented. Similarly, we see that $\mathcal{L}_-$ is orthocomplemented. It follows that $\mathcal{L} = \mathcal{L}_+[\dot{+}]\mathcal{L}_-$ is also orthocomplemented, cf. Corollary 1.2.6. In particular, $\mathcal{L}$ is closed. ❑

Let us explicitly mention the following fact.

REC13 *3.3.4 Remark.* Let $\langle \mathcal{K}, [.,.]\rangle$ be a Krein space, and let $\mathcal{L}$ be an orthocomplemented subspace of $\mathcal{K}$. Then the topology $\mathcal{L}$ carries as the Krein space $\langle \mathcal{L}, [.,.]|_{\mathcal{L}\times\mathcal{L}}\rangle$ coincides with the restriction to $\mathcal{L}$ of the topology of $\mathcal{K}$. This follows for example from Proposition 2.4.3, *(ii)*. //

In the following example we elaborate the equivalences in Theorem 3.3.3. This discussion points out many pecularities of Krein spaces.

EXC51 *3.3.5 Example.* Let $\mathcal{K} := \ell^2(\mathbb{N})$, and define an inner product $[.,.]$ on $\mathcal{K}$ by

$$\big[(\xi_j)_{j\in\mathbb{N}}, (\eta_j)_{j\in\mathbb{N}}\big] := \sum_{j=1}^{\infty} (-1)^j \xi_j \overline{\eta_j}\,.$$

Moreover, let $\mathcal{L}$ be the subspace

$$\mathcal{L} := \Big\{(\xi_j)_{j\in\mathbb{N}} \in \mathcal{K} :\ \xi_{2k} = \frac{2k}{2k-1}\xi_{2k-1}, k \in \mathbb{N}\Big\}\,.$$

$\langle \mathcal{K}, [.,.]\rangle$ is a Krein space (via definition): Put

$$\mathcal{K}_+ := \big\{(\xi_j)_{j\in\mathbb{N}} :\ \xi_k = 0, k \text{ odd}\big\},\ \mathcal{K}_- := \big\{(\xi_j)_{j\in\mathbb{N}} :\ \xi_k = 0, k \text{ even}\big\}\,.$$

Then, clearly, $\mathcal{K}_+$ is positive, $\mathcal{K}_-$ is negative, and $\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-$. Since $\langle \mathcal{K}_+, [.,.]\rangle \cong \langle \mathcal{K}_-, -[.,.]\rangle \cong \ell^2(\mathbb{N})$, we have found a fundamental decomposition of $\mathcal{K}$ with intrinsically complete components.

$\langle \mathcal{K}, [.,.]\rangle$ is a Krein space (via Gram operator): Denote by $(.,.)$ the usual $\ell^2(\mathbb{N})$-inner product and by $\|.\|$ the corresponding norm. Then $\langle \mathcal{K}, (.,.)\rangle$ is a Hilbert space. Since

$$\big|\big[(\xi_j)_{j\in\mathbb{N}}, (\eta_j)_{j\in\mathbb{N}}\big]\big| \leq \sum_{j=1}^{\infty} |\xi_j| \cdot |\eta_j| \leq \|(\xi_j)_{j\in\mathbb{N}}\| \cdot \|(\eta_j)_{j\in\mathbb{N}}\|\,,$$

the inner product $[.,.]$ is continuous with respect to $(.,.)$. The Gram operator G of $[.,.]$ with respect to $(.,.)$ acts as

$$G(\xi_j)_{j\in\mathbb{N}} = ((-1)^j \xi_j)_{j\in\mathbb{N}},\ (\xi_j)_{j\in\mathbb{N}} \in \mathcal{K}\,.$$

We see that G is $(.,.)$-unitary, and hence in particular boundedly invertible. Actually, $G^2 = I$.

$\mathcal{L}$ is closed and positive: Let $(\xi_j)_{j\in\mathbb{N}} \in \mathcal{L}$, then

$$\big[(\xi_j)_{j\in\mathbb{N}}, (\xi_j)_{j\in\mathbb{N}}\big] = \sum_{j=1}^{\infty}(-1)^j|\xi_j|^2 = \sum_{k=1}^{\infty}\Big[\Big(\frac{2k}{2k-1}\Big)^2 - 1\Big]|\xi_{2k-1}|^2 .$$

Hence $\mathcal{L}$ is positive. Let $\pi_k : (\xi_j)_{j\in\mathbb{N}} \to \xi_k$ denote the canonical projection onto the k-th component. Then π_k is continuous with respect to $\|.\|$. Put $h_k := \pi_{2k} - \frac{2k}{2k-1}\pi_{2k-1}$, then

$$\mathcal{L} = \bigcap_{k\in\mathbb{N}} \ker h_k .$$

Hence $\mathcal{L}$ is closed.

$\mathcal{L}$ is not uniformly positive: Consider the elements $(\xi_j^n)_{j\in\mathbb{N}}$, $n \in \mathbb{N}$, defined by

$$\xi_j^n := \begin{cases} 1 & , \quad j = 2n-1 \\ \frac{2n}{2n-1} & , \quad j = 2n \\ 0 & , \quad \text{otherwise} \end{cases}$$

Then

$$\big[(\xi_j^n)_{j\in\mathbb{N}}, (\xi_j^n)_{j\in\mathbb{N}}\big] = -1 + \Big(\frac{2n}{2n-1}\Big)^2 = \frac{4n-1}{(2n-1)^2} ,$$

$$\big((\xi_j^n)_{j\in\mathbb{N}}, (\xi_j^n)_{j\in\mathbb{N}}\big) = 1 + \Big(\frac{2n}{2n-1}\Big)^2 .$$

Hence there cannot exist a positive constant $\gamma > 0$ with $\gamma\|x\|^2 \leq [x,x]$, $x \in \mathcal{L}$.

$\mathcal{L}$ is not intrinsically complete: Consider the sequences $(\xi_j^n)_{j\in\mathbb{N}}$, $n \in \mathbb{N}$, defined by

$$\xi_j^n := \begin{cases} \frac{1}{\sqrt{j}} & , \quad j \leq 2n,\ j \text{ odd} \\ \frac{j}{\sqrt{(j-1)^3}} & , \quad j \leq 2n,\ j \text{ even} \\ 0 & , \quad \text{otherwise} \end{cases}$$

Then

$$\xi_{2k}^n = \frac{2k}{\sqrt{(2k-1)^3}} = \frac{2k}{2k-1}\frac{1}{\sqrt{2k_1}} = \frac{2k}{2k-1}\xi_{2k-1}^n,\ k \leq n ,$$

$$\xi_{2k}^n = 0 = \frac{2k}{2k-1}\xi_{2k-1}^n,\ k > n ,$$

and hence $(\xi_j^n)_{j\in\mathbb{N}} \in \mathcal{L}$. For $m > n$ we have

$$\big[(\xi_j^n)_{j\in\mathbb{N}} - (\xi_j^m)_{j\in\mathbb{N}}, (\xi_j^n)_{j\in\mathbb{N}} - (\xi_j^m)_{j\in\mathbb{N}}\big] = \sum_{k=1}^{\infty}\Big[\Big(\frac{2k}{2k-1}\Big)^2 - 1\Big]\big|\xi_{2k-1}^n - \xi_{2k-1}^m\big|^2 =$$

$$= \sum_{k=n+1}^{m}\Big[\Big(\frac{2k}{2k-1}\Big)^2 - 1\Big]\frac{1}{2k-1} \leq \sum_{k=n+1}^{m}\frac{4k-1}{(2k-1)^2}\frac{1}{2k-1} .$$

It follows that $((\xi_j^n)_{j\in\mathbb{N}})_{n\in\mathbb{N}}$ is a Cauchy sequence in $\langle\mathcal{L},[.,.]\rangle$.

Assume that $\mathcal{L}$ is intrinsically complete. Since $\mathcal{L}$ is closed with respect to $(.,.)$, this implies that the norms induced by $[.,.]$ and $(.,.)$ on $\mathcal{L}$ are equivalent. Hence, there exists $(\xi_j^0)_{j\in\mathbb{N}}$ with $\lim_{n\to\infty}(\xi_j^n)_{j\in\mathbb{N}} = (\xi_j^0)_{j\in\mathbb{N}}$ with respect to $(.,.)$. Since the projections onto single components are $(.,.)$-continuous, this implies that

$$\xi_j^0 = \begin{cases} \frac{1}{\sqrt{j}} &, \quad j \text{ odd} \\ \frac{j}{\sqrt{(j-1)^3}} , & \quad j \text{ even} \end{cases}$$

This sequence, however, does not belong to $\ell^2(\mathbb{N})$, and we have reached a contradiction.

$\mathcal{L}$ is not orthocomplemented: We show that the orthogonal complement of $\mathcal{L}$ is given as

$$\mathcal{L}^\perp = \big\{(\eta_j)_{j\in\mathbb{N}} \in \ell^2(\mathbb{N}):\ \eta_{2k} = \frac{2k-1}{2k}\eta_{2k-1},\, k\in\mathbb{N}\big\}\,. \tag{3.3.5}$$ C52

To this end assume first that $(\eta_j)_{j\in\mathbb{N}}$ satisfies $\eta_{2k} = \frac{2k-1}{2k}\eta_{2k-1}$, $k\in\mathbb{N}$. Then, for each $(\xi_j)_{j\in\mathbb{N}} \in \mathcal{L}$, we have

$$\big[(\xi_j)_{j\in\mathbb{N}}, (\eta_j)_{j\in\mathbb{N}}\big] = \sum_{j=1}^\infty (-1)^j \xi_j\overline{\eta_j} = \sum_{k=1}^\infty \big(\xi_{2k}\overline{\eta_{2k}} - \xi_{2k-1}\overline{\eta_{2k-1}}\big) =$$

$$= \sum_{k=1}^\infty \Big(\frac{2k}{2k-1}\xi_{2k}\frac{2k-1}{2k}\overline{\eta_{2k}} - \xi_{2k-1}\overline{\eta_{2k-1}}\Big) = 0\,.$$

Conversely, let $(\eta_j)_{j\in\mathbb{N}} \in \mathcal{L}^\perp$ be given. We have

$$(\xi_j)_{j\in\mathbb{N}} := \big(0,\ldots,0,\frac{2k-1}{2k}, \underset{\substack{\uparrow \\ 2k\text{-th place}}}{1}, 0,\ldots\big) \in \mathcal{L}\,,$$

and hence

$$\eta_{2k} - \frac{2k-1}{2k}\eta_{2k-1} = \big[(\eta_j)_{j\in\mathbb{N}}(\xi_j)_{j\in\mathbb{N}}\big] = 0\,.$$

This establishes the equality (3.3.5).

Consider the element $(\zeta_j)_{j\in\mathbb{N}}$ defined as

$$\zeta_j := \begin{cases} \frac{1}{j}\,, & j \text{ even} \\ 0\,, & j \text{ odd} \end{cases}$$

and assume that $(\zeta_j)_{j\in\mathbb{N}} = (\xi_j)_{j\in\mathbb{N}} + (\eta_j)_{j\in\mathbb{N}}$ with some $(\xi_j)_{j\in\mathbb{N}} \in \mathcal{L}$ and $(\eta_j)_{j\in\mathbb{N}} \in \mathcal{L}^\perp$. Then

$$\xi_{2k-1} + \eta_{2k-1} = 0,\ \xi_{2k} + \eta_{2k} = \frac{1}{2k}, \quad k\in\mathbb{N}\,,$$

and we obtain, using the definition of $\mathcal{L}$ and (3.3.5),

$$\frac{1}{2k} = \frac{2k}{2k-1}\xi_{2k-1} + \frac{2k-1}{2k}\eta_{2k-1} = \Big(\frac{2k}{2k-1} - \frac{2k-1}{2k}\Big)\xi_{2k-1} =$$

$$= \frac{4k-1}{(2k-1)2k}\xi_{2k-1}\,.$$

It follows that

$$\xi_{2k-1} = \frac{2k-1}{4k-1},\ \xi_{2k} = \frac{2k}{4k-1},\quad k\in\mathbb{N}\,.$$

We have reached a contradiction, since this sequence is not summable. //

3.4 Isometric mappings

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert space. Then each isometric map ϕ, defined on some subspace $\operatorname{dom}\phi$ of H_1, is continuous and possesses an isometric continuation of $\overline{\operatorname{dom}\phi}$ onto $\overline{\operatorname{ran}\phi}$. In the indefinite situation, this statement is no longer true.

EXC76 *3.4.1 Example.* Let $\langle\mathcal{K},[.,.]\rangle$ be a Krein space with $\operatorname{ind}_-\mathcal{K} = \operatorname{ind}_+\mathcal{K} = \infty$. We are going to construct a dense linear subspace D and a linear, bijective, and $[.,.]$-isometric map of D onto itself which is not continuous with respect to the topology of $\mathcal{K}$.

Choose a fundamental decomposition $\mathfrak{J} = (\mathcal{K}_+,\mathcal{K}_-)$ of $\mathcal{K}$, choose $(.,.)_{\mathfrak{J}}$-orthonormal sequences $(e_n)_{n\in\mathbb{N}}$, $e_n\in\mathcal{K}_+$, and $(f_n)_{n\in\mathbb{N}}$, $f_n\in\mathcal{K}_-$, and choose a sequence $(\gamma_n)_{n\in\mathbb{N}}$, $\gamma_n\in(0,1)$ with $\lim_{n\to\infty}\gamma_n = 1$.

Set $D_0 := \overline{\operatorname{span}\{\hat{e}_n : n\in\mathbb{N}\}}[\dot{+}]\overline{\operatorname{span}\{\hat{f}_n : n\in\mathbb{N}\}}$, and

$$D := D_0^\perp[\dot{+}]\operatorname{span}\{\hat{e}_n : n\in\mathbb{N}\}[\dot{+}]\operatorname{span}\{\hat{f}_n : n\in\mathbb{N}\}\,. \tag{3.4.1}$$ C66

Since $\operatorname{span}\{\hat{e}_n : n\in\mathbb{N}\}\subseteq\mathcal{K}_+$ and $\operatorname{span}\{\hat{f}_n : n\in\mathbb{N}\}\subseteq\mathcal{K}_-$, the space D_0 is orthocomplemented. In fact, we have $D_0^\perp = (D_0^\perp\cap\mathcal{K}_+)[\dot{+}](D_0^\perp\cap\mathcal{K}_-)$ and

$$D_0^\perp\cap\mathcal{K}_+ = \mathcal{K}_+(-)_{\mathfrak{J}}\operatorname{span}\{\hat{e}_n : n\in\mathbb{N}\},\ D_0^\perp\cap\mathcal{K}_- = \mathcal{K}_-(-)_{\mathfrak{J}}\operatorname{span}\{\hat{f}_n : n\in\mathbb{N}\}\,.$$

In particular, we see that D is dense in $\mathcal{K}$. Moreover, the three summands in (3.4.1) are also pairwise orthogonal with respect to $(.,.)_{\mathfrak{J}}$, and we have

$$[e_n,e_m] = \begin{cases}1\,, & n=m\\ 0\,, & n\neq m\end{cases},\quad [f_n,f_m] = \begin{cases}-1\,, & n=m\\ 0\quad, & n\neq m\end{cases},\quad [e_n,f_m]=0\,.$$

Set $S_n := \operatorname{span}\{e_n,f_n\}$, $n\in\mathbb{N}$, then the space D can be written as

$$D = D_0^\perp\,[\dot{+}]\,\Big(\underset{n\in\mathbb{N}}{[\dot{+}]}S_n\Big)\,, \tag{3.4.2}$$ C68

with all summands being pairwise orthogonal also with respect to $(.,.)_{\mathfrak{J}}$. Hence, a linear and $[.,.]_{\mathcal{K}}$–isometric map $U : D\to D$ will be well-defined by specifying linear and $[.,.]_{\mathcal{K}}$–isometric maps $U_0 : D_0^\perp\to D_0^\perp$ and $U_n : S_n\to S_n$, $n\in\mathbb{N}$, and letting U be defined componentwise.

On the component $D_0^\perp$, we use the map $U_0 := \operatorname{id}_{D_0^\perp}$. The space S_n is spanned by $\{e_n,f_n\}$, and this is an orthonormal system with respect to $(.,.)_{\mathcal{K}}$. Thus the map

$$\varphi : \alpha e_n + \beta f_n \mapsto \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

is an isometric isomorphism of $\langle S_n, (.,.)_{\mathcal{K}}\rangle$ onto the space $\mathbb{C}^2$ endowed with the euclidean inner product. Let $u_n : \mathbb{C}^2 \to \mathbb{C}^2$, $n \in \mathbb{N}$, be given by the matrix

$$u_n := \frac{1}{\sqrt{1-\gamma_n^2}} \begin{pmatrix} 1 & \gamma_n \\ \gamma_n & 1 \end{pmatrix}$$

and set

$$U_n := \varphi^{-1} \circ u_n \circ \varphi : \, S_n \to S_n \, .$$

We have

$$U_n e_n = \frac{1}{\sqrt{1-\gamma_n^2}} \big(e_n + \gamma_n f_n\big), \quad U_n f_n = \frac{1}{\sqrt{1-\gamma_n^2}} \big(\gamma_n e_n + f_n\big) \, ,$$

and hence

$$\big[U_n e_n, U_n e_n\big] = 1, \; \big[U_n e_n, U_n f_n\big] = 0, \; \big[U_n f_n, U_n f_n\big] = -1 \, .$$

This shows that U_n is isometric with respect to $[.,.]$.

Let $U : D \to D$ be the $[.,.]$–isometric map defined by linearity and the requirements that $U|_{D_0^+} = U_0$, and $U|_{S_n} = U_n$, $n \in \mathbb{N}$. Since $\det u_n = \sqrt{1-\gamma_n^2} \neq 0$, the map U is a bijection of D onto itself.

The eigenvalues of u_n are equal to

$$\lambda_{n,+} = \frac{1+\gamma_n^2}{\sqrt{1-\gamma_n^2}}, \; \lambda_{n,-} = \frac{1-\gamma_n^2}{\sqrt{1-\gamma_n^2}} \, .$$

These numbers are thus also eigenvalues of U. Since $\gamma_n \to 1$, we have $\lambda_{n,+} \to \infty$ and hence U cannot be bounded with respect to any norm on D. //

THC53 **3.4.2 Theorem.** *Let $\langle \mathcal{K}_1, [.,.]_1\rangle$ and $\langle \mathcal{K}_2, [.,.]_2\rangle$ be Krein spaces, and let*

$$\phi : \operatorname{dom}\phi \subseteq \mathcal{K}_1 \to \mathcal{K}_2$$

be isometric. Then the following hold:

(*i*) *If $\operatorname{dom}\phi$ is closed and nondegenerated, and $\overline{\operatorname{ran}\phi}$ is nondegenerated, then ϕ is continuous.*

(*ii*) *Assume that $\operatorname{dom}\phi = D_+[\dot{+}]_1 D_-$ with some subspaces $D_+ \in \operatorname{Sub}_{\gg 0}\mathcal{K}_1$ and $D_- \in \operatorname{Sub}_{\ll 0}\mathcal{K}_1$. Then ϕ is continuous if and only if $\phi(D_+) \in \operatorname{Sub}_{\gg 0}\mathcal{K}_2$ and $\phi(D_-) \in \operatorname{Sub}_{\ll 0}\mathcal{K}_2$.*

(*iii*) *Assume that $\operatorname{dom}\phi$ is orthocomplemented. Then ϕ is continuous if and only if $\operatorname{ran}\phi$ is orthocomplemented.*

(*iv*) *If $\operatorname{dom}\phi$ contains a maximal uniformly definite subspace of $\mathcal{K}_1$ and $\overline{\operatorname{ran}\phi}$ is orthocomplemented, then ϕ is continuous.*

Assume that one of the hypothesis listed in (ii)–(iv) which implies continuity of ϕ holds, and let $\hat{\phi} : \overline{\operatorname{dom}\phi} \to \mathcal{K}_2$ be the extension of ϕ by continuity. Then $\operatorname{ran}\hat{\phi} = \overline{\operatorname{ran}\phi}$.

Proof. For the proof of (i) we consider ϕ as a linear operator defined on the Banach space $\operatorname{dom}\phi$ and taking values in the Banach space $\overline{\operatorname{ran}\phi}$. Let $x_n \in \operatorname{dom}\phi$, $n \in \mathbb{N}$, and assume that

$$x_n \to x \in \operatorname{dom}\phi,\ \phi x_n \to z \in \overline{\operatorname{ran}\phi}\,.$$

If $y \in \operatorname{dom}\phi$, then

$$[\phi x, \phi y]_2 = [x,y]_1 = \lim_{n\to\infty}[x_n,y]_1 = \lim_{n\to\infty}[\phi x_n, \phi y]_2 = [y, \phi y]_2\,,$$

i.e. $\phi x - z \perp \operatorname{ran}\phi$. Thus also $\phi x - z \perp \overline{\operatorname{ran}\phi}$, and hence $\phi x = z$. The Closed Graph Theorem implies that ϕ is continuous.

We come to the proof of (ii). Let $\operatorname{dom}\phi = D_+[\dot{+}]_1 D_-$ with $D_+ \in \operatorname{Sub}_{\gg 0}\mathcal{K}_1$, $D_- \in \operatorname{Sub}_{\ll 0}\mathcal{K}_1$. By Theorem 3.2.2 we can choose a fundamental decomposition $\mathfrak{J}_1 = (\mathcal{K}_+^1, \mathcal{K}_-^1)$ of $\mathcal{K}_1$ with $D_+ \subseteq \mathcal{K}_+^1$, $D_- \subseteq \mathcal{K}_-^1$.

Assume first that $\phi(D_+) \in \operatorname{Sub}_{\gg 0}\mathcal{K}_2$ and $\phi(D_-) \in \operatorname{Sub}_{\ll 0}\mathcal{K}_2$. Choose a fundamental decomposition $\mathfrak{J}_2 = (\mathcal{K}_+^2, \mathcal{K}_-^2)$ of $\mathcal{K}_2$ with $\phi(D_+) \subseteq \mathcal{K}_+^2$, $\phi(D_-) \subseteq \mathcal{K}_-^2$. Let $x, y \in \operatorname{dom}\phi$, and write $x = x_+ + x_-$, $y = y_+ + y_-$ according to the decomposition $\operatorname{dom}\phi = D_+[\dot{+}]_1 D_-$. Then

$$(\phi x, \phi y)_{\mathfrak{J}_2} = \big(\phi x_+ + \phi x_-, \phi y_+ + \phi y_-\big)_{\mathfrak{J}_2} = [\phi x_+, \phi y_+] - [\phi x_-, \phi y_-] =$$

$$= [x_+, y_+] - [x_-, y_-] = (x,y)_{\mathfrak{J}_1}\,,$$

i.e. ϕ is $(.,.)_{\mathfrak{J}_1}$-to-$(.,.)_{\mathfrak{J}_2}$–isometric. This, however, implies that ϕ is $\|.\|_{\mathfrak{J}_1}$-to-$\|.\|_{\mathfrak{J}_2}$–continuous. Let $\hat{\phi} : \overline{\operatorname{dom}\phi} \to \mathcal{K}_2$ be the extension of ϕ by continuity. Then $\hat{\phi}$ is again $(.,.)_{\mathfrak{J}_1}$-to-$(.,.)_{\mathfrak{J}_2}$–isometric. Thus $\operatorname{ran}\hat{\phi}$ is $(.,.)_{\mathfrak{J}_2}$–complete and hence closed in $\mathcal{K}_2$.

Conversely, assume that ϕ is continuous. Then there exists some constant $\gamma > 0$ such that $\|\phi x\|_{\mathfrak{J}_2} \leq \gamma \|x\|_{\mathfrak{J}_1}$, $x \in \operatorname{dom}\phi$, where $\mathfrak{J}_2$ denotes some fundamental symmetry of $\mathcal{K}_2$. We obtain

$$\frac{1}{\gamma^2}\|\phi x\|_{\mathfrak{J}_2}^2 \leq \|x\|_{\mathfrak{J}_1}^2 = [x,x]_1 = [\phi x, \phi x]_2,\ x \in D_+\,,$$

$$\frac{1}{\gamma^2}\|\phi x\|_{\mathfrak{J}_2}^2 \leq \|x\|_{\mathfrak{J}_1}^2 = -[x,x]_1 = -[\phi x, \phi x]_2,\ x \in D_-\,.$$

It follows that $\phi(D_+)$ is uniformly positive and $\phi(D_-)$ is uniformly negative.

For the proof of (iii) assume that $\operatorname{dom}\phi$ is orthocomplemented.

Assume first that ϕ is continuous. By Theorem 3.3.3 we can decompose $\operatorname{dom}\phi$ as $\operatorname{dom}\phi = D_+[\dot{+}]_1 D_-$ with some subspaces $D_+ \in \operatorname{Sub}_{\gg 0}\mathcal{K}_1$, $D_- \in \operatorname{Sub}_{\ll 0}\mathcal{K}_1$, which are closed in $\mathcal{K}_1$. Let $\mathfrak{J}_1$ be a fundamental decomposition of $\mathcal{K}_1$, then D_+ and D_- are complete with respect to $\|.\|_{\mathfrak{J}_1}$. However, uniform definiteness implies that the norms $\|.\|_{\mathfrak{J}_1}$ and $[.,.]^{\frac{1}{2}}$ on D_+ or $\|.\|_{\mathfrak{J}_1}$ and $(-[.,.])^{\frac{1}{2}}$ on D_-, respectively, are equivalent. Thus D_+ and D_- are intrinsically complete. Since ϕ is isometric, $\phi(D_+)$ and $\phi(D_-)$ are also intrinsically complete. An application of the already proved item (ii) yields that $\phi(D_+)$ and $\phi(D_-)$ are uniformly definite. Hence, again by equivalence of the respective norms, D_+ and D_- are also complete with respect to the norm $\|.\|_{\mathfrak{J}_2}$, where $\mathfrak{J}_2$ is some fundamental decomposition of $\mathcal{K}_2$. Thus $\phi(D_+)$ and $\phi(D_-)$ are closed in $\mathcal{K}_2$, and Theorem 3.3.3 implies that $\operatorname{ran}\phi = \phi(D_+)[\dot{+}]\phi(D_-)$ is orthocomplemented.

Conversely, assume that $\operatorname{ran}\phi$ is orthocomplemented. In view of Theorem 3.3.3, we may apply the already proved item (i), to conclude that ϕ is continuous. By the previous paragraph, the condition in (ii) which ensures continuity is satisfied. Hence, by what we already proved, $\operatorname{ran}\hat{\phi}$ is closed in $\mathcal{K}_2$.

We come to the proof of (iv). For definiteness assume that $\operatorname{dom}\phi$ contains a maximal uniformly positive subspace D_+ of $\mathcal{K}_1$. Then $\mathfrak{J} := (D_+, D_+^\perp)$ is a fundamental decomposition of $\mathcal{K}_1$. Since $D_+ \subseteq \operatorname{dom}\phi$, we have

$$\operatorname{dom}\phi = D_+[\dot{+}](D_+^\perp \cap \operatorname{dom}\phi) .$$

Clearly, the subspace $D_- := D_+^\perp \cap \operatorname{dom}\phi$ is uniformly negative. Set $R_+ := \phi(D_+)$ and $R_- := \phi(D_-)$. We are going to show that R_+ and R_- are uniformly definite.

The space R_+ is, as isometric image of the intrinsically complete space D_+, also intrinsically complete. Assume that $x \in \overline{R_+}$, and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in R_+ with $x_n \to x$ in the norm of $\mathcal{K}_2$. Then $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the norm of $\mathcal{K}_2$ and hence also in the norm induced by $[.,.]|_{R_+\times R_+}$. By intrinsic completeness, there exists $y \in R_+$ such that $x_n \to y$ with respect to $[.,.]$. It follows that

$$[x,z] = \lim_{n\to\infty}[x_n,z] = [y,z], \quad z \in R_+ .$$

Since $x, y \in \overline{R_+}$, and hence are both orthogonal to R_-, it follows that

$$[x,z] = [y,z], \quad z \in \overline{R_+[\dot{+}]R_-} = \overline{\operatorname{ran}\phi} .$$

Since $\overline{\operatorname{ran}\phi}$ is orthocomplemented, and hence in particular nondegenerated, this implies $x = y$. We conclude that R_+ is closed in the norm of $\mathcal{K}_2$. In conjunction with intrinsic completeness this gives $R_+ \in \operatorname{Sub}_{\gg 0}\mathcal{K}_2$.

Since $\overline{\operatorname{ran}\phi}$ is orthocomplemented, it is itself a Krein space and the topology it carries as such coincides with the topology it inherits from $\mathcal{K}_2$. Hence, for a subspace of $\overline{\operatorname{ran}\phi}$, uniform definiteness in $\mathcal{K}_2$ is equal to uniform definiteness in $\overline{\operatorname{ran}\phi}$.

Since R_+ is closed and uniformly positive, we have (considering R_+ as a subspace of $\overline{\operatorname{ran}\phi}$)

$$\overline{\operatorname{ran}\phi} = R_+[\dot{+}]R_+^\perp .$$

Hence the topology of $\overline{\operatorname{ran}\phi}$ equals the product topology of its restrictions to R_+ and $R_+^\perp$, and we conclude that

$$R_+[\dot{+}]R_+^\perp = \overline{\operatorname{ran}\phi} = \overline{R_+[\dot{+}]R_-} = R_+[\dot{+}]\overline{R_-} .$$

Thus $R_+^\perp = \overline{R_-}$, and hence negative semidefinite. However, $\overline{\operatorname{ran}\phi}$ is nondegenerated, and it follows that $R_+^\perp$ is negative definite. Thus $(R_+, R_+^\perp)$ is a fundamental decomposition of $\overline{\operatorname{ran}\phi}$, and hence $R_+^\perp$ is uniformly negative. In turn, this implies that R_- is uniformly negative.

An application of the already proved item (ii) yields the assertion (iv). ❑

COC54 **3.4.3 Corollary.** *Let $\langle\mathcal{K}_1, [.,.]_1\rangle$ and $\langle\mathcal{K}_2, [.,.]_2\rangle$ be Krein spaces, and let $\phi : \mathcal{K}_1 \to \mathcal{K}_2$ be a linear and isometric map. Then the following are equivalent:*

(i) ϕ is a morphism.

(ii) $\operatorname{ran}\phi$ *is orthocomplemented.*

(iii) $\overline{\operatorname{ran}\phi}$ *is nondegenerated.*

In this case ϕ is closed, i.e. maps closed subsets of $\mathcal{K}_1$ to closed subsets of $\mathcal{K}_2$. The map ϕ is an isomorphism if and only if it is surjective.

Proof. The domain of ϕ is all of $\mathcal{K}_1$, and hence in particular orthcomplemented. Theorem 3.4.2, *(iii)*, gives *(i)* $\Leftrightarrow$ *(ii)*. Trivially, *(ii)* $\Rightarrow$ *(iii)*. The implication *(iii)* $\Rightarrow$ *(i)* follows from Theorem 3.4.2, *(i)*.

If ϕ is a morphism then, by what we just proved, its range is a Banach space with respect to the norm of $\mathcal{K}_2$. By the Open Mapping Theorem, ϕ maps open subsets of $\mathcal{K}_1$ to relatively open subsets of $\operatorname{ran}\phi$. Thus it maps closed subsets of $\mathcal{K}_1$ to closed subsets of $\mathcal{K}_2$.

As an isometry with nondegenerated domain, the map ϕ is in any case injective. If $\operatorname{ran}\phi = \mathcal{K}_2$ then, by the just proved equivalences, ϕ is a morphism. The same argument shows that ϕ^{-1} is a morphism, and hence ϕ is an isomorphism. ❑

We also obtain a corresponding version of the 1st Homomorphism Theorem.

COC55 **3.4.4 Corollary.** *Let $\langle\mathcal{K}_1,[.,.]_1\rangle$ and $\langle\mathcal{K}_2,[.,.]_2\rangle$ be Krein spaces, and let $\phi : \mathcal{K}_1 \to \mathcal{K}_2$ be a morphism. Then $\langle\operatorname{ran}\phi,[.,.]\rangle$ is a Krein space, and there exists a unique isomorphism $\hat\phi$ such that*

$$\begin{tikzcd}
\langle\mathcal{K}_1,[.,.]_1\rangle \arrow[r, "\phi"] \arrow[dr, dotted, "\hat\phi"'] & \langle\mathcal{K}_2,[.,.]_2\rangle \\
 & \langle\operatorname{ran}\phi,[.,.]_2\rangle \arrow[u, "\iota"']
\end{tikzcd}$$

Proof. By the previous Corollary 3.4.3, the space $\operatorname{ran}\phi$ is orthocomplemented, and hence by Theorem 3.3.3 itself a Krein space. Trivially, there exists a linear and isometric map $\hat\phi$ which makes the above diagram commute. Again by the previous corollary, $\hat\phi$ is an isomorphism. Uniqueness is clear. ❑

3.5 Krein space completions

The concept of completion, as known from the setting of Banach or Hilbert spaces, can be considered also in the setting of Krein spaces. The situation changes drastically. Completions need not exist, neither be unique.

DEC56 **3.5.1 Definition.** Let $\langle\mathcal{L},[.,.]\rangle$ be an inner product space. A pair $(\iota,\langle\mathcal{K},[.,.]\rangle)$ is called a *Krein space completion of* $\langle\mathcal{L},[.,.]\rangle$, if $\langle\mathcal{K},[.,.]\rangle$ is a Krein space, and $\iota : \mathcal{L} \to \mathcal{K}$ is an isometric map whose range is dense in $\mathcal{K}$.

Two completions $(\iota_1,\mathcal{K}_1)$ and $(\iota_2,\mathcal{K}_2)$ are called *isomorphic*, if there exist an isomorphism ϕ of $\mathcal{K}_1$ onto $\mathcal{K}_2$, such that $\phi\circ\iota_1 = \iota_2$, i.e. such that we have the diagram

$$\begin{tikzcd}
 & \mathcal{L} \arrow[dl, "\iota_1"'] \arrow[dr, "\iota_2"] & \\
\mathcal{K}_1 \arrow[rr, "\phi"'] & & \mathcal{K}_2
\end{tikzcd} \tag{3.5.1}$$ C57

In this case, we write $(\iota_1, \mathcal{K}_1) \cong (\iota_2, \mathcal{K}_2)$. //

Clearly, isomorphy of Krein space completions is an equivalence relation. Moreover, if $(\iota, \mathcal{K})$ is a Krein space completion of $\mathcal{L}$, and λ is an isomorphism of $\mathcal{K}$ onto some other Krein space $\tilde{\mathcal{K}}$, then $(\lambda \circ \iota, \tilde{\mathcal{K}})$ is a Krein space completion of $\mathcal{L}$ isomorphic to $(\iota, \mathcal{K})$.

On the set of all isomorphy classes of Krein space completions of a given inner product space $\mathcal{L}$, a partial order can be defined. This construction is based on the following simple geometric property.

LEC58 **3.5.2 Lemma.** *Let $\langle \mathcal{L}, [.,.] \rangle$ be an inner product space.*

(*i*) *If $(\iota, \mathcal{K})$ is a Krein space completion of $\mathcal{L}$, then $\ker \iota = \mathcal{L}^\circ$.*

(*ii*) *If $(\iota_1, \mathcal{K}_1)$ and $(\iota_2, \mathcal{K}_2)$ are Krein space completions of $\mathcal{L}$, then there exists a unique map $\iota_{12} : \operatorname{ran} \iota_1 \to \operatorname{ran} \iota_2$ with*

$$\begin{tikzcd} & \mathcal{L} \arrow[dl, "\iota_1"'] \arrow[dr, "\iota_2"] & \\ \operatorname{ran}\iota_1 \arrow[rr, "\iota_{12}"'] & & \operatorname{ran}\iota_2 \end{tikzcd} \tag{3.5.2}$$ C1

The map ι_{12} is isometric, and bijective.

Proof. Since $\operatorname{ran} \iota$ is dense in $\mathcal{K}$, we have $[\operatorname{ran} \iota]^\circ = \mathcal{K}^\perp \cap \operatorname{ran} \iota = \{0\}$. Since ι is isometric,

$$\mathcal{L}^\circ = \iota^{-1}\big([\operatorname{ran} \iota]^\circ\big) = \ker \iota \,,$$

cf. Lemma 1.1.9. This is (*i*). We come two the proof of (*ii*). Since $\ker \iota_1 = \mathcal{L}^\circ = \ker \iota_2$, a linear map ι_{12} is uniquely defined by (3.5.2). Clearly, ι_{12} is bijective. Since ι_1 and ι_2 are both isometric, also ι_{12} has this property. ❑

DEC61 **3.5.3 Definition.** Let $\langle \mathcal{L}, [.,.] \rangle$ be an inner product space, and let $(\iota_1, \mathcal{K}_1)$ and $(\iota_2, \mathcal{K}_2)$ be two Krein space completions of $\mathcal{L}$. Then we write $(\iota_1, \mathcal{K}_1) \succeq (\iota_2, \mathcal{K}_2)$, if ι_{12} is continuous, where $\operatorname{ran} \iota_j$ is endowed with the restriction of the topology of $\mathcal{K}_j$. //

Obviously, the relation $\succeq$ is reflexive and transitive.

LEC62 **3.5.4 Lemma.** *Let $\langle \mathcal{L}, [.,.] \rangle$ be an inner product space, and let $(\iota_1, \mathcal{K}_1)$ and $(\iota_2, \mathcal{K}_2)$ be two Krein space completions of $\mathcal{L}$. Then*

$$\Big((\iota_1, \mathcal{K}_1) \succeq (\iota_2, \mathcal{K}_2) \wedge (\iota_2, \mathcal{K}_2) \succeq (\iota_1, \mathcal{K}_1)\Big) \iff (\iota_1, \mathcal{K}_1) \cong (\iota_2, \mathcal{K}_2)$$

Proof. If $(\iota_1, \mathcal{K}_1) \cong (\iota_2, \mathcal{K}_2)$, and ϕ is as in Definition 3.5.1, then $\phi|_{\operatorname{ran} \iota_1} = \iota_{12}$ and $\phi^{-1}|_{\operatorname{ran} \iota_2} = \iota_{21}$. Hence ι_{12} and ι_{21} are both continuous.

Conversely, assume that ι_{12} and ι_{21} are both continuous. These maps are mutually inverse bijections between $\operatorname{ran} \iota_1$ and $\operatorname{ran} \iota_2$. Since $\overline{\operatorname{ran} \iota_j} = \mathcal{K}_j$, $j = 1, 2$, they can be extended by continuity to mutually inverse continuous bijections between $\mathcal{K}_1$ and $\mathcal{K}_2$. Clearly, these extensions are again isometric and make the diagram (3.5.1) commute. Thus $(\iota_1, \mathcal{K}_1) \cong (\iota_2, \mathcal{K}_2)$. ❑

From this statement we see that indeed $\succeq$ induces a partial order on the set of all isomorphy classes of Krein space completions of $\mathcal{L}$.

When considering the notion of completions, two questions suggest themselves: Let an inner product space $\langle\mathcal{L}, [.,.]\rangle$ be given.

(A) Does there exist a Krein space completion of $\mathcal{L}$?

(B) If there exists a Krein space completion of $\mathcal{L}$, is it unique (up to isomorphism)?

As we will see, the answer to both questions is in general negative, and it is a nontrivial task to describe the totality of Krein space completions. If $\langle\mathcal{L}, [.,.]\rangle$ is semidefinite, however, matters are plain and simple. Let us discuss this situation explicitly.

PRC59 **3.5.5 Proposition.** *Let $\langle\mathcal{L}, [.,.]\rangle$ be a positive semidefinite inner product space. Then the following hold:*

(*i*) *There exists a Krein space completion of $\mathcal{L}$.*

(*ii*) *In each Krein space completion $(\iota, \mathcal{K})$ of $\mathcal{L}$, the Krein space $\mathcal{K}$ is positive definite, i.e. is a Hilbert space.*

(*iii*) *Each two Krein space completions of $\mathcal{L}$ are isomorphic.*

Proof. The factor space $\mathcal{L}/\mathcal{L}^\circ$ is positive definite. Hence, there exists a Hilbert space $\langle\mathcal{H}, (.,.)_\mathcal{H}\rangle$ together with a linear map $\iota_\mathcal{H} : \mathcal{L}/\mathcal{L}^\circ \to \mathcal{H}$, such that $\operatorname{ran}\iota_\mathcal{H}$ is dense in $\mathcal{H}$ and

$$(\iota_\mathcal{H} x, \iota_\mathcal{H} y)_\mathcal{H} = (x, y), \quad x, y \in \mathcal{L}\,.$$

Denote by π the canonical projection of $\mathcal{L}$ onto $\mathcal{L}/\mathcal{L}^\circ$, then $(\iota_\mathcal{H} \circ \pi, \mathcal{H})$ is a Krein space completion of $\mathcal{L}$.

Next, let $(\iota, \mathcal{K})$ be any Krein space completion of $\mathcal{L}$. Then $\operatorname{ran}\iota$ is a dense and positive semidefinite subspace of $\mathcal{K}$. Thus $\mathcal{K}$ is itself positive semidefinite, and hence a Hilbert space.

Finally, let $(\iota_1, \mathcal{K}_1)$ and $(\iota_2, \mathcal{K}_2)$ be two Krein space completions of $\mathcal{L}$. The map ι_{12} is an isometry between the dense subspaces $\operatorname{ran}\iota_1$ and $\operatorname{ran}\iota_2$ of the Hilbert spaces $\mathcal{K}_1$ and $\mathcal{K}_2$. Hence, it extends to an isomorphism ϕ of $\mathcal{K}_1$ onto $\mathcal{K}_2$. Clearly, we have $\phi \circ \iota_1 = \iota_2$:

$$\begin{tikzcd}
 & \mathcal{L} \arrow[dl, "\iota_1"'] \arrow[dr, "\iota_2"] & \\
\operatorname{ran}\iota_1 \arrow[rr, "\iota_{12}"'] \arrow[d, phantom, "\cap" , "|" description] & & \operatorname{ran}\iota_2 \arrow[d, phantom, "\cap", "|" description] \\
\mathcal{K}_1 \arrow[rr, dashed, "\phi"'] & & \mathcal{K}_2
\end{tikzcd}$$

❑

In the situation of Proposition 3.5.5 we will also refer to the (up to isomorphism) unique Krein space completion of $\mathcal{L}$ as its *Hilbert space completion*.

Clearly, the statement analogous to Proposition 3.5.5 also holds for negative semidefinite inner product spaces. Instead of Hilbert spaces, thereby, one will obtain anti-Hilbert spaces, i.e. spaces $\langle \mathcal{H}, (.,.)_{\mathcal{H}} \rangle$ such that $\langle \mathcal{H}, -(.,.)_{\mathcal{H}} \rangle$ is a Hilbert space. Correspondingly, we will speak of the *anti-Hilbert space completion* of $\mathcal{L}$.

The existence part of Proposition 3.5.5 can be lifted easily to decomposable spaces. Uniqueness is a more delicate matter and need not prevail, cf. Example 3.5.16. In this place we give a sufficient condition for uniqueness.

PRC5 **3.5.6 Proposition.** *Let $\langle \mathcal{L}, [.,.] \rangle$ be a decomposable inner product space. Then the following hold:*

(*i*) *There exists a Krein space completion of $\mathcal{L}$.*

(*ii*) *If $\mathcal{L}$ has a fundamental decomposition with (at least) one intrinsically complete component, then each two Krein space completions of $\mathcal{L}$ are isomorphic.*

Proof. Let $\mathfrak{J} = (\mathcal{L}_+, \mathcal{L}_-)$ be a fundamental decomposition of $\mathcal{L}$, and let $(\iota_+, \mathcal{H}_+)$ and $(\iota_-, \mathcal{H}_-)$ be the Hilbert space and anti-Hilbert space completion of $\langle \mathcal{L}_+, [.,.] \rangle$ and $\langle \mathcal{L}_-, [.,.] \rangle$, respectively. Set

$$\mathcal{K}_{\mathfrak{J}} := \mathcal{H}_+ [\dot{+}] \mathcal{H}_-$$

and let $\iota_{\mathfrak{J}} : \mathcal{L} \to \mathcal{K}_{\mathfrak{J}}$ be defined as

$$\iota_{\mathfrak{J}}(x_+ + x_-) := \iota_+ x_+ + \iota_- x_-, \quad x_+ \in \mathcal{L}_+, x_- \in \mathcal{L}_- .$$

Then $\mathcal{K}_{\mathfrak{J}}$ is a Krein space, having

$$\tilde{\mathfrak{J}} := (\mathcal{H}_+, \mathcal{H}_-) \tag{3.5.3}$$ C6

as a fundamental decomposition. The map $\iota_{\mathfrak{J}}$ is clearly $[.,.]$-to-$[.,.]_{\mathcal{K}_{\mathfrak{J}}}$–isometric, and $\operatorname{ran} \iota_{\mathfrak{J}} = \operatorname{ran} \iota_+ + \operatorname{ran} \iota_-$ is dense in $\mathcal{K}_{\mathfrak{J}}$. Thus $(\iota_{\mathfrak{J}}, \mathcal{K}_{\mathfrak{J}})$ is a Krein space completion of $\mathcal{L}$.

For the proof of (*ii*), assume that $\mathfrak{J} = (\mathcal{L}_+, \mathcal{L}_-)$ is a fundamental decomposition of $\mathcal{L}$ with (at least) one intrinsically complete component. For definiteness assume that $\mathcal{L}_+$ is intrinsically complete. Then we may, in the construction of the preceeding paragraph, choose $\mathcal{H}_+ := \mathcal{L}_+$ and $\iota_+ := \mathrm{id}$. The Krein space completion $\mathcal{K}_{\mathfrak{J}}$ obtained in this way has $(\mathcal{L}_+, \mathcal{H}_-)$ as a fundamental decomposition. Let $(\iota, \mathcal{K})$ be any Krein space completion of $\mathcal{L}$, and consider the linear and isometric map $\eta : \operatorname{ran} \iota_{\mathfrak{J}} \to \operatorname{ran} \iota$ defined by the corresponding diagram (3.5.2). The domain of η is equal to $\mathcal{L}_+ [\dot{+}] \iota_-(\mathcal{L}_-)$, and hence contains the maximal uniformly positive subspace $\mathcal{L}_+$ of $\mathcal{K}_{\mathfrak{J}}$. The range of η is equal to $\operatorname{ran} \iota$, and hence is dense in $\mathcal{K}$. Theorem 3.4.2, (*iv*), implies that η extends to an isomorphism ϕ of $\mathcal{K}_{\mathfrak{J}}$ onto $\mathcal{K}$. We have the diagram

$$\begin{tikzcd}
& \mathcal{L} \arrow[dl, "\iota_{\mathfrak{J}}"'] \arrow[dr, "\iota_2"] & \\
\operatorname{ran} \iota_{\mathfrak{J}} \arrow[rr, "\eta"'] \arrow[d, phantom, "\rotatebox{-90}{$\subseteq$}"] & & \operatorname{ran} \iota \arrow[d, phantom, "\rotatebox{-90}{$\subseteq$}"] \\
\mathcal{K}_{\mathfrak{J}} \arrow[rr, dashed, "\phi"'] & & \mathcal{K}
\end{tikzcd}$$

and hence ϕ is an isomorphism between the completions $(\iota_{\mathfrak{J}}, \mathcal{K}_{\mathfrak{J}})$ and $(\iota, \mathcal{K})$. ❑

COC64 **3.5.7 Corollary.** *Let $\langle\mathcal{K},[.,.]\rangle$ be a Krein space. Then there exists (up to isomorphism) a unique Krein space completion of $\langle\mathcal{K},[.,.]\rangle$, namely $(\mathrm{id}_{\mathcal{K}},\langle\mathcal{K},[.,.]\rangle)$.* ❑

We will refer to the Krein space completion constructed in the proof of Proposition 3.5.6, (i), starting from a fundamental decomposition $\mathfrak{J}$ as the *Krein space completion of $\langle\mathcal{L},[.,.]\rangle$ induced by $\mathfrak{J}$*. In order to justify this terminology, note that different choices of the Hilbert space completions $(\iota_\pm,\mathcal{H}_\pm)$ in the above construction give rise to isomorphic Krein space completions. However, as we will see later, different fundamental decompositions may induce nonisomorphic completions, cf. Example 3.5.16.

Sometimes the following continuity property is of good use.

REC7 *3.5.8 Remark.* Let $\langle\mathcal{L},[.,.]\rangle$ be an inner product space, let $\mathfrak{J}=(\mathcal{L}_+,\mathcal{L}_-)$ be a fundamental decomposition of $\mathcal{L}$, let $(\iota_{\mathfrak{J}},\mathcal{K}_{\mathfrak{J}})$ be the Krein space completion induced by $\mathfrak{J}$, and let $\tilde{\mathfrak{J}}$ be the fundamental decomposition (3.5.3). Then the embedding $\iota_{\mathfrak{J}}$ is $(.,.)_{\mathfrak{J}}$-to-$(.,.)_{\tilde{\mathfrak{J}}}$–isometric. In particular, it is continuous with respect to the decomposition topology induced by $\mathfrak{J}$ on $\mathcal{L}$ and the Krein space topology on $\mathcal{K}_{\mathfrak{J}}$. //

Let us also observe that decomposability is not necessary in order that a Krein space completion exists.

EXC60 *3.5.9 Example.* Consider the inner product space $\langle\ell^2(\mathbb{Z}),[.,.]\rangle$, where $[.,.]$ is given by (2.3.1), i.e.

$$\big[(\xi_j)_{j\in\mathbb{Z}},(\eta_j)_{j\in\mathbb{Z}}\big]:=\sum_{j\in\mathbb{Z}}\xi_j\overline{\eta_{-j-1}}\,.$$

Let $(.,.)$ denote the usual $\ell^2(\mathbb{Z})$-inner product, then we can write

$$\big[(\xi_j)_{j\in\mathbb{Z}},(\eta_j)_{j\in\mathbb{Z}}\big]=\big(G(\xi_j)_{j\in\mathbb{Z}},(\eta_j)_{j\in\mathbb{Z}}\big)$$

where

$$G(\xi_j)_{j\in\mathbb{Z}}:=(\xi_{-j-1})_{j\in\mathbb{Z}},\ (\xi_j)_{j\in\mathbb{Z}}\in\ell^2(\mathbb{Z})\,.$$

Apparently, G is a unitary operator of $\ell^2(\mathbb{Z})$ onto itself. This, first of all, justifies the definition of $[.,.]$ and, moreover, tells us that G is boundedly invertible with respect to the norm of $\ell^2(\mathbb{Z})$. Theorem 3.1.5 implies that $\langle\ell^2(\mathbb{Z}),[.,.]\rangle$ is a Krein space.

The space $\mathcal{L}:=\mathbb{C}^{\mathbb{Z}}_{\mathrm{lf}}\cap\ell^2(\mathbb{Z})$ considered in Example 2.3.4 is a dense subspace of this Krein space. Hence, we may consider $\langle\ell^2(\mathbb{Z}),[.,.]\rangle$ as a Krein space completion of $\langle\mathcal{L},[.,.]\rangle$.

As we saw in Example 2.3.4, the space $\langle\mathcal{L},[.,.]\rangle$ is not decomposable. We conclude that not every Krein space completion necessarily must be induced by a fundamental decomposition. //

Krein space completions are related to certain compatible topologies on $\mathcal{L}$. In order to to describe this class, we need the following construction.

LEC72 **3.5.10 Lemma.** *Let $\langle\mathcal{L},[.,.]\rangle$ be an inner product space, and let $(.,.)$ be a positive semidefinite inner product on $\mathcal{L}$ with $\mathcal{L}^{(\circ)}=\mathcal{L}^{[\circ]}$. Moreover, denote by $(\iota_{\mathcal{H}},\langle\mathcal{H},(.,.)_{\mathcal{H}}\rangle)$ a Hilbert space completion of $\langle\mathcal{L},(.,.)\rangle$. Then the following hold:*

(*i*) *There exists an inner product* $[.,.]_{\mathcal{H}}$ *on* $\mathcal{H}$ *which is continuous with respect to* $(.,.)_{\mathcal{H}}$*, nondegenerated, and such that* $\iota_{\mathcal{H}}$ *is* $[.,.]$*-to-*$[.,.]_{\mathcal{H}}$*–isometric.*

(*ii*) *The space* $\langle\mathcal{H},[.,.]_{\mathcal{H}}\rangle$ *is decomposable, and each decomposition topology is coarser than the Hilbert space topology of* $\mathcal{H}$*.*

(*iii*) *If the construction in* (*i*) *is carried out with two Hilbert space completions* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *of* $\langle\mathcal{L},(.,.)\rangle$*, then there exists a linear and bijective map of* $\mathcal{H}_1$ *onto* $\mathcal{H}_2$*, which is* $[.,.]_{\mathcal{H}_1}$*-to-*$[.,.]_{\mathcal{H}_2}$ *and* $(.,.)_{\mathcal{H}_1}$*-to-*$(.,.)_{\mathcal{H}_2}$ *isometric.*

(*iv*) *If the construction in* (*i*) *is carried out with two inner products* $(.,.)_1$*,* $(.,.)_2$ *on* $\mathcal{L}$ *which induce the same topology, and two respective Hilbert space completions* $\mathcal{H}_j$ *of* $\langle\mathcal{L},(.,.)_j\rangle$*, then there exists a linear and bijective map of* $\mathcal{H}_1$ *onto* $\mathcal{H}_2$*, which is* $[.,.]_{\mathcal{H}_1}$*-to-*$[.,.]_{\mathcal{H}_2}$ *isometric and bicontinuous with respect to the respective Hilbert space topologies of* $\mathcal{H}_1$ *and* $\mathcal{H}_2$*.*

Proof. $\boxed{\text{insert proof}}$ ❑

Let $\mathcal{T}\in\operatorname{Top}_{\mathrm{ip}}\mathcal{L}$, assume that $\overline{\{0\}}^{\mathcal{T}}=\mathcal{L}^{[\circ]}$, and choose a positive semidefinite inner product $(.,.)$ on $\mathcal{L}$ which induces $\mathcal{T}$. Then $\mathcal{L}^{(\circ)}=\mathcal{L}^{[\circ]}$, and hence Lemma 3.5.10 is applicable. Denote a Hilbert space and inner product obtained in this way by $\langle\mathcal{H}_{\mathcal{T}},(.,.)_{\mathcal{T}}\rangle$ and $[.,.]_{\mathcal{T}}$. By Lemma 3.5.10, (*iv*), different choices of $(.,.)$ or of a Hilbert space completion, respectively, will give rise to ($[.,.]$-) isometrically and ($(.,.)$-) bicontinuously isomorphic spaces. Hence, the following notion is well-defined.

DEC71 **3.5.11 Definition.** Let $\langle\mathcal{L},[.,.]\rangle$ be an inner product space. Then we set

$$\operatorname{Top}_{\mathrm{com}}\mathcal{L}:=\big\{\mathcal{T}\in\operatorname{Top}_{\mathrm{ip}}\mathcal{L}:\ \overline{\{0\}}^{\mathcal{T}}=\mathcal{L}^{[\circ]}\text{ and }\langle\mathcal{H}_{\mathcal{T}},[.,.]_{\mathcal{T}}\rangle\text{ is a Krein space}\big\}\,.$$

//

The relation between Krein space completions of $\mathcal{L}$ and the class $\operatorname{Top}_{\mathrm{com}}\mathcal{L}$ of compatible topologies is established by the following construction.

DEC73 **3.5.12 Definition.** Let $\langle\mathcal{L},[.,.]\rangle$ be an inner product space, and let $(\iota,\mathcal{K})$ be a Krein space completion of $\mathcal{L}$. Then we denote by $\mathfrak{T}(\iota,\mathcal{K})$ the initial topology on $\mathcal{L}$ with respect to the map ι. //

THC74 **3.5.13 Theorem.** *Let* $\langle\mathcal{L},[.,.]\rangle$ *be an inner product space.*

(*i*) *The assignment* $\mathfrak{T}:(\iota,\mathcal{K})\mapsto\mathfrak{T}(\iota,\mathcal{K})$ *induces an order-isomorphism of the set of all isomorphy classes of Krein space completions of* $\mathcal{L}$ *onto* $\operatorname{Top}_{\mathrm{com}}\mathcal{L}$*.*

(*ii*) *For each* $\mathcal{T}\in\operatorname{Top}_{\mathrm{ip}}\mathcal{L}$ *with* $\overline{\{0\}}^{\mathcal{T}}=\mathcal{L}^{[\circ]}$ *there exists a Krein space completion* $(\iota,\mathcal{K})$ *of* $\mathcal{L}$ *with* $\mathfrak{T}(\iota,\mathcal{K})\subseteq\mathcal{T}$*.*

(*iii*) *The space* $\mathcal{L}$ *possesses a Krein space completion if and only if* $\operatorname{Top}_{\mathrm{ip}}\mathcal{L}\neq\emptyset$*.*

Proof. Let $(\iota,\langle\mathcal{K},[.,.]_{\mathcal{K}}\rangle)$ be a Krein space completion of $\langle\mathcal{L},[.,.]\rangle$, and choose a positive definite inner product $(.,.)_{\mathcal{K}}$ on $\mathcal{K}$ which induces the topology of $\mathcal{K}$. Set

$$(x,y):=(\iota x,\iota y)_{\mathcal{K}},\quad x,y\in\mathcal{L}\,,$$

then $(.,.)$ is a positive semidefinite inner product on $\mathcal{L}$. The topology induced by $(.,.)$ on $\mathcal{L}$ is the initial topology with respect to the map ι into the space $\mathcal{K}$ endowed with its Krein space topology, i.e. equal to $\mathfrak{T}(\iota,\mathcal{K})$.

Clearly, $\mathcal{L}^{(\circ)} = \ker\iota = \mathcal{L}^{[\circ]}$ and hence $\overline{\{0\}}^{\mathfrak{T}(\iota,\mathcal{K})} = \mathcal{L}^{[\circ]}$. Moreover, with an appropriate constant $\gamma > 0$,

$$\big|[x,x]\big| = \big|[\iota x,\iota x]_{\mathcal{K}}\big| \leq \gamma(\iota x,\iota x)_{\mathcal{K}} = \gamma(x,x), \quad x \in \mathcal{L}\,.$$

Hence, $\mathfrak{T}(\iota,\mathcal{K})$ is compatible. Finally, the map ι is $(.,.)$-to-$(.,.)_{\mathcal{K}}$–isometric and its range is dense in $\mathcal{K}$. Thus $(\iota,\langle\mathcal{K},(.,.)_{\mathcal{K}}\rangle)$ is a Hilbert space completion of $\langle\mathcal{L},(.,.)\rangle$. Since ι is also $[.,.]$-to-$[.,.]_{\mathcal{K}}$–isometric, the inner product defined on $\mathcal{K}$ by means of Lemma 3.5.10 coincides with $[.,.]_{\mathcal{K}}$, and we conclude that $\mathfrak{T}(\iota,\mathcal{K}) \in \operatorname{Top}_{\mathrm{com}}\mathcal{L}$.

Let $(\iota_1,\mathcal{K}_1)$ and $(\iota_2,\mathcal{K}_2)$ be two Krein space completions of $\mathcal{L}$. Then we have $\mathfrak{T}(\iota_1,\mathcal{K}_1) \supseteq \mathfrak{T}(\iota_2,\mathcal{K}_2)$ if and only if the identity map $\mathrm{id}_{\mathcal{L}}$ is $\mathfrak{T}(\iota_1,\mathcal{K}_1)$-to-$\mathfrak{T}(\iota_2,\mathcal{K}_2)$-continuous. Consider the diagram

$$\begin{tikzcd}
\langle\mathcal{L},\mathfrak{T}(\iota_1,\mathcal{K}_1)\rangle \arrow[r, "\mathrm{id}_{\mathcal{L}}"] \arrow[d, "\iota_1"'] & \langle\mathcal{L},\mathfrak{T}(\iota_2,\mathcal{K}_2)\rangle \arrow[d, "\iota_2"] \\
\operatorname{ran}\iota_1 \arrow[r, "\iota_{12}"'] & \operatorname{ran}\iota_2
\end{tikzcd}$$

If ι_{12} is continuous, then by the universal property of initial topologies also $\mathrm{id}_{\mathcal{L}}$ is continuous. Denote by $(.,.)_j$ the inner product defined on $\mathcal{L}$ by $(x,y)_j := (\iota_j x,\iota_j y)_j$, so that the topology $\mathfrak{T}(\iota_j,\mathcal{K}_j)$ is induced by $(.,.)_j$. Since ι_j is $(.,.)_j$-to-$(.,.)_{\mathcal{K}_j}$–isometric, it is continuous and open when $\operatorname{ran}\iota_j$ is considered with the restriction of the topology of $\mathcal{K}_j$. Moreover, $\iota_{12}^{-1}(O) = \iota_1(\iota_2^{-1}(O))$, $O \subseteq \operatorname{ran}\iota_2$. Hence $\mathfrak{T}(\iota_2,\mathcal{K}_2) \subseteq \mathfrak{T}(\iota_1,\mathcal{K}_1)$ implies that ι_{12} is continuous. Together, it follows that $(\iota_1,\mathcal{K}_1) \succeq (\iota_2,\mathcal{K}_2)$ if and only if $\mathfrak{T}(\iota_1,\mathcal{K}_1) \supseteq \mathfrak{T}(\iota_2,\mathcal{K}_2)$.

Up to now we have shown that the assignment $\mathfrak{T}$ induces an order isomorphism of the set of all isomorphy classes onto some subset of $\operatorname{Top}_{\mathrm{com}}\mathcal{L}$. Let $\mathcal{T} \in \operatorname{Top}_{\mathrm{com}}\mathcal{L}$ be given, and consider the Krein space $\langle\mathcal{H}_{\mathcal{T}},[.,.]_{\mathcal{T}}\rangle$ together with the map $\iota_{\mathcal{T}}$. Since $[.,.]_{\mathcal{T}}$ is nondegenerated, the Krein space topology of $\langle\mathcal{H}_{\mathcal{T}},[.,.]_{\mathcal{T}}\rangle$ must coincide with the Hilbert space topology induced on $\mathcal{H}_{\mathcal{T}}$ by $(.,.)_{\mathcal{T}}$. Thus $\operatorname{ran}\iota_{\mathcal{T}}$ is dense in the Krein space $\mathcal{H}_{\mathcal{T}}$. It is also $[.,.]$-to-$[.,.]_{\mathcal{T}}$–isometric, and we conclude that $(\iota_{\mathcal{T}},\langle\mathcal{H}_{\mathcal{T}},[.,.]_{\mathcal{T}}\rangle)$ is a Krein space completion of $\langle\mathcal{L},[.,.]\rangle$. Since $\iota_{\mathcal{T}}$ is $(.,.)$-to-$(.,.)_{\mathcal{T}}$–isometric, when $(.,.)$ denotes the inner product on $\mathcal{L}$ chosen in the construction of $\mathcal{H}_{\mathcal{T}}$, this completion is mapped by $\mathfrak{T}$ to the topology $\mathcal{T}$.

We come to the proof of *(ii)*. Let $\mathcal{T} \in \operatorname{Top}_{\mathrm{ip}}\mathcal{L}$ with $\overline{\{0\}}^{\mathcal{T}} = \mathcal{L}^{[\circ]}$ be given. Consider the inner product space $\langle\mathcal{H}_{\mathcal{T}},[.,.]_{\mathcal{T}}\rangle$. Let $\mathfrak{J}$ be a fundamental decomposition of $\mathcal{H}_{\mathcal{T}}$ whose components are closed in the Hilbert space topology of $\mathcal{H}_{\mathcal{T}}$, let $(\iota,\mathcal{K})$ be the Krein space completion induced by $\mathfrak{J}$, and let $\tilde{\mathfrak{J}}$ be the fundamental decomposition (3.5.3) of $\mathcal{K}$. The map ι is $(.,.)_{\mathfrak{J}}$-to-$(.,.)_{\tilde{\mathfrak{J}}}$–isometric and hence $\mathcal{T}_{\mathfrak{J}}$-to-$\mathcal{T}_{\tilde{\mathfrak{J}}}$–continuous. However, the Hilbert space topology of $\mathcal{H}_{\mathcal{T}}$ is finer than $\mathcal{T}_{\mathfrak{J}}$, cf. Proposition 2.4.2, *(iv)*. Thus ι is also $\mathcal{H}_{\mathcal{T}}$-to-$\mathcal{K}$–continuous. Since $\operatorname{ran}\iota$ is dense, therefore ι maps dense subsets of $\mathcal{H}_{\mathcal{T}}$ onto dense subsets of $\mathcal{K}$, in particular, $\operatorname{ran}(\iota\circ\iota_{\mathcal{T}})$ is dense in $\mathcal{K}$. Clearly, $\iota\circ\iota_{\mathcal{T}}$ is $[.,.]$-to-$[.,.]_{\mathcal{K}}$–isometric, and it follows that $(\iota\circ\iota_{\mathcal{T}},\mathcal{K})$ is a Krein space completion of $\langle\mathcal{L},[.,.]\rangle$. The map $\iota\circ\iota_{\mathcal{T}}$ is $\mathcal{T}$-to-$\mathcal{K}$–continuous, and hence $\mathfrak{T}(\iota,\mathcal{K})$ is coarser than $\mathcal{T}$.

For the proof of (iii) assume that $\operatorname{Top}_{\mathrm{ip}}\mathcal{L}\neq\emptyset$, and choose a positive semidefinite inner product $(.,.)$ on $\mathcal{L}$ which induces a compatible topology. Choose a linear subspace $\mathcal{L}_1$ of $\mathcal{L}$ such that $\mathcal{L}=\mathcal{L}_1\dot{+}\mathcal{L}^\circ$, and define an inner product $(.,.)_1$ on $\mathcal{L}$ by

$$(x_1+x_0,y_1+y_0)_1:=(x_1,y_1),\quad x_1,y_1\in\mathcal{L}_1,\ x_0,y_0\in\mathcal{L}^\circ\,.$$

We have, with some appropriate constant $\gamma>0$,

$$|[x_1{+}x_0,x_1{+}x_0]|=|[x_1,x_1]|\leq\gamma(x_1,x_1)=\gamma(x_1{+}x_0,x_1{+}x_0)_1,\quad x_1\in\mathcal{L}_1,x_0\in\mathcal{L}^\circ\,,$$

and hence $(.,.)_1$ induces a compatible topology on $\mathcal{L}$. This also implies that $\mathcal{L}^{(\circ)_1}\subseteq\mathcal{L}^{[\circ]}$. The inclusion $\mathcal{L}^{(\circ)_1}\supseteq\mathcal{L}^{[\circ]}$ is trivial. We have constructed an element $\mathcal{T}\in\operatorname{Top}_{\mathrm{ip}}\mathcal{L}$ with $\overline{\{0\}}^{\mathcal{T}}=\mathcal{L}^{[\circ]}$, and the already proved item (ii) implies in particular that there exists a Krein space completion of $\mathcal{L}$. ❑

Without further notice, we obtain that Krein space completions neither need to exist nor need to be unique.

EXC4 *3.5.14 Example.* The inner product space constructed in Example 2.3.1 has no compatible topologies at all, in particular $\operatorname{Top}_{\mathrm{ip}}\mathcal{L}=\emptyset$. //

COC11 **3.5.15 Corollary.** *Let $\langle\mathcal{L},[.,.]\rangle$ be an inner product space, and assume that $\mathcal{L}$ is decomposable. Let $\mathfrak{J}_1$ and $\mathfrak{J}_2$ be fundamental decompositions of $\mathcal{L}$, and let $(\iota_1,\mathcal{K}_1)$ and $(\iota_2,\mathcal{K}_2)$ be the Krein space completions of $\mathcal{L}$ induced by $\mathfrak{J}_1$ and $\mathfrak{J}_2$, respectively. Then $(\iota_1,\mathcal{K}_1)$ and $(\iota_2,\mathcal{K}_2)$ are isomorphic if and only if $\mathcal{T}_{\mathfrak{J}_1}=\mathcal{T}_{\mathfrak{J}_2}$.*

Proof. Since ι_j is $(.,.)_{\mathfrak{J}_j}$-to-$(.,.)_{\tilde{\mathfrak{J}}_j}$–isometric, we have $\mathfrak{T}(\iota_j,\mathcal{K}_j)=\mathcal{T}_{\mathfrak{J}_j}$. Hence, $\mathfrak{T}$ being an order isomorphism implies the present assertion. ❑

EXC12 *3.5.16 Example.* The inner product space constructed in Example 2.6.1 has two different decomposition topologies. Thus it has (at least) two nonisomorphic Krein space completions. //

THC75 **3.5.17 Theorem.** *Let $\langle\mathcal{L},[.,.]\rangle$ be an inner product space. Then each two Krein space completions of $\mathcal{L}$ are isomorphic if and only if for each $\mathcal{T}\in\operatorname{Top}_{\mathrm{ip}}\mathcal{L}$ with $\overline{\{0\}}^{\mathcal{T}}=\mathcal{L}^{[\circ]}$ the space $\langle\mathcal{H}_{\mathcal{T}},[.,.]_{\mathcal{T}}\rangle$ has a fundamental decomposition with (at least) one intrinsically complete component.*

Proof. The case that $\mathcal{L}$ does not possess a Krein space completion is trivial, since then $\operatorname{Top}_{\mathrm{ip}}\mathcal{L}=\emptyset$.

For the proof of sufficiency, assume that the stated condition holds true, and let $(\iota_1,\mathcal{K}_1)$ and $(\iota_2,\mathcal{K}_2)$ be two Krein space completions of $\mathcal{L}$. Choose positive definite inner products $(.,.)_j$ on $\mathcal{K}_j$ which induce the topology of $\mathcal{K}_j$, $j=1,2$, and define a positive semidefinite inner product on $\mathcal{L}$ by

$$(x,y):=(\iota_1x,\iota_1y)_1+(\iota_2x,\iota_2y)_2,\quad x,y\in\mathcal{L}\,.$$

Then, with some appropriate constant $\gamma>0$,

$$|[x,x]|=|[\iota_1x,\iota_1x]_1|\leq\gamma(\iota_1x,\iota_1x)_1\leq\gamma(x,x),\quad x,y\in\mathcal{L}\,,$$

and hence $[.,.]$ is continuous with respect to the topology induced by $(.,.)$. Moreover, we have $(x,x)=0$ if and only if $\iota_1x=0$ and $\iota_2x=0$. Remembering

$\ker \iota_1 = \ker \iota_2 = \mathcal{L}^{[\circ]}$, we conclude that $\mathcal{L}^{(\circ)} = \mathcal{L}^{[\circ]}$. According to Lemma 3.5.10, let $(\iota_{\mathcal{H}}, \langle \mathcal{H}, (.,.)_{\mathcal{H}} \rangle)$ be a Hilbert space completion of $\langle \mathcal{L}, (.,.) \rangle$ and let $[.,.]_{\mathcal{H}}$ be the correspondingly defined inner product on $\mathcal{H}$.

Since $\ker \iota_{\mathcal{H}} = \ker \iota_j = \mathcal{L}^{[\circ]}$, there exists a linear map $\eta_j : \operatorname{ran} \iota_j \subseteq \mathcal{H} \to \mathcal{K}_j$ with $\eta \circ \iota_{\mathcal{H}} = \iota_j$. For $x, y \in \mathcal{L}$ we have

$$(\iota_j x, \iota_j x)_j \leq (x,x) = (\iota_{\mathcal{H}} x, \iota_{\mathcal{H}} x)_{\mathcal{H}}, \quad [\iota_j x, \iota_j x]_j \leq [x,x] = [\iota_{\mathcal{H}} x, \iota_{\mathcal{H}} x]_{\mathcal{H}} \,.$$

We see that η_j is continuous and $[.,.]_{\mathcal{H}}$-to-$[.,.]_j$–isometric. Let $\phi_j : \mathcal{H} \to \mathcal{K}_j$ be the extension of η_j by continuity. Then also ϕ_j is $[.,.]_{\mathcal{H}}$-to-$[.,.]_j$–isometric. The range of ϕ_j certainly contains $\operatorname{ran} \iota_j$, and hence is dense in $\mathcal{K}_j$. Thus $(\phi_j, \mathcal{K}_j)$ is a Krein space completion of $\langle \mathcal{H}, [.,.]_{\mathcal{H}} \rangle$.

Our present assumption says that $\langle \mathcal{H}, [.,.]_{\mathcal{H}} \rangle$ has a fundamental decomposition with (at least) one intrinsically complete component. Thus Proposition 3.5.6, (ii), applies, and we conclude that there exists an isomorphism ϕ of $\mathcal{K}_1$ onto $\mathcal{K}_2$ with $\phi \circ \phi_1 = \phi_2$.

$$\begin{tikzcd}
 & & \mathcal{K}_1 \arrow[dd, bend left=60, "\phi"] \\
\mathcal{L} \arrow[urr, "\iota_1"] \arrow[r, "\iota_{\mathcal{H}}"] \arrow[drr, "\iota_2"'] & \operatorname{ran} \iota_{\mathcal{H}} \arrow[r, "\subseteq"] \arrow[ur, dashed, "\eta_1"] \arrow[dr, dashed, "\eta_2"'] & \mathcal{H} \arrow[u, dashed, "\phi_1"'] \arrow[d, dashed, "\phi_2"] \\
 & & \mathcal{K}_2
\end{tikzcd}$$

We see that also $\phi \circ \iota_1 = \iota_2$, and hence $(\iota_1, \mathcal{K}_1)$ and $(\iota_2, \mathcal{K}_2)$ are isomorphic as Krein space completions of $\langle \mathcal{L}, [.,.] \rangle$.

The proof of necessity is more involved. Assume that the stated condition does not hold. This means, there exists a positive semidefinite inner product $(.,.)$ on $\mathcal{L}$ with $\mathcal{L}^{(\perp)} = \mathcal{L}^{[\perp]}$, such that (notation as in Lemma 3.5.10) the components of a fundamental decomposition of $\langle \mathcal{H}, [.,.]_{\mathcal{H}} \rangle$ are both not intrinsically complete. Let G be the Gram operator of $[.,.]_{\mathcal{H}}$ with respect to $(.,.)_{\mathcal{H}}$, and denote by E the spectral measure of G. Our hypothesis says that for every $\epsilon > 0$ we have

$$E(0, \epsilon) \neq 0 \text{ and } E(-\epsilon, 0) \neq 0 \,. \tag{3.5.4}$$ C65

We are going to construct two nonisomorphic Krein space completions of $\mathcal{L}$. One completion to be used is obvious: Let $(\iota, \mathcal{K})$ be the Krein space completion of $\langle \mathcal{H}, [.,.]_{\mathcal{H}} \rangle$ induced by the fundamental decomposition $\mathfrak{J} := (\operatorname{ran} E(0,\infty), \operatorname{ran} E(-\infty, 0))$. Since $\iota_{\mathcal{K}}$ is $\mathcal{H}$-to-$\mathcal{K}$–continuous and has dense range, also $\operatorname{ran}(\iota_{\mathcal{K}} \circ \iota_{\mathcal{H}})$ is dense in $\mathcal{K}$. Thus $(\iota_{\mathcal{K}} \circ \iota_{\mathcal{H}}, \mathcal{K})$ is a Krein space completion of $\langle \mathcal{L}, [.,.] \rangle$.

In order to obtain another, nonisomorphic, completion, we employ the construction carried out in Example 3.4.1 with the Krein space $\mathcal{K}$. To this end, we need to specify the parameters $(e_n)_{n \in \mathbb{N}}$, $(f_n)_{n \in \mathbb{N}}$, and $(\gamma_n)_{n \in \mathbb{N}}$.

Step 1, Choice of μ_n, ν_n, e_n, f_n*:* Due to (3.5.4) we can choose sequences $(\mu_n)_{n \in \mathbb{N}}$ and $(\nu_n)_{n \in \mathbb{N}}$ of numbers $\mu_n, \nu_n \in (0,1)$ which monotonically decrease to zero, such that

$$\operatorname{ran} E\big((\mu_{n+1}^2, \mu_n^2]\big) \neq \{0\}, \ \operatorname{ran} E\big([-\nu_n^2, -\nu_{n+1}^2)\big) \neq \{0\} \,.$$

Choose elements $\tilde{e}_n \in \operatorname{ran} E((\mu_{n+1}^2, \mu_n^2])$ and $\tilde{f}_n \in \operatorname{ran} E([-\nu_n^2, -\nu_{n+1}^2))$ with

$$\big[\tilde{e}_n, \tilde{e}_n\big]_{\mathcal{H}} = 1,\ \big[\tilde{f}_n, \tilde{f}_n\big]_{\mathcal{H}} = -1\,,$$

and set $e_n := \iota_{\mathcal{K}} \tilde{e}_n$ and $f_n := \iota_{\mathcal{K}} \tilde{f}_n$. Moreover, set

$$\gamma_n := \min\{\sqrt{1-\mu_n^2}, \sqrt{1-\nu_n^2}\}\,.$$

Example 3.4.1 provides us with a dense linear subspace D of $\mathcal{K}$ and a linear, bijective, $[.,.]$-isometric, but not continuous map $U : D \to D$.

Step 2, The restriction $\lambda := U \circ \iota_{\mathcal{K}}|_H$: Set

$$H_+ := \operatorname{ran} E(0,\infty) \cap \operatorname{span}\{\tilde{e}_n : n \in \mathbb{N}\}^{(\perp)_{\mathcal{H}}}\,,$$

$$H_- := \operatorname{ran} E(-\infty,0) \cap \operatorname{span}\{\tilde{f}_n : n \in \mathbb{N}\}^{(\perp)_{\mathcal{H}}}\,,$$

then

$$\big(\operatorname{span}\{\tilde{e}_n : n \in \mathbb{N}\} + \operatorname{span}\{\tilde{f}_n : n \in \mathbb{N}\}\big)^{(\perp)_{\mathcal{H}}} = H_+ + H_-\,.$$

Hence, the subspace

$$H := H_+ + H_- + \operatorname{span}\{\tilde{e}_n : n \in \mathbb{N}\} + \operatorname{span}\{\tilde{f}_n : n \in \mathbb{N}\} \tag{3.5.5}$$ C70

is dense in $\mathcal{H}$.

Isometry of $\iota_{\mathcal{K}}$ implies that $\iota_{\mathcal{K}}(H_+ + H_-) \subseteq D_0^\perp$, and we conclude that $\iota_{\mathcal{K}}(H) \supseteq D$. Thus the composition

$$\lambda := U \circ \iota_{\mathcal{K}}|_H : H \subseteq \mathcal{H} \to \mathcal{K}$$

is well-defined.

Step 3, $\mathcal{H}$-to-$\mathcal{K}$–continuity of λ: The domain of λ decomposes as

$$H = (H_+ + H_-)[\dot{+}]_{\mathcal{H}} \Big(\underset{n\in\mathbb{N}}{[\dot{+}]_{\mathcal{H}}} \operatorname{span}\{\tilde{e}_n, \tilde{f}_n\} \Big)\,, \tag{3.5.6}$$ C67

and the summands on the right hand side are also pairwise orthogonal with respect to $(.,.)_{\mathcal{H}}$. Moreover, we have

$$\lambda(H_+ + H_-) \subseteq D_0^\perp, \quad \lambda\big(\operatorname{span}\{\tilde{e}_n, \tilde{f}_n\}\big) = \operatorname{span}\{e_n, f_n\},\ n \in \mathbb{N}\,.$$

Together with the fact that the decompositions (3.5.6) and (3.4.2) are $(.,.)_{\mathcal{H}}$–orthogonal and $(.,.)_{\mathcal{K}}$–orthogonal, respectively, it is thus enough to show that the restrictions $\lambda|_{H_+ + H_-}$ and $\lambda|_{\operatorname{span}\{\tilde{e}_n, \tilde{f}_n\}}$ are bounded and that their $(.,.)_{\mathcal{H}}$-to-$(.,.)_{\mathcal{K}}$–operator norms are uniformly bounded.

Clearly, we have $\lambda|_{H_+ + H_-} = \iota_{\mathcal{K}}|_{H_- + H_-}$, and hence $\|\lambda|_{H_- + H_-}\| \leq 1$. Since $\tilde{e}_n \in \operatorname{ran} E((\mu_{n+1}^2, \mu_n^2])$, we have

$$(\iota_{\mathcal{K}} e_n, \iota_{\mathcal{K}} e_n)_{\mathcal{K}} = (e_n, e_n)_{\mathfrak{J}} = [e_n, e_n]_{\mathcal{H}} =$$
$$= (G_{(.,.)} e_n, e_n) = \int_{\mu_{n+1}^2}^{\mu_n^2} t\, dE_{e_n, e_n} \leq \mu_n^2 (e_n, e_n)_{\mathcal{H}}\,.$$

Similarly, since $f_n \in \operatorname{ran} E([-\nu_n^2, -\nu_{n+1}^2))$,

$$(\iota_{\mathcal{K}} f_n, \iota_{\mathcal{K}} f_n)_{\mathcal{K}} = (f_n, f_n)_{\mathfrak{J}} = -[f_n, f_n]_{\mathcal{H}} =$$

$$= -(G_{(.,.)} f_n, f_n) = -\int_{-\nu_n^2}^{-\nu_{n+1}^2} t \, dE_{f_n, f_n} \leq \nu_n^2 (f_n, f_n)_{\mathcal{H}} \, .$$

Since $e_n (\perp)_{\mathcal{H}} f_n$ this implies that $\|\iota_{\mathcal{K}}|_{\operatorname{span}\{e_n, f_n\}}\| \leq \max\{\mu_n, \nu_n\}$.

The operator norm $\|.\|$ on $\mathbb{C}^{2\times 2}$ is equivalent to the maximum-entry norm. Hence there exists a constant $C > 0$ such that

$$\|u_n\| \leq \frac{C}{\sqrt{1-\gamma_n^2}} \, .$$

Note here that C does not depend on $n \in \mathbb{N}$. Since $\varphi : S_n \to \mathbb{C}$ is an isomorphism, it follows that

$$\|\lambda|_{\operatorname{span}\{e_n, f_n\}}\| \leq C \frac{\max\{\mu_n, \nu_n\}}{\sqrt{1-\gamma_n^2}} = C, \ n \in \mathbb{N} \, .$$

We have shown that λ is $(.,.)_{\mathcal{H}}$-to-$(.,.)_{\mathcal{K}}$–continuous.

Step 4, The nonisomorphic completion: Let $\eta : \mathcal{H} \to \mathcal{K}$ be the extension of λ by continuity. Since $\iota_{\mathcal{K}}$ is $[.,.]_{\mathcal{H}}$-to-$[.,.]_{\mathcal{K}}$–isometric, and all maps U_0, U_n, $n \in \mathbb{N}$, are $[.,.]_{\mathcal{K}}$–isometric, also the map λ is $[.,.]_{\mathcal{H}}$-to-$[.,.]_{\mathcal{K}}$–isometric. By continuity, thus also η has this property. Moreover, since $\iota_{\mathcal{K}}$ is continuous and has dense range, the space $\iota_{\mathcal{K}}(H)$ is dense in $\mathcal{K}$. However, we have $U(\iota_{\mathcal{K}}(H)) = \iota_{\mathcal{K}}(H)$, and hence $\operatorname{ran}\lambda$ is dense in $\mathcal{K}$. It follows that $(\eta, \mathcal{K})$ is a completion of $\langle \mathcal{H}, [.,.]_{\mathcal{H}} \rangle$.

Since η is $\mathcal{H}$-to-$\mathcal{K}$–continuous and has dense ranges, also $\operatorname{ran}(\iota_{\mathcal{K}} \circ \iota_{\mathcal{H}})$ is dense in $\mathcal{K}$. Thus $(\eta \circ \iota_{\mathcal{H}}, \mathcal{K})$ is a Krein space completion of $\langle \mathcal{L}, [.,.] \rangle$.

$$\begin{tikzcd}
\mathcal{K} & D \arrow[l, "\supseteq"'] \arrow[r, "U"] & D \arrow[r, "\subseteq"] & \mathcal{K} \\
\mathcal{H} \arrow[u, "\iota_{\mathcal{K}}"] \arrow[rrru, dotted, bend right, "\eta"'] & H \arrow[l, "\supseteq"'] \arrow[u, "\iota_{\mathcal{K}}"] \arrow[rru, "\lambda"] & & \\
\mathcal{L} \arrow[u, "\iota_{\mathcal{H}}"] \arrow[uu, bend left, "\iota_{\mathcal{K}}\circ\iota_{\mathcal{H}}"] \arrow[rrruu, bend right, "\eta\circ\iota_{\mathcal{H}}"'] & & &
\end{tikzcd}$$

Assume on the contrary that $(\iota_{\mathcal{K}} \circ \iota_{\mathcal{H}}, \mathcal{K}) \succeq (\eta \circ \iota_{\mathcal{H}}, \mathcal{K})$. Then there exists a continuous map $\phi : \mathcal{K} \to \mathcal{K}$ with

$$\begin{tikzcd}
& \mathcal{L} \arrow[dl, "\iota_{\mathcal{K}}\circ\iota_{\mathcal{H}}"'] \arrow[dr, "\iota'\circ\iota_{\mathcal{H}}"] & \\
\mathcal{K} \arrow[rr, "\phi"'] & & \mathcal{K}
\end{tikzcd}$$

This says that $(\phi \circ \iota_{\mathcal{K}})|_{\operatorname{ran}\iota_{\mathcal{H}}} = \eta|_{\operatorname{ran}\iota_{\mathcal{H}}}$. Since the maps $\iota_{\mathcal{K}}$ and η are continuous, it follows that actually $\phi \circ \iota_{\mathcal{K}} = \iota'$. Consider the linear map $U|_{\iota_{\mathcal{K}}(H)}$. Since the eigenvectors of U_n, $n \in \mathbb{N}$, belong to $\iota_{\mathcal{K}}(H)$, we conclude that $U|_{\iota_{\mathcal{K}}(H)}$ is unbounded. However, we have

$$U(\iota_{\mathcal{K}} x) = \iota'(x) = \phi(\iota_{\mathcal{K}} x), \quad x \in \iota_{\mathcal{K}}^{-1}(D) \supseteq H \, ,$$

and hence $U|_{\iota_{\mathcal{K}}(H)}$ is bounded. We have derived a contradiction, and conclude that

$$(\iota_{\mathcal{K}} \circ \iota_{\mathcal{H}}, \mathcal{K}) \not\leq (\iota' \circ \iota_{\mathcal{H}}, \mathcal{K}),$$

in particular these completions cannot be isomorphic. ❑

Chapter 4

Classes of complete TIPS. II. Pontryagin spaces

4.1 Definition of Pontryagin spaces

DED1 **4.1.1 Definition.** An inner product space $\langle \mathcal{P}, [.,.] \rangle$ is called a *Pontryagin space*, if

(PS1) $\mathcal{P}$ is nondegenerated and $\operatorname{ind}_- \mathcal{P} < \infty$.

(PS2) There exists a fundamental decomposition $(\mathcal{P}_+, \mathcal{P}_-)$ of $\mathcal{P}$ whose component $\mathcal{P}_+$ is intrinsically complete.

//

Appealing to Proposition 2.6.4, we see that this definition does not depend on the particular choice of the fundamental decomposition in (Ps2). Moreover, let us remark that, of course, a completely parallel theory could be developed for spaces with $\operatorname{ind}_+ \mathcal{P} < \infty$ instead of $\operatorname{ind}_- \mathcal{P} < \infty$.

Again we start with some immediate reformulations of the defining property of a Pontryagin space.

RED2 *4.1.2 Remark.* Let $\langle \mathcal{P}, [.,.] \rangle$ be an inner product space. Then the following are equivalent:

(i) $\langle \mathcal{P}, [.,.] \rangle$ is a Pontryagin space.

(ii) $\langle \mathcal{P}, [.,.] \rangle$ is a Krein space and $\operatorname{ind}_- \mathcal{P} < \infty$.

(iii) There exists a Hilbert space $\mathcal{H}_1$ and a finite-dimensional negative definite space $\mathcal{H}_2$, such that
$$\langle \mathcal{P}, [.,.] \rangle = \mathcal{H}_1 [\dot{+}] \mathcal{H}_2 \,,$$
where $\mathcal{H}_1 [\dot{+}] \mathcal{H}_2$ is endowed with the sum inner product.

(iv) $\mathcal{P}$ is nondegenerated, $\operatorname{ind}_- \mathcal{P} < \infty$, and $\mathcal{P}$ is complete with respect to $\mathcal{T}^{\curlywedge}$. For the use of the terminology 'complete with respect to $\mathcal{T}^{\curlywedge}$' compare Remark 2.6.8, *(ii)*.

//

RED3 *4.1.3 Remark.* A Pontryagin space is a particular instance of a Krein space. Hence:

(*i*) In view of Remark 3.1.3 and Definition 2.6.7, we have

$$\operatorname{Top}_{\mathrm{Hs}} \mathcal{P} = \operatorname{Top}_{\mathrm{Bs}} \mathcal{P} = \operatorname{Top}_{\mathrm{dec}} \mathcal{P} = \operatorname{Top}_{\min} \mathcal{P} = \{\mathcal{T}^{\curlywedge}\}.$$

Unless the contrary is explicitly stated, all topological notions will be understood with respect to this topology.

(*ii*) The dual space of $\mathcal{P}$ is given as

$$\langle \mathcal{P}, \mathcal{T}^{\curlywedge} \rangle' = \big\{ [.,y] : y \in \mathcal{P} \big\}.$$

//

The fact whether $\mathcal{P}$ is a Pontryagin space can also be characterized via Gram operators.

PRD4 **4.1.4 Proposition.** *Let $\langle \mathcal{P}, [.,.] \rangle$ be an inner product space. Then the following are equivalent:*

(*i*) *$\langle \mathcal{P}, [.,.] \rangle$ is a Pontryagin space.*

(*ii*) *There exists an inner product $(.,.)$ on $\mathcal{P}$ with $\mathcal{T}_{(.,.)} \in \operatorname{Top}_{\mathrm{Hs}} \mathcal{P}$, such that the Gram operator G of $[.,.]$ with respect to $(.,.)$ satisfies $0 \in \rho(G)$ and*

$$\dim \operatorname{ran} E(-\infty, 0) < \infty. \tag{4.1.1}$$ D5

Here E denotes the spectral measure of G as a bounded selfadjoint operator in the Hilbert space $\langle \mathcal{P}, (.,.) \rangle$.

(*iii*) *$\mathcal{P}$ is nondegenerated and there exists $\mathcal{T} \in \operatorname{Top}_{\mathrm{Hs}} \mathcal{P}$ and $\mathcal{M} \in \operatorname{Sub}_{>0} \mathcal{P}$ with $\dim \mathcal{P}/\mathcal{M} < \infty$, such that $\mathcal{M}$ is $\mathcal{T}$-closed and intrinsically complete.*

Proof. The equivalence of (*i*) and (*ii*) is obtained as a corollary of the corresponding result for Krein spaces, cf. Theorem 3.1.5. We know from this source that $\mathcal{P}$ is a Krein space if and only if the present item (*ii*) without the condition (4.1.1) holds. Thereby, a fundamental decomposition is given as $(\mathcal{P}_+, \mathcal{P}_-)$ with

$$\mathcal{P}_+ := \operatorname{ran} E(0,\infty),\ \mathcal{P}_- := \operatorname{ran} E(-\infty, 0).$$

Thus $\mathcal{P}$ is a Krein space with finite negative index, if and only if in addition $\dim \operatorname{ran} E(-\infty,0) < \infty$.

For the proof of (*i*) $\Rightarrow$ (*iii*), choose a fundamental decomposition $\mathfrak{J} = (\mathcal{P}_+, \mathcal{P}_-)$ of $\mathcal{P}$. Then the inner product $(.,.)_{\mathfrak{J}}$ and the subspace $\mathcal{P}_+$ has the desired properties. Note here that $(x,y)_{\mathfrak{J}} = [x,y]$, $x,y \in \mathcal{P}_+$, i.e. intrinsic completeness equals $\|.\|_{\mathfrak{J}}$-completeness.

Finally, assume that (*iii*) holds. Choose a positive definite inner product $(.,.)$ on $\mathcal{P}$ such that $\mathcal{T} = \mathcal{T}_{(.,.)}$, then $(.,.)$ is an inner product with $\mathcal{T}_{(.,.)} \in \operatorname{Top}_{\mathrm{Hs}} \mathcal{P}$. Let G be the Gram operators of $[.,.]$ with respect to $(.,.)$. Since $\mathcal{M}$ is $(.,.)$-closed, and hence itself a Hilbert space, there exists a Gram operator $G_{\mathcal{M}}$ of $[.,.]|_{\mathcal{M}\times\mathcal{M}}$ with respect to $(.,.)|_{\mathcal{M}\times\mathcal{M}}$. The space $\mathcal{M}$ is not only $(.,.)$-complete, but also intrinsically complete, i.e. complete with respect to the norm $[.,.]^{\frac{1}{2}}$.

Hence, the scalar products $(.,.)$ and $[.,.]$ induce equivalent norms. Therefore, $G_{\mathcal{M}} \geq \delta I$ for some $\delta > 0$.

Assume that $x \in \operatorname{ran} E(-\infty, \frac{\delta}{2}) \cap \mathcal{M}$, where E denotes the spectral measure of G. Then

$$\delta(x,x) \leq (G_{\mathcal{M}}x,x) = [x,x] = (Gx,x) \leq \frac{\delta}{2}(x,x)\,,$$

and it follows that $x = 0$. This shows that $\operatorname{ran} E(-\infty, \frac{\delta}{2}) \cap \mathcal{M} = \{0\}$, and hence

$$\dim \operatorname{ran} E(-\infty, \frac{\delta}{2}) \leq \dim \mathcal{P}/\mathcal{M} < \infty\,.$$

Since $\mathcal{P}$ is nondegenerated, we have $\ker G = \{0\}$. Hence, by discreteness of the spectrum in $(-\infty, \frac{\delta}{2}]$, $0 \in \rho(G)$. ❑

RED6 *4.1.5 Remark.* Let $\langle \mathcal{P}, [.,.] \rangle$ be a Pontryagin space, and let $(.,.)$ any inner product on $\mathcal{P}$ with $\mathcal{T}_{(.,.)} \in \operatorname{Top}_{\mathrm{Hs}} \mathcal{P}$. Then the corresponding Gram operator satisfies $0 \in \rho(G)$ and $\dim \operatorname{ran} E(-\infty, 0) < \infty$.

This follows, since in the proof of $(iii) \Rightarrow (ii)$ above, we actually have started with an arbitrary inner product $(.,.)$ on $\mathcal{P}$ with $\mathcal{T}_{(.,.)} \in \operatorname{Top}_{\mathrm{Hs}} \mathcal{P}$. //

Let us give another characterization of Pontryagin spaces which is of more intrinsic nature, and rather related to Remark 4.1.2, (iv). For the use of the term 'completeness', the same notice as in this place applies.

PRD7 **4.1.6 Proposition.** *Let $\langle \mathcal{P}, [.,.] \rangle$ be an inner product space. Then $\mathcal{P}$ is a Pontryagin space, if and only if $\mathcal{P}$ is nondegenerated and there exists $\mathcal{M} \in \operatorname{Sub}_{>0} \mathcal{P}$ with $\dim \mathcal{P}/\mathcal{M} < \infty$, such that $\mathcal{M}$ is complete with respect to $\mathcal{T}^{\curlywedge}|_{\mathcal{M}}$.*

Proof. If $\mathcal{P}$ is a Pontryagin space, choose a fundamental decomposition $\mathfrak{J} = (\mathcal{P}_+, \mathcal{P}_-)$, and set $\mathcal{M} := \mathcal{P}_+$. Then, clearly, $\mathcal{M} \in \operatorname{Sub}_{>0} \mathcal{P}$ and $\dim \mathcal{P}/\mathcal{M} < \infty$. Since $\|x\|_{\mathfrak{J}} = [x,x]^{\frac{1}{2}}$, $x \in \mathcal{P}_+$, the subspace $\mathcal{M}$ is moreover complete with respect $\|.\|_{\mathfrak{J}}$.

Conversely, assume that a subspace $\mathcal{M}$ with the stated properties exists. First of all, if $\mathcal{N} \in \operatorname{Sub}_{<0} \mathcal{P}$, then $\mathcal{N} \cap \mathcal{M} = \{0\}$, and hence $\dim \mathcal{N} \leq \dim \mathcal{P}/\mathcal{M}$. Thus

$$\operatorname{ind}_- \mathcal{P} \leq \dim \mathcal{P}/\mathcal{M} < \infty\,.$$

In particular, $\mathcal{P}$ is decomposable and the topology $\mathcal{T}^{\curlywedge}$ is well-defined.

Choose a fundamental decomposition $\mathfrak{J} = (\mathcal{P}_+, \mathcal{P}_-)$ of $\mathcal{P}$, and consider the positive definite inner product $(.,.)_{\mathfrak{J}}$. Since the subspace $\mathcal{M}$ is complete with respect to $(.,.)_{\mathfrak{J}}$, we have

$$\mathcal{M} (\dot{+})_{\mathfrak{J}} \mathcal{M}^{(\perp)_{\mathfrak{J}}} = \mathcal{P}\,.$$

Here we refer to the notice made in Remark 2.6.6. However, since

$$\dim \mathcal{M}^{(\perp)_{\mathfrak{J}}} = \dim \mathcal{P}/\mathcal{M} < \infty\,,$$

also the space $\mathcal{M}^{(\perp)_{\mathfrak{J}}}$ is complete with respect to $(.,.)_{\mathfrak{J}}$. Thus $\mathcal{P}$ is $(.,.)_{\mathfrak{J}}$-complete, and we conclude that $\mathcal{P}$ is a Pontryagin space, cf. Remark 4.1.2, (iv). ❑

RED8 *4.1.7 Remark.* Let $\langle \mathcal{P}_1, [.,.]_1\rangle$ and $\langle \mathcal{P}_2, [.,.]_2\rangle$ be Pontryagin spaces, and consider the direct sum $\mathcal{P} := \mathcal{P}_1 \dot{+} \mathcal{P}_2$ endowed with the sum inner product. Then $\langle \mathcal{K}, [.,.]\rangle$ is a Krein space, cf. Remark 3.1.8. However, we have $\operatorname{ind}_- \mathcal{P} = \operatorname{ind}_- \mathcal{P}_1 + \operatorname{ind}_- \mathcal{P}_2$, and hence $\mathcal{P}$ is a Pontryagin space. //

Pontryagin spaces are very well-behaved instances of Krein spaces. It is the following simple fact, which is responsible that many pecularities of Krein spaces disappear.

LED9 **4.1.8 Lemma.** *Let $\langle \mathcal{L}, [.,.], \mathcal{T}\rangle$ be a topological inner product space, and let $\mathcal{M} \in \operatorname{Sub} \mathcal{L}$. If $\mathcal{N} \in \operatorname{Sub}_{<0} \overline{\mathcal{M}}$, and $n \in \mathbb{N}$ with $n \leq \dim \mathcal{N}$, then there exists a subspace $\mathcal{N}' \in \operatorname{Sub}_{<0} \mathcal{M}$ with $\dim \mathcal{N}' = n$.*

In particular, if $\operatorname{ind}_- \overline{\mathcal{M}}$ is finite, then $\operatorname{ind}_- \mathcal{M} = \operatorname{ind}_- \overline{\mathcal{M}}$.

Proof. Choose a n-dimensional subspace $\mathcal{N}_1$ of $\mathcal{N}$ and write $\mathcal{N}_1 = \operatorname{span}\{x_1, \dots, x_n\}$. Then the matrix $A := ([x_i, x_j])_{i,j=1}^n$ is negative definite, i.e. all zeros of the polynomial

$$p(\lambda) := \det(A - \lambda I)$$

are negative.

Since a polynomial depends continuously on its coefficients in the topology of locally uniform convergence, there exists some $\epsilon > 0$ such that the polynomial $\det(A' - \lambda I)$ has exclusively negative zeros whenever $\|A' - A\| < \epsilon$. Here $\|.\|$ denotes some matrix norm.

Since $\mathcal{M}$ is dense in $\overline{\mathcal{M}}$, there exist elements $x_1', \dots, x_n' \in \mathcal{L}$, such that

$$\big\| \big([x_i', x_j']\big)_{i,j=1}^n - \big([x_i, x_j]\big)_{i,j=1}^n \big\| < \epsilon \,.$$

By what we said above, this implies that the matrix $([x_i', x_j'])_{i,j=1}^n$ is negative definite. The space

$$\mathcal{M} := \operatorname{span}\{x_1', \dots, x_n'\} \subseteq \mathcal{L}$$

is negative definite and has dimension n. ❑

COD10 **4.1.9 Corollary.** *Let $\langle \mathcal{P}, [.,.]\rangle$ be a Pontryagin space, and let $\mathcal{L}$ be a dense linear subspace. Then there exists a maximal negative subspace $\mathcal{M}$ with $\mathcal{M} \subseteq \mathcal{L}$.*

Proof. We have $\kappa := \operatorname{ind}_- \mathcal{L} < \infty$. By the above lemma, $\mathcal{P} = \overline{\mathcal{L}}$ cannot contain any negative subspace with dimension $\kappa+1$. Thus $\operatorname{ind}_- \mathcal{P} = \kappa$, and we conclude that each maximal negative subspace of $\mathcal{L}$ is already maximal negative in $\mathcal{P}$. ❑

As a first consequence, we obtain a characterization of the topology $\mathcal{T}^{\curlywedge}$, which is explicit in terms of the inner product $[.,.]$, i.e. does not refer to some fundamental decomposition.

PRD11 **4.1.10 Proposition.** *Let $\langle \mathcal{P}, [.,.]\rangle$ be a Pontryagin space, let $x_n \in \mathcal{P}$, $n \in \mathbb{N}$, and $x \in \mathcal{P}$. Then the following hold:*

(*i*) *We have $\lim_{n\to\infty} x_n = x$ with respect to $\mathcal{T}^{\curlywedge}$, if and only if there exists a dense subset D of $\mathcal{P}$, such that*

$$\lim_{n\to\infty} [x_n, x_n] = [x, x], \qquad \lim_{n\to\infty} [x_n, y] = [x, y],\ y \in D\,. \tag{4.1.2}$$ D12

(*ii*) *The sequence* $(x_n)_{n\in\mathbb{N}}$ *is a Cauchy sequence with respect to* $\mathcal{T}^\curlywedge$ *(i.e. with respect to some norm inducing* $\mathcal{T}^\curlywedge$*), if and only if there exists a maximal negative subspace* $\mathcal{M}$ *of* $\mathcal{P}$*, such that*

$$\lim_{n,m\to\infty}[x_n - x_m, x_n - x_m] = 0, \qquad \lim_{n,m\to\infty}[x_n - x_m, y] = 0,\ y \in \mathcal{M}\,. \tag{4.1.3}$$ D13

Proof. Let $\mathfrak{J} = (\mathcal{P}_+, \mathcal{P}_-)$ be any fundamental decomposition of $\mathcal{P}$. Then $|[x,y]| \leq \|x\|_\mathfrak{J}\|y\|_\mathfrak{J}$, $x, y \in \mathcal{P}$. Thus the stated conditions (4.1.2) and (4.1.3), respectively, are necessary. Thereby, we may take $D := \mathcal{P}$.

Conversely, assume that D is a dense subset of $\mathcal{P}$ such that (4.1.2) or (4.1.3) holds. Clearly, we may assume without loss of generality that D is a linear subspace of $\mathcal{P}$. According to Corollary 4.1.9, we can choose a maximal negative subspace $\mathcal{M}$ which is contained in D. Then $\mathfrak{J} := (\mathcal{M}^\perp, \mathcal{M})$ is a fundamental decomposition of $\mathcal{P}$.

Assume that (4.1.2) holds, and consider the sequences $(P_\mathfrak{J}^+ x_n)_{n\in\mathbb{N}}$ and $(P_\mathfrak{J}^- x_n)_{n\in\mathbb{N}}$. The second relation in (4.1.2), together with our choice of $\mathcal{M}$ as a subspace of D, gives

$$\lim_{n\to\infty}[P_\mathfrak{J}^- x_n, y] = \lim_{n\to\infty}[x_n, y] = [x,y] = [P_\mathfrak{J}^- x, y],\ y \in \mathcal{M}\,.$$

Since $\mathcal{M}$ is finite-dimensional and negative definite, this implies that

$$\lim_{n\to\infty} P_\mathfrak{J}^- x_n = P_\mathfrak{J}^- x \text{ w.r.t. } \|.\|_\mathfrak{J}|_\mathcal{M} = (-[.,.])^{\frac{1}{2}}\,.$$

Using the first relation in (4.1.2), it follows that

$$\lim_{n\to\infty}[P_\mathfrak{J}^+ x_n, P_\mathfrak{J}^+ x_n] = \lim_{n\to\infty}\Big([x_n, x_n] - [P_\mathfrak{J}^- x_n, P_\mathfrak{J}^- x_n]\Big) =$$
$$= [x,x] - [P_\mathfrak{J}^- x, P_\mathfrak{J}^- x] = [P_\mathfrak{J}^+ x, P_\mathfrak{J}^+ x]\,.$$

Moreover, by the second relation in (4.1.2),

$$\lim_{n\to\infty}[P_\mathfrak{J}^+ x_n, y] = \lim_{n\to\infty}[x_n, y] = [x,y] = [P_\mathfrak{J}^+ x, y],\ y \in D \cap \mathcal{M}^\perp\,.$$

Since $\mathcal{M} \subseteq D$ the set $D \cap \mathcal{M}^\perp$ is dense in the Hilbert space $\mathcal{M}^\perp$. Thus

$$\lim_{n\to\infty} P_\mathfrak{J}^+ x_n = P_\mathfrak{J}^+ x \text{ w.r.t. } \|.\|_\mathfrak{J}|_{\mathcal{M}^\perp} = [.,.]^{\frac{1}{2}}\,.$$

In total $\lim_{n\to\infty} x_n = x$ with respect to $\|.\|_\mathfrak{J}$.

Assume that (4.1.3) holds. We argue similarly, and show that both of $(P_\mathfrak{J}^+ x_n)_{n\in\mathbb{N}}$ and $(P_\mathfrak{J}^- x_n)_{n\in\mathbb{N}}$ are Cauchy sequences in the norm $\|.\|_\mathfrak{J}$. First,

$$\lim_{n,m\to\infty}[P_\mathfrak{J}^- x_n - P_\mathfrak{J}^- x_m, y] = 0,\ y \in \mathcal{M}\,.$$

Again finite dimensionality implies that $(P_\mathfrak{J}^+ x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the norm $(-[.,.])^{\frac{1}{2}}$. Next, we compute

$$\lim_{n,m\to\infty}[P_\mathfrak{J}^+ x_n - P_\mathfrak{J}^+ x_m, P_\mathfrak{J}^+ x_n - P_\mathfrak{J}^+ x_m] =$$
$$= \lim_{n,m\to\infty}\Big([x_n - x_m, x_n - x_m] - [P_\mathfrak{J}^- x_n - P_\mathfrak{J}^- x_m, P_\mathfrak{J}^- x_n - P_\mathfrak{J}^- x_m]\Big) = 0\,.$$

❑

4.2 Fundamental decompositions, orthocomplemented subspaces

Also the geometry of Pontryagin spaces is significantly simpler than the one of general Krein spaces. This fact origins from the following result.

LED14 **4.2.1 Lemma.** *Let $\langle\mathcal{P},[.,.]\rangle$ be a Pontryagin space, and let $\mathcal{M}\in\operatorname{Sub}\mathcal{P}$. Then*

(i) If $\mathcal{M}$ is closed and positive definite, then $\mathcal{M}$ is uniformly positive.

(ii) If $\mathcal{M}$ is negative definite, then $\mathcal{M}$ is uniformly negative.

Proof. Let $\mathfrak{J}=(\mathcal{P}_+,\mathcal{P}_-)$ be a fundamental decomposition of $\mathcal{P}$. Assume that $\mathcal{M}\in\operatorname{Sub}_{>0}\mathcal{P}$ and that $\mathcal{M}$ is closed, and consider the angular operator

$$\mathfrak{a}_{\mathfrak{J}}(\mathcal{M}):P_{\mathfrak{J}}^+\mathcal{M}\subseteq\mathcal{P}_+\to\mathcal{P}_-\,.$$

Then $\mathfrak{a}_{\mathfrak{J}}(\mathcal{M})$ is $\|.\|_{\mathfrak{J}}$-contractive, in fact

$$\|\mathfrak{a}_{\mathfrak{J}}(\mathcal{M})x\|_{\mathfrak{J}}^2=-\big[\mathfrak{a}_{\mathfrak{J}}(\mathcal{M})x,\mathfrak{a}_{\mathfrak{J}}(\mathcal{M})x\big]<[x,x]=\|x\|_{\mathfrak{J}}^2,\ x\in P_{\mathfrak{J}}^+\mathcal{M}\setminus\{0\}\,. \quad (4.2.1)$$ D15

Since $\mathcal{M}$ is closed, it is $\|.\|_{\mathfrak{J}}$-complete. By Lemma 2.6.5, also $P_{\mathfrak{J}}^+\mathcal{M}$ is $\|.\|_{\mathfrak{J}}$-complete. The subspace $\ker\mathfrak{a}_{\mathfrak{J}}(\mathcal{M})$ is a $\|.\|_{\mathfrak{J}}$-closed subspace of $P_{\mathfrak{J}}^+\mathcal{M}$, hence we may write $P_{\mathfrak{J}}^+\mathcal{M}=\ker\mathfrak{a}_{\mathfrak{J}}(\mathcal{M})[\dot+]\mathcal{F}$. Note here once more that $\|.\|_{\mathfrak{J}}|_{\mathcal{P}_+}$ is induced by the inner product $[.,.]$. We have

$$\dim\mathcal{F}=\dim\operatorname{ran}\mathfrak{a}_{\mathfrak{J}}(\mathcal{M})\leq\dim\mathcal{P}_-<\infty\,.$$

Thus the unit ball of $\mathcal{F}$ is compact, and we obtain an element $x_0\in\mathcal{F}$, $\|x_0\|_{\mathfrak{J}}\leq 1$, with

$$\sup_{x\in\mathcal{F},\|x\|_{\mathfrak{J}}\leq 1}\|\mathfrak{a}_{\mathfrak{J}}(\mathcal{M})x\|_{\mathfrak{J}}=\|\mathfrak{a}_{\mathfrak{J}}(\mathcal{M})x_0\|_{\mathfrak{J}}\,.$$

Remembering (4.2.1), it follows that

$$\|\mathfrak{a}_{\mathfrak{J}}(\mathcal{M})\|=\sup_{\substack{x\in P_{\mathfrak{J}}^+\mathcal{M}\\ \|x\|_{\mathfrak{J}}\leq 1}}\|\mathfrak{a}_{\mathfrak{J}}(\mathcal{M})x\|_{\mathfrak{J}}=\sup_{\substack{x\in\mathcal{F}\\ \|x\|_{\mathfrak{J}}\leq 1}}\|\mathfrak{a}_{\mathfrak{J}}(\mathcal{M})x\|_{\mathfrak{J}}=$$

$$=\|\mathfrak{a}_{\mathfrak{J}}(\mathcal{M})x_0\|_{\mathfrak{J}}\begin{cases}<\|x_0\|_{\mathfrak{J}}=1\,, & x_0\neq 0\\ 0\quad , & x_0=0\end{cases}.$$

This shows that $\mathcal{M}$ is uniformly positive.

The proof of (ii) is similar, even simpler. Let $\bar{\mathfrak{J}}$ be the orthogonal decomposition $\bar{\mathfrak{J}}:=(\mathcal{P}_-,\mathcal{P}_+)$, let $\mathcal{M}\in\operatorname{Sub}_{<0}\mathcal{P}$, and consider the angular operator

$$\mathfrak{a}_{\bar{\mathfrak{J}}}(\mathcal{M}):P_{\mathfrak{J}}^-\mathcal{M}\subseteq\mathcal{P}_-\to\mathcal{P}_+\,.$$

Then $\dim P_{\mathfrak{J}}^-\mathcal{M}\leq\dim\mathcal{P}_-<\infty$, and the same compactness argument as above will apply and yield that $\|\mathfrak{a}_{\bar{\mathfrak{J}}}(\mathcal{M})\|<1$. ❑

As an immediate corollary, we obtain the following Pontryagin space version of Theorem 3.2.2.

COD16 **4.2.2 Corollary.** *Let $\langle \mathcal{P}, [.,.] \rangle$ be a Pontryagin space, and let $\mathcal{L}_+, \mathcal{L}_- \in \operatorname{Sub}\mathcal{P}$. Then there exists a fundamental decomposition $\mathfrak{J} = (\mathcal{P}_+, \mathcal{P}_-)$ of $\mathcal{P}$ with*

$$\mathcal{L}_+ \subseteq \mathcal{P}_+ \quad \textit{and} \quad \mathcal{L}_- \subseteq \mathcal{P}_- \tag{4.2.2}$$ D17

if and only if

$$\mathcal{L}_+ \in \operatorname{Sub}_{\gg 0}\mathcal{P},\ \mathcal{L}_- \in \operatorname{Sub}_{<0}\mathcal{P}, \quad \mathcal{L}_+ \perp \mathcal{L}_- .$$

If we assume that $\mathcal{L}_+$ is closed, then there exists $\mathfrak{J}$ with (4.2.2) *if and only if $\mathcal{L}_+ \in \operatorname{Sub}_{>0}\mathcal{P}$, $\mathcal{L}_- \in \operatorname{Sub}_{<0}\mathcal{P}$, and $\mathcal{L}_+ \perp \mathcal{L}_-$.* ❑

The Pontryagin space version of Theorem 3.3.3 on characterizing orthocomplementedness is less evident, and actually a quite important result.

PRD18 **4.2.3 Proposition.** *Let $\langle \mathcal{P}, [.,.] \rangle$ be a Pontryagin space, and let $\mathcal{L} \in \operatorname{Sub}\mathcal{P}$. Then the following are equivalent:*

- (i) *$\mathcal{L}$ is orthocomplemented.*
- (ii) *$\mathcal{L}$ is closed and nondegenerated.*
- (iii) *$\mathcal{L}$ is decomposable, nondegenerated, and for each fundamental decomposition $\mathfrak{J}_{\mathcal{L}} = (\mathcal{L}_+, \mathcal{L}_-)$ of $\mathcal{L}$ the component $\mathcal{L}_+$ is closed in $\mathcal{P}$.*
- (iii') *We have $\mathcal{L} = \mathcal{L}_+[\dot{+}]\mathcal{L}_-$ with some subspaces $\mathcal{L}_+ \in \operatorname{Sub}_{>0}\mathcal{P}$ and $\mathcal{L}_- \in \operatorname{Sub}_{<0}\mathcal{P}$, where $\mathcal{L}_+$ is closed in $\mathcal{P}$.*
- (iv) *$\mathcal{L}$ is closed in $\mathcal{P}$ and $\langle \mathcal{L}, [.,.] \rangle$ is a Pontryagin space.*
- (iv') *The closure $\overline{\mathcal{L}}$ is nondegenerated and $\langle \mathcal{L}, [.,.] \rangle$ is a Pontryagin space.*

Proof. The implications indicated on the left are immediate from Theorem 3.3.3, those on the right are deduced from Theorem 3.3.3 with the help of Lemma 4.2.1:

$$\begin{tikzcd}
 & (iv) \arrow[d, Rightarrow, bend left] & \\
(iii') & (i) \arrow[l, Rightarrow] \arrow[r, Rightarrow] \arrow[d, Rightarrow] \arrow[u, Rightarrow, bend left] & (ii) \\
 & (iii) &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
(iii') \arrow[r, Rightarrow] & (i) \\
 & (iii) \arrow[u, Rightarrow]
\end{tikzcd}$$

Moreover, we have $(ii) \Rightarrow (iii')$ since, by Proposition 3.3.1, (i), a closed subspace has a fundamental decomposition with closed components. This establishes the equivalence of $(i), (ii), (iii)$, and (iii').

The implication $(iv) \Rightarrow (iv')$ is trivial. To finish the proof it is hence enough to show that (iv') implies that $\mathcal{L}$ is closed in $\mathcal{P}$. To this end we will employ Proposition 4.1.10. Let a sequence $(x_n)_{n\in\mathbb{N}}$ of points $x_n \in \mathcal{L}$ be given, and assume that $\lim_{n\to\infty} x_n = x$ in $\mathcal{P}$. Then, certainly,

$$\lim_{n,m\to\infty}[x_n - x_m, x_n - x_m] = 0, \quad \lim_{n\to\infty}[x_n - x_m, y] = 0,\ y \in \mathcal{L}.$$

Thus $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the Pontryagin space $\langle\mathcal{L},[.,.]\rangle$, and hence convergent, say $\lim_{n\to\infty} x_n = x'$ in $\mathcal{L}$.

By convergence in $\mathcal{P}$ on the one hand and convergence in $\mathcal{L}$ on the other, we have

$$[x,y] = \lim_{n\to\infty}[x_n,y] = [x',y],\ y\in\mathcal{L}\,,$$

$$[x,y] = \lim_{n\to\infty}[x_n,y] = 0 = [x',y],\ y\in\mathcal{L}^\perp\,.$$

Thus $[x-x',y]=0$ whenever $y\in\mathcal{L}+\mathcal{L}^\perp$. However,

$$(\mathcal{L}+\mathcal{L}^\perp)^\perp = \mathcal{L}^\perp\cap\mathcal{L}^{\perp\perp} = \overline{\mathcal{L}}^\perp\cap\overline{\mathcal{L}} = (\overline{\mathcal{L}})^\circ = \{0\}\,,$$

and hence $x-x'=0$. In particular $x'\in\mathcal{L}$. ❑

4.3 Isometric mappings, Completions

We saw that an isometric map defined on some subspace of a Krein space need not be continuous. This does not change in the Pontryagin space situation. However, we can give a quite useful condition which implies continuity.

PRD19 **4.3.1 Proposition.** *Let $\langle\mathcal{P}_1,[.,.]_1\rangle$ and $\langle\mathcal{P}_2,[.,.]_2\rangle$ be Pontryagin spaces, and let*

$$\phi:\operatorname{dom}\phi\subseteq\mathcal{P}_1\to\mathcal{P}_2$$

be isometric. If $\overline{\operatorname{ran}\phi}$ is nondegenerated, then ϕ is continuous. Its continuation $\tilde\phi$ by continuity is isometric and maps $\overline{\operatorname{dom}\phi}$ surjectively onto $\overline{\operatorname{ran}\phi}$.

Proof. Since $\overline{\operatorname{ran}\phi}$ is closed and nondegenerated, it is itself a Pontryagin space and the topology it carries as such coincides with the topology it inherits from $\mathcal{P}_2$. For the proof of the present assertion we may therefore consider ϕ as a map of $\operatorname{dom}\phi$ into $\overline{\operatorname{ran}\phi}$. Hence, assume throughout the following that $\operatorname{ran}\phi$ is dense in $\mathcal{P}_2$.

Let D_- be a maximal negative subspace of $\operatorname{dom}\phi$. Then $\mathcal{P}_1 = D_-^\perp[\dot+]D_-$ and $\operatorname{dom}\phi = (\operatorname{dom}\phi\cap D_-^\perp)[\dot+]D_-$. The orthogonal projections P_1 and P_2 of $\mathcal{P}_1$ onto $D_-^\perp$ and D_-, respectively, are continuous. Moreover, note that isometry of ϕ implies that the restriction $\phi|_{D_-}$ is injective, and that $\phi(\operatorname{dom}\phi\cap D_-^\perp)\perp\phi(D_-)$. Finally, let $\|.\|_1$ be a norm which induces the topology of $\mathcal{P}_1$, and let $C>0$ be such that $|[x,x]|\le C\|x\|_1^2$, $x\in\mathcal{P}_1$.

By Lemma 4.1.8, we have $\operatorname{ind}_-\mathcal{P}_2 = \operatorname{ind}_-\operatorname{ran}\phi = \operatorname{ind}_-\operatorname{dom}\phi$. Thus the image $R_- := \phi(D_-)$ is a maximal negative subspace of $\mathcal{P}_2$. Let $\|.\|_{\mathfrak{J}_2}$ be the norm induced by the fundamental decomposition $\mathfrak{J}_2 := (R_-^\perp, R_-)$ of $\mathcal{P}_2$. For $x\in\operatorname{dom}\phi$, we can then compute

$$\|\phi x\|_{\mathfrak{J}_2}^2 = \|\underbrace{\phi(P_1x)}_{\in R_-^\perp}+\underbrace{\phi(P_2x)}_{\in R_-}\|_{\mathfrak{J}_2}^2 = [\phi(P_1x),\phi(P_1x)]_2 - [\phi(P_2x),\phi(P_2x)]_2 =$$

$$= [P_1x,P_1x]_1 - [P_2x,P_2x]_1 \le C\big(\|P_1x\|_1^2+\|P_2x\|_1^2\big) \le C(\|P_1\|^2+\|P_2\|^2)\|x\|_1^2\,.$$

This proves continuity of ϕ.

Let $\tilde\phi:\overline{\operatorname{dom}\phi}\to\mathcal{P}_2$ be the continuation of ϕ by continuity. Clearly, $\tilde\phi$ is isometric. In order to show surjectivity of $\tilde\phi$, it is enough to show that $\operatorname{ran}\tilde\phi$ is closed in $\mathcal{P}_2$. Set $D := \overline{\operatorname{dom}\phi} \cap D_-^\perp$, then we have $\overline{\operatorname{dom}\phi} = D[\dot{+}]D_-$ and, by Lemma 4.1.8, D is positive semidefinite. Choose a closed subspace D_+ of D with $D = D_+[\dot{+}]D^\circ$. Then, by Lemma 4.2.1, D_+ is uniformly positive and hence a Hilbert space with respect to the inner product $[.,.]_1$. The image $R_+ := \tilde{\phi}(D_+)$ is thus complete with respect to the inner product $[.,.]_2$. However, $R_+ \subseteq R_-^\perp$, and hence the inner products $(.,.)_{\mathfrak{J}_2}$ and $[.,.]_2$ coincide on R_+. It follows that R_+ is closed in the topology of $\mathcal{P}_2$. Since

$$\operatorname{ran}\tilde{\phi} = R_+ + \underbrace{\tilde{\phi}(D^\circ + D_-)}_{\dim<\infty},$$

we finally obtain that $\operatorname{ran}\tilde{\phi}$ is closed in $\mathcal{P}_2$. ❑

COD20 **4.3.2 Corollary.** *Let $\langle \mathcal{P}_1, [.,.]_1\rangle$ and $\langle \mathcal{P}_2, [.,.]_2\rangle$ be Pontryagin spaces, and let*

$$\phi : \operatorname{dom}\phi \subseteq \mathcal{P}_1 \to \mathcal{P}_2$$

be isometric. Assume that $\operatorname{dom}\phi$ *is closed and* $\overline{\operatorname{ran}\phi}$ *is nondegenerated. Then ϕ is continuous and maps closed subsets of $\mathcal{P}_1$ to closed subsets of $\mathcal{P}_2$.*

Proof. By the above proposition, ϕ is continuous and maps $\operatorname{dom}\phi$ onto $\overline{\operatorname{ran}\phi}$. Hence, $\operatorname{ran}\phi = \overline{\operatorname{ran}\phi}$, i.e. $\operatorname{ran}\phi$ is closed. By the Open Mapping Theorem, ϕ maps open subsets of $\mathcal{P}_1$ to relatively open subsets of $\operatorname{ran}\phi$. Since $\operatorname{ran}\phi$ is closed, this implies that ϕ maps closed subsets of $\mathcal{P}_1$ to closed subsets of $\mathcal{P}_2$. ❑

COD21 **4.3.3 Corollary.** *Let $\langle \mathcal{P}_1, [.,.]_1\rangle$ and $\langle \mathcal{P}_2, [.,.]_2\rangle$ be Pontryagin spaces, and let $\phi : \operatorname{dom}\phi \subseteq \mathcal{P}_1 \to \mathcal{P}_2$ be isometric. Assume that* $\operatorname{dom}\phi$ *and* $\operatorname{ran}\phi$ *are dense in $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively. Then there exists an isomorphism $\tilde{\phi}$ of $\mathcal{P}_1$ onto $\mathcal{P}_2$, such that $\tilde{\phi}|_{\operatorname{dom}\phi} = \phi$.*

Proof. By density, $(\overline{\operatorname{dom}\phi})^\circ = (\operatorname{dom}\phi)^\circ = \{0\}$ and $(\overline{\operatorname{ran}\phi})^\circ = (\operatorname{ran}\phi)^\circ = \{0\}$. The map ϕ is a bijection of $\operatorname{dom}\phi$ onto $\operatorname{ran}\phi$. Its inverse $\psi := \phi^{-1} : \operatorname{ran}\phi \to \operatorname{dom}\phi$ is also isometric. The previous proposition may be applied to both, ϕ and ψ. We conclude that ϕ and ψ can be extended to continuous maps

$$\tilde{\phi} : \mathcal{P}_1 \to \mathcal{P}_2, \ \tilde{\psi} : \mathcal{P}_2 \to \mathcal{P}_1 .$$

Clearly, $\tilde{\phi}$ and $\tilde{\psi}$ are inverses of each other. ❑

RED22 *4.3.4 Remark.* Let us explicitly mention one instance when Proposition 4.3.1 will apply: *If, with the notation of Proposition 4.3.1, we have* $\operatorname{ind}_-\operatorname{dom}\phi = \operatorname{ind}_-\mathcal{P}_2$ *then* $\overline{\operatorname{ran}\phi}$ *is nondegenerated.*

To see this, choose a maximal negative subspace D_- of $\operatorname{dom}\phi$. Then $R_- := \phi(D_-)$ is maximal negative in $\mathcal{P}_2$. By Proposition 1.5.2, it is even maximal in $\operatorname{Sub}_{\leq 0}\mathcal{P}_2$. Assume that $\overline{\operatorname{ran}\phi}^\circ$ would contain a nonzero element x_0. Then the subspace $\mathcal{M} := R_- + \operatorname{span}\{x_0\}$ is a nonpositive subspace of $\mathcal{P}_2$ which properly contains R_-, and we have obtained a contradiction. //

Concerning completions, inner product spaces with finite negative index behave very well.

PRD23 **4.3.5 Proposition.** *Let $\langle \mathcal{L}, [.,.]\rangle$ be an inner product space with* $\operatorname{ind}_-\mathcal{L} < \infty$. *Then the following hold:*

(*i*) *There exists a Krein space completion of* $\mathcal{L}$. *In each Krein space completion* $(\iota, \mathcal{K})$ *of* $\mathcal{L}$, *the space* $\mathcal{K}$ *is a Pontryagin space with* $\operatorname{ind}_- \mathcal{K} = \operatorname{ind}_- \mathcal{L}$.

(*ii*) *Each two Krein space completions of* $\mathcal{L}$ *are isomorphic.*

Proof. Since $\operatorname{ind}_- \mathcal{L} < \infty$, $\mathcal{L}$ is decomposable. Hence, there exist Krein space completions of $\mathcal{L}$. Whenever $(\iota, \mathcal{K})$ is such, then $\iota(\mathcal{L})$ is dense in $\mathcal{K}$. Thus $\mathcal{K}$ cannot contain any negative definite subspace with dimension $\operatorname{ind}_- \mathcal{L} + 1$, cf. Lemma 4.1.8, and it follows that $\operatorname{ind}_- \mathcal{K} = \operatorname{ind}_- \mathcal{L} < \infty$.

Next, let $(\iota_1, \mathcal{K}_1)$ and $(\iota_2, \mathcal{K}_2)$ be two Krein space completions of $\mathcal{L}$. Then $\mathcal{K}_1$ and $\mathcal{K}_2$ are Pontryagin spaces. Since $\ker \iota_1 = \mathcal{L}^\circ = \ker \iota_2$, there exists a linear and isometric map $\phi : \operatorname{ran} \iota_1 \to \operatorname{ran} \iota_2$ such that $\phi \circ \iota_1 = \iota_2$. Since $\operatorname{ran} \iota_1$ and $\operatorname{ran} \iota_2$ are dense subspaces of the Pontryagin spaces $\mathcal{K}_1$ and $\mathcal{K}_2$, and the map ϕ is isometric, we may apply Proposition 4.3.1. This gives an isomorphism $\Phi : \mathcal{K}_1 \to \mathcal{K}_2$ with $\Phi \circ \iota_1 = \iota_2$:

$$\begin{tikzcd}
& \mathcal{L} \arrow[dl, "\iota_1"'] \arrow[dr, "\iota_2"] & \\
\operatorname{ran} \iota_1 \arrow[rr, "\phi"'] \arrow[d, phantom, sloped, "\subseteq"] & & \operatorname{ran} \iota_2 \arrow[d, phantom, sloped, "\subseteq"] \\
\mathcal{K}_1 \arrow[rr, dashed, "\Phi"'] & & \mathcal{K}_2
\end{tikzcd}$$

❑

We will refer to a Krein space completion $(\iota, \mathcal{K})$ of a space $\langle \mathcal{L}, [.,.] \rangle$ with $\operatorname{ind}_- \mathcal{K} < \infty$ as a *Pontryagin space completion* of $\mathcal{L}$.

RED24 *4.3.6 Remark.* Let $\langle \mathcal{L}, [.,.] \rangle$ be an inner product space. Trivially, the existence of Pontryagin space completion $(\iota, \mathcal{P})$ of $\mathcal{L}$ implies that $\operatorname{ind}_- \mathcal{L} < \infty$. Hence, the space $\mathcal{L}$ admits a Pontryagin space completion if and only if $\operatorname{ind}_- \mathcal{L} < \infty$. //

Let $\langle \mathcal{L}, [.,.] \rangle$ be an inner product space with $\operatorname{ind}_- \mathcal{L} < \infty$. Then, by Proposition 4.3.5, (*ii*), a Pontryagin space completion $(\iota, \mathcal{P})$ of $\mathcal{L}$ is an object intrinsically determined (up to isomorphism) by $\mathcal{L}$. Thus also the topological dual space of a Pontryagin space completion of $\mathcal{L}$ has this property. Let us make this precise.

PRD25 **4.3.7 Proposition.** *Let* $\langle \mathcal{L}, [.,.] \rangle$ *be an inner product space with* $\operatorname{ind}_- \mathcal{L} < \infty$, *and let* $(\iota, \mathcal{P})$ *be a Pontryagin space completion of* $\mathcal{L}$. *Then*

$$\iota^* \mathcal{P}' = \mathcal{L}^\curlywedge .$$

Here $\mathcal{P}'$ *denotes the topological dual of* $\mathcal{P}$, *and* ι^* *denotes the (algebraic) adjoint of* ι, *that is* $\iota^* : \mathcal{P}^* \to \mathcal{L}^*$ *and* $\iota^* f = f \circ \iota$.

Proof. Choose a fundamental decomposition $\mathfrak{J} := (\mathcal{L}_+, \mathcal{L}_-)$ of $\mathcal{L}$, and let $(\tilde{\iota}, \tilde{\mathcal{P}})$ be the Pontryagin space completion of $\mathcal{L}$ induced by this fundamental decomposition. As we also saw in this place, the decomposition topology $\mathcal{T}_{\mathfrak{J}} = \mathcal{T}^\curlywedge$ is the initial topology with respect to the map $\tilde{\iota}$. Hence, $\iota^* \mathcal{P}' = \mathcal{L}^\curlywedge$.

By Proposition 4.3.5, (*ii*), there exists an isomorphism $\phi : \mathcal{P} \to \tilde{\mathcal{P}}$ with $\tilde{\iota} = \phi \circ \iota$. Passing to adjoints gives $\tilde{\iota}^* = \iota^* \circ \phi^*$. Since ϕ is in particular a homeomorphism, we have $\phi^*(\tilde{\mathcal{P}}') = \mathcal{P}'$. Thus

$$\iota^* \mathcal{P}' = \iota^* \circ \phi^*(\tilde{\mathcal{P}}') = \tilde{\iota}^*(\tilde{\mathcal{P}}') = \mathcal{L}^\curlywedge .$$

❑

4.4 Degenerated subspaces

If $\mathcal{L}$ is a closed but degenerated subspace of a Pontryagin space $\mathcal{P}$, then $\mathcal{L}$ cannot be orthocomplemented already for the simple reason that

$$(\mathcal{L}+\mathcal{L}^\perp)^\perp = \mathcal{L}^\perp \cap \mathcal{L}^{\perp\perp} \supseteq \mathcal{L}^\circ .$$

However, unlike in the general case of Krein spaces, it turns out that existence of a nontrivial isotropic part is the only obstacle. We will in this section establish the proper analogue of the decomposition $\mathcal{P} = \mathcal{L}[\dot+]\mathcal{L}^\perp$ for degenerated subspaces of $\mathcal{P}$.

THD26 **4.4.1 Theorem.** *Let $\langle \mathcal{P}, [.,.]\rangle$ be a Pontryagin space, and let $\mathcal{L}$ be a closed and degenerated subspace of $\mathcal{P}$. Then*

(*i*) *There exists a closed and nondegenerated subspace $\mathcal{L}_1$ of $\mathcal{P}$ with $\mathcal{L} = \mathcal{L}_1[\dot+]\mathcal{L}^\circ$.*

(*ii*) *Whenever $\mathcal{L}_1$ and $\mathcal{L}_2$ are closed subspaces of $\mathcal{P}$ with $\mathcal{L} = \mathcal{L}_1[\dot+]\mathcal{L}^\circ$ and $\mathcal{L}^\perp = \mathcal{L}_2[\dot+]\mathcal{L}^\circ$, then there exists $\mathcal{M} \in \operatorname{Sub}_0 \mathcal{P}$ with $\mathcal{L}^\circ \# \mathcal{M}$ and*

$$\mathcal{P} = \mathcal{L}_1[\dot+](\mathcal{L}^\circ \dot+ \mathcal{M})[\dot+]\mathcal{L}_2 . \tag{4.4.1}$$ D27

(*iii*) *If $\mathcal{M} \in \operatorname{Sub}_0 \mathcal{P}$ with $\mathcal{L}^\circ \# \mathcal{M}$, then there exist unique closed subspaces $\mathcal{L}_1$ and $\mathcal{L}_2$ of $\mathcal{P}$, such that $\mathcal{L} = \mathcal{L}_1[\dot+]\mathcal{L}^\circ$, $\mathcal{L}^\perp = \mathcal{L}_2[\dot+]\mathcal{L}^\circ$, and* (4.4.1).

Proof. For (*i*) it is enough to note that $\mathcal{L}^\circ$ is finite dimensional, since this ensures the existence of a closed complement of $\mathcal{L}^\circ$ in $\mathcal{L}$. Thereby $[.,.]$-orthogonality is trivially satisfied.

Let $\mathcal{L}_1, \mathcal{L}_2$ be given as in (*ii*). Then, clearly, $\mathcal{L}_1$ and $\mathcal{L}_2$ are nondegenerated, and hence orthocomplemented. Moreover, $\mathcal{L}_1 \perp \mathcal{L}_2$ and $\mathcal{L}_1 \cap \mathcal{L}_2 = \{0\}$. Thus also $\mathcal{L}_1[\dot+]\mathcal{L}_2$ is orthocomplemented, i.e.

$$\mathcal{P} = (\mathcal{L}_1[\dot+]\mathcal{L}_2)[\dot+](\mathcal{L}_1^\perp \cap \mathcal{L}_2^\perp) .$$

Since $\mathcal{L}_1^\perp \cap \mathcal{L}_2^\perp$ is nondegenerated and contains the neutral subspace $\mathcal{L}^\circ$, we find a neutral subspace $\mathcal{M} \subseteq \mathcal{L}_1^\perp \cap \mathcal{L}_2^\perp$ with $\mathcal{L}^\circ \# \mathcal{M}$. Thereby, $\dim \mathcal{M} = \dim \mathcal{L}^\circ < \infty$. Thus $\mathcal{L}^\circ \dot+ \mathcal{M}$ is a finite dimensional, and hence closed, subspace of the Pontryagin space $\mathcal{L}_1^\perp \cap \mathcal{L}_2^\perp$. Assume that $x \in \mathcal{L}_1^\perp \cap \mathcal{L}_2^\perp$, $x \perp (\mathcal{L}^\circ \dot+ \mathcal{M})$. Then

$$x \in (\mathcal{L}_1 + \mathcal{L}^\circ)^\perp \cap (\mathcal{L}_2 + \mathcal{L}^\circ)^\perp \cap \mathcal{M}^\perp = \mathcal{L}^\perp \cap \mathcal{L}^{\perp\perp} \cap \mathcal{M}^\perp = \mathcal{L}^\circ \cap \mathcal{M}^\perp = \{0\} .$$

It follows that

$$\mathcal{L}_1^\perp \cap \mathcal{L}_2^\perp = \mathcal{L}^\circ \dot+ \mathcal{M} ,$$

and this is (4.4.1).

We come to the proof of (*iii*). Let $\mathcal{M} \in \operatorname{Sub}_0 \mathcal{P}$ with $\mathcal{L}^\circ \# \mathcal{M}$ be given. Define

$$\mathcal{L}_1 := \mathcal{L} \cap (\mathcal{L}^\circ \dot+ \mathcal{M})^\perp,\ \mathcal{L}_2 := \mathcal{L}^\perp \cap (\mathcal{L}^\circ \dot+ \mathcal{M})^\perp .$$

Clearly, $\mathcal{L}_1$ and $\mathcal{L}_2$ are closed, and $\mathcal{L}_1 \subseteq \mathcal{L}$, $\mathcal{L}_2 \subseteq \mathcal{L}^\perp$. Since $\mathcal{L}_1, \mathcal{L}_2 \perp \mathcal{M}$, we also have $\mathcal{L}_1 \cap \mathcal{L}^\circ = \mathcal{L}_2 \cap \mathcal{L}^\circ = \{0\}$.

In order to show that $\mathcal{L} = \mathcal{L}_1[\dot{+}]\mathcal{L}^\circ$, let $z \in \mathcal{L}$ be given. The space $\mathcal{L}^\circ\dot{+}\mathcal{M}$ is nondegenerated and finite dimensional, and hence orthocomplemented. Thus we may write

$$z = x + y \text{ with some } x \in \mathcal{L}^\circ\dot{+}\mathcal{M}, y \in (\mathcal{L}^\circ\dot{+}\mathcal{M})^\perp .$$

Since $x = z - y \perp \mathcal{L}^\circ$ and $x \in \mathcal{L}^\circ\dot{+}\mathcal{M}$, it follows that $x \in \mathcal{L}^\circ$. This, in turn, implies $y = z - x \in \mathcal{L}$, and hence $y \in \mathcal{L}_1$. We conclude that $z \in \mathcal{L}_1 + \mathcal{L}^\circ$. The relation $\mathcal{L}^\perp = \mathcal{L}_2[\dot{+}]\mathcal{L}^\circ$ is seen in the same way.

Applying the already proved item (ii) with the subspaces $\mathcal{L}_1$ and $\mathcal{L}_2$, gives $\mathcal{M}' \in \operatorname{Sub}_0 \mathcal{P}$ with $\mathcal{L}^\circ \# \mathcal{M}'$ and

$$\mathcal{P} = \mathcal{L}_1[\dot{+}](\mathcal{L}^\circ\dot{+}\mathcal{M}')[\dot{+}]\mathcal{L}_2 .$$

However, we have $\mathcal{L}^\circ\dot{+}\mathcal{M}' = \mathcal{L}_1^\perp \cap \mathcal{L}_2^\perp \supseteq \mathcal{L}^\circ\dot{+}\mathcal{M}$. Since both spaces have the same finite dimension, namely $2 \dim \mathcal{L}^\circ$, it follows that $\mathcal{L}^\circ\dot{+}\mathcal{M}' = \mathcal{L}^\circ\dot{+}\mathcal{M}$. This shows that (4.4.1) holds with $\mathcal{M}$.

In order to see uniqueness, assume that $\mathcal{L}_1'$ and $\mathcal{L}_2'$ are closed subspaces of $\mathcal{P}$ with $\mathcal{L} = \mathcal{L}_1'[\dot{+}]\mathcal{L}^\circ$, $\mathcal{L}^\perp = \mathcal{L}_2'[\dot{+}]\mathcal{L}^\circ$, and $\mathcal{P} = \mathcal{L}_1'[\dot{+}](\mathcal{L}^\circ\dot{+}\mathcal{M})[\dot{+}]\mathcal{L}_2'$. Then

$$\mathcal{L}_1' \subseteq \mathcal{L} \cap (\mathcal{L}^\circ\dot{+}\mathcal{M})^\perp = \mathcal{L}_1,\ \mathcal{L}_2' \subseteq \mathcal{L}^\perp \cap (\mathcal{L}^\circ\dot{+}\mathcal{M})^\perp = \mathcal{L}_2 .$$

Since the relation (4.4.1) also holds with $\mathcal{L}_1, \mathcal{L}_2$, this implies that actually $\mathcal{L}_1' = \mathcal{L}_1$ and $\mathcal{L}_2' = \mathcal{L}_2$. ❏

Chapter 5

Classes of complete TIPS. III. Almost Pontryagin spaces

5.1 Definition of aPs

DEE1 **5.1.1 Definition.** A topological inner product space $\langle\mathcal{A},[.,.],\mathcal{T}\rangle$ is called an *almost Pontryagin space*, if

(APS1) $\operatorname{ind}_0\mathcal{A}<\infty$, $\operatorname{ind}_-\mathcal{A}<\infty$.

(APS2) $\mathcal{T}\in\operatorname{Top}_{\mathrm{Bs}}\mathcal{A}$.

(APS3) There exists a fundamental decomposition $\mathfrak{J}=(\mathcal{A}_+,\mathcal{A}_-)$ of $\mathcal{A}$, such that $\mathcal{A}_+$ is $\mathcal{T}$-closed and intrinsically complete.

//

Note that the topology $\mathcal{T}$ and the fundamental decomposition $(\mathcal{A}_+,\mathcal{A}_-)$ in (APS3) are related. Thus the choice of $(\mathcal{A}_+,\mathcal{A}_-)$ in (APS3) is not arbitrary.

The following statement is an immediate reformulation of this definition.

REE2 *5.1.2 Remark.* Let $\langle\mathcal{A},[.,.],\mathcal{T}\rangle$ be a topological inner product space. Then $\langle\mathcal{A},[.,.],\mathcal{T}\rangle$ is an almost Pontryagin space, if and only if there exists a Hilbert space $\mathcal{H}_1$, a finite-dimensional negative definite space $\mathcal{H}_2$, and a finite-dimensional neutral space $\mathcal{H}_3$, such that

$$\langle\mathcal{A},[.,.],\mathcal{T}\rangle=\mathcal{H}_1[\dot{+}]\mathcal{H}_2[\dot{+}]\mathcal{H}_3\,.$$

Here $\mathcal{H}_1[\dot{+}]\mathcal{H}_2[\dot{+}]\mathcal{H}_3$ is endowed with the sum inner product and the product topology, where $\mathcal{H}_1$ and $\mathcal{H}_2$ carry the topologies induced by their inner products, and $\mathcal{H}_3$ carries the euclidean topology. //

PRE3 **5.1.3 Proposition.** *Let $\langle\mathcal{A},[.,.],\mathcal{T}\rangle$ be a topological inner product space. Then the following are equivalent:*

(i) $\langle\mathcal{A},[.,.],\mathcal{T}\rangle$ is an almost Pontryagin space.

(*ii*) *There exists an inner product* $(.,.)$ *on* $\mathcal{A}$ *with* $\mathcal{T}_{(.,.)} = \mathcal{T}$*, such that* $\mathcal{T}_{(.,.)} \in \operatorname{Top}_{\mathrm{Hs}} \mathcal{A}$*, and the Gram operator of* $[.,.]$ *with respect to* $(.,.)$ *satisfies*

$$\dim \operatorname{ran} E(-\infty, \delta) < \infty$$

for some $\delta > 0$*. Here, again,* E *denotes the spectral measure of* G *as operator in* $\langle \mathcal{A}, (.,.) \rangle$*.*

(*iii*) *We have* $\mathcal{T} \in \operatorname{Top}_{\mathrm{Hs}} \mathcal{A}$ *and there exists* $\mathcal{M} \in \operatorname{Sub}_{>0} \mathcal{A}$ *with* $\dim \mathcal{A}/\mathcal{M} < \infty$*, such that* $\mathcal{M}$ *is* $\mathcal{T}$*-closed and intrinsically complete.*

Proof. We show that $(i) \Rightarrow (iii)$. Assume that $\langle \mathcal{A}, [.,.], \mathcal{T} \rangle$ is an almost Pontryagin space, and choose a fundamental decomposition $\mathfrak{J} = (\mathcal{A}_+, \mathcal{A}_-)$ of $\mathcal{A}$ such that $\mathcal{A}_+$ is $\mathcal{T}$-closed and intrinsically complete. Since $\mathcal{A} = \mathcal{A}_+ [\dot{+}] \mathcal{A}_- [\dot{+}] \mathcal{A}^\circ$, and each component is $\mathcal{T}$-closed, the topology $\mathcal{T}$ is equal to the product topology $\mathcal{T}|_{\mathcal{A}_+} \times \mathcal{E}_{\mathcal{A}_-} \times \mathcal{E}_{\mathcal{A}^\circ}$, where $\mathcal{E}_{\mathcal{A}_-}$ and $\mathcal{E}_{\mathcal{A}^\circ}$ denote the euclidean topologies on the respective finite dimensional spaces. However, since $\mathcal{M}$ is intrinsically complete, the topology $\mathcal{T}|_{\mathcal{A}_+}$ is equal to the topology induced by the inner product $[.,.]|_{\mathcal{A}_+ \times \mathcal{A}_+}$. Altogether, $\mathcal{T}$ is induced by the inner product

$$(x, y) := [P_{\mathfrak{J}}^+ x, P_{\mathfrak{J}}^+ y] - [P_{\mathfrak{J}}^- x, P_{\mathfrak{J}}^- y] + [P_0 x, P_0 y]_0, \ x, y \in \mathcal{A},$$

where we have set $P_0 := I - P_{\mathfrak{J}}^+ - P_{\mathfrak{J}}^-$, and where $[.,.]_{\mathcal{E}}$ denotes any positive definite inner product on $\mathcal{A}^\circ$. We conclude that $\mathcal{T} \in \operatorname{Top}_{\mathrm{Hs}} \mathcal{A}$. Set $\mathcal{M} := \mathcal{A}_+$, then $\mathcal{M}$ has all the properties required in (iii).

The proof of the implication $(iii) \Rightarrow (ii)$ proceeds word by word as the proof of the corresponding implication in Proposition 4.1.4, just deleting the last two lines. Assume that (ii) holds. Choose $\epsilon \in (0, \delta)$ such that $\sigma(G) \cap (0, \epsilon) = \emptyset$, and set

$$\mathcal{A}_+ := \operatorname{ran} E(\epsilon, \infty), \ \mathcal{A}_- := \operatorname{ran} E(-\infty, 0).$$

Then $(\mathcal{A}_+, \mathcal{A}_-)$ is a fundamental decomposition of $\mathcal{A}$, and

$$\dim \mathcal{A}_-, \dim \mathcal{A}^\circ \leq \dim \operatorname{ran} E(-\infty, \delta) < \infty.$$

As we have shown in the proof of Theorem 3.1.5, cf. (3.1.1), the inner products $(.,.)|_{\mathcal{A}_+ \times \mathcal{A}_+}$ and $[.,.]|_{\mathcal{A}_+ \times \mathcal{A}_+}$ give rise to equivalent norms. Since $\mathcal{A}_+$ is $\mathcal{T}$-closed, it is thus also intrinsically complete. ❑

For the same reason as in Remark 4.1.5, we obtain the following statement.

REE4 *5.1.4 Remark.* Let $\langle \mathcal{A}, [.,.], \mathcal{T} \rangle$ be an almost Pontryagin space, and let $(.,.)$ be any inner product on $\mathcal{A}$ with $\mathcal{T}_{(.,.)} = \mathcal{T}$. Then the corresponding Gram operator satisfies $\dim \operatorname{ran} E(-\infty, \delta) < \infty$ for some $\delta > 0$. //

As a corollary we obtain the following statements.

COE5 **5.1.5 Corollary.**

(*i*) *If* $\langle \mathcal{P}, [.,.] \rangle$ *is a Pontryagin space, then* $\langle \mathcal{P}, [.,.], \mathcal{T}^\curlywedge \rangle$ *is an almost Pontryagin space.*

(*ii*) *If* $\langle \mathcal{A}, [.,.], \mathcal{T} \rangle$ *is an almost Pontryagin space and is nondegenerated, then* $\langle \mathcal{A}, [.,.] \rangle$ *is a Pontryagin space and* $\mathcal{T} = \mathcal{T}^\curlywedge$*.*

Proof. Let $\langle\mathcal{P},[.,.]\rangle$ be a Pontryagin space, and let $\mathfrak{J}=(\mathcal{P}_+,\mathcal{P}_-)$ be a fundamental decomposition of $\mathcal{P}$. Then $\mathcal{P}_+=\mathcal{P}_-^\perp$ is $\mathcal{T}_\mathfrak{J}$-closed and intrinsically complete. Moreover, $\mathcal{T}^\curlywedge=\mathcal{T}_\mathfrak{J}$ and $\mathcal{P}$ is complete with respect to $\mathcal{T}^\curlywedge$. We see that $\langle\mathcal{P},[.,.],\mathcal{T}^\curlywedge\rangle$ is an almost Pontryagin space.

The assertion (ii) follows immediately from the equivalences '$(i)\iff(iii)$' in Proposition 4.1.4 and Proposition 5.1.3, respectively, and the fact that for nondegenerated spaces $|\operatorname{Top}_{\mathrm{Bs}}\mathcal{A}|\leq 1$, cf. Remark 4.1.3, (i). ❑

Unlike for nondegenerated spaces, in the presence of a nontrivial isotropic part, we may have different compatible Banach space topologies. In fact, the topology of an infinite dimensional degenerated almost Pontryagin space $\langle\mathcal{A},[.,.],\mathcal{T}\rangle$ is never uniquely determined by the inner product space $\langle\mathcal{A},[.,.]\rangle$.

EXE6 *5.1.6 Example.* Let $\langle\mathcal{A},[.,.],\mathcal{T}\rangle$ be an almost Pontryagin space with $\dim\mathcal{A}=\infty$ and $\operatorname{ind}_0\mathcal{A}>0$. Choose a fundamental decomposition $\mathfrak{J}=(\mathcal{A}_+,\mathcal{A}_-)$ where $\mathcal{A}_+$ is $\mathcal{T}$-closed and intrinsically complete, so that $\mathcal{T}|_{\mathcal{A}_+}=\mathcal{T}_\mathfrak{J}|_{\mathcal{A}_+}$. Moreover, choose a norm $\|.\|$ which induced by a Hilbert space inner product on $\mathcal{A}$, which induces $\mathcal{T}$ and satisifes $\|x\|=\|x\|_\mathfrak{J}$, $x\in\mathcal{A}_++\mathcal{A}_-$.

Let $h\in\mathcal{A}^\circ\setminus\{0\}$, and let f be a linear functional $f:\mathcal{A}_+\to\mathbb{C}$ which is not $\mathcal{T}|_{\mathcal{A}_+}$-continuous. Define a map $\phi:\mathcal{A}\to\mathcal{A}$ as

$$\phi(x):=x+f(P_\mathfrak{J}^+x)h,\ x\in\mathcal{A}\,.$$

Then ϕ is obviously isometric and satisfies $\phi|_{\mathcal{A}_-+\mathcal{A}^\circ}=\mathrm{id}_{\mathcal{A}_-+\mathcal{A}^\circ}$. In particular, $\ker\phi\cap(\mathcal{A}_-+\mathcal{A}^\circ)=\{0\}$.

If $x\in\mathcal{A}_+$ is given, then

$$\phi\big(x-f(x)h\big)=x-f(x)h+f\big(\underbrace{P_\mathfrak{J}^+(x-f(x)h)}_{=x}\big)h=x\,,$$

and we conclude that ϕ is surjective. If $x\in\ker\phi$, then

$$0=\phi(x)=x-f(P_\mathfrak{J}^+x)h\,,$$

and hence $x\in\mathcal{A}^\circ$. This implies that $x\in\ker\phi\cap\mathcal{A}^\circ=\{0\}$, and we conclude that ϕ is injective.

Let $\mathcal{T}':=\phi^{-1}(\mathcal{T})$, then $\mathcal{T}'$ is induced by the norm $\|x\|':=\|\phi x\|$, $x\in\mathcal{A}$, and $\mathcal{A}$ is complete with respect to $\|.\|'$. Moreover, we have

$$|[x,y]|=|[\phi x,\phi y]|\leq\alpha\|\phi x\|\cdot\|\phi y\|=\alpha\|x\|'\cdot\|y\|',\ x,y\in\mathcal{A}\,,$$

and hence $\mathcal{T}'\in\operatorname{Top}_{\mathrm{Hs}}\mathcal{A}$. The subspace $\mathcal{M}:=\phi^{-1}(\mathcal{A}_+)$ is $\mathcal{T}'$-closed, has finite codimension, and

$$[x,x]^{\frac12}=[\phi x,\phi x]^{\frac12}=\|\phi x\|=\|x\|',\ x\in\mathcal{M}\,.$$

Hence $\mathcal{M}$ is also intrinsically complete. We conclude that $\langle\mathcal{A},[.,.],\mathcal{T}'\rangle$ is an almost Pontryagin space, and that ϕ is an isomorphism of $\langle\mathcal{A},[.,.],\mathcal{T}'\rangle$ onto $\langle\mathcal{A},[.,.],\mathcal{T}\rangle$.

However, since $(\phi|_{\mathcal{A}_+}-\mathrm{id}_{\mathcal{A}_+})x=f(x)h$, $x\in\mathcal{A}_+$, the map $\phi|_{\mathcal{A}_+}$ and hence also ϕ cannot be $\mathcal{T}$-to-$\mathcal{T}$-continuous. Thus $\mathcal{T}'\neq\mathcal{T}$. //

Nevertheless, of course, the fact whether or not an inner product space $\langle\mathcal{A},[.,.]\rangle$ can be made into an almost Pontryagin space, is an intrinsic property. The next statement is the analogue to Proposition 4.1.6.

PRE7 **5.1.7 Proposition.** *Let $\langle\mathcal{L},[.,.]\rangle$ be an inner product space. Then the following are equivalent:*

(i) There exists a vector topology $\mathcal{T}$ such that $\langle\mathcal{L},[.,.],\mathcal{T}\rangle$ is an almost Pontryagin space.

(ii) There exists a subspace $\mathcal{M}\in\operatorname{Sub}_{>0}\mathcal{L}$ with $\dim\mathcal{L}/\mathcal{M}<\infty$, such that $\mathcal{M}$ is complete with respect to $\mathcal{T}^{\curlywedge}|_{\mathcal{M}}$.

Note here that the existence of a positive subspace with finite codimension implies $\operatorname{ind}_-\mathcal{L}<\infty$, and hence ensures that $\mathcal{T}^{\curlywedge}$ is well-defined. Moreover, if $\mathcal{M}$ is positive, then certainly $\mathcal{M}\cap\mathcal{L}^\circ=\{0\}$, and hence for each fundamental decomposition $\mathfrak{J}$ of $\mathcal{L}$ the seminorm $p_{\mathfrak{J}}|_{\mathcal{M}}$ is a norm.

(iii) $\mathcal{L}/\mathcal{L}^\circ$ is a Pontryagin space and $\operatorname{ind}_0\mathcal{L}<\infty$.

If one (and hence all) of these conditions hold, then for each $\mathcal{T}\in\operatorname{Top}_{\mathrm{Bs}}\mathcal{L}$, the triple $\langle\mathcal{L},[.,.],\mathcal{T}\rangle$ is an almost Pontryagin space.

Proof.
(i) $\Rightarrow$ (ii): Choose a fundamental decomposition $\mathfrak{J}=(\mathcal{L}_+,\mathcal{L}_-)$ of $\mathcal{L}$ such that $\mathcal{L}_+$ is intrinsically complete. Then the subspace $\mathcal{M}:=\mathcal{M}_+$ has all the required properties.

(ii) $\Rightarrow$ (iii): Clearly, $\mathcal{L}/\mathcal{L}^\circ$ is nondegenerated. Let $\mathcal{M}$ be a subspace as in *(ii)*, let $\mathfrak{J}=(\mathcal{L}_+,\mathcal{L}_-)$ be a fundamental decomposition of $\mathcal{L}$, and denote by $\pi:\mathcal{L}\to\mathcal{L}/\mathcal{L}^\circ$ the canonical projection. Then, since π is isometric, surjective, and $\ker\pi=\mathcal{L}^\circ$, the pair $\mathfrak{J}_\sim:=(\pi(\mathcal{L}_+),\pi(\mathcal{L}_-))$ is a fundamental decomposition of $\mathcal{L}/\mathcal{L}^\circ$. We have

$$P^\pm_{\mathfrak{J}_\sim}\circ\pi=\pi\circ P^\pm_{\mathfrak{J}}\,,$$

and hence can compute

$$p_{\mathfrak{J}_\sim}(\pi x)^2=[P^+_{\mathfrak{J}_\sim}\pi x,P^+_{\mathfrak{J}_\sim}\pi x]-[P^-_{\mathfrak{J}_\sim}\pi x,P^-_{\mathfrak{J}_\sim}\pi x]=$$

$$=[\pi P^+_{\mathfrak{J}}x,\pi P^+_{\mathfrak{J}}x]-[\pi P^-_{\mathfrak{J}}x,\pi P^-_{\mathfrak{J}}x]=[P^+_{\mathfrak{J}}x,P^+_{\mathfrak{J}}x]-[P^-_{\mathfrak{J}}x,P^-_{\mathfrak{J}}x]=p_{\mathfrak{J}}(x)^2,\ x\in\mathcal{L}\,.$$

Consider the subspace $\mathcal{M}_\sim:=\pi(\mathcal{M})$. The map $\pi|_{\mathcal{M}}$ is a bijection of $\mathcal{M}$ onto $\pi(\mathcal{M})$. Since π is isometric, $\mathcal{M}_\sim\in\operatorname{Sub}_{>0}\mathcal{L}/\mathcal{L}^\circ$. Moreover, clearly,

$$\dim(\mathcal{L}/\mathcal{L}^\circ)/\mathcal{M}_\sim\leq\dim\mathcal{L}/\mathcal{M}<\infty\,.$$

Finally, since $\mathcal{M}$ is complete with respect to the norm $p_{\mathfrak{J}}|_{\mathcal{M}}$, this implies that $\pi(\mathcal{M})$ is complete with respect to the norm $p_{\mathfrak{J}_\sim}$. We conclude from Proposition 4.1.6 that $\mathcal{L}/\mathcal{L}^\circ$ is a Pontryagin space.

(iii) $\Rightarrow$ (i): Let $\mathfrak{J}_\sim=(\mathcal{L}_+^\sim,\mathcal{L}_-^\sim)$ be a fundamental decomposition of $\mathcal{L}/\mathcal{L}^\circ$, and choose $\mathcal{L}_+,\mathcal{L}_-\in\operatorname{Sub}\mathcal{L}$, such that

$$\pi^{-1}\mathcal{L}_+^\sim=\mathcal{L}_+\dot{+}\mathcal{L}^\circ,\quad\pi^{-1}\mathcal{L}_-^\sim=\mathcal{L}_-\dot{+}\mathcal{L}^\circ\,.$$

Since $\ker\pi=\mathcal{L}^\circ$, we have $\mathcal{L}_+\in\operatorname{Sub}_{>0}\mathcal{L}$ and $\mathcal{L}_-\in\operatorname{Sub}_{<0}\mathcal{L}$. Moreover, $\dim\mathcal{L}_-=\dim\mathcal{L}_-^\sim<\infty$. Since $\pi|_{\mathcal{L}_+}$ is a bijective isometry of $\mathcal{L}_+$ onto $\mathcal{L}_+^\sim$, and $\mathcal{L}_+^\sim$ is intrinsically complete, also $\mathcal{L}_+$ has this property. We have

$$\mathcal{L}=\mathcal{L}_+[\dot{+}]\mathcal{L}_-[\dot{+}]\mathcal{L}^\circ\,.$$

and hence $\mathcal{L}$ becomes an almost Pontryagin space with the inner product $[.,.]$ and the topology $\mathcal{T}$ which is the product topology of $\mathcal{T}_{[.,.]|_{\mathcal{L}_+\times\mathcal{L}_+}}$, $\mathcal{T}_{-[.,.]|_{\mathcal{L}_-\times\mathcal{L}_-}}$, and the euclidean topology on $\mathcal{L}^\circ$, cf. Remark 5.1.2.

Finish of proof: To see the last assertion, let $\mathcal{T} \in \mathrm{Top}_{\mathrm{Bs}}\,\mathcal{L}$ be given. Choose a subspace $\mathcal{M}$ as in (ii), then $\pi(\mathcal{M})$ is a closed and positive subspace of the Pontryagin space $\mathcal{L}/\mathcal{L}^\circ$. By Corollary 4.2.2, there exists a fundamental decomposition $(\mathcal{P}_+,\mathcal{P}_-)$ of $\mathcal{L}/\mathcal{L}^\circ$ with $\pi(\mathcal{M}) \subseteq \mathcal{P}_+$. The subspace $\pi^{-1}(\mathcal{P}_+)$ of $\mathcal{L}$ is closed. By Theorem 2.5.10 we have $\mathcal{T} \supseteq \mathcal{T}^\curlywedge$, and hence $\pi^{-1}(\mathcal{P}_+)$ is also $\mathcal{T}$-closed. Choose a $\mathcal{T}$-closed subspace $\hat{\mathcal{L}}_+$ with $\pi^{-1}(\mathcal{P}_+) = \hat{\mathcal{L}}_+\dot{+}\mathcal{L}^\circ$, and choose a negative subspace $\hat{\mathcal{L}}_-$ with $\pi(\hat{\mathcal{L}}_-) = \mathcal{P}_-$. Then $(\hat{\mathcal{L}}_+,\hat{\mathcal{L}}_-)$ is a fundamental decomposition of $\mathcal{L}$. The map $\pi|_{\hat{\mathcal{L}}_+}$ is an $[.,.]$-isometric bijection of $\hat{\mathcal{L}}_+$ onto $\mathcal{P}_+$, and hence $\hat{\mathcal{L}}_+$ is intrinsically complete. We see that $\langle\mathcal{L},[.,.],\mathcal{T}\rangle$ is an almost Pontryagin space. ❏

COE8 **5.1.8 Corollary.** *Let $\langle\mathcal{A},[.,.],\mathcal{T}\rangle$ be an almost Pontryagin space. Then in each fundamental decomposition $(\mathcal{A}_+,\mathcal{A}_-)$ of $\mathcal{A}$, the component $\mathcal{A}_+$ is intrinsically complete.*

Proof. Let π denote the canonical projection of $\mathcal{A}$ onto $\mathcal{A}/\mathcal{A}^\circ$. Then $(\pi(\mathcal{A}_+),\pi(\mathcal{A}_1))$ is a fundamental decomposition of the Pontryagin space $\mathcal{A}/\mathcal{A}^\circ$, and hence $\pi(\mathcal{A}_+)$ is intrinsically complete. However, $\pi|_{\mathcal{A}_+}$ maps $\mathcal{A}_+$ bijectively and isometrically onto $\pi(\mathcal{A}_+)$, and hence $\mathcal{A}_+$ is intrinsically complete. ❏

The appropriate notion of a 'structur-preserving' map between two almost Pontryagin spaces differs from the one in the setting of topological inner product spaces. It turns out that requiring a map ϕ to be linear, isometric, and continuous, is for several purposes too weak.

DEE9 **5.1.9 Definition.** Let $\langle\mathcal{A}_1,[.,.]_1,\mathcal{T}_1\rangle$ and $\langle\mathcal{A}_2,[.,.]_2,\mathcal{T}_2\rangle$ be almost Pontryagin spaces. Then ϕ is called a (aPs–) *morphism of $\mathcal{A}_1$ to $\mathcal{A}_2$*, if ϕ is linear map of $\mathcal{A}_1$ into $\mathcal{A}_2$, which is isometric and continuous, and for which $\operatorname{ran}\phi$ is closed in $\mathcal{A}_2$. //

The following observation is simple but important.

LEE10 **5.1.10 Lemma.** *Let $\langle\mathcal{A}_1,[.,.]_1,\mathcal{T}_1\rangle$ and $\langle\mathcal{A}_2,[.,.]_2,\mathcal{T}_2\rangle$ be almost Pontryagin spaces, let $\phi:\mathcal{A}_1\to\mathcal{A}_2$ be a morphism. Then ϕ maps closed linear subspaces of $\mathcal{A}_1$ to closed linear subspaces of $\mathcal{A}_2$.*

Proof. Since $\operatorname{ran}\phi$ is $\mathcal{T}$-closed, $\operatorname{ran}\phi$ is a Banach space with respect to a norm inducing $\mathcal{T}$. By the Open Mapping Theorem, ϕ maps open subsets of $\mathcal{A}_1$ to relatively open subsets of $\operatorname{ran}\phi$. Let $\mathcal{M}$ be a closed subspace of $\mathcal{A}_1$. Since $\ker\phi\subseteq\mathcal{A}_1^\circ$, we have $\dim\ker\phi\le\operatorname{ind}_0\mathcal{A}_1<\infty$. Thus also the subspace $\mathcal{M}+\ker\phi$ is closed. However, we have

$$\big[\phi(\mathcal{M})\big]^c = \big[\phi(\mathcal{M}+\ker\phi)\big]^c = \phi\big([\mathcal{M}+\ker\phi]^c\big)\,,$$

and hence the space $\phi(\mathcal{M})$ is relatively closed in $\operatorname{ran}\phi$. Since $\operatorname{ran}\phi$ is $\mathcal{T}$-closed $\phi(\mathcal{M})$ is thus also $\mathcal{T}$-closed. ❏

COE11 **5.1.11 Corollary.** *Let $\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3$ be almost Pontryagin spaces. The composition $\phi_2\circ\phi_1:\mathcal{A}_1\to\mathcal{A}_3$ of two morphisms $\phi_1:\mathcal{A}_1\to\mathcal{A}_2$ and $\phi_2:\mathcal{A}_2\to\mathcal{A}_3$ is a morphism. The identity map $\mathrm{id}_\mathcal{A}:\mathcal{A}\to\mathcal{A}$ is a morphism.* ❏

Let us note that a linear, isometric, and continuous map $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ is an isomorphism, if and only if it is bijective. In other words, ϕ is an aPs-isomorphism if and only if it is a bijective TIPS-morphism. Moreover, let us state that, if $\mathcal{A}_1$ and $\mathcal{A}_2$ are nondegenerated, then $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ is an aPs-morphism if and only if it is a KS-morphism, cf. Corollary 3.4.3.

5.2 Subspaces, products, factors

In this section we investigate some natural constructions which can be carried out with almost Pontryagin spaces.

PRE12 **5.2.1 Proposition.** *Let $\langle \mathcal{A}, [.,.], \mathcal{T} \rangle$ be an almost Pontryagin space, and let $\mathcal{B}$ be a closed linear subspace of $\mathcal{A}$. Then $\langle \mathcal{B}, [.,.]|_{\mathcal{B}\times\mathcal{B}}, \mathcal{T}|_{\mathcal{B}} \rangle$ is an almost Pontryagin space. We have*

$$\operatorname{ind}_- \mathcal{B} \leq \operatorname{ind}_- \mathcal{A}, \quad \operatorname{ind}_0 \mathcal{B} \leq \operatorname{ind}_0 \mathcal{A} + \big(\operatorname{ind}_- \mathcal{A} - \operatorname{ind}_- \mathcal{B}\big). \tag{5.2.1}$$ E48

The inclusion map $\iota : \mathcal{B} \to \mathcal{A}$ is a morphism. Let $\langle \mathcal{C}, [.,.]_{\mathcal{C}}, \mathcal{T}_{\mathcal{C}} \rangle$ be an almost Pontryagin space, and let $\phi : \mathcal{C} \to \mathcal{B}$. Then ϕ is a morphism if and only if $\iota \circ \phi : \mathcal{C} \to \mathcal{A}$ is such.

Proof. Since $\mathcal{T} \in \operatorname{Top}_{\mathrm{Hs}} \mathcal{A}$ and $\mathcal{B}$ is $\mathcal{T}$-closed, also $\mathcal{T}|_{\mathcal{B}} \in \operatorname{Top}_{\mathrm{Hs}} \mathcal{B}$. Let $\mathcal{M}$ be a $\mathcal{T}$-closed, positive, and intrinsically complete subspace of $\mathcal{A}$ with finite codimension in $\mathcal{A}$, and set $\mathcal{N} := \mathcal{M} \cap \mathcal{B}$. Then $\mathcal{N}$ is $\mathcal{T}$-closed and, in particular, thus $\mathcal{T}|_{\mathcal{B}}$-closed and $\mathcal{T}|_{\mathcal{M}}$-closed. Since $\mathcal{M}$ is intrinsically complete, the topology $\mathcal{T}_0$ induced on $\mathcal{M}$ by $[.,.]$ is equal to $\mathcal{T}|_{\mathcal{M}}$. Hence $\mathcal{N}$ is $\mathcal{T}_0$-closed, and therefore intrinsically complete. Clearly, $\dim \mathcal{B}/\mathcal{N} \leq \dim \mathcal{A}/\mathcal{M} < \infty$. We conclude that $\langle \mathcal{B}, [.,.], \mathcal{T}|_{\mathcal{B}} \rangle$ is an almost Pontryagin space.

insert proof of (5.2.1)

In order to see the last assertion, it is enough to refer to Corollary 5.1.11 and Proposition 2.7.1. ❑

PRE13 **5.2.2 Proposition.** *Let $\langle \mathcal{A}_i, [.,.]_i, \mathcal{T}_i \rangle$, $i = 1, \dots, n$, be almost Pontryagin spaces, and define*

$$\mathcal{L} := \prod_{i=1}^n \mathcal{L}_i, \ [x,y] := \sum_{i=1}^n [\pi_i x, \pi_i y]_i, \ \mathcal{T} := \prod_{i=1}^n \mathcal{T}_i \,,$$

where π_i denotes the canonical projection of $\mathcal{A}$ onto $\mathcal{A}_i$. Then $\langle \mathcal{A}, [.,.], \mathcal{T} \rangle$ is an almost Pontryagin space. We have

$$\operatorname{ind}_- \mathcal{A} = \sum_{i=1}^n \operatorname{ind}_- \mathcal{A}_i, \quad \operatorname{ind}_0 \mathcal{A} = \sum_{i=1}^n \operatorname{ind}_0 \mathcal{A}_i \,.$$

In fact, if $\mathcal{M}_i$, $i = 1, \dots, n$, are maximal negative subspaces of $\mathcal{A}_i$, then $\mathcal{M} := \prod_{i=1}^n \mathcal{M}_i$ is a maximal negative subspace of $\mathcal{A}$. Moreover, $\mathcal{A}^\circ = \prod_{i=1}^n \mathcal{A}_i^\circ$.

Denote by $\iota_i : \mathcal{A}_i \to \mathcal{A}$, $i = 1, \dots, n$, the canonical embeddings

$$\iota_i(x) := (0, \dots, \underset{\substack{\uparrow \\ i\text{-th place}}}{x}, \dots, 0) \,.$$

Then ι_i is a morphism. Let $\langle \mathcal{C}, [.,.]_{\mathcal{C}}, \mathcal{T}_{\mathcal{C}} \rangle$ be an almost Pontryagin space, and let $\phi : \mathcal{C} \to \mathcal{A}$. Then ϕ is a morphism if and only if ϕ is isometric and $\pi_i \circ \phi$, $i = 1, \dots, n$, are all continuous and map closed subspaces to closed subspaces.

Proof. We argue completely similar as in Proposition 5.2.1, using $\mathcal{N} := \prod_{i=1}^n \mathcal{M}_i$ with subspaces $\mathcal{M}_i \subseteq \mathcal{A}_i$, $i = 1, \dots, n$, which are $\mathcal{T}_i$-closed, positive, intrinsically complete, and have finite codimension in $\mathcal{A}_i$. ❑

PRE14 **5.2.3 Proposition.** *Let $\langle \mathcal{A}, [.,.], \mathcal{T} \rangle$ be an almost Pontryagin space, and let $\mathcal{B}$ be a linear subspace of $\mathcal{A}$ with $\mathcal{B} \subseteq \mathcal{A}^\circ$. Then an inner product $[.,.]_\sim$ on $\mathcal{A}/\mathcal{B}$ is well-defined by*

$$[\pi x, \pi y]_\sim := [x, y], \ x, y \in \mathcal{A},$$

where π denotes the canonical projection. The triple $\langle \mathcal{A}/\mathcal{B}, [.,.]_\sim, \mathcal{T}/\mathcal{B} \rangle$, where $\mathcal{T}/\mathcal{B}$ denotes the quotient topology, is an almost Pontryagin space. We have

$$\operatorname{ind}_- \mathcal{A}/\mathcal{B} = \operatorname{ind}_- \mathcal{A}, \quad \operatorname{ind}_0 \mathcal{A}/\mathcal{B} = \operatorname{ind}_0 \mathcal{A} - \dim \mathcal{B}.$$

The canonical projection $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{B}$ is a morphism. Let $\langle \mathcal{C}, [.,.]_{\mathcal{C}}, \mathcal{T}_{\mathcal{C}} \rangle$ be a topological inner product space, and let $\phi : \mathcal{A}/\mathcal{B} \to \mathcal{C}$. Then ϕ is a morphism if and only if $\phi \circ \pi$ is such.

Proof. Let $(\mathcal{A}_+, \mathcal{A}_-)$ be a fundamental decomposition of $\mathcal{A}$, with $\mathcal{A}_+$ being $\mathcal{T}$-closed and intrinsically complete. Then $(\pi(\mathcal{A}_+), \pi(\mathcal{A}_-))$ is a fundamental decomposition of $\mathcal{A}/\mathcal{B}$. Since $\pi|_{\mathcal{A}_+}$ maps $\mathcal{A}_+$ bijectively and isometrically onto $\pi(\mathcal{A}_+)$, the subspace $\pi(\mathcal{A}_+)$ is intrinsically complete. Since $\pi^{-1}(\pi(\mathcal{A}_+)) = \mathcal{A}_+ + \mathcal{A}^\circ$ is $\mathcal{T}$-closed, $\pi(\mathcal{A}_+)$ is $\mathcal{T}/\mathcal{B}$-closed. We conclude that $\langle \mathcal{A}/\mathcal{B}, [.,.]_\sim, \mathcal{T}/\mathcal{B} \rangle$ is an almost Pontryagin space.

The remaining assertions are immediate. ❑

Also a corresponding version of the 1st Homomorphism Theorem is valid.

COE15 **5.2.4 Corollary.** *Let $\langle \mathcal{A}_1, [.,.]_1, \mathcal{T}_1 \rangle$ and $\langle \mathcal{A}_2, [.,.]_2, \mathcal{T}_2 \rangle$ be almost Pontryagin spaces, and let $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ be a morphism. Then there exists a unique isomorphism $\hat{\phi}$ such that*

$$\begin{tikzcd}
\langle \mathcal{A}_1, [.,.]_1, \mathcal{T}_1 \rangle \arrow[r, "\phi"] \arrow[d, "\pi"'] & \langle \mathcal{A}_2, [.,.]_2, \mathcal{T}_2 \rangle \\
\langle \mathcal{A}_1 / \ker \phi, [.,.]_{1,\sim}, \mathcal{T} / \ker \phi \rangle \arrow[r, dotted, "\hat{\phi}"'] & \langle \operatorname{ran} \phi, [.,.]_2|_{\operatorname{ran} \phi \times \operatorname{ran} \phi}, \mathcal{T}_2|_{\operatorname{ran} \phi} \rangle \arrow[u, "\iota"']
\end{tikzcd}$$

Proof. As we just showed that space in the lower row of this diagram actually are almost Pontryagin spaces. For existence and uniqueness of $\hat{\phi}$, it is enugh to refer to Corollary 2.7.4, and recall that a bijective TIPS-morphism is an aPs-isomorphism. ❑

Next, we turn to orthogonal couplings. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be almost Pontryagin spaces, and let α be a linear subspace of $\mathcal{A}_1^\circ \times \mathcal{A}_2^\circ$. Then, by the previous statements, also $\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2$ is an almost Pontryagin space. The corresponding version of Proposition 1.7.7 now reads as follows.

PRE16 **5.2.5 Proposition.** *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be almost Pontryagin spaces, and let $\mathcal{A}$ be an almost Pontryagin space together with morphisms $\iota_j' : \mathcal{A}_j \to \mathcal{A}$, $j = 1, 2$, such that $\iota_1'(\mathcal{A}_1) \perp \iota_2'(\mathcal{A}_2)$. Then the map $\psi : \mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2 \to \mathcal{A}$ in Proposition 1.7.7 is a morphism.*

Proof. We already know that ψ is isometric. Continuity of ψ follows easily from continuity of ι_j', cf. the diagram (1.7.3). We need to show that $\operatorname{ran}\psi$ is closed in $\mathcal{A}$.

As closed subspaces of the almost Pontryagin space $\mathcal{A}$, both of $\operatorname{ran}\iota_1'$ and $\operatorname{ran}\iota_2'$ are themselves almost Pontryagin spaces. Hence there exist closed and intrinsically complete subspaces $\mathcal{M}_j$ of $\operatorname{ran}\iota_j'$, $j = 1, 2$, with finite codimension in $\operatorname{ran}\iota_j'$ Since $\operatorname{ran}\iota_1' \perp \operatorname{ran}\iota_2'$, also $\mathcal{M}_1 \perp \mathcal{M}_2$, in particular $\mathcal{M}_1 \cap \mathcal{M}_2 = \{0\}$. Their sum $\mathcal{M} := \mathcal{M}_{[\dot{+}]}\mathcal{M}_2$ is thus also intrinsically complete. As the orthogonal sum of two uniformly positive subspaces, $\mathcal{M}$ is itself uniformly positive. Thus $\mathcal{M}$ is also complete, and hence closed, in the norm of $\mathcal{A}$. Since $\mathcal{M}$ has finite codimension in $\operatorname{ran}\iota_1' + \operatorname{ran}\iota_2'$, it follows that $\operatorname{ran}\iota_1' + \operatorname{ran}\iota_2'$ is closed in the norm of $\mathcal{A}$. ❑

REE17 *5.2.6 Remark.* Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be almost Pontryagin spaces, and let α be a bijective map between some subspaces $\operatorname{dom}\alpha$ and $\operatorname{ran}\alpha$ of $\mathcal{A}_1^\circ$ and $\mathcal{A}_2^\circ$, respectively. The space $\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2$ can also be described explicitly. To this end choose closed subspaces $\mathcal{A}_{1,r}$ and $\mathcal{A}_{2,r}$ such that

$$\mathcal{A}_1 = \mathcal{A}_{1,r}[\dot{+}]\mathcal{A}_1^\circ,\ \mathcal{A}_2 = \mathcal{A}_{2,r}[\dot{+}]\mathcal{A}_2^\circ\,,$$

choose D_1 and D_2 such that

$$\mathcal{A}_1^\circ = D_1 \dot{+} \operatorname{dom}\alpha,\ \mathcal{A}_2^\circ = D_2 \dot{+} \operatorname{ran}\alpha\,,$$

and set $D := \operatorname{ran}\alpha$. Consider the almost Pontryagin space

$$\mathcal{A} := \mathcal{A}_{1,r}[\dot{+}]\big(D_1 \dot{+} D \dot{+} D_2\big)[\dot{+}]\mathcal{A}_{2,r} \tag{5.2.2}$$ E47

where the inner product and topology on $\mathcal{A}_{1,r}$ and $\mathcal{A}_{2,r}$ is the one inherited from $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively, and where $D_1 \dot{+} D \dot{+} D_2$ is neutral and endowed with the euclidean topology. Moreover, define $\iota_1' : \mathcal{A}_1 \to \mathcal{A}$ by

$$\iota_1'|_{\mathcal{A}_{1,r}\dot{+}D_1} := \mathrm{id},\ \iota_1'|_{\operatorname{dom}\alpha} := -\alpha\,,$$

and let $\iota_2' : \mathcal{A}_2 \to \mathcal{A}$ be the identity map. Then ι_1' and ι_2' are morphisms. Moreover, it is apparent from their definition that $\iota_1'(\mathcal{A}_1) \perp \iota_2'(\mathcal{A}_2)$ and $\iota_1'(\mathcal{A}_1) + \iota_2'(\mathcal{A}_2) = \mathcal{A}$.

By Proposition 1.7.7 there exists $\hat\alpha \subseteq \mathcal{A}_1^\circ \times \mathcal{A}_2^\circ$ and an isomorphism $\psi : \mathcal{A}_1 \boxplus_{\hat\alpha} \mathcal{A}_2 \to \mathcal{A}$ with

$$\begin{tikzcd}
\mathcal{A}_1 \arrow[r, "\iota_1^{\hat\alpha}"] \arrow[dr, "\iota_1'"'] & \mathcal{A}_1 \boxplus_{\alpha'} \mathcal{A}_2 \arrow[d, dotted, "\psi"] & \mathcal{A}_2 \arrow[l, "\iota_2^{\hat\alpha}"'] \arrow[dl, "\iota_2'"] \\
 & \mathcal{A} &
\end{tikzcd}$$

Thereby the linear subspace $\hat\alpha$ is given as $\hat\alpha = \{(x_1, x_2) \in \mathcal{A}_1^\circ \times \mathcal{A}_2^\circ : \iota_1'(x_1) = \iota_2'(x_2)\}$. Write $x_1 = a_1 + b_1$ according to the decomposition $\mathcal{A}_1^\circ = D_1 \dot{+} \operatorname{dom}\alpha$, and let $x_2 = a_2 + b_2$ according to $\mathcal{A}_2 = D_2 \dot{+} \operatorname{ran}\alpha$. Then $\iota_1'(x_1) = a_1 - \alpha(b_1)$ and $\iota_2'(x_2) = a_2 + b_2$. Hence we have $\iota_1'(x_1) = \iota_2'(x_2)$ if and only if $a_1 = a_2 = 0$ and $b_2 = \alpha(b_1)$. This, in turn, is equivalent to $(x_1, x_2) \in \alpha$.

We see that $\hat{\alpha} = \alpha$, and hence ψ is actually an isomorphism between $\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2$ and $\mathcal{A}$, i.e. $\mathcal{A}$ can be regarded as a concrete realization of $\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2$. //

5.3 The canonical Pontryagin space extension

There is a natural way to associate with a given almost Pontryagin space a Pontryagin space by means of a factorization process. Namely, for an almost Pontryagin space $\mathcal{A}$ the space $\mathfrak{P}(\mathcal{A}) := \mathcal{A}/\mathcal{A}^\circ$ is a Pontryagin space, cf. Proposition 5.2.3, Corollary 5.1.5.

It is an important observation that there is also a natural way to associate with a given almost Pontryagin space $\mathcal{A}$ a Pontryagin space $\mathfrak{P}_{\text{ext}}(\mathcal{A})$ by means of an extension process.

NRE18 *5.3.1. Construction of* $\mathfrak{P}_{\text{ext}}(\mathcal{A})$*:* Let $\mathcal{A}$ be an almost Pontryagin space. Choose a closed subspace $\mathcal{B}$ of $\mathcal{A}$ such that $\mathcal{A} = \mathcal{B}[\dot{+}]\mathcal{A}^\circ$. Since $\mathcal{B}$ is a closed and nondegenerated subspace of $\mathcal{A}$, it is itself a Pontryagin space. Let C be a linear space with $\dim C = \operatorname{ind}_0 \mathcal{A} =: \Delta$, and choose bases $\{a_1, \dots, a_\Delta\}$and $\{c_1, \dots, c_\Delta\}$ of $\mathcal{A}^\circ$ and C, respectively. Set

$$\mathfrak{P}_{\text{ext}}(\mathcal{A}) := \mathcal{A} \dot{+} C = \mathcal{B} \dot{+} \mathcal{A}^\circ \dot{+} C\,,$$

and define on this linear space an inner product $[.,.]$ by the requirements

$$[.,.]|_{\mathcal{A}\times\mathcal{A}} = [.,.]_{\mathcal{A}}, \quad \mathcal{B} \perp C, \quad [a_i, c_i] = \delta_{ij}, \quad C \text{ neutral}.$$

As the direct and orthogonal sum of two Pontryagin spaces, $\langle \mathfrak{P}_{\text{ext}}(\mathcal{A}), [.,.] \rangle$ is a Pontryagin space. Moreover, the natural embedding ι_{ext} of $\mathcal{A}$ into $\mathfrak{P}_{\text{ext}}(\mathcal{A})$ is isometric and has closed range, i.e. is a morphism. Clearly, ι_{ext} is injective and $\dim \mathcal{P}_{\text{ext}}(\mathcal{A})/\mathcal{A} = \operatorname{ind}_0 \mathcal{A}$. //

Ad hoc the space $\mathfrak{P}_{\text{ext}}(\mathcal{A})$ depends on the choice of $\mathcal{B}$ and the respective bases of $\mathcal{A}^\circ$ and C. But actually we will shortly see that $\mathfrak{P}_{\text{ext}}(\mathcal{A})$ and ι_{ext} are uniquely determined up to isomorphisms by their properties that $\mathfrak{P}_{\text{ext}}(\mathcal{A})$ is a Pontryagin space, ι_{ext} is an injective morphism, and $\dim \mathcal{P}_{\text{ext}}(\mathcal{A})/\mathcal{A} = \operatorname{ind}_0 \mathcal{A}$, cf. Remark 5.3.5. We will refer to $\mathfrak{P}_{\text{ext}}(\mathcal{A})$ as the *canonical Pontryagin space extension of* $\mathcal{A}$, and to ι_{ext} as the *extension embedding* of $\mathcal{A}$ into its canonical Pontryagin space extension.

Morphisms between almost Pontryagin spaces can be extended to morphisms between their Pontryagin space extensions.

PRE19 **5.3.2 Proposition.** *Let $\mathcal{A}_1, \mathcal{A}_2$ be almost Pontryagin spaces, and let $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ be a morphism. Let spaces $\mathfrak{P}_{\text{ext}}(\mathcal{A}_1/\ker\phi)$ and $\mathfrak{P}_{\text{ext}}(\mathcal{A}_2)$ be constructed as in 5.3.1 from some subspaces $\mathcal{B}_1 \subseteq \mathcal{A}_1/\ker\phi$ and $\mathcal{B}_2 \subseteq \mathcal{A}_2$, respectively. Then there exists a morphism $\tilde{\phi} : \mathfrak{P}_{\text{ext}}(\mathcal{A}_1/\ker\phi) \to \mathfrak{P}_{\text{ext}}(\mathcal{A}_2)$, such that*

$$\begin{tikzcd}
\mathcal{A}_1 \arrow[r, "\pi"] \arrow[d, "\phi"'] & \mathcal{A}_1/\ker\phi \arrow[r, "\iota_{\text{ext}}"] & \mathfrak{P}_{\text{ext}}(\mathcal{A}_1/\ker\phi) \arrow[d, dotted, "\tilde{\phi}"] \\
\mathcal{A}_2 \arrow[rr, "\iota_{\text{ext}}"'] & & \mathfrak{P}_{\text{ext}}(\mathcal{A}_2)
\end{tikzcd} \tag{5.3.1}$$ E23

Proof. By the 1st Homomorphism Theorem, there exists an injective morphism $\phi' : \mathcal{A}_1/\ker\phi \to \mathcal{A}_2$ such that

$$\begin{tikzcd} \mathcal{A}_1 \arrow[r, "\pi"] \arrow[d, "\phi"'] & \mathcal{A}_1/\ker\phi \arrow[dl, dotted, "\phi'"] \\ \mathcal{A}_2 & \end{tikzcd}$$

cf. Corollary 5.2.4. Hence we may assume without loss of generality that ϕ is injective.

The subspace $(\iota_{\text{ext}} \circ \phi)(\mathcal{B}_1)$ of $\mathfrak{P}_{\text{ext}}(\mathcal{A}_2)$ is closed and nondegenerated. Moreover, $(\iota_{\text{ext}} \circ \phi)(\mathcal{A}_1^\circ)$ is a neutral subspace of $(\iota_{\text{ext}} \circ \phi)(\mathcal{B}_1)^\perp$. Applying Theorem 4.4.1 with the closed subspace

$$\mathcal{L} := (\iota_{\text{ext}} \circ \phi)(\mathcal{B}_1)[\dot{+}](\iota_{\text{ext}} \circ \phi)(\mathcal{A}_1^\circ) \subseteq \mathfrak{P}_{\text{ext}}(\mathcal{A}_2)\,,$$

we obtain a subspace $\mathcal{M} \subseteq (\iota_{\text{ext}} \circ \phi)(\mathcal{B}_1)^\perp$, such that $(\iota_{\text{ext}} \circ \phi)(\mathcal{A}_1^\circ)\#\mathcal{M}$.

Let $\{a_1, \ldots, a_\Delta\}$ and $\{c_1, \ldots, c_\Delta\}$ be the bases of $\mathcal{A}_1^\circ$ and C_1 used in the construction of $\mathfrak{P}_{\text{ext}}(\mathcal{A}_1)$. The set $\{\iota_{\text{ext}} \circ \phi(a_1), \ldots, \iota_{\text{ext}} \circ \phi(a_\Delta)\}$ is a basis of $\iota_{\text{ext}} \circ \phi(\mathcal{A}_1^\circ)$. By Lemma 1.6.3, there exists a basis $\{b_1, \ldots, b_\Delta\}$ of $\mathcal{M}$ such that

$$[\iota_{\text{ext}} \circ \phi(a_j), b_k] = \delta_{jk}, \quad j, k = 1, \ldots, \Delta\,.$$

With these notations define $\tilde{\phi} : \mathfrak{P}_{\text{ext}}(\mathcal{A}_1) \to \mathfrak{P}_{\text{ext}}(\mathcal{A}_2)$ by

$$\tilde{\phi}|_{\iota_{\text{ext}}(\mathcal{A}_1)} := \iota_{\text{ext}} \circ \phi \circ \iota_{\text{ext}}^{-1}, \quad \tilde{\phi}(c_j) := b_j, \quad j = 1, \ldots, \Delta\,.$$

The restriction $\tilde{\phi}|_{\iota_{\text{ext}}(\mathcal{A}_1)}$ is continuous and maps closed subspaces of $\iota_{\text{ext}}(\mathcal{A}_1)$ to closed subspaces of $\mathfrak{P}_{\text{ext}}(\mathcal{A}_2)$. Since $\iota_{\text{ext}}(\mathcal{A}_1)$ is a closed subspace with finite codimension in $\mathfrak{P}_{\text{ext}}(\mathcal{A}_1)$, the map $\tilde{\phi}$ inherits these properties from its restriction. It is straightforward to check that $\tilde{\phi}$ is isometric. Finally, the fact that (5.3.1) commutes is built into the definition. ❏

REE20 *5.3.3 Remark.* The extension $\tilde{\phi}$ in Proposition 5.3.2 is in general not unique. For example, whenever $\mathcal{P}$ is a Pontryagin space with

$$(\iota_{\text{ext}} \circ \phi)(\mathcal{A}_1) \subseteq \mathcal{P} \subseteq \mathfrak{P}_{\text{ext}}(\mathcal{A}_2)\,,$$

the choice of $\tilde{\phi}$ can be made such that $\operatorname{ran}\tilde{\phi} \subseteq \mathcal{P}$. //

COE21 **5.3.4 Corollary.** *Let $\mathcal{A}$ be an almost Pontryagin space, and let $\mathfrak{P}_{\text{ext}}(\mathcal{A})$ be constructed as in 5.3.1 from some subspace $\mathcal{B}$. Moreover, let $\mathcal{P}$ be a Pontryagin space, let $\iota : \mathcal{A} \to \mathcal{P}$ an injective morphism, and assume that* $\dim \mathcal{P}/\iota(\mathcal{A}) = \operatorname{ind}_0 \mathcal{A}$. *Then there exists an isomorphism of $\lambda : \mathfrak{P}_{\text{ext}}(\mathcal{A}) \to \mathcal{P}$ such that*

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\iota_{\text{ext}}"] \arrow[d, "\iota"'] & \mathfrak{P}_{\text{ext}}(\mathcal{A}) \arrow[dl, dotted, "\lambda"] \\ \mathcal{P} & \end{tikzcd}$$

Proof. Since $\mathcal{P}$ is a Pontryagin space, we have $\mathfrak{P}_{\text{ext}}(\mathcal{P}) = \mathcal{P}$ and $\iota_{\text{ext}} = \text{id}$. Proposition 5.3.2 applied with the map $\iota : \mathcal{A} \to \mathcal{P}$ gives a morphism $\lambda : \mathfrak{P}_{\text{ext}}(\mathcal{A}) \to \mathcal{P}$. Since a morphism between Pontryagin spaces is injective, we conclude from $\lambda(\iota_{\text{ext}}(\mathcal{A})) = \iota(\mathcal{A})$ and

$$\dim \mathcal{P}/\iota(\mathcal{A}) = \operatorname{ind}_0 \mathcal{A} = \dim \mathfrak{P}_{\text{ext}}(\mathcal{A})/\iota_{\text{ext}}(\mathcal{A}),$$

that λ is bijective, and hence an isomorphism. ❑

REE22 *5.3.5 Remark.* We obtain from Corollary 5.3.4 that the canonical Pontryagin space extension does not depend on the choice of the space $\mathcal{B}$ in its construction 5.3.1. This independence includes the embedding ι_{ext}. More exactly, let $\mathcal{A}$ be an almost Pontryagin space, and let $\mathcal{B}$ and $\tilde{\mathcal{B}}$ be two subspaces qualifed for being used in 5.3.1. Denote the correspondingly constructed Pontryagin space extensions of $\mathcal{A}$ by $\mathfrak{P}_{\text{ext}}(\mathcal{A})$ and $\tilde{\mathfrak{P}}_{\text{ext}}(\mathcal{A})$ and let ι_{ext} and $\tilde{\iota}_{\text{ext}}$ be the corresponding embeddings. An application of Corollary 5.3.4 with $\mathcal{P} := \tilde{\mathfrak{P}}_{\text{ext}}(\mathcal{A})$ and $\iota := \tilde{\iota}_{\text{ext}}$ gives an isomorphism $\lambda : \mathfrak{P}_{\text{ext}}(\mathcal{A}) \to \tilde{\mathfrak{P}}_{\text{ext}}(\mathcal{A})$ which satisfies $\tilde{\iota}_{\text{ext}} = \lambda \circ \iota_{\text{ext}}$. //

The following result shows that Pontryagin space extension is compatible with orthogonal coupling.

PRE24 **5.3.6 Proposition.** *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be almost Pontryagin spaces and let α be a bijective function between subspaces of $\mathcal{A}_1^\circ$ and $\mathcal{A}_2^\circ$. Then there exist morphisms $\tilde{\iota}_1^\alpha$ and $\tilde{\iota}_2^\alpha$, such that*

$$\begin{tikzcd}
\mathcal{A}_1 \arrow[rr, "\iota_1^\alpha"] \arrow[d, "\iota_{\text{ext}}"'] & & \mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2 \arrow[d, "\iota_{\text{ext}}"] & & \mathcal{A}_2 \arrow[ll, "\iota_2^\alpha"'] \arrow[d, "\iota_{\text{ext}}"] \\
\mathfrak{P}_{\text{ext}}(\mathcal{A}_1) \arrow[rr, dotted, "\tilde{\iota}_1^\alpha"'] & & \mathfrak{P}_{\text{ext}}(\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2) & & \mathfrak{P}_{\text{ext}}(\mathcal{A}_2) \arrow[ll, dotted, "\tilde{\iota}_2^\alpha"]
\end{tikzcd} \tag{5.3.2}$$ E25

The choice of $\tilde{\iota}_1^\alpha$ and $\tilde{\iota}_2^\alpha$ can be made such that $\tilde{\iota}_1^\alpha(\mathfrak{P}_{\text{ext}}(\mathcal{A}_1)) \cap \tilde{\iota}_2^\alpha(\mathfrak{P}_{\text{ext}}(\mathcal{A}_2))$ is a nondegenerated subspace of $\mathfrak{P}_{\text{ext}}(\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2)$ with dimension $2\dim(\operatorname{dom}\alpha)$ which contains $(\iota_{\text{ext}} \circ \iota_1^\alpha)(\operatorname{dom}(\alpha))$.

Proof. The existence of $\tilde{\iota}_1^\alpha$ and $\tilde{\iota}_2^\alpha$ which satisfy (5.3.2) is immediate from Proposition 5.3.2. We have to show that they can be chosen so to satisfy the stated additional requirement. To this end we use the description of $\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2$ given in Remark 5.2.6, cf. (5.2.2). With the notation introduced there, choose bases

$$\{\gamma_1^1, \dots, \gamma_{n_1}^1\},\ \{\gamma_1, \dots, \gamma_n\},\ \{\gamma_2^2, \dots, \gamma_{n_2}^2\}$$

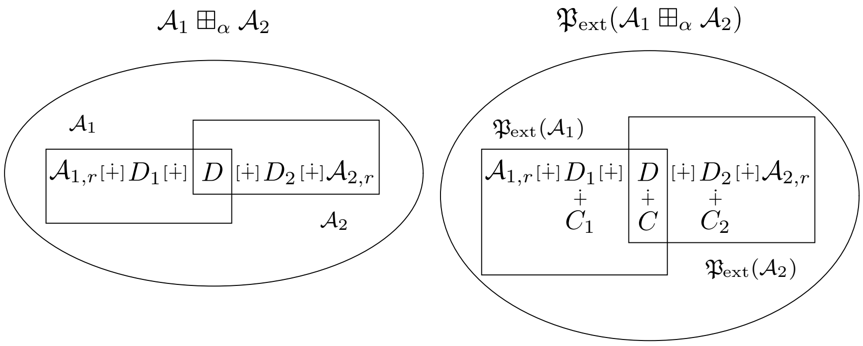
of D_1, D and D_2, respectively, and use for the construction in Proposition 5.3.2 the basis

$$\{\gamma_1^1, \dots, \gamma_{n_1}^1, -\alpha^{-1}(\gamma_1), \dots, -\alpha^{-1}(\gamma_n), \gamma_2^2, \dots, \gamma_{n_2}^2\}$$

of $(\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2)^\circ$. ❑

REE26 *5.3.7 Remark.* Since, in the situation of Proposition 5.3.6, the mappings ι_1^α and ι_2^α are both injective, we can think of $\mathfrak{P}_{\text{ext}}(\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2)$ as the biggest of all the six spaces in (5.3.2) which contains all the others. Note here that all extension embeddings ι_{ext} are by definition injective and that $\tilde{\iota}_1^\alpha, \tilde{\iota}_1^\alpha$ are morphisms whose domain is nondegenerated and are thus also injective. //

NRE27 *5.3.8. :* Thinking in terms of the concrete construction of $\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2$ given in Remark 5.2.6 and in terms of the construction 5.3.1 of Pontryagin spaces extensions, we can picture the situation present in Proposition 5.3.6 as follows:



Thereby we have $C\#D$, $C_1\#D_1$, $C_2\#D_2$, and the spaces $\mathcal{B}$, $\mathcal{B}_1$ and $\mathcal{B}_2$, used in the construction of $\mathfrak{P}_{\text{ext}}(\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2)$, $\mathfrak{P}_{\text{ext}}(\mathcal{A}_1)$ and $\mathfrak{P}_{\text{ext}}(\mathcal{A}_2)$, respectively, are

$$\mathcal{B} = \mathcal{A}_{1,r}[\dot{+}]\mathcal{A}_{2,r}, \quad \mathcal{B}_1 = \mathcal{A}_{1,r}, \ \mathcal{B}_2 = \mathcal{A}_{2,r}$$

Moreover, we see that

$$\mathfrak{P}_{\text{ext}}(\mathcal{A}_1) \cap \mathfrak{P}_{\text{ext}}(\mathcal{A}_2) = D\dot{+}C\,.$$

//

5.4 Fundamental decompositions, Orthocomplements, Isometries

The fact that we can map an almost Pontryagin space into or onto a Pontryagin space, often allows us to employ Pontryagin space results. In this section we give some results of this kind. We start with a geometric lemma.

LEE28 **5.4.1 Lemma.** *Let $\langle \mathcal{A}, [.,.], \mathcal{T} \rangle$ be an almost Pontryagin space, and let $\mathcal{M}$ be a linear subspace of $\mathcal{A}$. Moreover, denote by $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{A}^\circ$ the canonical projection. Then*

$$\pi(\mathcal{M})^\perp = \pi(\mathcal{M}^\perp), \quad \pi(\mathcal{M}^\circ) = \pi(\mathcal{M})^\circ\,.$$

Proof. The first relation is clear, since π is isometric and surjective. Moreover, the inclusion '$\subseteq$' in the second relation is trivial. Assume that $x \in \mathcal{A}$ and $\pi(x) \in \pi(\mathcal{M})^\circ$. Then $x \in (\mathcal{M} + \mathcal{A}^\circ) \cap \mathcal{M}^\perp$, and hence we may choose $x_1 \in \mathcal{M}$ with $\pi(x_1) = \pi(x)$. Clearly, also $x_1 \in \mathcal{M}^\perp$, i.e. $x_1 \in \mathcal{M}^\circ$. This gives the inclusion '$\supseteq$' in the second asserted relation. ❑

As a first consequence, we obtain:

COE29 **5.4.2 Corollary.** *Let $\langle \mathcal{A}, [.,.], \mathcal{T} \rangle$ be an almost Pontryagin space, and let $\mathcal{M}$ be a linear subspace of $\mathcal{A}$. Then the following hold.*

(*i*) $\mathcal{M}^{\perp\perp} = \overline{\mathcal{M}} + \mathcal{A}^\circ$.

(ii) $(\mathcal{M}^\perp)^\circ = \overline{\mathcal{M}}^\circ + \mathcal{A}^\circ$.

(iii) $\mathcal{M}^\perp = \mathcal{A}^\circ$ *if and only if* $\mathcal{M} + \mathcal{A}^\circ$ *is dense in* $\mathcal{A}$.

Proof. Again let $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{A}^\circ$ be the canonical projection. Consider the subspace $\pi(\mathcal{M})$. Using Corollary 3.3.2, we obtain

$$\pi(\mathcal{M}^{\perp\perp}) = \pi(\mathcal{M})^{\perp\perp} = \overline{\pi(\mathcal{M})} = \pi(\overline{\mathcal{M}})\,.$$

Since $\mathcal{A}^\circ \subseteq \mathcal{M}^{\perp\perp}$, taking inverse images gives (i). In order to see (ii), we compute

$$(\mathcal{M}^\perp)^\circ = \mathcal{M}^\perp \cap \mathcal{M}^{\perp\perp} = \overline{\mathcal{M}}^\perp \cap \big(\overline{\mathcal{M}} + \mathcal{A}^\circ\big) = \overline{\mathcal{M}}^\circ + \mathcal{A}^\circ\,.$$

Finally, $\mathcal{M} + \mathcal{A}^\circ$ is dense in $\mathcal{A}$ if and only if $\pi(\mathcal{M})$ is dense in $\mathcal{A}/\mathcal{A}^\circ$. The latter is, by Corollary 3.3.2, equivalent to $\pi(\mathcal{M})^\perp = \{0\}$. Since $\pi(\mathcal{M})^\perp = \pi(\mathcal{M}^\perp)$, this just means that $\mathcal{M}^\perp \subseteq \mathcal{A}^\circ$. ❑

Next we turn to fundamental decompositions of an almost Pontryagin space $\langle \mathcal{A}, [.,.], \mathcal{T} \rangle$ compatible with the topology $\mathcal{T}$. To this end, we need the corresponding notion of uniform definiteness.

DEE30 **5.4.3 Definition.** Let $\langle \mathcal{A}, [.,.], \mathcal{T} \rangle$ be an almost Pontryagin space, and let $\mathcal{M} \in \operatorname{Sub}\mathcal{A}$. Moreover, let $\|.\|$ be a norm on $\mathcal{A}$ which induces $\mathcal{T}$. Then $\mathcal{M}$ is called *uniformly positive*, if there exists a constant $\gamma > 0$ such that

$$[x,x] \geq \gamma \|x\|^2,\ x \in \mathcal{M}\,.$$

The subspace $\mathcal{M}$ is called *uniformly negative*, if there exists a constant $\gamma > 0$ such that

$$-[x,x] \geq \gamma \|x\|^2,\ x \in \mathcal{M}\,.$$

The set of all uniformly positive subspaces of $\mathcal{K}$ will be denoted by $\operatorname{Sub}_{\gg 0}\mathcal{A}$, the set of all uniformly negative ones by $\operatorname{Sub}_{\ll 0}\mathcal{A}$. //

Let us explicitly note that the $\mathcal{T}$-closure of a uniformly positive (negative) subspace of $\mathcal{A}$ is again uniformly positive (negative, respectively).

PRE31 **5.4.4 Proposition.** *Let* $\langle \mathcal{A}, [.,.], \mathcal{T} \rangle$ *be an almost Pontryagin space, and let* $\mathcal{L}_+, \mathcal{L}_- \in \operatorname{Sub}\mathcal{A}$. *Then there exists a fundamental decomposition* $\mathfrak{J} = (\mathcal{A}_+, \mathcal{A}_-)$ *of* $\mathcal{A}$, *with* $\mathcal{A}_+$ *being* $\mathcal{T}$*-closed and*

$$\mathcal{L}_+ \subseteq \mathcal{A}_+ \quad \textit{and} \quad \mathcal{L}_- \subseteq \mathcal{A}_-\,,$$

if and only if

$$\mathcal{L}_+ \in \operatorname{Sub}_{\gg 0}\mathcal{A},\ \mathcal{L}_- \in \operatorname{Sub}_{<0}\mathcal{A}, \quad \mathcal{L}_+ \perp \mathcal{L}_-\,.$$

Proof. Assume that $\mathcal{L}_+$ and $\mathcal{L}_-$ satisfy the stated conditions. We have to construct a fundamental decomposition with the required properties. Since the closure $\overline{\mathcal{L}_+}$ of $\mathcal{L}_+$ is again uniformly positive, we may assume without loss of generality that $\mathcal{L}_+$ is closed. Then $\pi(\mathcal{L}_+)$ is a closed and positive subspace of $\mathcal{A}/\mathcal{A}^\circ$. Moreover, $\pi(\mathcal{L}_-)$ is negative and $\pi(\mathcal{L}_-) \perp \pi(\mathcal{L}_+)$. By Corollary 4.2.2, there exists a fundamental decomposition $(\mathcal{P}_+, \mathcal{P}_-)$ of $\mathcal{A}/\mathcal{A}^\circ$ with

$$\pi(\mathcal{L}_+) \subseteq \mathcal{P}_+, \quad \pi(\mathcal{L}_-) \subseteq \mathcal{P}_-\,.$$

The subspace $\pi^{-1}(\mathcal{P}_+)$ is closed in $\mathcal{A}$, it contains the closed subspace $\mathcal{L}_+$ and the finite dimensional subspace $\mathcal{A}^\circ$, and their intersection equals $\{0\}$. Since $\mathcal{T} \in \operatorname{Top}_{\mathrm{Hs}} \mathcal{A}$, there exists a subspace $\mathcal{M}$ such that $\mathcal{L}_+ \dot{+} \mathcal{M}$ is closed and $\pi^{-1}(\mathcal{P}_+) = \mathcal{L}_+ \dot{+} \mathcal{A}^\circ \dot{+} \mathcal{M}$. Set $\mathcal{A}_+ := \mathcal{L}_+ \dot{+} \mathcal{M}$, and choose a negative subspace $\mathcal{A}_-$ with $\pi(\mathcal{A}_-) = \mathcal{P}_-$. Then $(\mathcal{A}_+, \mathcal{A}_-)$ is a fundamental decomposition of $\mathcal{A}$, and $\mathcal{A}_+$ is closed.

The converse follows since each positive, closed, and intrinsically complete subspace of $\mathcal{A}$ is certainly uniformly positive, and in turn this property is inherited by subspaces. ❏

PRE32 **5.4.5 Proposition.** *Let $\langle \mathcal{A}, [.,.], \mathcal{T} \rangle$ be an almost Pontryagin space, and let $\mathcal{M} \in \operatorname{Sub} \mathcal{A}$. Then $\mathcal{M}$ is orthocomplemented if and only if $\mathcal{M} + \mathcal{A}^\circ$ is $\mathcal{T}$-closed and $\mathcal{M}^\circ \subseteq \mathcal{A}^\circ$.*

Proof. Let π denote the canonical projection of $\mathcal{A}$ onto $\mathcal{A}/\mathcal{A}^\circ$. Since $\pi(\mathcal{M})^\perp = \pi(\mathcal{M}^\perp)$ and $\mathcal{A}^\circ \subseteq \mathcal{M}^\perp$, we have

$$\mathcal{M} + \mathcal{M}^\perp = \mathcal{A} \iff \pi(\mathcal{M}) + \pi(\mathcal{M})^\perp = \mathcal{A}/\mathcal{A}^\circ .$$

By Proposition 4.2.3, $\pi(\mathcal{M})$ is orthocomplemented if and only if it is closed and nondegenerated. However, $\pi(\mathcal{M})$ is closed if and only if $\mathcal{M} + \mathcal{A}^\circ$ is closed. By Lemma 5.4.1, the space $\pi(\mathcal{M})$ is nondegenerated if and only if $\mathcal{M}^\circ \subseteq \mathcal{A}^\circ$. ❏

PRE33 **5.4.6 Proposition.** *Let $\langle \mathcal{A}_1, [.,.]_1, \mathcal{T}_1 \rangle$ and $\langle \mathcal{A}_2, [.,.]_2, \mathcal{T}_2 \rangle$ be almost Pontryagin spaces, and let $\phi : \operatorname{dom} \phi \subseteq \mathcal{A}_1 \to \mathcal{A}_2$ be isometric. If $\overline{\operatorname{ran} \phi}$ is nondegenerated, then ϕ is continuous. Its continuation $\tilde{\phi}$ by continuity is a morphism of $\overline{\operatorname{dom} \phi}$ onto $\overline{\operatorname{ran} \phi}$.*

Proof. Let $\iota^j_{\mathrm{ext}} : \mathcal{A}_j \to \mathfrak{P}_{\mathrm{ext}}(\mathcal{A}_j)$, $j = 1, 2$, be the respective extension embeddings. Consider the map

$$\psi := \iota^2_{\mathrm{ext}} \circ \phi \circ (\iota^1_{\mathrm{ext}})^{-1} : \iota^1_{\mathrm{ext}}(\operatorname{dom} \phi) \subseteq \mathfrak{P}_{\mathrm{ext}}(\mathcal{A}_1) \to \mathfrak{P}_{\mathrm{ext}}(\mathcal{A}_2)$$

Then ψ is isometric, and $\operatorname{ran} \psi = \iota^2_{\mathrm{ext}}(\operatorname{ran} \phi)$. Thus

$$\overline{\operatorname{ran} \psi} = \overline{\iota^2_{\mathrm{ext}}(\operatorname{ran} \phi)} = \iota^2_{\mathrm{ext}}(\overline{\operatorname{ran} \phi}) ,$$

and hence $\overline{\operatorname{ran} \psi}$ is nondegenerated. By Proposition 4.3.1, ψ is continuous and its extension $\hat{\psi}$ by continuity maps $\overline{\operatorname{dom} \psi}$ onto $\overline{\operatorname{ran} \psi}$. Since $\overline{\operatorname{dom} \psi} \subseteq \iota^1_{\mathrm{ext}}(\mathcal{A}_1)$ and $\overline{\operatorname{ran} \psi} \subseteq \iota^2_{\mathrm{ext}}(\mathcal{A}_2)$, we may consider the map $\hat{\phi} := (\iota^2_{\mathrm{ext}})^{-1} \circ \hat{\psi} \circ \iota^1_{\mathrm{ext}}$. It is defined on $\overline{\operatorname{dom} \phi}$, is isometric, continuous, and its range is equal to $\overline{\operatorname{ran} \phi}$. Thus it is a morphism of $\overline{\operatorname{dom} \phi}$ onto $\overline{\operatorname{ran} \phi}$. ❏

PRE34 **5.4.7 Proposition.** *Let $\langle \mathcal{A}, [.,.], \mathcal{T} \rangle$ be an almost Pontryagin space, let $\mathcal{L}$ be a closed subspace of $\mathcal{A}$, and let $\mathcal{L}^1$ be a subspace with $\mathcal{L}^\circ = \mathcal{L}^1 \dot{+} (\mathcal{L}^\circ \cap \mathcal{A}^\circ)$. Then*

(*i*) *There exist closed and nondegenerated subspaces $\mathcal{L}_1$ and $\mathcal{L}_2$ of $\mathcal{A}$, such that $\mathcal{L} = \mathcal{L}_1 [\dot{+}] \mathcal{L}^\circ$ and $\mathcal{L}^\perp = \mathcal{L}_2 [\dot{+}] \mathcal{L}^1 [\dot{+}] \mathcal{A}^\circ$.*

(*ii*) *Whenever $\mathcal{L}_1$ and $\mathcal{L}_2$ have the properties stated in* (*i*)*, there exists $\mathcal{N} \in \operatorname{Sub}_0 \mathcal{A}$ with $\mathcal{L}^1 \# \mathcal{N}$ and*

$$\mathcal{A} = \mathcal{L}_1 [\dot{+}] (\mathcal{L}^1 \dot{+} \mathcal{N}) [\dot{+}] \mathcal{L}_2 [\dot{+}] \mathcal{A}^\circ . \qquad (5.4.1)$$ E35

(*iii*) *Whenever* $\mathcal{N} \in \operatorname{Sub}_0 \mathcal{A}$ *with* $\mathcal{L}^1 \# \mathcal{N}$, *there exist subspaces* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *as in* (*i*), *such that* (5.4.1) *holds.*

Proof. By finite-dimensionality, there exist closed complements $\mathcal{L}_1$ of $\mathcal{L}^\circ$ in $\mathcal{L}$. By Corollary 5.4.2, (*ii*), we have $(\mathcal{L}^\perp)^\circ = \mathcal{L}^\circ + \mathcal{A}^\circ$. Hence, we may choose for $\mathcal{L}_2$ any closed complement of $\mathcal{L}^\circ + \mathcal{A}^\circ$ in $\mathcal{L}^\perp$. This shows (*i*).

For the proof of (*ii*), let $\mathcal{L}_1$ and $\mathcal{L}_2$ be given. Let π denoten the canonical projection of $\mathcal{A}$ onto $\mathcal{A}/\mathcal{A}^\circ$, and consider the closed subspace $\pi(\mathcal{L})$ of the Pontryagin space $\mathcal{A}/\mathcal{A}^\circ$. We have

$$\pi(\mathcal{L})^\circ = \pi(\mathcal{L}^\circ) = \pi(\mathcal{L}^1)\,,$$

and

$$\pi(\mathcal{L}) = \pi(\mathcal{L}_1)[\dot{+}]\pi(\mathcal{L}^1), \quad \pi(\mathcal{L})^\perp = \pi(\mathcal{L}^\perp) = \pi(\mathcal{L}_2)[\dot{+}]\pi(\mathcal{L}^1)\,.$$

Theorem 4.4.1, (*ii*), furnishes us with a neutral subspace $\mathcal{M}$ of $\mathcal{A}/\mathcal{A}^\circ$ with $\mathcal{L}^1 \# \mathcal{M}$ and

$$\mathcal{A}/\mathcal{A}^\circ = \pi(\mathcal{L}_1)[\dot{+}]\big(\pi(\mathcal{L}^1)\dot{+}\mathcal{M}\big)[\dot{+}]\pi(\mathcal{L}_2)\,.$$

Choose $\mathcal{N} \subseteq \mathcal{A}$ with $\dim \mathcal{N} = \dim \mathcal{M}$ and $\pi(\mathcal{N}) = \mathcal{M}$. Then

$$\mathcal{L}_1[\dot{+}]\big(\mathcal{L}^1\dot{+}\mathcal{N}\big)[\dot{+}]\mathcal{L}_2$$

is a nondegenerated subspace of $\mathcal{A}$ which is mapped by π onto $\mathcal{A}/\mathcal{A}^\circ$. Thus the desired decomposition (5.4.1) of $\mathcal{A}$ holds.

$\boxed{\downarrow\downarrow \text{ fix: } \mathcal{L}_1\dot{+}(\mathcal{A}^\circ \cap \mathcal{L}) = \mathcal{L} \cap \pi^{-1}(\mathcal{L}_1)}$

Finally, let $\mathcal{N}$ be given as in (*iii*). Put $\mathcal{M} := \pi(\mathcal{N})$, then $\mathcal{M}$ is a neutral subspace of $\mathcal{A}/\mathcal{A}^\circ$ and $\mathcal{M} \# \pi(\mathcal{L})^\circ$. Again employing Theorem 4.4.1 for the subspace $\pi(\mathcal{L})$, we find closed subspaces $\hat{\mathcal{L}}_1$ and $\hat{\mathcal{L}}_2$ of $\mathcal{A}/\mathcal{A}^\circ$ such that

$$\hat{\mathcal{L}}_1[\dot{+}]\big(\pi(\mathcal{L}^1)\dot{+}\mathcal{M}\big)[\dot{+}]\hat{\mathcal{L}}_2, \quad \pi(\mathcal{L}) = \hat{\mathcal{L}}_1[\dot{+}]\pi(\mathcal{L}^1),\ \pi(\mathcal{L})^\perp = \hat{\mathcal{L}}_2[\dot{+}]\pi(\mathcal{L}^1)\,.$$

Set $\mathcal{L}_1 := \mathcal{L} \cap \pi^{-1}(\hat{\mathcal{L}}_1)$ and let $\mathcal{L}_2$ be a closed complement of $\mathcal{A}^\circ$ in $\mathcal{L}^\perp \cap \pi^{-1}(\hat{\mathcal{L}}_2)$.

Since $\hat{\mathcal{L}}_1 \subseteq \pi(\mathcal{L})$, we have $\pi(\mathcal{L}_1) = \hat{\mathcal{L}}_1$ and hence $\mathcal{L} = \mathcal{L}_1[\dot{+}]\mathcal{L}^\circ$. Since $\hat{\mathcal{L}}_2 \subseteq \pi(\mathcal{L})^\perp = \pi(\mathcal{L}^\perp)$, we have $\pi(\mathcal{L}_2) = \pi(\mathcal{L}^\perp \cap \pi^{-1}(\hat{\mathcal{L}}_2)) = \hat{\mathcal{L}}_2$. It follows that the decomposition (5.4.1) holds. ❑

5.5 Almost Pontryagin space completions

DEE36 **5.5.1 Definition.** Let $\langle \mathcal{L}, [.,.] \rangle$ be an inner product space. A pair $(\iota, \mathcal{A})$ is called an aPs-*completion* of $\mathcal{L}$, if $\mathcal{A}$ is an almost Pontryagin space, and ι is an isometric map whose range is dense in $\mathcal{A}$.

Two completions $(\iota_1, \mathcal{A}_1)$ and $(\iota_2, \mathcal{A}_2)$ are called *isomorphic*, if there exist an isomorphism ϕ of $\mathcal{A}_1$ onto $\mathcal{A}_2$, such that $\phi \circ \iota_1 = \iota_2$, i.e. such that we have the diagram

$$\begin{tikzcd} & \mathcal{L} \arrow[dl, "\iota_1"'] \arrow[dr, "\iota_2"] & \\ \mathcal{A}_1 \arrow[rr, "\phi"'] & & \mathcal{A}_2 \end{tikzcd} \tag{5.5.1}$$ E46

In this case, we write $(\iota_1, \mathcal{A}_1) \cong (\iota_2, \mathcal{A}_2)$. //

If $\langle\mathcal{L},[.,.]\rangle$ has an aPs-completion, then clearly $\operatorname{ind}_-\mathcal{L}<\infty$. Conversely, if $\operatorname{ind}_-\mathcal{L}<\infty$, then by Proposition 4.3.5 there exists even a Pontryagin space completion of $\mathcal{L}$. We conclude that $\langle\mathcal{L},[.,.]\rangle$ admits an aPs-completion if and only if $\operatorname{ind}_-\mathcal{L}<\infty$.

Unlike in the case of Pontryagin space completions, aPs-completions are not uniquely determined up to isomorphism. However, their totality can be described in a neat way. Let us introduce an order relation on the set of isomorphy classes of aPs-completions of a given inner product space $\langle\mathcal{L},[.,.]\rangle$.

DEE37 **5.5.2 Definition.** Let $\langle\mathcal{L},[.,.]\rangle$ be an inner product space, and let $(\iota_1,\mathcal{A}_1)$ and $(\iota_2,\mathcal{A}_2)$ be two aPs-completions of $\mathcal{L}$. Then we write $(\iota_1,\mathcal{A}_1)\succeq(\iota_2,\mathcal{A}_2)$, if there exists a surjective morphism π_2^1 of $\mathcal{A}_1$ onto $\mathcal{A}_2$, such that $\pi_2^1\circ\iota_1=\iota_2$. //

The relation $\succeq$ is obviously reflexive and transitive. By density of $\iota_j(\mathcal{L})$ in $\mathcal{A}_j$, $j=1,2$, and continuity of the involved maps, we have

$$\Big((\iota_1,\mathcal{A}_1)\succeq(\iota_2,\mathcal{A}_2)\wedge(\iota_2,\mathcal{A}_2)\succeq(\iota_1,\mathcal{A}_1)\Big)\iff(\iota_1,\mathcal{A}_1)\cong(\iota_2,\mathcal{A}_2)$$

Hence indeed $\succeq$ induces a partial order on the set of all isomorphy classes of aPs-completions of $\mathcal{L}$.

REE38 *5.5.3 Remark.* Since the image of a dense set under a surjective and continuous map is again dense, we may also proceed the other way. If $(\iota_1,\mathcal{A}_1)$ is an aPs-completion of $\mathcal{L}$, $\mathcal{A}_2$ is an almost Pontryagin space, and π is a surjective morphism of $\mathcal{A}_1$ onto $\mathcal{A}_2$, then $(\pi\circ\iota_1,\mathcal{A}_2)$ is an aPs-completion of $\mathcal{L}$ and $(\iota_1,\mathcal{A}_1)\succeq(\pi\circ\iota_1,\mathcal{A}_2)$. //

DEE39 **5.5.4 Definition.** Let $\langle\mathcal{L},[.,.]\rangle$ be an inner product space, and let $(\iota,\mathcal{A})$ be an aPs-completion of $\mathcal{L}$. Then we denote by $\mathfrak{L}(\iota,\mathcal{A})$ the linear subspace

$$\mathfrak{L}(\iota,\mathcal{A}):=\iota^*\mathcal{A}'$$

of the algebraic dual $\mathcal{L}^*$ of $\mathcal{L}$. Here $\mathcal{A}'$ denotes the topological dual of $\mathcal{A}$, and ι^* denotes the (algebraic) adjoint of ι, that is $\iota^*:\mathcal{A}^*\to\mathcal{L}^*$ and $\iota^*f=f\circ\iota$. //

Passing to adjoints in the diagram (5.5.1), shows that $(\iota_1,\mathcal{A}_1)\cong(\iota_2,\mathcal{A}_2)$ implies $\mathfrak{L}(\iota_1,\mathcal{A}_1)=\mathfrak{L}(\iota_2,\mathcal{A}_2)$. Hence $\mathfrak{L}$ induces a map of isomorphy classes of aPs-completions to linear subspaces of $\mathcal{L}^*$.

THE40 **5.5.5 Theorem.** *Let $\langle\mathcal{L},[.,.]\rangle$ be an inner product space with $\operatorname{ind}_-\mathcal{L}<\infty$. Then the assignment $\mathfrak{L}$ induces an order-isomorphism of the set of all aPs-completions of $\mathcal{L}$ modulo isomorphism onto the set of all linear subspaces of $\mathcal{L}^*$ which contain $\mathcal{L}^\curlywedge$ with finite codimension. Thereby,*

$$\dim\big(\mathfrak{L}(\iota,\mathcal{A})/\mathcal{L}^\curlywedge\big)=\operatorname{ind}_0\mathcal{A}\,. \tag{5.5.2}$$ E41

Proof.
Step 1: Let $(\iota_1,\mathcal{A}_1)$ and $(\iota_2,\mathcal{A}_2)$ be two aPs-completions of $\mathcal{L}$ with $(\iota_1,\mathcal{A}_1)\succeq(\iota_2,\mathcal{A}_2)$. We are going to show that

$$\mathfrak{L}(\iota_1,\mathcal{A}_1)\supseteq\mathfrak{L}(\iota_2,\mathcal{A}_2),\quad\dim\big(\mathfrak{L}(\iota_1,\mathcal{A}_1)/\mathfrak{L}(\iota_2,\mathcal{A}_2)\big)=\operatorname{ind}_0\mathcal{A}_1-\operatorname{ind}_0\mathcal{A}_2\,.$$

Let $\pi : \mathcal{A}_1 \to \mathcal{A}_2$ be a surjective morphism with $\pi \circ \iota_1 = \iota_2$. Passing to adjoints yields

$$\begin{tikzcd} & \mathcal{L} \arrow[dl, "\iota_1"'] \arrow[dr, "\iota_2"] & \\ \mathcal{A}_1 \arrow[rr, "\pi"'] & & \mathcal{A}_2 \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} & \mathcal{L}^* & \\ \mathcal{A}_1^* \arrow[ur, "\iota_1^*"] & & \mathcal{A}_2^* \arrow[ul, "\iota_2^*"'] \arrow[ll, "\pi^*"] \end{tikzcd}$$

Since π is continuous, we have $\pi^*\mathcal{A}_2' \subseteq \mathcal{A}_1'$. It readily follows that

$$\mathfrak{L}(\iota_2, \mathcal{A}_2) = \iota_2^*\mathcal{A}_2' = \iota_1^*\pi^*\mathcal{A}_2' \subseteq \iota_1^*\mathcal{A}_1' = \mathfrak{L}(\iota_1, \mathcal{A}_1)\,.$$

We need to compute codimension. Since $\operatorname{ran}\iota_1$ is dense in $\mathcal{A}_1$, the restriction of ι_1^* to $\mathcal{A}_1'$ is injective. Thus

$$\dim\big(\mathfrak{L}(\iota_1, \mathcal{A}_1)/\mathfrak{L}(\iota_2, \mathcal{A}_2)\big) = \dim\big(\iota_1^*\mathcal{A}_1'/\iota_1^*\pi^*\mathcal{A}_2'\big) = \dim\big(\mathcal{A}_1'/\pi^*\mathcal{A}_2'\big)\,.$$

Since π is surjective, by the Closed Range Theorem, $\pi^*\mathcal{A}_2'$ is a w^*-closed subspace of $\mathcal{A}_1'$. It follows that

$$\pi^*\mathcal{A}_2' = \overline{\pi^*\mathcal{A}_2'}^{w^*} = (\ker\pi)^\perp\,, \tag{5.5.3}$$ E42

and hence

$$\dim\big(\mathcal{A}_1'/\pi^*\mathcal{A}_2'\big) = \dim\big(\mathcal{A}_1'/(\ker\pi)^\perp\big) = \dim\,(\ker\pi)'\,.$$

Since π is isometric, we have $\ker\pi \subseteq \mathcal{A}_1^\circ$. In particular, $\ker\pi$ is finite dimensional, and therefore

$$\dim\,(\ker\pi)' = \dim\ker\pi\,.$$

The relation $\ker\pi \subseteq \mathcal{A}_1^\circ$ also shows that $\ker\pi = \ker(\pi|_{\mathcal{A}_1^\circ})$. Since π is surjective, we have $\pi^{-1}(\mathcal{A}_2^\circ) = \mathcal{A}_1^\circ$, and hence $\pi|_{\mathcal{A}_1^\circ}$ maps $\mathcal{A}_1^\circ$ surjectively onto $\mathcal{A}_2^\circ$. It follows that

$$\dim\ker\pi = \dim\ker(\pi|_{\mathcal{A}_1^\circ}) = \dim\mathcal{A}_1^\circ - \dim\mathcal{A}_2^\circ\,.$$

Putting together these relations, the desired formula follows.

Step 2: From Step 1 it is easy to deduce (5.5.2). Let $(\iota, \mathcal{A})$ be an aPs-completion of $\mathcal{L}$. Denote by $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{A}^\circ$ the canonical projection, then π is a surjective morphism. Hence, $(\pi \circ \iota, \mathcal{A}/\mathcal{A}^\circ)$ is also an aPs-completion and $(\iota, \mathcal{A}) \succeq (\pi\circ\iota, \mathcal{A}/\mathcal{A}^\circ)$, cf. Remark 5.5.3. However, since $\mathcal{A}/\mathcal{A}^\circ$ is nondegenerated $(\pi \circ \iota, \mathcal{A}/\mathcal{A}^\circ)$, actually is a Pontryagin space completion of $\mathcal{L}$. By Proposition 4.3.7, $\mathfrak{L}(\pi \circ \iota, \mathcal{A}/\mathcal{A}^\circ) = \mathcal{L}^\curlywedge$, and we obtain from Step 1 that $\mathfrak{L}(\iota, \mathcal{A})$ contains $\mathcal{L}^\curlywedge$ with codimension $\operatorname{ind}_0\mathcal{A}^\circ$.

Step 3: Let $(\iota_1, \mathcal{A}_1)$ and $(\iota_2, \mathcal{A}_2)$ be aPs-completions of $\mathcal{L}$ with $\mathfrak{L}(\iota_1, \mathcal{A}_1) \supseteq \mathfrak{L}(\iota_2, \mathcal{A}_2)$. We are going to show that $(\iota_1, \mathcal{A}_1) \succeq (\iota_2, \mathcal{A}_2)$. Let $f \in \mathcal{A}_2'$ be given. Then there exists $\tilde{f} \in \mathcal{A}_1'$ with $\iota_1^*\tilde{f} = \iota_2^* f$. Since $\iota_1^*|_{\mathcal{A}_1'}$ is injective, the element $\tilde{f}$ is uniquely determined by this property. Hence, a map $\Lambda : \mathcal{A}_2' \to \mathcal{A}_1'$ is well-defined by

$$\iota_1^*(\Lambda f) = \iota_2^* f, \quad f \in \mathcal{A}_2'\,.$$

Clearly, Λ is linear.

We will apply the Closed Graph Theorem to show that Λ is bounded. To check the necessary hypothesis, let a sequence $(f_n)_{n\in\mathbb{N}}$ of functionals $f_n \in \mathcal{A}_2'$ be given, and assume that $f_n \to f$ in $\mathcal{A}_2'$ and $\Lambda f_n \to g$ in $\mathcal{A}_1'$. Then, in particular, for each fixed $x \in \mathcal{L}$

$$\begin{array}{l} (\iota_2^* f_n)x = f_n(\iota_2 x) \longrightarrow f(\iota_2 x) = (\iota_2^* f)x = \iota_1^*(\Lambda f)x \\ \quad \parallel \\ \iota_1^*(\Lambda f_n)x = (\Lambda f_n)(\iota_1 x) \longrightarrow g(\iota_1 x) = (\iota_1^* g)x \end{array}$$

Since $\iota_1^*|_{\mathcal{A}_1'}$ is injective, this implies that $\Lambda f = g$. It follows that indeed Λ is bounded.

Let $\|.\|_1$ and $\|.\|_2$ be norms on $\mathcal{A}_1$ and $\mathcal{A}_2$ which induce their respective topologies. Moreover, let $\|.\|_1'$ and $\|.\|_2'$ be the corresponding operator norms on $\mathcal{A}_1'$ and $\mathcal{A}_2'$. We compute for $x \in \mathcal{L}$

$$\|\iota_2 x\|_2 = \sup\Big\{ \,|\underbrace{f(\iota_2 x)}_{\substack{\parallel \\ (\iota_2^* f)x = \iota_1^*(\Lambda f)x = (\Lambda f)(\iota_1 x)}}| : \; f \in \mathcal{A}_2',\, \|f\|_2' \leq 1 \Big\} =$$

$$= \sup\Big\{ |\tilde{f}(\iota_1 x)| : \; \tilde{f} \in \underbrace{\Lambda\big(\{f \in \mathcal{A}_2' : \|f\|_2' \leq 1\}\big)}_{\subseteq \{\tilde{f} \in \mathcal{A}_1' :\, \|\tilde{f}\|_1' \leq \|\Lambda\|\}} \Big\} \leq \|\Lambda\| \cdot \|\iota_1 x\|_1 \,. \qquad (5.5.4)$$ E43

It follows that $\ker \iota_1 \subseteq \ker \iota_2$, and therefore the map $\iota_2 \circ \iota_1^{-1} : \operatorname{ran} \iota_1 \to \mathcal{A}_2$ is well-defined. Moreover, again by (5.5.4), it is bounded. Let $\pi : \mathcal{A}_1 \to \mathcal{A}_2$ be its extension by continuity. Then π is isometric and its range is dense in $\mathcal{A}_2$.

Let $\pi_j : \mathcal{A}_j \to \mathcal{A}_j/\mathcal{A}_j^\circ$, $j = 1, 2$, denote the canonical projections. Since $(\pi_1 \circ \iota_1, \mathcal{A}_1/\mathcal{A}_1^\circ)$ and $(\pi_2 \circ \iota_2, \mathcal{A}_2/\mathcal{A}_2^\circ)$ are both Pontryagin space completions of $\mathcal{L}$, there exists an isomorphism ϕ of $\mathcal{A}_2/\mathcal{A}_2^\circ$ onto $\mathcal{A}_1/\mathcal{A}_1^\circ$ with $\phi \circ (\pi_2 \circ \iota_2) = \pi_1 \circ \iota_1$. Altogether, in left of the below diagrams, each outer triangle commutes. Passing to adjoints then gives the outer triangles in the right diagram.

$$\begin{tikzcd}
\mathcal{A}_1 \arrow[rr, "\pi"] \arrow[dd, "\pi_1"'] \arrow[dr, phantom, "\#"] & \arrow[d, phantom, "\#"] & \mathcal{A}_2 \arrow[dd, "\pi_2"] \\
\arrow[r, phantom, "\#"] & \mathcal{L} \arrow[ul, "\iota_1"'] \arrow[ur, "\iota_2"] \arrow[dl, "\pi_1\circ\iota_1"] \arrow[dr, "\pi_2\circ\iota_2"'] \arrow[r, phantom, "\#"] \arrow[d, phantom, "\#"] & \\
\mathcal{A}_1/\mathcal{A}_1^\circ & & \mathcal{A}_2/\mathcal{A}_2^\circ \arrow[ll, "\phi"]
\end{tikzcd}
\qquad \rightsquigarrow \qquad
\begin{tikzcd}
\mathcal{A}_1' \arrow[dr, "\iota_1^*"] & \arrow[d, phantom, "\#"] & \mathcal{A}_2' \arrow[ll, "\pi'"'] \arrow[dl, "\iota_2^*"'] \\
\arrow[r, phantom, "\#"] & \mathcal{L} \arrow[r, phantom, "\#"] \arrow[d, phantom, "\#"] & \\
(\mathcal{A}_1/\mathcal{A}_1^\circ)' \arrow[uu, "\pi_1'"] \arrow[ur, "\iota_1^*\circ\pi_1'"'] \arrow[rr, "\phi'"'] & & (\mathcal{A}_2/\mathcal{A}_2^\circ)' \arrow[uu, "\pi_2'"'] \arrow[ul, "\iota_2^*\circ\pi_2'"]
\end{tikzcd}$$

We see that $\iota_1^* \circ \pi_1' = \iota_1^* \circ (\pi' \circ \pi_2' \circ \phi')$, and injectivity of $\iota_1^*|_{\mathcal{A}_1'}$ implies $\pi_1' = \pi' \circ \pi_2' \circ \phi'$. In particular, $\operatorname{ran} \pi_1' \subseteq \operatorname{ran} \pi' \subseteq \mathcal{A}_1'$. However, as we saw in (5.5.3) applied with $\mathcal{A}_1/\mathcal{A}_1^\circ$ in place of $\mathcal{A}_2$, $\operatorname{ran} \pi_1'$ is a closed subspace of $\mathcal{A}_1'$ with finite codimension. It follows that $\operatorname{ran} \pi'$ is closed in $\mathcal{A}_1'$. By the Closed Range Theorem, thus $\operatorname{ran} \pi$ is closed in $\mathcal{A}_1$, and hence π is surjective. Therefore π is a morphism and we have shown that $(\iota_1, \mathcal{A}_1) \succeq (\iota_2, \mathcal{A}_2)$.

Step 4: So far, we have seen that $\mathfrak{L}$ maps aPs-completions into the set of all subspaces of $\mathcal{L}^*$ which contain $\mathcal{L}^\curlywedge$ with finite codimension, that actually (5.5.2) holds, and that

$$(\iota_1, \mathcal{A}_1) \succeq (\iota_2, \mathcal{A}_2) \iff \mathfrak{L}(\iota_1, \mathcal{A}_1) \supseteq \mathfrak{L}(\iota_2, \mathcal{A}_2)\,.$$

In particular, $\mathfrak{L}(\iota_1,\mathcal{A}_1)=\mathfrak{L}(\iota_2,\mathcal{A}_2)$ if and only if $(\iota_1,\mathcal{A}_1)$ and $(\iota_2,\mathcal{A}_2)$ are isomorphic.

In order to complete the proof of Theorem 5.5.5, it remains to show that for each given subspace $\mathcal{M}$ with $\mathcal{L}^\curlywedge\subseteq\mathcal{M}$ and $\dim\mathcal{M}/\mathcal{L}^\curlywedge<\infty$, there exists an aPs-completion $(\iota,\mathcal{A})$ of $\mathcal{L}$ with $\mathfrak{L}(\iota,\mathcal{A})=\mathcal{M}$. To this end, set $n:=\dim\mathcal{M}/\mathcal{L}^\curlywedge$ and choose $f_1,\dots,f_n\in\mathcal{L}^*$ such that $\mathcal{M}=\operatorname{span}(\mathcal{L}^\curlywedge\cup\{f_1,\dots,f_n\})$. Let $(\iota_\mathcal{P},\mathcal{P})$ denote a Pontryagin space completion of $\mathcal{L}$. Then we define

$$\mathcal{A}:=\mathcal{P}[\dot{+}]\mathbb{C}^n\,,$$

$$[x+\xi,y+\eta]_\mathcal{A}:=[x,y]_\mathcal{P},\quad x,y\in\mathcal{P},\ \xi,\eta\in\mathbb{C}^n\,,$$
$$\iota x:=\iota_\mathcal{P}x+\big(f_1(x),\dots,f_n(x)\big),\quad x\in\mathcal{L}\,.$$

Moreover, the space $\mathcal{A}$ is endowed with the product topology $\mathcal{T}_\mathcal{A}$ of the topology $\mathcal{P}$ carries as a Pontryagin space and the euclidean topology on $\mathbb{C}^n$. Then $\langle\mathcal{A},[.,.]_\mathcal{A},\mathcal{T}_\mathcal{A}\rangle$ is, as the direct and orthogonal sum of two almost Pontryagin spaces, itself an almost Pontryagin space. We have

$$\mathcal{A}^\circ=\{0\}[\dot{+}]\mathbb{C}^n\,,$$

and thus $\operatorname{ind}_0\mathcal{A}=n=\dim\mathcal{M}/\mathcal{L}^\curlywedge$.

Denote by $[.,.]_1$ the sum inner product on $\mathcal{A}$ of the inner product of $\mathcal{P}$ and the euclidean inner product of $\mathbb{C}^n$. Then $\langle\mathcal{A},[.,.]_1\rangle$ is, as the direct and orthogonal sum of two Pontryagin spaces, a Pontryagin space. Clearly, its Pontryagin space topology equals $\mathcal{T}_\mathcal{A}$. Hence, $\langle\mathcal{A},\mathcal{T}_\mathcal{A}\rangle'=\langle\mathcal{A},[.,.]_1\rangle'$, and we conclude that each $\mathcal{T}_\mathcal{A}$-continuous linear functional $f:\mathcal{A}\to\mathbb{C}$ can be represented as

$$f(x+\xi)=[x+\xi,\iota_\mathcal{P}x(f)+\xi(f)]_1,\quad x\in\mathcal{P},\xi\in\mathbb{C}^n\,,$$

with some $x(f)\in\mathcal{P}$ and $\xi(f)\in\mathbb{C}^n$. In particular, for each $x\in\mathcal{L}$,

$$\begin{aligned}f(\iota x)=\big[\iota_\mathcal{P}x+(f_1(x),\dots,f_n(x)),\iota_\mathcal{P}x(f)+\xi(f)\big]_1=\\=[\iota_\mathcal{P}x,x(f)]_\mathcal{P}+\sum_{j=1}^n\xi(f)_jf_j(x)\,.\end{aligned}\tag{5.5.5}$$ E44

Due to Proposition 4.3.7, the first summand is $\mathcal{L}^\curlywedge$-continuous.

Let $f\in\mathcal{A}'$ with $f(\operatorname{ran}\iota)=0$ be given. Then we have

$$[\iota_\mathcal{P}x,x(f)]_\mathcal{P}+\sum_{j=1}^n\xi(f)_jf_j(x)=0,\quad x\in\mathcal{L}\,.$$

Thus $\sum_{j=1}^n\xi(f)_jf_j\in\mathcal{L}^\curlywedge$, and hence $\xi(f)=0$. Since $\operatorname{ran}\iota_\mathcal{P}$ is dense in $\mathcal{P}$, thus also $x(f)=0$, and together $f=0$. It follows that $\operatorname{ran}\iota$ is dense in $\mathcal{A}$. We have shown that $(\iota,\mathcal{A})$ is an aPs-completion of $\mathcal{L}$.

By (5.5.5), we have $\mathcal{A}'\subseteq\mathcal{M}$. Together with (5.5.2) and the fact that $\operatorname{ind}_0\mathcal{A}=\dim\mathcal{M}/\mathcal{L}^\curlywedge$, this implies that actually $\mathcal{A}'=\mathcal{M}$, i.e. $\mathfrak{L}(\iota,\mathcal{A})=\mathcal{M}$. ❑

COE45 **5.5.6 Corollary.** *Let $(\iota_1,\mathcal{A}_1)$ and $(\iota_2,\mathcal{A}_2)$ be two aPs-completions of an inner product space $\langle\mathcal{L},[.,.]\rangle$. Then $(\iota_1,\mathcal{A}_1)\succeq(\iota_2,\mathcal{A}_2)$ if and only if $\ker\iota_1\subseteq\ker\iota_2$ and $\iota_2\circ\iota_1^{-1}:\operatorname{ran}\iota_1\to\operatorname{ran}\iota_2$ is bounded.*

Proof. If $(\iota_1, \mathcal{A}_1) \succeq (\iota_2, \mathcal{A}_2)$, then the map π_2^1 guaranteed by the definition of $\succeq$ is linear, bounded, and satisfies $\pi_2^1 \circ \iota_1 = \iota_2$. The required properties of ι_1 and ι_2 follow. Conversely, assume that $\ker \iota_1 \subseteq \ker \iota_2$ and $\iota_2 \circ \iota_1^{-1} : \operatorname{ran} \iota_1 \to \operatorname{ran} \iota_2$ is bounded. Let $\pi : \mathcal{A}_1 \to \mathcal{A}_2$ be the extension by continuity of $\iota_2 \circ \iota_1^{-1}$, then $\iota_2^* = \iota_1^* \circ \pi'$ and hence

$$\mathfrak{L}(\iota_2, \mathcal{A}_2) = \iota_2^* \mathcal{A}_2' = \big(\iota_1^* \circ \pi'\big) \mathcal{A}_2' \subseteq \iota_1^* \mathcal{A}_1' = \mathfrak{L}(\iota_1, \mathcal{A}_1)\,.$$

❑

Chapter 6

Reproducing kernel spaces

6.1 Reproducing kernel Krein spaces

DEF1 **6.1.1 Definition.** Let Ω be a set, $\langle \mathfrak{V}, [.,.]_{\mathfrak{V}} \rangle$ a Krein space, and $\langle \mathcal{K}, [.,.] \rangle$ another Krein space. Then $\mathcal{K}$ is called a *reproducing kernel Krein space of* $\mathfrak{V}$*-valued functions on* Ω [1] , if

(rk1) The elements of $\mathcal{K}$ are functions of Ω into $\mathfrak{V}$.

(rk2) For each $w \in \Omega$, the point evaluation map

$$\chi_w : \begin{cases} \mathcal{K} & \to & \mathfrak{V} \\ F & \mapsto & F(w) \end{cases}$$

is linear and continuous.

//

If $\mathcal{K}$ is a reproducing kernel Krein space with $\operatorname{ind}_- \mathcal{K} < \infty$, then we will speak of a *reproducing kernel Pontryagin space of* $\mathfrak{V}$*-valued functions on* Ω.

Let us remark that the axiom (rk2) could also be formulated in two parts as follows:

(rk2$_\mathbf{a}$) The linear operations on $\mathcal{K}$ are given by pointwise addition and pointwise scalar multiplication.

(rk2$_\mathbf{b}$) The topology of $\mathcal{K}$ is finer than the restriction to $\mathcal{K}$ of the topology of pointwise convergence on $\mathfrak{V}^\Omega$.

Let $\langle \mathcal{K}, [.,.] \rangle$ be a reproducing kernel Krein space of $\mathfrak{V}$-valued functions on Ω. Then, for each $w \in \Omega$ and $f \in \mathfrak{V}$, the linear functional $F \mapsto [F(w), f]_{\mathfrak{V}}$ is continuous. Hence, there exists an element $K_{w,f} \in \mathcal{K}$ such that

$$[F(w), f]_{\mathfrak{V}} = [F, K_{w,f}], \quad F \in \mathcal{K}. \qquad (6.1.1)$$ F3

[1] If we do not want or do not need to be specific about the domain set Ω and the value space $\mathfrak{V}$, we will shorter speak of a reproducing kernel Krein space.

DEF2 **6.1.2 Definition.** Let $\langle\mathcal{K},[.,.]\rangle$ be a reproducing kernel Krein space of $\mathfrak{V}$-valued functions on Ω. Then the function $K:\Omega\times\Omega\to\mathfrak{V}^{\mathfrak{V}}$ defined by

$$K(w,z)f := K_{w,f}(z),\quad w,z\in\Omega,\ f\in\mathfrak{V}\,,$$

is called the *kernel function*[2] of $\mathcal{K}$. //

We will frequently use the notation $K(w,.)$ to denote the function $z\mapsto K(w,z)$, $z\in\Omega$. With this notation we have $K(w,.)f=K_{w,f}$. Thus the defining relation (6.1.1) of $K_{w,f}$ rewrites, more suggestively, as

$$[F(w),f]_{\mathfrak{V}} = [F,K(w,.)f],\quad F\in\mathcal{K},\ w\in\Omega,\ f\in\mathfrak{V}\,.$$

Let us note that, in particular,

$$[K(w,z)f,g]_{\mathfrak{V}} = [K(w,.)f,K(z,.)g],\quad z,w\in\Omega,\ f,g\in\mathfrak{V}\,.$$

In order to formulate the basic properties of kernel functions, we need to introduce Krein space adjoints. Let $\langle\mathcal{K}_1,[.,.]_1\rangle$ and $\langle\mathcal{K}_2,[.,.]_2\rangle$ be Krein spaces. Then we denote by $\mathcal{B}(\mathcal{K}_1,\mathcal{K}_2)$ the linear space of all continuous linear operators of $\mathcal{K}_1$ into $\mathcal{K}_2$. We will also write $\mathcal{B}(\mathcal{K})$ for $\mathcal{B}(\mathcal{K},\mathcal{K})$.

If $A\in\mathcal{B}(\mathcal{K}_1,\mathcal{K}_2)$, then there exists a unique operator $A^*\in\mathcal{B}(\mathcal{K}_2,\mathcal{K}_1)$ which satisfies

$$[Ax_1,x_2]_2 = [x_1,A^*x_2]_1,\quad x_1\in\mathcal{K}_1, x_2\in\mathcal{K}_2\,. \tag{6.1.2}$$ F5

To see existence, choose fundamental decompositions $\mathfrak{J}_1$ and $\mathfrak{J}_2$ of $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively. Let $A^{(*)}$ denote the Hilbert space adjoint of A considered as a bounded operator of $\langle\mathcal{K}_1,(.,.)_{\mathfrak{J}_1}\rangle$ into $\langle\mathcal{K}_2,(.,.)_{\mathfrak{J}_2}\rangle$. Then

$$A^* := J_1A^{(*)}J_2$$

satisfies (6.1.2). Since the inner product $[.,.]_1$ is nondegenerated, A^* is uniquely determined by (6.1.2). The operator A^* is called the *adjoint* of A. Of course, it depends on the inner products under consideration[3].

LEF4 **6.1.3 Lemma.** *Let $\langle\mathcal{K},[.,.]\rangle$ be a reproducing kernel Krein space of $\mathfrak{V}$-valued functions on Ω, and let K be the kernel function of $\mathcal{K}$. Then, for each $w,z\in\Omega$, we have $K(w,z)\in\mathcal{B}(\mathfrak{V})$ and $K(w,z)^*=K(z,w)$.*

Proof. Let $w\in\Omega$, $\alpha,\beta\in\mathbb{C}$ and $f,g\in\mathfrak{V}$, then the relation (6.1.1) gives

$$[F,\alpha K_{w,f}+\beta K_{w,g}] = \overline{\alpha}[F,K_{w,f}]+\overline{\beta}[F,K_{w,g}] = \overline{\alpha}[F(w),f]_{\mathfrak{V}}+\overline{\beta}[F(w),g]_{\mathfrak{V}} =$$

$$= [F(w),\alpha f+\beta g]_{\mathfrak{V}} = [F,K_{w,\alpha f+\beta g}]\,.$$

This shows that $K(w,z)$ is linear.

To prove continuity, choose fundamental decompositions $\mathfrak{J}$ and $\mathfrak{J}'$ of $\mathcal{K}$ and $\mathfrak{V}$, respectively. Denote by $\|\chi_w\|$ the $\|.\|_{\mathfrak{J}}$-to-$\|.\|_{\mathfrak{J}'}$–operator norm of χ_w. Then we have

$$|(K_{w,f},F)_{\mathfrak{J}}| = |[K_{w,f},JF]| = |[f,(JF)(w)]_{\mathfrak{V}}| \le \|f\|_{\mathfrak{J}'}\cdot\|(JF)(w)\|_{\mathfrak{J}'} \le$$

[2] Or *reproducing kernel*

[3] If necessary, we will thus more precisely speak of the $[.,.]_1$-to-$[.,.]_2$–adjoint of A.

$$\leq \|f\|_{\mathfrak{J}'} \cdot \|\chi_w\| \cdot \|JF\|_{\mathfrak{J}} = \|f\|_{\mathfrak{J}'} \cdot \|\chi_w\| \cdot \|F\|_{\mathfrak{J}}\,,$$

It follows that $\|K_{w,f}\|_{\mathfrak{J}} \leq \|f\|_{\mathfrak{J}'} \cdot \|\chi_w\|$, and hence

$$\|K(w,z)f\|_{\mathfrak{J}'} = \|K_{w,f}(z)\|_{\mathfrak{J}'} \leq \|\chi_z\| \cdot \|\chi_w\| \cdot \|f\|_{\mathfrak{J}'}\,.$$

This shows that $K(w,z)$ is continuous. In fact, the $\|.\|_{\mathfrak{J}'}$-to-$\|.\|_{\mathfrak{J}'}$–operator norm of $K(w,z)$ does not exceed $\|\chi_z\| \cdot \|\chi_w\|$.

It remains to compute adjoints:

$$[f, K(w,z)^*g]_{\mathfrak{V}} = [K(w,z)f, g]_{\mathfrak{V}} = [K(w,.)f, K(z,.)g] = \overline{[K(z,.)f, K(w,.)g]} =$$
$$= \overline{[K(z,w)g, f]_{\mathfrak{V}}} = [f, K(z,w)g]_{\mathfrak{V}}\,.$$

Since f and g were arbitrary, it follows that $K(w,z)^* = K(z,w)$. ❑

The property of being a reproducing kernel Krein space is inherited by orthocomplemented subspaces.

LEF8 **6.1.4 Lemma.** *Let $\mathcal{K}$ be a reproducing kernel Krein space, and let $\mathcal{K}_1$ be an orthocomplemented subspace. Then $\mathcal{K}_1$ and $\mathcal{K}_1^\perp$ are both reproducing kernel Krein spaces. If K, K_1, and $K_1^\perp$ denote the kernel functions of $\mathcal{K}$, $\mathcal{K}_1$, and $\mathcal{K}_1^\perp$, respectively, then we have*

$$K(w,z) = K_1(w,z) + K_1^\perp(w,z), \quad z,w \in \Omega\,. \tag{6.1.3}$$ F11

Proof. First of all $\mathcal{K}_1$ is itself a Krein space. Its Krein space topology coincides with the restriction of the topology of $\mathcal{K}$, and hence point evaluation is continuous. The same argument applies to $\mathcal{K}_1^\perp$, and we conclude that both, $\mathcal{K}_1$ and $\mathcal{K}_1^\perp$, are reproducing kernel Krein spaces.

Denote by P the orthogonal projection of $\mathcal{K}$ onto $\mathcal{K}_1$. Then, for $F \in \mathcal{K}_1$, we have

$$[F(w), f]_{\mathfrak{V}} = [F, K(w,.)f] = [F, PK(w,.)f], \quad w \in \Omega, f \in \mathfrak{V}\,.$$

Hence, the kernel function K_1 of $\mathcal{K}_1$ is given as

$$K_1(w,z)f = [PK(w,.)f](z)\,.$$

Similarly we obtain that $K_1^\perp(w,z)f = [(I-P)K(w,.)f](z)$, and (6.1.3) follows. ❑

6.2 Kernel functions

DEF18 **6.2.1 Definition.** Let Ω be a set and $\langle \mathfrak{V}, [.,.]_{\mathfrak{V}} \rangle$ a Krein space. A function $K : \Omega \times \Omega \to \mathcal{B}(\mathfrak{V})$ which satisfies $K(w,z)^* = K(z,w)$, $w,z \in \Omega$, is called a *$\mathfrak{V}$-valued kernel on Ω*[4]. //

In the previous section we have associated to each reproducing kernel Krein space a kernel. A converse question suggests itself: Assume that a $\mathfrak{V}$-valued kernel K on Ω is given, does there exist a reproducing kernel Krein space having K as its reproducing kernel? The answer to this question, and a corresponding uniqueness question, depends on the geometry of an inner product space constructed from the given function K.

We denote by $\mathrm{Fin}(\Omega, \mathfrak{V})$ the linear space of all functions from Ω into $\mathfrak{V}$ which have finite support.

[4]Less specific we will speak of a *kernel*

DEF6 **6.2.2 Definition.** Let Ω be a set, $\langle\mathfrak{V},[.,.]_{\mathfrak{V}}\rangle$ a Krein space, and K a $\mathfrak{V}$-valued kernel on Ω. Then set

$$[\eta,\mu]_K := \sum_{z,w\in\Omega} \big[K(w,z)\eta(w),\mu(z)\big]_{\mathfrak{V}}, \quad \eta,\mu\in \operatorname{Fin}(\Omega,\mathfrak{V})\,.$$

Moreover, let $\iota_K : \operatorname{Fin}(\Omega,\mathfrak{V})\to\mathfrak{V}^\Omega$ be defined as

$$\iota_K : \eta\mapsto \sum_{w\in\Omega} K(w,.)\eta(w)\,.$$

//

The fact that $[.,.]_K$ is indeed an inner product is immediate from the properties of kernels.

THF10 **6.2.3 Theorem.** *Let K be a $\mathfrak{V}$-valued kernel on Ω.*

(i) If $\langle\mathcal{K},[.,.]\rangle$ is a reproducing kernel Krein space with kernel function K, then $(\iota_K,\mathcal{K})$ is a Krein space completion of $\langle\operatorname{Fin}(\Omega,\mathfrak{V}),[.,.]_K\rangle$.

(ii) Each isomorphy class of Krein space completions of $\langle\operatorname{Fin}(\Omega,\mathfrak{V}),[.,.]_K\rangle$ contains an element $(\iota,\mathcal{K})$ where $\mathcal{K}$ is a reproducing kernel Krein space with kernel function K and $\iota=\iota_K$.

(iii) If $\mathcal{K}_1$ and $\mathcal{K}_2$ are both reproducing kernel Krein spaces with kernel function K, then $(\iota_K,\mathcal{K}_1)$ and $(\iota_K,\mathcal{K}_2)$ are isomorphic as completions of $\langle\operatorname{Fin}(\Omega,\mathfrak{V}),[.,.]_K\rangle$ if and only if $\mathcal{K}_1=\mathcal{K}_2$ as sets of functions, and in turn if and only if $\mathcal{K}_1=\mathcal{K}_2$ as Krein spaces.

Proof. To show (i) assume that $\langle\mathcal{K},[.,.]\rangle$ is a reproducing kernel Krein space having K as its kernel function. First of all, $K(w,.)\eta(w)=K_{w,\eta(w)}$, cf. Definition 6.1.2, and hence ι_K maps $\operatorname{Fin}(\Omega,\mathfrak{V})$ into $\mathcal{K}$. Next, let $\eta,\mu\in\operatorname{Fin}(\Omega,\mathfrak{V})$ be given. Then

$$[\eta,\mu]_K=\sum_{z,w\in\Omega}[K(w,z)\eta(w),\mu(z)]_{\mathfrak{V}}=$$

$$=\sum_{z,w\in\Omega}[K(w,.)\eta(w),K(z,.)\mu(z)]=\Big[\sum_{w\in\Omega}K(w,.)\eta(w),\sum_{z\in\Omega}K(z,.)\mu(z)\Big]\,,$$

i.e. ι_K is isometric. Finally, if $F\in\mathcal{K}$ is orthogonal to $\operatorname{ran}\iota_K$, then for all $w\in\Omega$ and $f\in\mathfrak{V}$

$$[F(w),f]_{\mathfrak{V}}=[F,K(w,.)f]=0\,,$$

and thus $F=0$. This says that $\operatorname{ran}\iota_K$ is dense in $\mathcal{K}$. We have shown that $(\iota_K,\mathcal{K})$ is a Krein space completion of $\langle\operatorname{Fin}(\Omega,\mathfrak{V}),[.,.]_K\rangle$.

Item (ii) contains the most involved assertion of the present theorem. Assume that $(\iota,\mathcal{K})$ is a Krein space completion of $\langle\operatorname{Fin}(\Omega,\mathfrak{V}),[.,.]_K\rangle$. We need to construct a reproducing kernel Krein space $\tilde{\mathcal{K}}$ whose kernel function equals K, such that $(\iota_K,\tilde{\mathcal{K}})$ is isomorphic to $(\iota,\mathcal{K})$.

Step 1; The operators $V(w)$: If $M\subseteq\Omega$, let $\chi_M:\Omega\to\{0,1\}$ denote the indicator function of the set M. For each $w\in\Omega$ we consider the map

$$V(w):\left\{\begin{array}{ccc}\mathfrak{V}&\to&\mathcal{K}\\ f&\mapsto&\iota(f\chi_{\{w\}})\end{array}\right. \tag{6.2.1}$$ F13

Clearly, $V(w)$ is linear. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of elements of $\mathfrak{V}$ which tends to the limit $f \in \mathfrak{V}$. Then, for each $g \in \mathfrak{V}$ and $z \in \Omega$, we have

$$[V(w)f_n, \iota(g\chi_{\{z\}})] = [\iota(f_n\chi_{\{w\}}), \iota(g\chi_{\{z\}})] = [f_n\chi_{\{w\}}, g\chi_{\{z\}}]_K =$$
$$= [K(w,z)f_n, g]_{\mathfrak{V}} \to [K(w,z)f, g]_{\mathfrak{V}} = [V(w)f, \iota(g\chi_{\{z\}})]\,.$$

Since $\operatorname{ran}\iota$ is dense in $\mathcal{K}$, this implies that $V(w)$ has closed graph, and hence that $V(w) \in \mathcal{B}(\mathfrak{V}, \mathcal{K})$.

For later use, let us state explicitly that

$$\operatorname{span} \bigcup_{w\in\Omega} \operatorname{ran} V(w) = \operatorname{ran}\iota\,. \tag{6.2.2}$$

F20

Step 2; The embedding Λ*:* For each $w \in \Omega$ and $x \in \mathcal{K}$ the linear functional $f \mapsto [V(w)f, x]_{\mathcal{K}}$ is continuous. Hence, there exists a unique element $\Lambda_{w,x} \in \mathfrak{V}$ such that

$$[V(w)f, x] = [f, \Lambda_{w,x}]_{\mathfrak{V}}, \quad f \in \mathfrak{V}\,.$$

We define a map $\Lambda : \mathcal{K} \to \mathfrak{V}^{\Omega}$ by

$$(\Lambda x)(w) := \Lambda_{w,x}, \quad x \in \mathcal{K}, w \in \Omega\,.$$

Clearly, Λ is linear. Let us show that Λ is injective: If $\Lambda x = 0$, then for all $w \in \Omega$ and $f \in \mathfrak{V}$ we have $[V(w)f, x] = 0$. Thus, by (6.2.2), the element x is orthogonal to $\operatorname{ran}\iota$, and hence must be equal to 0.

Step 3; The reproducing kernel space $\tilde{\mathcal{K}}$*:* Set $\tilde{\mathcal{K}} := \operatorname{ran}\Lambda$, and define an inner product $[.,.]_\sim$ on $\tilde{\mathcal{K}}$ by requiring Λ to be isometric. Then $\langle\tilde{\mathcal{K}}, [.,.]_\sim\rangle$ is a Krein space and Λ is an isomorphism of $\mathcal{K}$ onto $\tilde{\mathcal{K}}$. It follows that $(\Lambda\circ\iota, \tilde{\mathcal{K}})$ is a Krein space completion of $\langle \operatorname{Fin}(\Omega, \mathfrak{V}), [.,.]_K\rangle$ which is isomorphic to $(\iota, \mathcal{K})$.

Let $F \in \tilde{\mathcal{K}}$, $w \in \Omega$, and $f \in \mathfrak{V}$ be given, and let $x \in \mathcal{K}$ be such that $F = \Lambda x$. Then we can compute

$$\begin{aligned}[F(w), f]_{\mathfrak{V}} &= [(\Lambda x)(w), f]_{\mathfrak{V}} = [x, V(w)f] = [x, \iota(f\chi_{\{w\}})] = \\ &= \big[\Lambda x, (\Lambda\circ\iota)(f\chi_{\{w\}})\big]_\sim = \big[F, (\Lambda\circ\iota)(f\chi_{\{w\}})\big]_\sim\,.\end{aligned} \tag{6.2.3}$$

F21

Hence, for each fixed $f \in \mathfrak{V}$ the functional $F \mapsto [F(w), f]_{\mathfrak{V}}$ is continuous. By the Principle of Uniform Boundedness, the point evaluation map $\chi_w : F \mapsto F(w)$ on $\tilde{\mathcal{K}}$ is continuous. This says that $\langle\tilde{\mathcal{K}}, [.,.]_\sim\rangle$ is a reproducing kernel Krein space. Moreover, if $\tilde{K}$ denotes the kernel function of $\tilde{\mathcal{K}}$, we see from (6.2.3) that

$$\tilde{K}(w,.)f = (\Lambda\circ\iota)(f\chi_{\{w\}}), \quad w \in \Omega, f \in \mathfrak{V}\,. \tag{6.2.4}$$

F22

Using this relation, we can compute

$$[\tilde{K}(w,z)f, g]_{\mathfrak{V}} = [\tilde{K}(w,.)f, \tilde{K}(z,.)g]_\sim = \big[(\Lambda\circ\iota)(f\chi_{\{w\}}), (\Lambda\circ\iota)(g\chi_{\{z\}})\big]_\sim =$$
$$= \big[\iota(f\chi_{\{w\}}), \iota(g\chi_{\{z\}})\big] = [f\chi_{\{w\}}, g\chi_{\{z\}}]_K = [K(w,z)f, g]_{\mathfrak{V}}\,.$$

It follows that $\tilde{K}(w,z) = K(w,z)$ and, by (6.2.4), that $\Lambda\circ\iota = \iota_K$. This finishes the proof of (*ii*).

It remains to establish (*iii*). Assume that $(\iota_K, \mathcal{K}_1)$ and $(\iota_K, \mathcal{K}_2)$ are isomorphic, and let $\phi : \mathcal{K}_1 \to \mathcal{K}_2$ be an isomorphism with $\phi\circ\iota_K = \iota_K$. Let $w \in \Omega$ be fixed. The maps $\chi_w \circ \phi : \mathcal{K}_1 \to \mathfrak{V}$ and $\chi_w : \mathcal{K}_1 \to \mathfrak{V}$ are both continuous and coincide on $\operatorname{ran} \iota_K$. Thus they coincide on all of $\mathcal{K}_1$. Since $w \in \Omega$ was arbitrary, this shows that ϕ acts as the identity, and hence $\mathcal{K}_1$ and $\mathcal{K}_2$ are equal as Krein spaces. Conversely, if $\mathcal{K}_1 = \mathcal{K}_2$ as sets of functions, then the identity map id is a linear bijection of $\mathcal{K}_1$ onto $\mathcal{K}_2$. Since point evaluation is continuous in both spaces, it has closed graph, and hence is bicontinuous. Clearly, $\iota_K = \mathrm{id} \circ \iota_K$. By continuity of inner products, this equality also implies that id is isometric. ❑

From our knowledge about Krein space completions, cf. Theorem 3.5.13, (*iii*), and Theorem 3.5.17, we immediately obtain the following statements.

COF17

6.2.4 Corollary. *Let K be a $\mathfrak{V}$-valued kernel on Ω.*

(*i*) *There exists a reproducing kernel Krein space with kernel function K if and only if* $\mathrm{Top}_{\mathrm{ip}}\langle \mathrm{Fin}(\Omega, \mathfrak{V}), [.,.]_K\rangle \neq \emptyset$.

(*ii*) *Assume that* $\mathrm{Top}_{\mathrm{ip}}\langle \mathrm{Fin}(\Omega, \mathfrak{V}), [.,.]_K\rangle \neq \emptyset$. *Then there exists exactly one reproducing kernel Krein space with kernel function K if and only if for each* $\mathcal{T} \in \mathrm{Top}_{\mathrm{ip}}\langle \mathrm{Fin}(\Omega, \mathfrak{V}), [.,.]_K\rangle$ *with* $\overline{\{0\}}^{\mathcal{T}} = \mathrm{Fin}(\Omega, \mathfrak{V})^{[\circ]_K}$ *the space* $\langle \mathcal{H}_{\mathcal{T}}, [.,.]_{K,\mathcal{T}}\rangle$ *is semicompletely decomposable.*

❑

Also the Pontryagin space situation is immediately settled. If K is a $\mathfrak{V}$-valued kernel on Ω, then we write

$$\mathrm{ind}_- K := \mathrm{ind}_- \langle \mathrm{Fin}(\Omega, \mathfrak{V}), [.,.]_K\rangle\,.$$

Proposition 4.3.5 now gives:

COF19

6.2.5 Corollary. *Let K be a $\mathfrak{V}$-valued kernel on Ω, and assume that* $\mathrm{ind}_- K < \infty$. *Then there exists a unique reproducing kernel Krein $\mathcal{K}$ space having K as its kernel function. The space $\mathcal{K}$ is a Pontryagin space whose negative index equals* $\mathrm{ind}_- K$. ❑

Due to this corollary, the following notation is well-defined.

DEF9

6.2.6 Definition. Let K be a $\mathfrak{V}$-valued kernel on Ω with $\mathrm{ind}_- K < \infty$. Then the reproducing kernel Pontryagin space with kernel K will be denoted by $\mathfrak{K}(K)$, and we will speak of the *reproducing kernel Pontryagin space generated by K* //

Existence of a space having K as its reproducing kernel can be characterized in different ways than in the immediate transcription Corollary 6.2.4, (*i*).

PRF12

6.2.7 Proposition. *Let K be a $\mathfrak{V}$-valued kernel on Ω. Then the following are equivalent:*

(*i*) *There exists a reproducing kernel Krein space with kernel function K.*

(*ii*) *There exist kernels K_+ and K_- with* $\mathrm{ind}_- K_+ = \mathrm{ind}_- K_- = 0$, *such that* $K = K_+ - K_-$ *and* $\mathfrak{K}(K_+) \cap \mathfrak{K}(K_-) = 0$.

(*iii*) *There exist kernels K_+ and K_- with* $\mathrm{ind}_- K_+ = \mathrm{ind}_- K_- = 0$, *such that* $K = K_+ - K_-$.

(*iv*) *There exists a kernel* L *with* $\operatorname{ind}_- L = 0$ *such that* $|[\eta,\eta]_K| \leq [\eta,\eta]_L$, $\eta \in \operatorname{Fin}(\Omega,\mathfrak{V})$.

Proof. Assume that $\mathcal{K}$ is a reproducing kernel Krein space with kernel function K. Choose a fundamental decomposition $\mathfrak{J} = (\mathcal{K}_+,\mathcal{K}_-)$ of $\mathcal{K}$, then $\mathcal{K}_+$ is an orthocomplemented subspace of $\mathcal{K}$ and $\mathcal{K}_+^{\perp} = \mathcal{K}_-$. Denote by K_+ the kernel function of $\mathcal{K}_+$, and set $K_- := -K_+^{\perp}$. Since $\mathcal{K}_+$ is positive definite and $\mathcal{K}_-$ is negative definite, we have $\operatorname{ind}_- K_+ = \operatorname{ind}_- K_- = 0$. Moreover, by (6.1.3), we have $K = K_+ - K_-$. Finally, $\mathfrak{K}(K_+) = \mathcal{K}_+$ (as Krein spaces) and $\mathfrak{K}(K_-) = \mathcal{K}_-$ (as sets; inner product changed sign). Hence $\mathfrak{K}(K_+) \cap \mathfrak{K}(K_-) = \emptyset$.

The implication (*ii*) $\Rightarrow$ (*iii*) is trivial. Next, assume that $K = K_1 - K_2$ with $\operatorname{ind}_- K_1 = \operatorname{ind}_- K_2 = 0$. Set $L := K_1 + K_2$, then

$$|[\eta,\eta]_K| = |[\eta,\eta]_{K_1} - [\eta,\eta]_{K_2}| \leq [\eta,\eta]_{K_1} + [\eta,\eta]_{K_2} = [\eta,\eta]_L, \quad \eta \in \operatorname{Fin}(\Omega,\mathfrak{V})\,.$$

Finally, assume that L is as in (*iv*). Then $[.,.]_L$ is a positive semidefinite inner product on $\operatorname{Fin}(\Omega,\mathfrak{V})$, and $[.,.]_K$ is continuous with respect to the topology induced by $[.,.]_L$. Thus $\operatorname{Top}_{\mathrm{ip}}\langle \operatorname{Fin}(\Omega,\mathfrak{V}), [.,.]_K\rangle \neq \emptyset$, and hence $\langle \operatorname{Fin}(\Omega,\mathfrak{V}), [.,.]_K\rangle$ possesses a Krein space completion, cf. Theorem 3.5.13, (*iii*). ❏

DEF14 **6.2.8 Definition.** Let K be a $\mathfrak{V}$-valued kernel on Ω. A Krein space $\langle \mathcal{K}, [.,.]\rangle$ together with a map $V : \Omega \to \mathcal{B}(\mathfrak{V},\mathcal{K})$ which satisfy

$$K(w,z) = V(z)^* V(w),\ z,w \in \Omega, \quad \text{and} \quad \mathcal{K} = \operatorname{cls} \bigcup_{w\in\Omega} \operatorname{ran} V(w) \tag{6.2.5}$$ F16

is called a *Kolmogoroff decomposition* of K. //

PRF15 **6.2.9 Proposition.** *Let* K *be a* $\mathfrak{V}$*-valued kernel on* Ω*. Then there exists a reproducing kernel Krein space with kernel function* K *if and only if there exists a Kolmogoroff decomposition of* K*.*

Proof. Assume first that $\mathcal{K}$ is a reproducing kernel Krein space with kernel function K. Then $(\iota_K, \mathcal{K})$ is a completion of $\langle \operatorname{Fin}(\Omega,\mathfrak{V}), [.,.]_K\rangle$. Let $V(w)$ be the map defined in (6.2.1). By the definition of ι_K, this map acts as $V(w)f = K(w,.)f$, $f \in \mathfrak{V}$. Thus, for $f,g \in \mathfrak{V}$,

$$[V(z)^*V(w)f,g] = [V(w)f,V(z)g] = [K(w,.)f, K(z,.)g] = [K(w,z)f,g]_{\mathfrak{V}}\,,$$

and we conclude that $K(w,z) = V(z)^*V(w)$. Moreover, by (6.2.2), the linear span of $\bigcup_{w\in\Omega} \operatorname{ran} V(w)$ is dense in $\mathcal{K}$. Together this says that $\mathcal{K}$ and $V : w \mapsto V(w)$ is a Kolmogoroff decomposition of K.

Conversely, assume that $\mathcal{K}$ and $V : \Omega \to \mathcal{B}(\mathfrak{V},\mathcal{K})$ are given subject to (6.2.5). Consider the linear map ι defined by

$$\iota : \begin{cases} \operatorname{Fin}(\Omega,\mathfrak{V}) & \to \quad \mathcal{K} \\ \qquad\qquad \eta & \mapsto \quad \sum\limits_{w\in\Omega} V(w)\eta(w) \end{cases}$$

We have

$$[\eta,\mu]_K = \sum_{z,w\in\Omega} [\underbrace{K(w,z)}_{=V(z)^*V(w)} \eta(w), \mu(z)]_{\mathfrak{V}} = \sum_{z,w\in\Omega} [V(w)\eta(w), V(z)\mu(z)] =$$

$$= \Big[\sum_{w\in\Omega} V(w)\eta(w), \sum_{z\in\Omega} V(z)\mu(z)\Big]_{\mathcal{K}} ,$$

i.e. ι is isometric. Moreover, by the second condition in (6.2.5), $\operatorname{ran}\iota$ is dense in $\mathcal{K}$. This says that $(\iota, \mathcal{K})$ is a Krein space completion of $\langle \operatorname{Fin}(\Omega, \mathfrak{V}), [.,.]_K\rangle$. ❑

Also concerning uniqueness we can obtain a more specific statement than Corollary 6.2.4, (*ii*).

PRF7

6.2.10 Proposition. *Let K be a $\mathfrak{V}$-valued kernel on Ω, and assume that there exists a reproducing kernel Krein space with kernel function K. Then there exists exactly one such space, if and only if the condition of Corollary 6.2.4 holds for all topologies induced by inner products $[.,.]_L$ with a kernel L, such that $|[\eta,\eta]_K| \leq [\eta,\eta]_L$, $\eta \in \operatorname{Fin}(\Omega, \mathfrak{V})$, and $\operatorname{Fin}(\Omega, \mathfrak{V})^{[\circ]_L} = \operatorname{Fin}(\Omega, \mathfrak{V})^{[\circ]_K}$.*

Proof. The present condition is weaker than the one formulated in Corollary 6.2.4, (*ii*), since the same property is required only for a smaller class of topologies. Hence, it is enough to show that the present condition implies uniqueness.

Step 1: Assume that $\mathcal{K}$ is a reproducing kernel Krein space with kernel function K, and consider a fundamental decomposition $\mathfrak{J} = (\mathcal{K}_+, \mathcal{K}_-)$ of $\mathcal{K}$. Then $\mathcal{K}_\pm$ are reproducing kernel Krein spaces, and their respective kernel functions are given by

$$K_\pm(w,.)f = P_{\mathfrak{J}}^\pm K(w,.)f .$$

Set $L := K_+ - K_-$, then L is a $\mathfrak{V}$-valued kernel function on Ω and $\operatorname{ind}_- L = 0$. Moreover, we compute

$$\begin{aligned}
[f\chi_{\{w\}}, g\chi_{\{z\}}]_L = [L(w,z)f, g]_{\mathfrak{V}} = [K_+(w,z)f, g]_{\mathfrak{V}} - [K_-(w,z)f, g]_{\mathfrak{V}} = \\
= [K_+(w,.)f, K_+(z,.)g] - [K_-(w,.)f, K_-(z,.)g] = \\
= [P_{\mathfrak{J}}^+ K(w,.)f, P_{\mathfrak{J}}^+ K(z,.)g] - [P_{\mathfrak{J}}^- K(w,.)f, P_{\mathfrak{J}}^- K(z,.)g] = \\
= (K(w,.)f, K(z,.)g)_{\mathfrak{J}} = \big(\iota_K(f\chi_{\{w\}}), \iota_K(g\chi_{\{z\}})\big)_{\mathfrak{J}} .
\end{aligned}$$

It follows that

$$\begin{aligned}
\big|[f\chi_{\{w\}}, f\chi_{\{w\}}]_K\big| = \big|\big[\iota_K(f\chi_{\{w\}}), \iota_K(f\chi_{\{w\}})\big]\big| \leq \\
\leq \big(\iota_K(f\chi_{\{w\}}), \iota_K(f\chi_{\{w\}})\big) = [f\chi_{\{w\}}, f\chi_{\{w\}}]_L .
\end{aligned}$$

Step 2: Assume that we are given two reproducing kernel Krein spaces $\mathcal{K}_1$ and $\mathcal{K}_2$ with kernel function K. Choose fundamental decompositions $\mathfrak{J}_j$ of $\mathcal{K}_j$, $j = 1, 2$, and let L_j be the kernels defined correspondingly as in Step 1.

Now inspect the proof of sufficiency in Theorem 3.5.17. There the assumption on semicomplete decomposability is applied with the inner product

$$(\eta, \mu) := (\iota_K\eta, \iota_K\mu)_{\mathfrak{J}_1} + (\iota_K\eta, \iota_K\mu)_{\mathfrak{J}_2}, \quad \eta, \mu \in \operatorname{Fin}(\Omega, \mathfrak{V}) .$$

However, by the above computation, we have

$$(f\chi_{\{w\}}, g\chi\{z\}) = [f\chi_{\{w\}}, g\chi_{\{z\}}]_{L_1+L_2} ,$$

and hence $(.,.) = [.,.]_{L_1+L_2}$. Clearly, $L_1 + L_2$ is a kernel and satisfies all the properties required in the condition of the present proposition. ❑

Part II

Spectral Theory

Chapter 7

Linear Relations

7.1 Algebraic operations

DEI7 **7.1.1 Definition.** Let $\mathcal{L}$ and $\mathcal{M}$ be linear spaces. A subset T of $\mathcal{L}\times\mathcal{M}$ is called a *linear relation* of $\mathcal{L}$ into $\mathcal{M}$, if it is a linear subspace of $\mathcal{L}\times\mathcal{M}$. Explicitly this is

$$\begin{aligned} (f_1,g_1),(f_2,g_2)\in T \quad &\Rightarrow \quad (f_1+f_2,g_1+g_2)\in T\,, \\ (f,g)\in T,\lambda\in\mathbb{C} \quad &\Rightarrow \quad (\lambda f,\lambda g)\in T\,. \end{aligned}$$

The set of all linear relations of $\mathcal{L}$ into $\mathcal{M}$ will be denoted by $\mathrm{LR}(\mathcal{L},\mathcal{M})$. In case $\mathcal{L}=\mathcal{M}$ we will write abbreviatory $\mathrm{LR}(\mathcal{L})$ instead of $\mathrm{LR}(\mathcal{L},\mathcal{L})$. //

The reader probably wonders why we invent the new name $\mathrm{LR}(\mathcal{L},\mathcal{M})$ for what was previously called $\mathrm{Sub}(\mathcal{L}\times\mathcal{M})$, remember Definition 1.1.2. In the present part we rather put emphasize on the operator theoretic viewpoint than on linear algebra: If D is a linear subspace of $\mathcal{L}$ and $T: D\to\mathcal{M}$ is a linear operator, then we may identify T with its *graph*

$$\operatorname{graph} T := \big\{(x,y)\in\mathcal{L}\times\mathcal{M}:\ x\in D, y=Tx\big\}\,,$$

and this is a linear subspace of $\mathcal{L}\times\mathcal{M}$. In this way, linear operators can be regarded as linear relations, and we will interchangably think of T as a map or as a subspace.

The operator theoretic viewpoint on linear relations motivates the following definitions.

DEI8 **7.1.2 Definition.** Let $\mathcal{L}$ and $\mathcal{M}$ be linear spaces, and denote by $\pi_1:\mathcal{L}\times\mathcal{M}\to\mathcal{L}$ and $\pi_2:\mathcal{L}\times\mathcal{M}\to\mathcal{M}$ the canonical projections. For $T\in\mathrm{LR}(\mathcal{L},\mathcal{M})$ we set

$$\begin{aligned} \operatorname{dom} T &:= \pi_1(T)=\{f\in\mathcal{L}:\ \exists\, g\in\mathcal{M}:(f,g)\in T\}, \\ \operatorname{ran} T &:= \pi_2(T)=\{g\in\mathcal{M}:\ \exists\, f\in\mathcal{L}:(f,g)\in T\}, \\ \ker T &:= \pi_1(\pi_2|_T)^{-1}(\{0\})=\{f\in\mathcal{L}:\ (f,0)\in T\}, \\ \operatorname{mul} T &:= \pi_2(\pi_1|_T)^{-1}(\{0\})=\{g\in\mathcal{M}:\ (0,g)\in T\}, \end{aligned}$$

and speak of the *domain*, *range*, *kernel*, and *multivalued part*, of T. //

Let us remark that a linear relation T is (the graph of) a linear operator if and only if $\operatorname{mul} T = \{0\}$. In this case, $\operatorname{dom} T$, $\operatorname{ran} T$, and $\ker T$ have their usual meaning.

Also algebraic operations for linear relations can be defined by taking operations with linear operators as a model.

DEI9

7.1.3 Definition. Let $\mathcal{L}$, $\mathcal{M}$, and $\mathcal{N}$ be linear spaces.

(*i*) If $T, S \in \mathrm{LR}(\mathcal{L}, \mathcal{M})$, we set

$$T + S := \Big\{(f,g) \in \mathcal{L} \times \mathcal{M} :\ \exists g_1, g_2 \in \mathcal{M} \text{ with } (f,g_1) \in T, (f,g_2) \in S,\ g = g_1 + g_2\Big\}.$$

Moreover, we denote $0_{\mathcal{L}} := \{(f,g) \in \mathcal{L} \times \mathcal{M} : g = 0\} = \mathcal{L} \times \{0\}$. Explicit notation of $\mathcal{L}$ will be dropped unless necessary.

(*ii*) If $T \in \mathrm{LR}(\mathcal{L}, \mathcal{M})$ and $\lambda \in \mathbb{C}$, we set

$$\lambda \cdot T := \big\{(f,g) \in \mathcal{L} \times \mathcal{M} :\ \exists g_0 \in \mathcal{M} : (f,g_0) \in T, g = \lambda g_0\big\}.$$

We will often write λT instead of $\lambda \cdot T$.

(*iii*) If $T \in \mathrm{LR}(\mathcal{L}, \mathcal{M})$ and $S \in \mathrm{LR}(\mathcal{M}, \mathcal{N})$, we set

$$S \circ T := \big\{(f,h) \in \mathcal{L} \times \mathcal{N} :\ \exists g \in \mathcal{M} : (f,g) \in T, (g,h) \in S\big\}.$$

We will often write ST instead of $S \circ T$.

If $S, T \in \mathrm{LR}(\mathcal{L})$, we say that T and S *commute*, if $S \circ T = T \circ S$.

(*iv*) If $T \in \mathrm{LR}(\mathcal{L}, \mathcal{M})$, we set

$$T^{-1} := \big\{(g,f) \in \mathcal{M} \times \mathcal{L} :\ (f,g) \in T\big\}.$$

Moreover, we denote $I_{\mathcal{L}} := \{(f,g) \in \mathcal{L} \times \mathcal{L} : g = f\}$. Again explicit notation of $\mathcal{L}$ will be dropped unless necessary. If $\lambda \in \mathbb{C}$, we will often write just λ instead of λI.

//

It goes without saying that these operations do produce linear relations, so that we have maps

$$+ : \mathrm{LR}(\mathcal{L}, \mathcal{M}) \times \mathrm{LR}(\mathcal{L}, \mathcal{M}) \to \mathrm{LR}(\mathcal{L}, \mathcal{M}) \qquad \cdot : \mathbb{C} \times \mathrm{LR}(\mathcal{L}, \mathcal{M}) \to \mathrm{LR}(\mathcal{L}, \mathcal{M})$$

$$\circ : \mathrm{LR}(\mathcal{L}, \mathcal{M}) \times \mathrm{LR}(\mathcal{M}, \mathcal{N}) \to \mathrm{LR}(\mathcal{L}, \mathcal{N}) \qquad .^{-1} : \mathrm{LR}(\mathcal{L}, \mathcal{M}) \to \mathrm{LR}(\mathcal{M}, \mathcal{L})$$

A word of caution is in order: The presently defined sum '+' in $\mathrm{LR}(\mathcal{L}, \mathcal{M})$ is not the same as the sum of T and S in $\mathrm{Sub}(\mathcal{L} \times \mathcal{M})$. Remember that

$$\underset{\substack{\uparrow \\ \text{sum in} \\ \mathrm{Sub}(\mathcal{L}\times\mathcal{M})}}{T + S} = \operatorname{span}\big(T \cup S\big) = \Big\{(f,g) \in \mathcal{L} \times \mathcal{M} :\ \exists f_1, f_2 \in \mathcal{L}, g_1, g_2 \in \mathcal{M} \text{ with } (f_1,g_1) \in T, (f_2,g_2) \in S,\ f = f_1 + f_2, g = g_1 + g_2\Big\}$$

Therefore we will, if confusion may occur, write $\operatorname{span}(T \cup S)$ for the sum in $\operatorname{Sub}(\mathcal{L} \times \mathcal{M})$, and use the symbol '$+$' for the sum in $\operatorname{LR}(\mathcal{L}, \mathcal{M})$.

The use of the direct sum symbol '$\dot{+}$', however, is unambigous: For linear relations $T, S \in \operatorname{LR}(\mathcal{L}, \mathcal{M})$ writing $T \dot{+} S$ still means that $T \cap S = \{0\}$ and denotes $\operatorname{span}(T \cup S)$ in this case. The same applies to the orthogonal sum symbols '$[+]$' or '$[\dot{+}]$'.

The set $\operatorname{LR}(\mathcal{L}, \mathcal{M})$ endowed with the operations '$+$' and '$\cdot$' does not form a linear space, let alone $\langle \operatorname{LR}(\mathcal{L}), +, \cdot, \circ \rangle$ is an algebra. Let us provide a list of computation rules. Although mainly obvious, let us for completeness also write down proofs explicitly.

NTI10 **7.1.4. *Computation rules. I. Additive structure:***

(*i*) *The operation '$+$' on* $\operatorname{LR}(\mathcal{L}, \mathcal{M})$ *is associative and commutative, and* $0_{\mathcal{L}}$ *is a neutral element. Explicitly, this is*

$$(T+S)+R = T+(S+R), \quad T+S = S+T, \quad T+0 = 0+T = T\,.$$

(*ii*) *If* $\operatorname{mul} T \neq \{0\}$, *then* T *has no additive inverse.*

Proof. For (*i*) we compute

$$\begin{aligned}(T+S)+R = \Big\{&(f,g) \in \mathcal{L} \times \mathcal{M} :\ \exists\, g_1, g_2 \in \mathcal{M} \text{ with} \\ &\qquad (f,g_1) \in T+S, (f,g_2) \in R,\ g = g_1 + g_2 \Big\} = \\ = \Big\{&(f,g) \in \mathcal{L} \times \mathcal{M} :\ \exists\, g_{11}, g_{12}, g_2 \in \mathcal{M} \text{ with} \\ &\qquad (f,g_{11}) \in T, (f,g_{12}) \in S, (f,g_2) \in R,\ g = \underbrace{(g_{11} + g_{12}) + g_2}_{=g_{11}+(g_{12}+g_2)} \Big\} = \\ = T + (S&+R)\,,\end{aligned}$$

$$\begin{aligned}T+S = \Big\{&(f,g) \in \mathcal{L} \times \mathcal{M} :\ \exists\, g_1, g_2 \in \mathcal{M} \text{ with} \\ &\qquad (f,g_1) \in T, (f,g_2) \in S,\ g = \underbrace{g_1 + g_2}_{=g_2+g_1} \Big\} = \\ = S + T&\,,\end{aligned}$$

$$\begin{aligned}T+0 = \Big\{&(f,g) \in \mathcal{L} \times \mathcal{M} :\ \exists\, g_1, g_2 \in \mathcal{M} \text{ with} \\ = T\,. &\qquad (f,g_1) \in T, (f,g_2) \in 0,\ g = \underbrace{g_1 + g_2}_{=g_1+0=g_1} \Big\}\end{aligned}$$

For (*ii*), it is enough to note that always $\{0\} \times \operatorname{mul} T \subseteq T + S$. If T has an additive inverse, i.e. there exists an element $S \in \operatorname{LR}(\mathcal{L}, \mathcal{M})$ with $T + S = 0$, hence $\operatorname{mul} T = \{0\}$. ❑

NTI11 **7.1.5. *Computation rules. II. Scalar multiplication:***

(*i*) $0 \cdot T = \operatorname{dom} T \times \{0\} = 0_{\operatorname{dom} T}$, *and* $1 \cdot T = T$.

(*ii*) $\lambda \cdot (T+S) = (\lambda \cdot T) + (\lambda \cdot S)$.

(iii) $(\lambda+\mu)\cdot T \subseteq (\lambda\cdot T)+(\mu\cdot T)$. *Thereby strict inequality holds if and only if* $\lambda=-\mu\neq 0$ *and* $\operatorname{mul} T\neq\{0\}$.

(iv) $(\lambda\mu)\cdot T=\lambda\cdot(\mu\cdot T)$.

Proof. For (i) compute

$$0\cdot T=\Big\{(f,g)\in\mathcal{L}\times\mathcal{M}:\ \exists\, g_0\in\mathcal{M}\text{ with }(f,g_0)\in T\ g=\underbrace{0g_0}_{=0}\Big\}=0_{\operatorname{dom}T}$$

$$1\cdot T=\Big\{(f,g)\in\mathcal{L}\times\mathcal{M}:\ \exists\, g_0\in\mathcal{M}\text{ with }(f,g_0)\in T\ g=\underbrace{1g_0}_{=g_0}\Big\}=T$$

For (ii) compute

$$\begin{aligned}\lambda\cdot(T+S)&=\Big\{(f,g)\in\mathcal{L}\times\mathcal{M}:\ \exists\, g_0\in\mathcal{M}\text{ with}\\&\qquad\qquad (f,g_0)\in T+S,\ g=\lambda g_0\Big\}=\\&=\Big\{(f,g)\in\mathcal{L}\times\mathcal{M}:\ \exists\, g_1,g_2\in\mathcal{M}\text{ with}\\&\qquad\qquad (f,g_1)\in T,(f,g_2)\in S,\ g=\underbrace{\lambda(g_1+g_2)}_{\lambda g_1+\lambda g_2}\Big\}=\\&=\lambda T+\lambda S\end{aligned}$$

The inclusion asserted in (iii) follows in the same way:

$$\begin{aligned}(\lambda+\mu)\cdot T&=\Big\{(f,g)\in\mathcal{L}\times\mathcal{M}:\ \exists\, g_0\in\mathcal{M}\text{ with}\\&\qquad\qquad (f,g_0)\in T,\ g=\underbrace{(\lambda+\mu)g_0}_{=\lambda g_0+\mu g_0}\Big\}\subseteq\\&\subseteq\Big\{(f,g)\in\mathcal{L}\times\mathcal{M}:\ \exists\, g_1,g_2\in\mathcal{M}\text{ with}\\&\qquad\qquad (f,g_1)\in T,(f,g_2)\in T,\ g=\lambda g_1+\mu g_2\Big\}=\\&=\lambda T+\mu T\end{aligned}$$

Let us check all cases to find out when in this relation equality holds: If $\lambda=\mu=0$, then both sides are equal to $0_{\operatorname{dom}T}$. Next, assume that $\lambda+\mu\neq 0$, and consider an elements $(f,g_1),(f,g_2)\in T$. Then we can write

$$\lambda g_1+\mu g_2=(\lambda+\mu)\big(g_1+\frac{\mu}{\lambda+\mu}\underbrace{(g_2-g_1)}_{\in\operatorname{mul}T}\big)\,,$$

and see that $(f,\lambda g_1+\mu g_2)\in(\lambda+\mu)T$. Again equality holds. If $\operatorname{mul}T=\{0\}$, then in the expression for $\lambda T+\mu T$ we must have $g_1=g_2$, and thus also in this case equality holds. It remains to have a look at the case that $\lambda=-\mu\neq 0$ and $\operatorname{mul}T\neq\{0\}$. Then, however, the left sides equals $0_{\operatorname{dom}T}$ whereas the right side certainly contains $\{0\}\times\operatorname{mul}T$. Thus, in this case, equality does not hold.

Finally, we compute

$$\begin{aligned}(\lambda\mu)\cdot T&=\Big\{(f,g)\in\mathcal{L}\times\mathcal{M}:\ \exists\, g_0\in\mathcal{M}\text{ with }(f,g_0)\in T,\ g=\underbrace{(\lambda\mu)g_0}_{=\lambda(\mu g_0)}\Big\}=\\&=\lambda\cdot(\mu\cdot T)\end{aligned}$$

❑

NTI12 **7.1.6. *Computation rules. III. Composition:***

(*i*) *The operation '$\circ$' is associative. Explicitly, this is* $T\circ(S\circ R) = (T\circ S)\circ R$.

(*ii*) $(T\circ S)^{-1} = S^{-1}\circ T^{-1}$.

(*iii*) *Let* $T\in \mathrm{LR}(\mathcal{L},\mathcal{M})$. *Then*

$$\begin{aligned} T\circ T^{-1} &= I_{\operatorname{ran} T}\dot{+}(\{0\}\times \operatorname{mul} T) = \\ &= \big\{(g_1,g_2)\in\mathcal{M}\times\mathcal{M}:\ g_1,g_2\in\operatorname{ran}T, g_1-g_2\in\operatorname{mul}T\big\} \\ T^{-1}\circ T &= I_{\operatorname{dom} T}\dot{+}(\ker T\times\{0\}) = \\ &= \big\{(f_1,f_2)\in\mathcal{L}\times\mathcal{L}:\ f_1,f_2\in\operatorname{dom}T, f_1-f_2\in\ker T\big\} \end{aligned}$$

(*iv*) $\lambda\cdot T = (\lambda\cdot I_{\mathcal{M}})\circ T = T\circ(\lambda\cdot I_{\mathcal{L}})$. *In particular,* $I_{\mathcal{L}}$ *is a neutral element in* $\langle \mathrm{LR}(\mathcal{L}),\circ\rangle$.

(*v*) $(T+S)\circ R\subseteq (T\circ R)+(S\circ R)$. *If* $\operatorname{mul}R=\{0\}$, *then equality holds.*

(*vi*) $(R\circ T)+(R\circ S)\subseteq R\circ(T+S)$. *If* $\operatorname{dom}R=\mathcal{M}$, *then equality holds.*

(*vii*) *If* R *commutes with* T *and* S, *then* $(T+S)\circ R\subseteq R\circ(T+S)$.

(*viii*) *If* R *commutes with* T, S, *and* $T+S$, *then* $(T+S)\circ R=(T\circ R)+(S\circ R)$.

Proof.

$$\begin{aligned} &(T\circ S)\circ R = \Big\{(f,k)\in\mathcal{L}\times\mathcal{P}:\ \exists g\in\mathcal{M} \text{ with } (f,g)\in R, (g,k)\in T\circ S\Big\} = \\ &= \Big\{(f,k)\in\mathcal{L}\times\mathcal{P}:\ \exists g\in\mathcal{M}, h\in\mathcal{N} \text{ with} \\ &\qquad\qquad (f,g)\in R, (g,h)\in S, (h,k)\in T\Big\} = \\ &= T\circ(S\circ R)\,, \end{aligned}$$

$$\begin{aligned} &(T\circ S)^{-1} = \Big\{(f,h)\in\mathcal{N}\times\mathcal{L}:\ (h,f)\in T\circ S\Big\} = \\ &= \Big\{(f,h)\in\mathcal{N}\times\mathcal{L}:\ \exists g\in\mathcal{M} \text{ with } (h,g)\in S, (g,f)\in T\Big\} = \\ &= S^{-1}\circ T^{-1}\,, \end{aligned}$$

$$\begin{aligned} &T\circ T^{-1} = \Big\{(g_1,g_2)\in\mathcal{M}\times\mathcal{M}:\ \exists f\in\mathcal{L} \text{ with } (g_1,f)\in T^{-1}, (f,g_2)\in T\Big\} = \\ &= \Big\{(g_1,g_2)\in\mathcal{M}\times\mathcal{M}:\ g_1,g_2\in\operatorname{ran}T, g_1-g_2\in\operatorname{mul}T\Big\}\,, \end{aligned}$$

$$\begin{aligned} &T^{-1}\circ T = \Big\{(f_1,f_2)\in\mathcal{L}\times\mathcal{L}:\ \exists g\in\mathcal{M} \text{ with } (f_1,g)\in T, (g,f_2)\in T^{-1}\Big\} = \\ &= \Big\{(f_1,f_2)\in\mathcal{L}\times\mathcal{L}:\ f_1,f_2\in\operatorname{dom}T, f_1-f_2\in\ker T\Big\}\,, \end{aligned}$$

$$\begin{aligned}
&\lambda\cdot T=\Big\{(f,g)\in\mathcal{L}\times\mathcal{M}:\ \exists g_0\in\mathcal{M}\text{ with }(f,g_0)\in T, g=\lambda g_0\Big\}=\\
&=\Big\{(f,g)\in\mathcal{L}\times\mathcal{M}:\ \exists g_0\in\mathcal{M}\text{ with }(f,g_0)\in T,(g_0,g)\in\lambda\cdot I_{\mathcal{M}}, g=\lambda g_0\Big\}=\\
&=(\lambda\cdot I_{\mathcal{M}})\circ T=\\
&=\Big\{(f,g)\in\mathcal{L}\times\mathcal{M}:\ \exists g_0\in\mathcal{M}\text{ with }(f,\lambda f)\in\lambda\cdot I_{\mathcal{L}},(\lambda f,\lambda g_0), g=\lambda g_0\in T\Big\}=\\
&=T\circ(\lambda\cdot I_{\mathcal{L}})\,,
\end{aligned}$$

$$\begin{aligned}
&(T+S)\circ R=\Big\{(f,h)\in\mathcal{L}\times\mathcal{N}:\ \exists g\in\mathcal{M}\text{ with }(f,g)\in R,(g,h)\in T+S\Big\}=\\
&=\Big\{(f,h)\in\mathcal{L}\times\mathcal{N}:\ \exists g\in\mathcal{M},h_1,h_2\in\mathcal{N}\text{ with}\\
&\qquad\qquad (f,g)\in R,(g,h_1)\in T,(g,h_2)\in S,h=h_1+h_2\Big\}\subseteq\\
&\subseteq\Big\{(f,h)\in\mathcal{L}\times\mathcal{N}:\ \exists g_1,g_2\in\mathcal{M},h_1,h_2\in\mathcal{N}\text{ with}\\
&\qquad\qquad (f,g_1)\in R,(g_1,h_1)\in T,(f,g_2)\in R,(g_2,h_2)\in S,h=h_1+h_2\Big\}=\\
&=T\circ R+S\circ R\,,
\end{aligned}$$

If $\operatorname{mul} R=\{0\}$, then the elements g_1,g_2 in the last but one line must coincide, and hence equality holds.

$$\begin{aligned}
&R\circ T+R\circ S=\Big\{(f,h)\in\mathcal{L}\times\mathcal{N}:\ \exists h_1,h_2\in\mathcal{N}\text{ with}\\
&\qquad\qquad (f,h_1)\in R\circ T,(f,h_2)\in R\circ S,h=h_1+h_2\Big\}=\\
&=\Big\{(f,h)\in\mathcal{L}\times\mathcal{N}:\ \exists g_1,g_2\in\mathcal{M},h_1,h_2\in\mathcal{N}\text{ with}\\
&\qquad\qquad (f,g_1)\in T,(f,g_2)\in S,(g_1,h_1)\in R,(g_2,h_2)\in R,h=h_1+h_2\Big\}\subseteq\\
&\subseteq\Big\{(f,h)\in\mathcal{L}\times\mathcal{N}:\ \exists g\in\mathcal{M}\text{ with }(f,g)\in T+S,(g,h)\in R\Big\}=\\
&=R\circ(T+S)\,,
\end{aligned}$$

The inclusion of the set in second line in the set in the third line thereby follows by setting $g:=g_1+g_2$. Assume that $\operatorname{dom} R=\mathcal{M}$, and let f,g,h be such that $(f,g)\in T+S$ and $(g,h)\in R$. Choose $g_1,g_2\in\mathcal{M}$ with $(f,g_1)\in T$, $(f,g_2)\in S$, and $g=g_1+g_2$, and choose $h_1\in\mathcal{N}$ with $(g_1,h_1)\in R$. Then

$$(g_2,h-h_1)=(g,h)-(g_1,h_1)\in R\,,$$

and it follows that (f,h) belongs to the set written in the second line. Thus, in case $\operatorname{dom} R=\mathcal{M}$, equality holds.

The assertions (*vii*) and (*viii*) follow easily from (*v*) and (*vi*). Assume that R commutes with S and T. Then we can compute

$$(T+S)\circ R\subseteq T\circ R+S\circ R=R\circ T+R\circ S\subseteq R\circ(T+S)\,.$$

If, in addition, R commutes with $T+S$, then the last term equals $(T+S)\circ R$, and hence throughout the above chain of inequalities the equality sign must hold. ❑

NTI22 **7.1.7. *Computation rules. IV. Miscellaneous:***

(*i*) *We have* $\operatorname{dom}(T^{-1}) = \operatorname{ran} T$*,* $\operatorname{ran}(T^{-1}) = \operatorname{dom} T$*, and* $\ker(T^{-1}) = \operatorname{mul} T$*,* $\operatorname{mul}(T^{-1}) = \ker T$*.*

(*ii*) *We have* $\operatorname{dom}(ST) \subseteq \operatorname{dom} T$*,* $\operatorname{ran}(ST) \subseteq \operatorname{ran} S$*, and* $\ker T \subseteq \ker(ST)$*,* $\operatorname{mul}(S) \subseteq \operatorname{mul}(ST)$*.*

(*iii*) *If* $R \subseteq S$*, then*

$$R + T \subseteq S + T, \quad \lambda \cdot R \subseteq \lambda \cdot S, \quad R^{-1} \supseteq S^{-1},$$

$$R \circ T \subseteq S \circ T, \quad T \circ R \subseteq T \circ S.$$

(*iv*) $\operatorname{span}(R \circ S \cup R \circ T) \subseteq R \circ \operatorname{span}(S \cup T)$*. If* $\operatorname{ran} T \subseteq \operatorname{dom} R$ *or* $\operatorname{ran} S \subseteq \operatorname{dom} R$*, then equality holds.*

(*v*) $\operatorname{span}(S \circ R \cup T \circ R) \subseteq \operatorname{span}(S \cup T) \circ R$*. If* $\operatorname{dom} T \subseteq \operatorname{ran} R$ *or* $\operatorname{dom} S \subseteq \operatorname{ran} R$*, then equality holds.*

Proof. Let τ be the map

$$\tau : \begin{cases} \mathcal{L} \times \mathcal{M} & \to & \mathcal{M} \times \mathcal{L} \\ (f, g) & \mapsto & (g, f) \end{cases}$$

Then (π_1 and π_2 denote the respective projections onto first and second components)

$$\pi_1 \circ \tau = \pi_2, \quad \pi_2 \circ \tau = \pi_1.$$

This immediately gives (*i*). The first two inclusions in (*ii*) are obvious, the third inclusion follows since $(0, 0) \in S$, the fourth since $(0, 0) \in T$. The inclusions in (*iii*) are obvious from the definitions.

We come to the proof of (*iv*) and (*v*).

$$\begin{aligned} &\operatorname{span}\big(R \circ S \cup R \circ T\big) = \Big\{(f, h) \in \mathcal{L} \times \mathcal{N} :\ \exists f_1, f_2 \in \mathcal{L}, h_1, h_2 \in \mathcal{N} \text{ with} \\ &\qquad (f_1, h_1) \in R \circ S, (f_2, h_2) \in R \circ T, f = f_1 + f_2, h = h_1 + h_2\Big\} = \\ &= \Big\{(f, h) \in \mathcal{L} \times \mathcal{N} :\ \exists f_1, f_2 \in \mathcal{L}, g_1, g_2 \in \mathcal{M}, h_1, h_2 \in \mathcal{N} \text{ with} \\ &\qquad (f_1, g_1) \in S, (g_1, h_1) \in R, (f_2, g_2) \in T, (g_2, h_2) \in R, \\ &\qquad\qquad f = f_1 + f_2, h = h_1 + h_2\Big\} \subseteq \\ &\subseteq \Big\{(f, h) \in \mathcal{L} \times \mathcal{N} :\ \exists g \in \mathcal{M} \text{ with } (f, g) \in \operatorname{span}(S \cup T), (g, h) \in R\Big\} = \\ &= \operatorname{span}(S \cup T) \circ R. \end{aligned}$$

Thereby, the inclusion follows on setting $g := g_1 + g_2$. Assume that $\operatorname{ran} T \subseteq \operatorname{dom} R$; the case that $\operatorname{ran} S \subseteq \operatorname{dom} R$ follows in the same way. Let f, g, h with $(f, g) \in \operatorname{span}(S \cup T)$, $(g, h) \in R$, be given. Then we can find $f_1, f_2 \in \mathcal{L}$ and $g_1, g_2 \in \mathcal{M}$ such that

$$(f_1, g_1) \in S, (f_2, g_2) \in T, \quad f = f_1 + f_2, g = g_1 + g_2.$$

Choose $h_2 \in \mathcal{N}$ such that $(g_2, h_2) \in R$, and set $g_1 := g - g_2$, $h_1 := h - h_2$. Then $(g_1, h_1) \in R$, and we see that in the above inclusion the equality sign holds.

$$\begin{aligned}
\operatorname{span}\big(S \circ R \cup T \circ R\big) = \Big\{&(f,h) \in \mathcal{L} \times \mathcal{N} :\ \exists f_1, f_2 \in \mathcal{L}, h_1, h_2 \in \mathcal{N} \text{ with} \\
&(f_1,h_1) \in S \circ R, (f_2,h_2) \in T \circ R, f = f_1 + f_2, h = h_1 + h_2\Big\} = \\
= \Big\{(f,h) \in \mathcal{L} \times \mathcal{N} :&\ \exists f_1, f_2 \in \mathcal{L}, g_1, g_2 \in \mathcal{M}, h_1, h_2 \in \mathcal{N} \text{ with} \\
&(f_1,g_1) \in R, (g_1,h_1) \in S, (f_2,g_2) \in R, (g_2,h_2) \in T, \\
&f = f_1 + f_2, h = h_1 + h_2\Big\} \subseteq \\
\subseteq \Big\{(f,h) \in \mathcal{L} \times \mathcal{N} :&\ \exists g \in \mathcal{M} \text{ with } (f,g) \in R, (g,h) \in \operatorname{span}(S \cup T)\Big\} = \\
= \operatorname{span}(S \cup T) \circ R\,.&
\end{aligned}$$

Again the inclusion follows on setting $g := g_1 + g_2$. Assume that $\operatorname{dom} T \subseteq \operatorname{ran} R$; the case that $\operatorname{dom} S \subseteq \operatorname{ran} R$ follows in the same way. Let f, g, h with $(f, g) \in R$, $(g, h) \in \operatorname{span}(S \cup T)$, be given. Choose $g_1, g_2 \in \mathcal{M}$ and $h_1, h_2 \in \mathcal{N}$ with

$$(g_1,h_1) \in S, (g_2,h_2) \in T, \quad g = g_1 + g_2, h = h_1 + h_2\,.$$

Next, choose $f_2 \in \mathcal{L}$ such that $(f_2, g_2) \in R$, and set $f_1 := f - f_2$, $g_1 := g - g_2$. Then $(f_1, g_1) \in R$, and we see that in the above inclusion the equality sign holds. ❑

Let $T \in \operatorname{LR}(\mathcal{L})$. Then powers T^n of T are defined for $n \in \mathbb{Z}$ in the usual way as

$$T^n := \begin{cases} \underbrace{T \circ \ldots \circ T}_{n \text{ times}} &, \quad n \in \mathbb{N} \\ I_{\mathcal{L}} &, \quad n = 0 \\ \underbrace{T^{-1} \circ \ldots \circ T^{-1}}_{-n \text{ times}}, & \quad n \in -\mathbb{N} \end{cases}$$

If $n \in \mathbb{N}$ and $a_n \in \mathbb{C}$, we may consider the expression $\sum_{n=0}^{N} a_n T^n$. This assignment, however, is not fully compatible with algebraic operations. For example, we have

$$T - T = \operatorname{dom} T \times \operatorname{mul} T\,,$$

which is not equal to $0_{\mathcal{L}}$ unless T is an everywhere defined operator. Or

$$0 \cdot T = 0_{\operatorname{dom} T}\,,$$

which is not equal to $0_{\mathcal{L}}$ if $\operatorname{dom} T \neq \mathcal{L}$.

7.2 Fractional linear transformations

Our next aim is to set up a functional calculus for fractional linear transformations. For $T \in \operatorname{LR}(\mathcal{L})$ and $M = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathbb{C}^{2\times 2}$, we denote

$$\phi_M(T) := \big\{(\gamma g + \delta f, \alpha g + \beta f) :\ (f,g) \in T\big\}\,.$$

Clearly, $\phi_M(T)$ is a linear relation. Thinking in terms of vectors instead of pairs, we could equally well write ('$.^T$' denotes the transpose of a vector or a matrix)

$$\phi_M(T) = \Big\{ \Big[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} M \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot (f,g)^T \Big]^T : (f,g) \in T \Big\}.$$

From this it is obvious that the assignment $M \mapsto \phi_M(T)$ is multiplicative in the sense that

$$\phi_{M_1 M_2}(T) = \phi_{M_1}\big(\phi_{M_2}(T)\big), \quad M_1, M_2 \in \mathbb{C}^{2\times 2}. \tag{7.2.1}$$ I30

Moreover, we have

$$\phi_M(T) = \phi_{\lambda M}(T), \ \lambda \in \mathbb{C} \setminus \{0\}. \tag{7.2.2}$$ I25

Mostly, we will work with $\phi_M(T)$ when M is a matrix having nonzero determinant. However, sometimes it happens that the case $\det M = 0$ occurs. In order to treat these somewhat exceptional cases, it is the best to explicitly write down $\phi_M(T)$ for all possibilities and thus make available a list to refer to:

$$\phi_{\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)}(T) = 0_{\{0\}}, \quad \phi_{\left(\begin{smallmatrix} 0 & 0 \\ 0 & \delta \end{smallmatrix}\right)}(T) = 0_{\operatorname{ran} T}, \ \delta \neq 0, \quad \phi_{\left(\begin{smallmatrix} 0 & 0 \\ \gamma & 0 \end{smallmatrix}\right)}(T) = 0_{\operatorname{dom} T}, \ \gamma \neq 0$$

$$\phi_{\left(\begin{smallmatrix} 0 & 0 \\ \gamma & \delta \end{smallmatrix}\right)}(T) = 0_{\operatorname{span}(\operatorname{ran} T \cup \operatorname{dom} T)}, \ \gamma, \delta \neq 0, \quad \phi_{\left(\begin{smallmatrix} 0 & \beta \\ 0 & 0 \end{smallmatrix}\right)}(T) = 0^{-1}_{\operatorname{ran} T}, \ \beta \neq 0$$

$$\phi_{\left(\begin{smallmatrix} \alpha & 0 \\ 0 & 0 \end{smallmatrix}\right)}(T) = 0^{-1}_{\operatorname{dom} T}, \ \beta \neq 0, \quad \phi_{\left(\begin{smallmatrix} \alpha & \beta \\ 0 & 0 \end{smallmatrix}\right)}(T) = 0^{-1}_{\operatorname{span}(\operatorname{ran} T \cup \operatorname{dom} T)}, \ \alpha, \beta \neq 0$$

$$\phi_{\left(\begin{smallmatrix} 0 & \lambda\delta \\ 0 & \delta \end{smallmatrix}\right)}(T) = \lambda I_{\operatorname{ran} T}, \ \lambda, \delta \neq 0, \quad \phi_{\left(\begin{smallmatrix} \lambda\gamma & 0 \\ \gamma & 0 \end{smallmatrix}\right)}(T) = \lambda I_{\operatorname{dom} T}, \ \lambda, \gamma \neq 0$$

$$\phi_{\left(\begin{smallmatrix} \lambda\gamma & \lambda\delta \\ \gamma & \delta \end{smallmatrix}\right)}(T) = \lambda I_{\operatorname{span}(\operatorname{ran} T \cup \operatorname{dom} T)}, \ \lambda, \gamma, \delta \neq 0$$

If $\det M \neq 0$, the relation $\phi_M(T)$ can be expressed via the algebraic operations on $\operatorname{LR}(\mathcal{L})$. We will denote by $\operatorname{GL}(2,\mathbb{C})$ the group of all 2×2-matrices with complex entries having nonzero determinant.

LEI31 **7.2.1 Lemma.** *Let $T \in \operatorname{LR}(\mathcal{L})$ and $M = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \operatorname{GL}(2,\mathbb{C})$. Then*

$$\phi_M(T) = \begin{cases} \frac{\alpha}{\gamma} - \frac{\det M}{\gamma^2}(T + \frac{\delta}{\gamma})^{-1}, & \gamma \neq 0 \\ \frac{\alpha}{\delta} T + \frac{\beta}{\delta} & , \quad \gamma = 0 \end{cases}$$

If T is an everywhere defined operator and $\gamma T + \delta$ is bijective, then

$$\phi_M(T) = (\alpha T + \beta)(\gamma T + \delta)^{-1}, \tag{7.2.3}$$ I38

in particular $\phi_M(T)$ is an everywhere defined operator.

Proof. The matrix M can be written as

$$M = \begin{cases} \begin{pmatrix} 1 & \frac{\alpha}{\gamma} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{-\det M}{\gamma} & 0 \\ 0 & \gamma \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & \frac{\delta}{\gamma} \\ 0 & 1 \end{pmatrix}, & \gamma \neq 0 \\ \begin{pmatrix} 1 & \frac{\beta}{\delta} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} & , \quad \gamma = 0 \end{cases} \tag{7.2.4}$$ I20

In view of (7.2.1), it remains to note that

$$\phi_{\left(\begin{smallmatrix}\alpha & 0\\ 0 & \delta\end{smallmatrix}\right)}(T) = \big\{(\delta f, \alpha g) : (f,g)\in T\big\} = \frac{\alpha}{\delta}\cdot T, \quad \alpha,\delta\neq 0\,,$$

$$\phi_{\left(\begin{smallmatrix}1 & \beta\\ 0 & 1\end{smallmatrix}\right)}(T) = \big\{(f, g+\beta f) : (f,g)\in T\big\} = T+\beta\,,$$

$$\phi_{\left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)}(T) = \big\{(g,f) : (f,g)\in T\big\} = T^{-1}\,.$$

Assume that T is an everywhere defined operator and $\gamma T+\delta$ is bijective. If $\gamma=0$, (7.2.3) is trivial. Hence, assume that $\gamma\neq 0$. If $\alpha=0$, again, (7.2.3) is trivial. Assume that moreover $\alpha\neq 0$, then

$$\alpha T+\beta = \alpha(T+\frac{\delta}{\gamma}-\frac{\delta}{\gamma}+\beta) = \alpha(T+\frac{\delta}{\gamma}) - \frac{\alpha\delta}{\gamma}+\beta = a(T+\frac{\delta}{\gamma}) - \frac{\det M}{\gamma}\,,$$

and hence

$$(\alpha T+\beta)(\gamma T+\delta)^{-1} = \frac{1}{\gamma}(\alpha T+\beta)(T+\frac{\delta}{\gamma})^{-1} = \frac{\alpha}{\gamma} - \frac{\det M}{\gamma^2}(T+\frac{\delta}{\gamma})^{-1} = \phi_M(T)\,.$$

❑

Concerning relational sums and products we have the following computation rules.

LEI36

7.2.2 Lemma. *Let $T\in \mathrm{LR}(\mathcal{L})$ and $M,N\in\mathbb{C}^{2\times 2}$.*

(i) If $(0,1)M=(0,1)N$, then

$$\phi_M(T)+\phi_N(T) = \operatorname{span}\Big(\phi_{\left(\begin{smallmatrix}(1,0)(M+N)\\ (0,1)M\end{smallmatrix}\right)}(T)\cup\big(\{0\}\times\operatorname{mul}\phi_M(T)\big)\Big) = \\ = \operatorname{span}\Big(\phi_{\left(\begin{smallmatrix}(1,0)(M+N)\\ (0,1)M\end{smallmatrix}\right)}(T)\cup\big(\{0\}\times\operatorname{mul}\phi_N(T)\big)\Big)$$

(ii) If $(0,1)M=(1,0)N$, then

$$\phi_M(T)\circ\phi_N(T) = \operatorname{span}\Big(\phi_{\left(\begin{smallmatrix}(1,0)M\\ (0,1)N\end{smallmatrix}\right)}(T)\cup\big(\{0\}\times\operatorname{mul}\phi_M(T)\big)\Big) = \\ = \operatorname{span}\Big(\phi_{\left(\begin{smallmatrix}(1,0)M\\ (0,1)N\end{smallmatrix}\right)}(T)\cup\big(\ker\phi_N(T)\times\{0\}\big)\Big)$$

Proof. Write $M=\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$, $N=\left(\begin{smallmatrix}\lambda & \mu\\ \gamma & \delta\end{smallmatrix}\right)$, and set $P:=\left(\begin{smallmatrix}\alpha+\lambda & \beta+\mu\\ \gamma & \delta\end{smallmatrix}\right)$. Let $(x,y)\in\phi_M(T)+\phi_N(T)$, and choose $(f_1,g_1),(f_2,g_2)\in T$ with

$$x=\gamma f_1+\delta g_1=\gamma g_2+\delta f_2,\quad y=(\alpha g_1+\beta f_1)+(\lambda g_2+\mu f_2)\,.$$

Then

$$\big(0,\alpha(g_1-g_2)+\beta(f_1-f_2)\big) = (\gamma f_1+\delta g_1,\alpha g_1+\beta f_1)-(\gamma g_2+\delta f_2,\alpha g_2+\beta f_2)\in\phi_M(T)\,,$$

and hence

$$(x,y) = \big(\gamma g_2+\delta f_2,(\alpha+\lambda)g_2+(\beta+\mu)f_2\big)+\big(0,\alpha(g_1-g_2)+\beta(f_1-f_2)\big) \\ \in\operatorname{span}\Big(\phi_P(T)\cup\big(\{0\}\times\operatorname{mul}\phi_M(T)\big)\Big)$$

Similarly,

$$\big(0,\lambda(g_2-g_1)+\mu(f_2-f_1)\big)=(\gamma g_2+\delta f_2,\lambda g_2+\mu f_2)-(\gamma f_1+\delta g_1,\lambda g_1+\mu f_1)\in\phi_N(T)\,,$$

and hence

$$\begin{aligned}(x,y)=\big(\gamma g_1+\delta f_1,(\alpha+\lambda)g_1+(\beta+\mu)f_1\big)+\big(0,\lambda(g_2-g_1)+\mu(f_2-f_1)\big)\\ \in\operatorname{span}\Big(\phi_P(T)\cup\big(\{0\}\times\operatorname{mul}\phi_N(T)\big)\Big)\end{aligned}$$

The reverse inclusion '$\supseteq$' in (i) is obvious.

For the proof of (ii) write $M=\begin{pmatrix}\alpha&\beta\\\gamma&\delta\end{pmatrix}$, $N=\begin{pmatrix}\gamma&\delta\\\lambda&\mu\end{pmatrix}$, and set $P:=\begin{pmatrix}\alpha&\beta\\\lambda&\mu\end{pmatrix}$. If $(x,y)\in\phi_P(T)$, then there exists $(f,g)\in T$ with

$$(x,y)=(\lambda g+\mu f,\alpha g+\beta f)\,.$$

Thus, setting $z:=\gamma g+\delta f$, we have

$$(x,z)=(\lambda g+\mu f,\gamma g+\delta f)\in\phi_N(T),\quad (z,y)=(\gamma g+\delta f,\alpha g+\beta f)\in\phi_M(T)\,.$$

We see that $\phi_P(T)\subseteq\phi_M(T)\circ\phi_N(T)$. By the computation rule 7.1.7, (ii), we also have

$$\{0\}\times\operatorname{mul}\phi_M(T)\subseteq\phi_M(T)\circ\phi_N(T),\quad \ker\phi_N(T)\times\{0\}\subseteq\phi_M(T)\circ\phi_N(T)\,.$$

Together, this shows that the inclusion '$\supseteq$' in the desired equality holds. To see the reverse inequality, assume that $(x,y)\in\phi_M(T)\circ\phi_N(T)$. Then there exist $(f_1,g_1),(f_2,g_2)\in T$ with

$$x=\lambda g_1+\mu f_1,\quad \gamma g_1+\delta f_1=\gamma g_2+\delta f_2,\quad y=\alpha g_2+\beta f_2\,.$$

Set

$$x':=\lambda g_2+\mu f_2,\quad y':=\alpha g_1+\beta f_1\,,$$

then $(x,y')\in\phi_P(T)$, $(0,y-y')\in\phi_M(T)$, and $(x',y)\in\phi_P(T)$, $(x-x',0)\in\phi_N(T)$. Thus we also have the inequality '$\subseteq$'. ❑

We can lift the assignment $M\mapsto\phi_M(T)$ to a functional calculus for fractional linear transformations. First some notation. If X and Y are analytic manifolds, we denote by $\mathbb{H}(X,Y)$ the set of all analytic maps of X into Y. Recall that the composition of analytic maps is again analytic.

The set $\mathbb{H}(X,\mathbb{C})$ is nothing else but the set of all analytic functions defined on X; and we will write $\mathbb{H}(X)$ instead of $\mathbb{H}(X,\mathbb{C})$. Recall that $\mathbb{H}(X)$ becomes a $\mathbb{C}$-algebra if endowed with the pointwise defined algebraic operations, and can be endowed with a complete metric which induces the topology of locally uniform convergence.

A first example of an analytic manifold (not equal to an open subset of $\mathbb{C}$), and we will almost exclusively be concerned with this example, is the one-point compactification $\mathbb{C}_\infty:=\mathbb{C}\cup\{\infty\}$ of $\mathbb{C}$. It becomes an analytic manifold when endowed with the analytic structure comprised of the charts

$$\phi_1:\left\{\begin{array}{ccc}\mathbb{C}_\infty\setminus\{\infty\}&\to&\mathbb{C}\\ z&\mapsto&z\end{array}\right.\qquad \phi_2:\left\{\begin{array}{ccc}\mathbb{C}_\infty\setminus\{0\}&\to&\mathbb{C}\\ z&\mapsto&\begin{cases}\frac{1}{z}&z\neq\infty\\ 0&z=\infty\end{cases}\end{array}\right. \tag{7.2.5}$$ I15

If G is an open subset of the complex plane, then $\mathbb{H}(G, \mathbb{C}_\infty)$ is nothing else but the set of all meromorphic functions on G. Moreover, let us note that $\mathbb{H}(\mathbb{C}_\infty, \mathbb{C}_\infty)$ is equal to the set of all rational functions (extended to maps from $\mathbb{C}_\infty$ to itself in the usual way).

For $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}(2, \mathbb{C})$, we denote by $\phi_M : \mathbb{C}_\infty \to \mathbb{C}_\infty$ the *fractional linear transformation*

$$\phi_M(z) := \begin{cases} \frac{\alpha z+\beta}{\gamma z+\delta} & z \neq \infty, \gamma z + \delta \neq 0 \\ \infty & z \neq \infty, \gamma z + \delta = 0 \text{ or } z = \infty, \gamma = 0 \\ \frac{\alpha}{\gamma} & z = \infty, \gamma \neq 0 \end{cases}$$

The assignment $M \mapsto \phi_M$ is a homomorphism of the group $\mathrm{GL}(2, \mathbb{C})$ onto the group of invertible elements $\mathbb{H}(\mathbb{C}_\infty, \mathbb{C}_\infty)^*$ of the semigroup $\langle \mathbb{H}(\mathbb{C}_\infty, \mathbb{C}_\infty), \circ \rangle$. Its kernel is equal to $\{\lambda I : \lambda \in \mathbb{C}\}$.

If $T \in \mathrm{LR}(\mathcal{L})$, we thus have a homomorphism of $\mathbb{H}(\mathbb{C}_\infty, \mathbb{C}_\infty)^*$ into $\mathrm{LR}(\mathcal{L})$, remember (7.2.2):

$$\begin{tikzcd} & \mathrm{GL}(2,\mathbb{C}) \arrow[dl, "M \mapsto \phi_M(z)"'] \arrow[dr, "M \mapsto \phi_M(T)"] & \\ \mathbb{H}(\mathbb{C}_\infty, \mathbb{C}_\infty)^* \arrow[rr, dashed, "\phi_M(z) \mapsto \phi_M(T)"'] & & \mathrm{LR}(\mathcal{L}) \end{tikzcd}$$

The set $\mathbb{H}(\mathbb{C}_\infty, \mathbb{C}_\infty)^*$ is not closed with respect to pointwise sums and products. In fact, for $M, N \in \mathrm{GL}(2, \mathbb{C})$,

$$\phi_M + \phi_N \in \mathbb{H}(\mathbb{C}_\infty, \mathbb{C}_\infty)^* \iff$$
$$(0,1)M, (0,1)N \text{ lin.dep.}, \ (1,0)(M+N), (0,1)M \text{ lin.indep.}$$

and in this case

$$\phi_M + \phi_N = \phi_{\left(\begin{smallmatrix} (1,0)(\lambda M+N) \\ (1,0)N \end{smallmatrix}\right)}$$

where $\lambda \in \mathbb{C}$ is such that $\lambda(0,1)M = (0,1)N$.

$$\phi_M \cdot \phi_N \in \mathbb{H}(\mathbb{C}_\infty, \mathbb{C}_\infty)^* \iff$$
$$\begin{aligned} \text{either} \quad & (0,1)M, (1,0)N \text{ lin.dep.}, \ (1,0)M, (0,1)N \text{ lin.indep.} \\ \text{or} \quad & (1,0)M, (0,1)N \text{ lin.dep.}, \ (0,1)M, (1,0)N \text{ lin.indep.} \end{aligned}$$

and in this case

$$\phi_M \cdot \phi_N = \phi_{\left(\begin{smallmatrix} \lambda(1,0)M \\ (0,1)N \end{smallmatrix}\right)} \text{ or } \phi_M \cdot \phi_N = \phi_{\left(\begin{smallmatrix} (1,0)N \\ \lambda(0,1)M \end{smallmatrix}\right)}$$

where $\lambda \in \mathbb{C}$ is such that $\lambda(0,1)M = (1,0)N$ or $\lambda(1,0)M = (0,1)N$, respectively.

In these cases, the functional calculus $\phi_M(z) \mapsto \phi_M(T)$ is compatible in the sense of Lemma 7.2.2.

7.3 Resolvent and spectrum

Let $T \in \mathrm{LR}(\mathcal{L})$. Then T is called *resolvable*, if T^{-1} is an everywhere defined operator, i.e. if $\operatorname{dom} T^{-1} = \mathcal{L}$ and $\operatorname{mul} T^{-1} = \{0\}$. It is practical to note that, by the computation rules 7.1.6, (iii), a linear relation T is resolvable if and only if

$$T^{-1}T \subseteq I_{\mathcal{L}} \subseteq TT^{-1}.$$

DEI33 **7.3.1 Definition.** Let $T \in \mathrm{LR}(\mathcal{L})$ and $z \in \mathbb{C}_\infty$. Then we say that z belongs to the *resolvent set* $\rho(T)$ of T, if either $z \in \mathbb{C}$ and $(T - z)$ is resolvable or $z = \infty$ and T^{-1} is resolvable. The assignment '$z \mapsto (T-z)^{-1}$' maps $\rho(T)$ into the subset of $\mathrm{LR}(\mathcal{L})$ consisting of all everywhere defined linear operators, and is called the *resolvent* of T. //

The definition of $\rho(T)$ can be reformulated immediately as

$$z \in \rho(T) \iff \begin{cases} \operatorname{ran}(T-z) = \mathcal{L} \text{ and } \ker(T-z) = \{0\}, & z \in \mathbb{C} \\ \operatorname{dom} T = \mathcal{L} \text{ and } \operatorname{mul} T = \{0\} & , \quad z = \infty \end{cases}$$

$$\iff \begin{cases} (T-z)^{-1}(T-z) \subseteq I_{\mathcal{L}} \subseteq (T-z)(T-z)^{-1}, & z \in \mathbb{C} \\ TT^{-1} \subseteq I_{\mathcal{L}} \subseteq T^{-1}T & , \quad z = \infty \end{cases}$$

Finally, note that $\ker(T-z)^{-1}$ and $\operatorname{ran}(T-z)^{-1}$ do not depend on z. In fact,

$$\ker(T-z)^{-1} = \operatorname{mul} T, \ \operatorname{ran}(T-z)^{-1} = \operatorname{dom} T, \quad z \in \mathbb{C}.$$

It is an important fact that an operator valued function is the resolvent of a linear relation if and only if it satisfies a functional equation.

PRI35 **7.3.2 Proposition.** *Let $T \in \mathrm{LR}(\mathcal{L})$, and denote by $R(z) := (T-z)^{-1}$, $z \in \rho(T)$, the resolvent of T. Then the* resolvent identity

$$R(z) - R(w) = (z-w)R(z)R(w), \quad z, w \in \rho(T) \cap \mathbb{C} \tag{7.3.1}$$ I37

holds. In particular, $R(z)$ and $R(w)$ commute.

Conversely, assume that $D \subseteq \mathbb{C}$ is nonempty, and $R : D \to \mathrm{LR}(\mathcal{L})$ is a function whose values are everywhere defined operators and which satisfies (7.3.1) for all $z, w \in D$. Then, for all $z, w \in D$, the operators $(I + (z-w)R(z))$ and $(I + (w-z)R(w))$ are mutually inverse bijections of $\mathcal{L}$ onto itself. There exists a linear relation T with $\rho(T) \supseteq D$ and $R(z) = (T-z)^{-1}$, $z \in D$. This relation is uniquely determined by the facts that $\rho(T) \cap D \neq \emptyset$ and $(T-z)^{-1} = R(z)$, $z \in \rho(T) \cap D$.

Proof. Let $T \in \mathrm{LR}(\mathcal{L})$ and $z, w \in \rho(T)$, $z \neq w$, be given. Set

$$M := \begin{pmatrix} 0 & 1 \\ 1 & -z \end{pmatrix}, \quad N := \begin{pmatrix} 1 & -z \\ 1 & -w \end{pmatrix},$$

then $\phi_M(T) = R(z)$ and, using Lemma 7.2.1, $\phi_N(T) = I + (w-z)R(w)$. Since $\operatorname{mul} R(z) = \{0\}$, Lemma 7.2.2 gives

$$R(z)\big(I + (w-z)R(w)\big) = \phi_{\begin{pmatrix} 0 & 1 \\ 1 & -w \end{pmatrix}}(T) = R(w).$$

Since $R(z)$ and $R(w)$ are everywhere defined operators, (7.3.1) follows.

Assume now that $R : D \to \mathrm{LR}(\mathcal{L})$ is given and subject to the stated conditions. Since $R(z)$ is always an everywhere defined operator, we can compute

$$\big(I + (z-w)R(z)\big)\big(I + (w-z)R(w)\big) =$$

$$= I + (z-w)R(z) + (w-z)R(w) + (z-w)(w-z)R(z)R(w) =$$

$$= I + (z-w)R(z) + (w-z)R(w) + (w-z)\big(R(z) - R(w)\big) = I\,.$$

For $z \in D$, set

$$T_z := z + R(z)^{-1} = \phi_{\begin{pmatrix} z & 1 \\ 1 & 0 \end{pmatrix}}(R(z))\,,$$

then $(T_z - z)^{-1} = \phi_{\begin{pmatrix} 0 & 1 \\ 1 & -z \end{pmatrix}}(T_z) = R(z)$. If $w \in D$, $w \neq z$, then due to (7.3.1) and Lemma 7.2.1 we have

$$R(z) = R(w)\big(I+(z-w)R(z)\big) = R(w)\big(I+(w-z)R(w)\big)^{-1} = \phi_{\begin{pmatrix} 1 & 0 \\ w-z & 1 \end{pmatrix}}\big(R(w)\big)\,.$$

It follows that

$$T_z = \phi_{\begin{pmatrix} z & 1 \\ 1 & 0 \end{pmatrix}}\Big(\phi_{\begin{pmatrix} 1 & 0 \\ w-z & 1 \end{pmatrix}}\big(R(w)\big)\Big) = \phi_{\begin{pmatrix} w & 1 \\ 1 & 0 \end{pmatrix}}\big(R(w)\big) = T_w\,.$$

Hence, a relation T is well-defined by $T := T_z$, $z \in D$. As we already observed, $(T - z)^{-1} = R(z)$, $z \in D$.

The uniqueness assertion is clear, since T can be recovered from $(T - z)^{-1}$ as $T = z + [(T - z)^{-1}]^{-1}$. ❑

COI29 **7.3.3 Corollary.** *Let $T \in \mathrm{LR}(\mathcal{L})$, and let $D \subseteq \mathbb{C}$ with $D \cap \rho(T) \neq \emptyset$. Assume that there exists a function $\tilde{R}$ of D into the set of all everywhere defined operators which satisfies the resolvent identity for all $z, w \in D$, and extends the resolvent of T, i.e. $\tilde{R}(z) = (T-z)^{-1}$, $z \in D\cap\rho(T)$. Then $\rho(T) \supseteq D$ and $\tilde{R}(z) = (T-z)^{-1}$, $z \in D$.*

Proof. By Proposition 7.3.2, there exists a relation $\tilde{T}$ with $\rho(\tilde{T}) \supseteq D$ and $\tilde{R}(z) = (\tilde{T} - z)^{-1}$, $z \in D$. If $z \in D \cap \rho(T)$, thus $(\tilde{T} - z)^{-1} = (T - z)^{-1}$, and hence $\tilde{T} = T$. ❑

We conclude this algebraic discussion of resolvents with showing that resolvability transfers to products.

LEI34 **7.3.4 Lemma.** *Let $T_1, \dots, T_n \in \mathrm{LR}(\mathcal{L})$, and denote by T the product $T := T_1 \circ \ldots \circ T_n$. If each relation T_i is resolvable, then also T has this property. Conversely, if the relations T_i pairwise commute, then T being resolvable implies that each T_i also is.*

Proof. Due to associativity of compositions the case of an arbitrary finite number of factors will follow by induction once the case of two factors has been shown. Assume that T_1 and T_2 are resolvable. Then

$$(T_1T_2)^{-1}(T_1T_2) = T_2^{-1}\underbrace{T_1^{-1}T_1}_{\subseteq I_{\mathcal{L}}}T_2 \subseteq T_2^{-1}T_2 \subseteq I_{\mathcal{L}}\,,$$

$$(T_1T_2)(T_1T_2)^{-1} = T_1 \underbrace{T_2T_2^{-1}}_{\supseteq I_{\mathcal{L}}} T_1^{-1} \supseteq T_1T_1^{-1} \supseteq I_{\mathcal{L}} .$$

For the converse assume that T_1 and T_2 commute. If T_1 is not resolvable, then either $\ker T_1 \neq \{0\}$ or $\operatorname{ran} T_1 \neq \mathcal{L}$. By the composition rules 7.1.7, (*ii*), we have $\ker(T_2T_1) \neq \{0\}$ in the first case, and $\operatorname{ran}(T_1T_2) \neq \mathcal{L}$ in the second. Since T_1 and T_2 commute, in both cases T_1T_2 is not resolvable. If T_2 is not resolvable the same argument applies. ❑

DEI50 **7.3.5 Definition.** Let $T \in \mathrm{LR}(\mathcal{L})$. The complement $\sigma(T) := \mathbb{C}_\infty \setminus \rho(T)$ is called the *spectrum* of T.

A point $z \in \mathbb{C}$ is said to be an *eigenvalue* of T, if $\ker(T - z) \neq \{0\}$; the point $z = \infty$ is said to be such, if $\operatorname{mul}(T) \neq \{0\}$. The set of all eigenvalues of T is called the *point spectrum* of T, and is denoted by $\sigma_p(T)$.

The *root manifold* $E_z(T)$ of T at a point $z \in \mathbb{C}_\infty$ is defined as

$$E_z(T) := \begin{cases} \bigcup_{n\in\mathbb{N}} \ker(T - z)^n , & z \in \mathbb{C} \\ \bigcup_{n\in\mathbb{N}} \operatorname{mul} T^n , & z = \infty \end{cases}$$

An eigenvalue z of T is called *semi-simple*, if $E_z(T) = \ker(T - z)$ in case $z \in \mathbb{C}$ or $E_z(T) = \operatorname{mul} T$ in case $z = \infty$. //

Note that, clearly, $\sigma_p(T) \subseteq \sigma(T)$. Moreover, since $\ker(T - z)^{n+1} \supseteq \ker(T - z)^n$ and $\operatorname{mul} T^{n+1} \supseteq \operatorname{mul} T^n$, $n \in \mathbb{N}$, the root manifold $E_z(T)$ is always a linear subspace of $\mathcal{L}$.

As a first observation, let us show that

$$z \in \sigma_p(T) \iff E_z(T) \neq \{0\} .$$

Thereby the implication '$\Rightarrow$' is trivial. To see the converse, assume that $E_z(T) \neq \{0\}$, i.e. that for some $n \in \mathbb{N}$ we have $\ker(T - z)^{-n} \neq \{0\}$ or $\operatorname{mul} T^n \neq \{0\}$, respectively. Then there exist elements

$$(f_0, f_1), \ldots, (f_{n-1}, f_n) \in T - z, \text{ with } f_0 \neq 0, f_n = 0 ,$$

if $z \in \mathbb{C}$, or

$$(f_0, f_1), \ldots, (f_{n-1}, f_n) \in T, \text{ with } f_0 = 0, f_n \neq 0 ,$$

if $z = \infty$. In the first case, there must exist $i \in \{1, \ldots, n\}$ such that $f_{i-1} \neq 0$ and $f_i = 0$, and we conclude that $\ker(T - z) \neq \{0\}$. In the second case, there exists $i \in \{1, \ldots, n\}$ such that $f_{i-1} = 0$ and $f_i \neq 0$, and thus $\operatorname{mul} T \neq \{0\}$.

A less simple, and very important, result is the *Spectral Mapping Theorem* for fractional linear transformations:

PRI32 **7.3.6 Proposition.** *Let $T \in \mathrm{LR}(\mathcal{L})$ and $M \in \mathrm{GL}(2, \mathbb{C})$. Then*

$$\sigma(\phi_M(T)) = \phi_M(\sigma(T)) .$$

Moreover,

$$E_{\phi_M(z)}\big(\phi_M(T)\big) = E_z(T) ,$$

in particular, $\sigma_p(\phi_M(T)) = \phi_M(\sigma_p(T))$.

Proof. In view of the factorization (7.2.4) of a matrix $M \in \mathrm{GL}(2,\mathbb{C})$, it is sufficient to consider the cases that ($\lambda \in \mathbb{C}\setminus\{0\}$)

$$M_1 = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

We have $\phi_{M_1}(T) = T + \lambda$, and hence

$$\phi_{M_1}(T) - \phi_{M_1}(z) = (T+\lambda) - (z+\lambda) = T - z\,.$$

Thus a point $z \in \mathbb{C}$ belongs to the spectrum of T if and only if $\phi_{M_1}(z)$ belongs to the spectrum of $\phi_{M_1}(T)$. Since either both of T and $T+\lambda$, or non of them, is an everywhere defined operators, we also have $\infty \in \sigma(\phi_{M_1}(T))$ if and only if $\infty \in \sigma(T)$. The equality $E_z(T) = E_{\phi_{M_1}(z)}(\phi_{M_1}(T))$ also follows immediately.

The case of M_2 is similarly simple. We have $\phi_{M_2}(T) = \lambda T$, and hence

$$\phi_{M_2}(T) - \phi_{M_2}(z) = \lambda T - \lambda z = \lambda(T-z)\,.$$

Again, we see that $\sigma(\phi_{M_2}(T)) = \phi_{M_2}(\sigma(T))$ and $E_{\phi_{M_2}(z)}(\phi_{M_2}(T)) = E_z(T)$.

The case of M_3 requires a bit more calculation. Let $z \in \mathbb{C}\setminus\{0\}$, then

$$\big(T^{-1} - \frac{1}{z}\big)^{-1} = \phi_{\left(\begin{smallmatrix} 0 & 1 \\ 1 & -\frac{1}{z} \end{smallmatrix}\right)}\big(T^{-1}\big) = \phi_{\left(\begin{smallmatrix} 0 & 1 \\ 1 & -\frac{1}{z} \end{smallmatrix}\right)\cdot\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)}(T) =$$
$$= \phi_{\left(\begin{smallmatrix} 1 & 0 \\ -\frac{1}{z} & 1 \end{smallmatrix}\right)}(T) = -z - z^2(T-z)^{-1}\,.$$

Thus, $(T^{-1} - \frac{1}{z})^{-1}$ is an everywhere defined operator if and only if $(T-z)^{-1}$ is, i.e. $\sigma(T^{-1}) \cap (\mathbb{C}\setminus\{0\}) = [\sigma(T) \cap (\mathbb{C}\setminus\{0\})]^{-1}$. Since $\ker(T-z) = \mathrm{mul}(-z - z^2(T-z)^{-1})$, this relation also implies that

$$\ker\big(T^{-1} - \frac{1}{z}\big) = \ker(T-z)\,.$$

In order to show equality of root manifolds, we use induction on n to show that $\ker(T^{-1} - \frac{1}{z})^n = \ker(T-z)^n$, $n \in \mathbb{N}$. Let $n > 1$ and $f \in \ker(T^{-1} - \frac{1}{z})^n$ be given. Then there exists $g \in \ker(T^{-1} - \frac{1}{z})^{n-1}$ with $(f,g) \in T^{-1} - \frac{1}{z}$. It follows by the inductive hypothesis that $g \in \ker(T-z)^{n-1}$, and by the above computation $(\frac{f+zg}{-z^2}, g) \in T - z$. Thus $f + zg \in \ker(T-z)^n$, and hence also $f \in \ker(T-z)^n$. The reverse inclusion follows in the same way.

It remains to consider the case that $z = 0$ or $z = \infty$. However, $\infty \in \rho(T)$ just means that T is an everywhere defined operator, and this is nothing else but $0 \in \rho(T^{-1})$. The same argument applies with T^{-1} in place in T, and hence we have $0 \in \sigma(T^{-1})$ if and only if $\infty \in \sigma(T)$ and $\infty \in \sigma(T^{-1})$ if and only if $0 \in \sigma(T)$. The assertion on root manifolds follows equally simple. We have

$$E_0(T) = \bigcup_{n\in\mathbb{N}} \ker T = \bigcup_{n\in\mathbb{N}} \mathrm{mul}(T^{-1}) = E_\infty(T^{-1})\,,$$

and, using T^{-1} in place of T, $E_\infty(T) = E_0(T^{-1})$. ❑

In the analysis of a linear relation, the notion of invariant subspaces or reducing decompositions is of importance, since it allows to split the given relation in smaller parts.

DEI26 **7.3.7 Definition.** Let $\mathcal{L}$ be a linear space, and $\mathcal{M}_1, \ldots, \mathcal{M}_n$ be linear subspaces of $\mathcal{L}$ with $\mathcal{L} = \mathcal{M}_1 \dot{+} \ldots \dot{+} \mathcal{M}_n$. Moreover, let $T \in \mathrm{LR}(\mathcal{L})$. Then we say that the decomposition $\mathcal{L} = \mathcal{M}_1 \dot{+} \ldots \dot{+} \mathcal{M}_n$ *reduces* T, if

$$T = \big(T \cap \mathcal{M}_1^2\big) \dot{+} \ldots \dot{+} \big(T \cap \mathcal{M}_n^2\big) . \tag{7.3.2}$$ I28

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Note that, trivially, the inclusion '$\supseteq$' in (7.3.2) always holds. Moreover, if $\infty \in \rho(T)$, then (7.3.2) is equivalent to

$$T(\mathcal{M}_i) \subseteq \mathcal{M}_i, \quad i = 1, \ldots, n .$$

PRI27 **7.3.8 Proposition.** *Let $\mathcal{L}$ be a linear space, and $\mathcal{M}_1, \ldots, \mathcal{M}_n$ be linear subspaces of $\mathcal{L}$ with $\mathcal{L} = \mathcal{M}_1 \dot{+} \ldots \dot{+} \mathcal{M}_n$. Moreover, let $T \in \mathrm{LR}(\mathcal{L})$. Then the following are equivalent:*

(*i*) *The decomposition $\mathcal{L} = \mathcal{M}_1 \dot{+} \ldots \dot{+} \mathcal{M}_n$ reduces T.*

(*ii*) *For all $z \in \rho(T) \cap \mathbb{C}$ we have $(T - z)^{-1}\mathcal{M}_i \subseteq \mathcal{M}_i$, $i = 1, \ldots, n$.*

(*iii*) *There exists $z \in \rho(T) \cap \mathbb{C}$ with $(T - z)^{-1}\mathcal{M}_i \subseteq \mathcal{M}_i$, $i = 1, \ldots, n$.*

In this case, we have

$$\rho(T) = \bigcap_{i=1}^{n} \rho\big(T \cap \mathcal{M}_i^2\big) .$$

Proof. Assume that $\mathcal{L} = \mathcal{M}_1 \dot{+} \ldots \dot{+} \mathcal{M}_n$ reduces T, and let $z \in \rho(T) \cap \mathbb{C}$. Let $f \in \mathcal{M}_j$, and set $g := (T - z)^{-1} f$. Write $g = \sum_{i=1}^n g_i$ with $g_i \in \mathcal{M}_i$, and set $f_i := 0$, $i \neq j$, and $f_j := f$. We have $(g, f + zg) \in T$, and hence $(g, f + zg) = \sum_{i=1}^n (h_i, k_i)$ with $(h_i, k_i) \in T \cap \mathcal{M}_i^2$. It follows that $h_i = g_i$ and $k_i = f_i + zg_i$. For $i \neq j$, we have $f_i = 0$, and hence $k_i = zg_i$. This implies that $(g_i, zg_i) \in T$, and since $z \in \rho(T)$ therefore $g_i = 0$. We see that $g \in \mathcal{M}_j$.

The implication '(*ii*) $\Rightarrow$ (*iii*)' is trivial. Assume that (*iii*) holds, and pick $z \in \rho(T) \cap \mathbb{C}$ with $(T - z)^{-1}\mathcal{M}_i \subseteq \mathcal{M}_i$, $i = 1, \ldots, n$. We need to show the inclusion '$\subseteq$' in (7.3.2). Let $(f, g) \in T$ be given, and write $f = \sum_{i=1}^n f_i$, $g = \sum_{i=1}^n g_i$ with $f_i, g_i \in \mathcal{M}_i$. We have $(g - zf, f) \in (T - z)^{-1}$, and hence

$$f = (T - z)^{-1}(g - zf) = \sum_{i=1} (T - z)^{-1}(g_i - zf_i) .$$

Since $(T - z)^{-1}(g_i - zf_i) \in \mathcal{M}_i$, this implies that $(T - z)^{-1}(g_i - zf_i) = f_i$. In other words, $(g_i - zf_i, f_i) \in (T - z)^{-1}$ or $(f_i, g_i) \in T$. We see that

$$(f, g) = \sum_{i=1}^{n} (f_i, g_i) \in \big(T \cap \mathcal{M}_1^2\big) \dot{+} \ldots \dot{+} \big(T \cap \mathcal{M}_n^2\big) .$$

We have shown '(*iii*) $\Rightarrow$ (*i*)', and hence established the equivalence of the conditions (*i*)–(*iii*).

Finally, if $\mathcal{L} = \mathcal{M}_1 \dot{+} \ldots \dot{+} \mathcal{M}_n$ reduces T, then we have

$$(T - z)^{-1} = \big(T \cap \mathcal{M}_1^2 - z\big)^{-1} \dot{+} \ldots \dot{+} \big(T \cap \mathcal{M}_n^2 - z\big)^{-1}, \quad z \in \mathbb{C} ,$$

and hence $(T - Z)^{-1}$ is an everywhere defined operator if and only of each of $(T \cap \mathcal{M}_i^2 - z)^{-1}$ has this property. This says that $\rho(T) \cap \mathbb{C} = \bigcap_{i=1}^n [\rho(T \cap \mathcal{M}_i^2) \cap \mathbb{C}]$. Due to (7.3.2), T is an everywhere defined operator if and only if each relation $T \cap \mathcal{M}_i^2$ is. ❑

7.4 Adjoints

DEI42 **7.4.1 Definition.** Let $\langle\mathcal{L},[.,.]_{\mathcal{L}}\rangle$ and $\langle\mathcal{M},[.,.]_{\mathcal{M}}\rangle$ be inner product spaces, and let $T\in\mathrm{LR}(\mathcal{L},\mathcal{M})$. Then we define the *adjoint* of T as

$$T^* := \big\{(k,h)\in\mathcal{M}\times\mathcal{L} :\ [f,h]_{\mathcal{L}}-[g,k]_{\mathcal{M}}=0,(f,g)\in T\big\}\,.$$

//

Clearly, T^* is a linear relation of $\mathcal{M}$ into $\mathcal{L}$. If T and T^* are both everywhere defined operators, the definition of T^* reduces to the usual definition of the adjoint operator

$$[Tx,y]_{\mathcal{M}}=[x,T^*y]_{\mathcal{L}},\quad x\in\mathcal{L},y\in\mathcal{M}\,.$$

The following viewpoint on adjoints is practical, since it allows us to use geometry.

REI43 *7.4.2 Remark.* Consider the inner product space $\langle\mathcal{L},[.,.]_{\mathcal{L}}\rangle\times\langle\mathcal{M},-[.,.]_{\mathcal{M}}\rangle$, that is the product space $\mathcal{L}\times\mathcal{M}$ endowed with the difference inner product

$$[\![(f,g),(h,k)]\!]_{\mathcal{L}\times\mathcal{M}} := [f,h]_{\mathcal{L}}-[g,k]_{\mathcal{M}},\quad (f,g),(h,k)\in\mathcal{L}\times\mathcal{M}\,.$$

Explicit notation of $\mathcal{L}$ and $\mathcal{M}$ will be dropped when no confusion is possible. If $T\in\mathrm{LR}(\mathcal{L},\mathcal{M})$, then

$$T^*=\big(T^{[\![\perp]\!]}\big)^{-1}\,.$$

//

NTI44 **7.4.3. *Computation rules. V. Adjoints:***

(*i*) *If $T\subseteq S$, then $T^*\supseteq S^*$. We always have $\mathcal{M}^\circ\times\mathcal{L}^\circ\subseteq T^*$. Moreover, $I^*=\operatorname{span}\{I\cup(\mathcal{L}^\circ\times\mathcal{L}^\circ)\}$.*

(*ii*) *For $M=\begin{pmatrix}\alpha&\beta\\\gamma&\delta\end{pmatrix}\in\mathbb{C}^{2\times 2}$ set $\overline{M}:=\begin{pmatrix}\overline{\alpha}&\overline{\beta}\\\overline{\gamma}&\overline{\delta}\end{pmatrix}$. Then we have $\phi_M(T)^*\supseteq\phi_{\overline{M}}(T^*)$. If $M\in\mathrm{GL}(2,\mathbb{C})$, then equality holds.*

(*iii*) *We have $T\subseteq T^{**}$ and $T^{***}=T^*$.*

(*iv*) *$T^*+S^*\subseteq(T+S)^*$. If $\operatorname{dom}S\supseteq\operatorname{dom}T$ and $\operatorname{dom}S^*=\mathcal{M}$, then equality holds.*

(*v*) *$T^*S^*\subseteq(ST)^*$. If $\operatorname{dom}S\supseteq\operatorname{ran}T$ and $\operatorname{dom}S^*=\mathcal{N}$, then equality holds.*

(*vi*) $\ker T^*=\operatorname{ran}T^\perp$, $\operatorname{mul}T^*=\operatorname{dom}T^\perp$.

Proof. Taking orthogonal complements reverses inclusions, taking inverses preserves them. Hence the first assertion in (*i*) follows. The second one is obvious from the definition of T^*. Moreover, the inclusion '$\supseteq$' in the asserted formula for I^* is clear. To see the reverse inclusion, let $(k,h)\in I^*$ be given. We can write

$$(k,h)=(k,k)+(0,h-k)\,.$$

However, for all $f\in\mathcal{L}$, we have

$$[f,h-k]=[f,h]-[f,k]=0\,,$$

i.e. $h - k \in \mathcal{L}$.

For the proof of (ii), let $M \in \mathbb{C}^{2\times 2}$ be given. Let $(f,g) \in T$ and $(k,h) \in T^*$, then

$$\llbracket(\gamma g + \delta f, \alpha g + \beta f), (\overline{\alpha} h + \overline{\beta} k, \overline{\gamma} h + \overline{\delta} k)\rrbracket = [\gamma g + \delta f, \overline{\alpha} h + \overline{\beta} k] -$$

$$-[\alpha g + \beta f, \overline{\gamma} h + \overline{\delta} k] = (\alpha\delta - \gamma\beta)\big([f,h] - [g,k]\big) = 0\,.$$

This says that

$$\phi_M(T)\llbracket\perp\rrbracket\big(\phi_{\overline{M}}(T^*)\big)^{-1}$$

i.e. $\phi_M(T)^* = (\phi_M(T)^{\llbracket\perp\rrbracket})^{-1} \supseteq \phi_{\overline{M}}(T^*)$.

If $M \in \mathrm{GL}(2,\mathbb{C})$, we may compute

$$T^* = \big[\phi_{M^{-1}}\big(\phi_M(T)\big)\big]^* \supseteq \phi_{\overline{M^{-1}}}\big(\phi_M(T)^*\big)\,.$$

However, $\overline{M^{-1}} = (\overline{M})^{-1}$, and it follows that $\phi_{\overline{M}}(T^*) \supseteq \phi_M(T)^*$.

We come to the proof of (iii). We have $(T^{\llbracket\perp\rrbracket})^{-1} = (T^{-1})^{\llbracket\perp\rrbracket}$, and hence

$$T \subseteq \big(T^{\llbracket\perp\rrbracket}\big)^{\llbracket\perp\rrbracket} = T^{**}, \quad T^{\llbracket\perp\rrbracket} = T^{\llbracket\perp\rrbracket\llbracket\perp\rrbracket\llbracket\perp\rrbracket} = \big(T^{***}\big)^{-1}\,.$$

We turn to (iv).

$$\begin{aligned}(T+S)^* &= \Big\{(k,h) \in \mathcal{M}\times\mathcal{L} :\ \forall\,(f,g) \in T+S :\ [f,h] - [g,k] = 0\Big\} = \\ &= \Big\{(k,h) \in \mathcal{M}\times\mathcal{L} :\ \forall\, f, g_1, g_2 \text{ with } (f,g_1) \in T, (f,g_2) \in S : \\ &\qquad\qquad [f,h] - [g_1,k] - [g_2,k] = 0\Big\}\end{aligned}$$

$$\begin{aligned}T^* + S^* &= \Big\{(k,h) \in \mathcal{M}\times\mathcal{L} :\ \exists\, h_1, h_2 \in \mathcal{L} \text{ with} \\ &\qquad\qquad (k,h_1) \in T^*, (k,h_2) \in S^*, h = h_1 + h_2\Big\} = \\ &= \Big\{(k,h) \in \mathcal{M}\times\mathcal{L} :\ \exists\, h_1, h_2 \in \mathcal{L} \text{ with } h = h_1 + h_2, \\ &\qquad \forall\,(f_1,g_1) \in T, (f_2,g_2) \in S : [f_1,h_1] - [g_1,k] = 0, [f_2,h_2] - [g_2,k] = 0\Big\}\end{aligned}$$

Let $(k,h) \in T^* + S^*$, and choose h_1, h_2 be as in the above description of this relation. For all f, g_1, g_2 with $(f,g_1) \in T, (f,g_2) \in S$ we thus have $[f,h_1] - [g_1,k] = [f,h_2] - [g_2,k] = 0$. Summing up shows $(k,h) \in (T+S)^*$.

Assume that $\operatorname{dom} S \supseteq \operatorname{dom} T$ and $\operatorname{dom} S^* = \mathcal{M}$, and let $(k,h) \in (T+S)^*$ be given. Choose h_2 with $(k,h_2) \in S^*$, and set $h_1 := h - h_2$. Let $(f,g_1) \in T$, and choose g_2 with $(f,g_2) \in S$. Then we have

$$0 = [f,h] - [g_1,k] - [g_2,k] = [f,h_1] - [g_1,k] + \underbrace{[f,h_2] - [g_2,k]}_{=0},$$

and we see that $(k,h_1) \in T^*$. Thus $(k,h) \in T^* + S^*$.

For the proof of (v) we proceed similarly.

$$\begin{aligned}(ST)^* &= \Big\{(k,j) \in \mathcal{N}\times\mathcal{L} :\ \forall\,(f,l) \in ST :\ [f,j] - [l,k] = 0\Big\} \\ &= \Big\{(k,j) \in \mathcal{N}\times\mathcal{L} :\ \forall\, f, g, l \text{ with } (f,g) \in T, (g,l) \in S : [f,j] - [l,k] = 0\Big\}\end{aligned}$$

$$\begin{aligned}T^*S^* &= \Big\{(k,j)\in\mathcal{N}\times\mathcal{L}:\ \exists\, h\in\mathcal{M} \text{ with } (k,h)\in S^*, (h,j)\in T^*\Big\} = \\ &= \Big\{(k,j)\in\mathcal{N}\times\mathcal{L}:\ \exists\, h\in\mathcal{M} \text{ with} \\ &\qquad \forall\,(f_1,g_1)\in T, (g_2,l_2)\in S : [f_1,j]-[g_1,h]=0, [g_2,h]-[l_2,k]=0\Big\}\end{aligned}$$

Let $(k,j)\in T^*S^*$, and choose h as in the description of this relation. Let f,g,l with $(f,g)\in T$ and $(g,l)\in S$ be given. Then $[f,j]-[g,h]=0$ and $[g,h]-[l,k]=0$. Summing up gives $(k,j)\in(ST)^*$.

Assume that $\operatorname{dom} S\supseteq \operatorname{ran} T$ and $\operatorname{dom} S^*=\mathcal{N}$, and let $(k,j)\in(ST)^*$ be given. Choose h with $(k,h)\in S^*$. Let $(f_1,g_1)\in T$ be given, and choose l_2 such that $(g_1,l_2)\in S$. Then $(f_1,l_2)\in ST$, and hence $[f_1,j]-[l_2,k]=0$. Using $[g_1,h]-[l_2,k]=0$, gives $[f_1,j]-[g_1,h]=0$. We conclude that $(h,j)\in T^*$, and hence that $(k,j)\in T^*S^*$.

For the first assertion in (vi) note that

$$(k,0)\in T^* \iff \forall\,(f,g)\in T : \underbrace{[f,0]}_{=0}-[g,k]=0\,.$$

The second one follows since

$$(0,h)\in T^* \iff \forall\,(f,g)\in T : [f,h]-\underbrace{[g,0]}_{=0}=0\,.$$

❑

The spectra of T and T^* are closely related. In the present, purely algebraic, setting, we have the following result.

LEI45 **7.4.4 Lemma.** *Let $T\in\operatorname{LR}(\mathcal{L})$ and $z,w\in\mathbb{C}$, then*

$$E_z(T)\perp E_w(T^*),\quad z\neq\overline{w}\,.$$

Proof. We have to show that for all $n,m\in\mathbb{N}_0$

$$\ker(T-z)^n\perp\ker(T^*-w)^m,\quad z\neq\overline{w}\,. \tag{7.4.1}$$ I16

We use induction. If one of n or m equals zero, (7.4.1) is trivial since by definition $(T-z)^0=(T^*-w)^0=I$.

Assume now that $n,m\geq 1$ are given. Let $f\in\ker(T-z)^n$ and $g\in\ker(T^*-w)^m$, and choose f_i, g_j, $i,j\in\mathbb{N}$, with $f_i=0$, $i\geq n$, and $g_j=0$, $j\geq m$, such that

$$(f,f_1),(f_i,f_{i+1})\in T-z,\ (g,g_1),(g_j,g_{j+1})\in T^*-w,\quad i,j\in\mathbb{N}\,.$$

Then we have $f_1\in\ker(T-z)^{n-1}$ and $g_1\in\ker(T^*-w)^{m-1}$, and hence $[f,g_1]=[f_1,g]=0$. It follows that

$$0=[f,g_1+wg]-[f_1+zf,g]=(z-w)[f,g]\,,$$

and hence that $[f,g]=0$. ❑

In many situations relations occur which are related with their adjoint.

DEI46 **7.4.5 Definition.** Let $\langle \mathcal{L}, [.,.]_{\mathcal{L}} \rangle$ be an inner product space, let $T \in \mathrm{LR}(\mathcal{L})$, and $M \in \mathrm{GL}(2, \mathbb{C})$. Then we say that T is *M-selfadjoint*, if

$$T^* = \phi_M(T)\,.$$

//

The following simple fact helps to switch between different cases of M-selfadjoint relations, $M \in \mathrm{GL}(2, \mathbb{C})$.

LEI17 **7.4.6 Lemma.** *Let $M, N \in \mathrm{GL}(2, \mathbb{C})$ and let $T \in \mathrm{LR}(\mathcal{L})$. Then T is M-selfadjoint if and only if $\phi_N(T)$ is $\overline{N} M N^{-1}$-selfadjoint.*

Proof. We compute

$$\phi_N(T)^* = \phi_{\overline{N}}(T^*) = \phi_{\overline{N}}\big(\phi_M(T)\big) = \phi_{\overline{N} M N^{-1}}\big(\phi_N(T)\big)\,.$$

□

EXI19 *7.4.7 Example.* Two cases are of particular importance.

(i) A relation $A \in \mathrm{LR}(\mathcal{L})$ is I-selfadjoint, if and only if $A^* = A$. In this case we say that A is *selfadjoint*.

(ii) A relation $U \in \mathrm{LR}(\mathcal{L})$ is $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$-selfadjoint, if and only if $U^* = U^{-1}$. In this case we will that U is *unitary*.

The fractional linear transformation which switches between selfadjoint and unitary relations is known as the *Cayley transform*: For $\mu \in \mathbb{C} \setminus \mathbb{R}$, set $C_\mu := \left(\begin{smallmatrix} 1 & -\overline{\mu} \\ 1 & -\mu \end{smallmatrix}\right)$. We have

$$\overline{C_\mu} \cdot I \cdot (C_\mu)^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,,$$

and hence a relation T is selfadjoint, if and only if $\phi_{C_\mu}(T)$ is unitary. //

7.5 Linear Relations in a Banach space

DEI1 **7.5.1 Definition.** Let $\mathcal{X}$ and $\mathcal{Y}$ be topological vector spaces. A linear relation $T \in \mathrm{LR}(\mathcal{X}, \mathcal{Y})$ is called *closed*, if it is a closed subspace of $\mathcal{X} \times \mathcal{Y}$ with respect to the product topology. The set of all closed linear relations of $\mathcal{X}$ to $\mathcal{Y}$ will be denoted as $\mathrm{CLR}(\mathcal{X}, \mathcal{Y})$. If $\mathcal{X} = \mathcal{Y}$, we will write $\mathrm{CLR}(\mathcal{X})$ instead of $\mathrm{CLR}(\mathcal{X}, \mathcal{X})$. //

The set of all continuous linear operators of $\mathcal{X}$ into $\mathcal{Y}$ is denoted by $\mathcal{B}(\mathcal{X}, \mathcal{Y})$. Clearly, the graph of a continuous operator is closed, and hence we may regard $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ as a subset of $\mathrm{CLR}(\mathcal{X}, \mathcal{Y})$. Conversely, if $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces, then by the Closed Graph Theorem each $T \in \mathrm{CLR}(\mathcal{X}, \mathcal{Y})$ with $\operatorname{dom} T = \mathcal{X}$ and $\operatorname{mul} T = \{0\}$ belongs to $\mathcal{B}(\mathcal{X}, \mathcal{Y})$.

First we check how algebraic operations are compatible with closure. The symbol 'Clos' thereby always denotes the closure with respect to the appropriate topologies.

NTI2 **7.5.2. *Computation rules. VI. Closure:*** *Let $\mathcal{X}$ and $\mathcal{Y}$ be topological vector spaces, and $T \in \mathrm{LR}(\mathcal{X}, \mathcal{Y})$.*

(*i*) *We have*

$$\mathrm{Clos}(T+S) = (\mathrm{Clos}\, T) + S,\ \ S \in \mathcal{B}(\mathcal{X}, \mathcal{Y})\,, \tag{7.5.1}$$ I3

$$\begin{gathered}\mathrm{Clos}(\lambda{\cdot}T) = \lambda{\cdot}\mathrm{Clos}\, T,\ \ \lambda \in \mathbb{C}\setminus\{0\}, \quad \mathrm{Clos}(T^{-1}) = (\mathrm{Clos}\, T)^{-1}, \\ \phi_M(\mathrm{Clos}\, T) = \mathrm{Clos}\,\phi_M(T),\ \ M \in \mathrm{GL}(2,\mathbb{C}),\ \ \textit{assuming } \mathcal{X} = \mathcal{Y}\,. \end{gathered} \tag{7.5.2}$$ I4

In particular, if $T \in \mathrm{CLR}(\mathcal{X}, \mathcal{Y})$, then the linear relations

$$\begin{gathered} T+S,\ \ S \in \mathcal{B}(\mathcal{X}, \mathcal{Y}), \quad \lambda\cdot T,\ \ \lambda \in \mathbb{C}\setminus\{0\}, \quad T^{-1}, \\ \phi_M(T),\ \ M \in \mathrm{GL}(2,\mathbb{C})\ \ \textit{assuming } \mathcal{X} = \mathcal{Y}\,, \end{gathered}$$

are again closed.

(*ii*) *Let in addition $\mathcal{W}$ be a topological vector space. If $T \in \mathrm{CLR}(\mathcal{X}, \mathcal{Y})$ and $S \in \mathcal{B}(\mathcal{W}, \mathcal{X})$, then also $T \circ S$ is closed. If $T \in \mathrm{LR}(\mathcal{X}, \mathcal{Y})$, then*

$$\mathrm{Clos}(T \circ S) \subseteq (\mathrm{Clos}\, T) \circ S\,, \tag{7.5.3}$$ I5

where equality holds if $\mathrm{Clos}(\mathrm{dom}\, T) \subseteq \mathrm{ran}\, S$ *and* $S^{-1} \in \mathcal{B}(\mathrm{ran}\, S, \mathcal{W})$.

(*iii*) *Let in addition $\mathcal{Z}$ be a topological vector space. If $T \in \mathrm{CLR}(\mathcal{X}, \mathcal{Y})$ and $S \in \mathrm{LR}(\mathcal{Y}, \mathcal{Z})$ with $S^{-1} \in \mathcal{B}(\mathcal{Z}, \mathcal{Y})$, then $S \circ T$ is closed.*

Proof. We start with showing (7.5.1). For $S \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ define a map

$$\tau_S : \left\{ \begin{array}{ccc} \mathcal{X}\times\mathcal{Y} & \to & \mathcal{X}\times\mathcal{Y} \\ (x,y) & \mapsto & (x, y+Sx) \end{array} \right.$$

Then τ_S is continuous. Moreover, $\tau_S \circ \tau_{-S} = \tau_{-S} \circ \tau_S = \mathrm{id}$, and hence τ_S is a homeomorphism. However, $\tau_S(T) = T+S$, and hence

$$\mathrm{Clos}(T+S) = \mathrm{Clos}(\tau_S(T)) = \tau_S(\mathrm{Clos}\, T) = (\mathrm{Clos}\, T) + S\,.$$

The assertions in (7.5.2) are proved with exactly the same argument, using the homeomorphisms

$$\tau_\lambda : \left\{ \begin{array}{ccc} \mathcal{X}\times\mathcal{Y} & \to & \mathcal{X}\times\mathcal{Y} \\ (x,y) & \mapsto & (x, \lambda y) \end{array} \right. ,\ \lambda \in \mathbb{C}\setminus\{0\}, \quad \tau : \left\{ \begin{array}{ccc} \mathcal{X}\times\mathcal{Y} & \to & \mathcal{Y}\times\mathcal{X} \\ (x,y) & \mapsto & (y,x) \end{array} \right. ,$$

$$\tau_M : \left\{ \begin{array}{ccc} \mathcal{X}\times\mathcal{X} & \to & \mathcal{X}\times\mathcal{X} \\ (x,y) & \mapsto & (\gamma y + \delta x, \alpha y + \beta x) \end{array} \right. , \quad M = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{GL}(2,\mathbb{C})\,.$$

Next, let $S \in \mathcal{B}(\mathcal{W}, \mathcal{X})$ be given. We consider the map

$$\tau^S : \left\{ \begin{array}{ccc} \mathcal{W}\times\mathcal{Y} & \to & \mathcal{X}\times\mathcal{Y} \\ (w,y) & \mapsto & (Sw, y) \end{array} \right. \tag{7.5.4}$$ I6

Then τ^S is continuous. We have

$$\begin{aligned} T \circ S &= \big\{(w,y) \in \mathcal{W}\times\mathcal{Y} :\ \exists\, x \in \mathcal{X} : (z,x) \in S, (x,y) \in T\big\} = \\ &= \big\{(w,y) \in \mathcal{W}\times\mathcal{Y} :\ (Sx, y) \in T\big\} = (\tau^S)^{-1}(T)\,. \end{aligned} \tag{7.5.5}$$ I13

Hence, T being closed implies that $T\circ S$ is closed. If T is arbitrary, thus the right side of (7.5.3) is a closed set which contains $T\circ S$, and hence also $\operatorname{Clos}(T\circ S)$.

If $S^{-1}\in\mathcal{B}(\operatorname{ran} S,\mathcal{W})$, then S is a homeomorphism of W onto $\operatorname{ran} S$. If $\operatorname{Clos}(\operatorname{dom} T)\subseteq\operatorname{ran} S$, we certainly may consider T as a linear relation of $\operatorname{ran} S$ to Y. Let τ^S be the map (7.5.4) where S is considered as a linear relation of $\mathcal{W}$ into $\operatorname{ran} S$ instead of $\mathcal{X}$. Then it has an inverse, namely

$$\tau^{S^{-1}}:\begin{cases}\operatorname{ran} S\times\mathcal{Y} & \to \quad \mathcal{W}\times\mathcal{Y}\\ \quad (x,y) & \mapsto \quad (S^{-1}x,y)\end{cases}$$

and this map is also continuous. By the computation (7.5.5), we have $T\circ S=(\tau^S)^{-1}(T)=\tau^{S^{-1}}(T)$, and hence

$$\operatorname{Clos}(T\circ S)=\operatorname{Clos}(\tau^{S^{-1}}(T))=\tau^{S^{-1}}(\operatorname{Clos} T)=(\operatorname{Clos} T)\circ S\,.$$

Note here that $\operatorname{dom}\operatorname{Clos} T\subseteq\operatorname{Clos}(\operatorname{dom} T)\subseteq\operatorname{ran} S$, and hence the application of $\tau^{S^{-1}}$ is possible.

Finally, let $S\in\operatorname{LR}(\mathcal{Y},\mathcal{Z})$ with $S^{-1}\in\mathcal{B}(\mathcal{Z},\mathcal{Y})$ and $T\in\operatorname{CLR}(\mathcal{X},\mathcal{Y})$ be given. Then T^{-1} is closed, and hence also $T^{-1}\circ S^{-1}$ is. This, in turn, implies that $S\circ T=(T^{-1}\circ S^{-1})^{-1}$ is closed. ❑

Let us note that the case $\lambda=0$ really needs to be excluded in (7.5.2): We have

$$\operatorname{Clos}(0\cdot T)=\operatorname{Clos} 0_{\operatorname{dom} T}=0_{\operatorname{Clos}(\operatorname{dom} T)},\quad 0\cdot\operatorname{Clos} T=0_{\operatorname{dom}\operatorname{Clos} T}\,,$$

and $\operatorname{dom}\operatorname{Clos} T\subseteq\operatorname{Clos}(\operatorname{dom} T)$, but equality need not hold.

Next, we turn to resolvent and spectrum of a closed linear relation in a Banach space. Let $\mathcal{X}$ be a Banach space, $T\in\operatorname{CLR}(\mathcal{X})$, and $z\in\mathbb{C}_\infty$. Then, by the Closed Graph Theorem,

$$z\in\rho(T)\iff\begin{cases}(T-z)^{-1}\in\mathcal{B}(\mathcal{X}), & z\in\mathbb{C}\\ T\in\mathcal{B}(\mathcal{X}), & z=\infty\end{cases}$$

For a closed subset K of $\mathbb{C}_\infty$ and $z\in\mathbb{C}$, we set

$$d(z,K):=\inf_{w\in K}|z-w|\,,$$

where we understand $|z-\infty|:=\infty$.

PRI14 **7.5.3 Proposition.** *Let $\mathcal{X}$ be a Banach space and $T\in\operatorname{CLR}(\mathcal{X})$. Then $\rho(T)$ is an open subset of $\mathbb{C}_\infty$. We have*

$$\|(T-z)^{-1}\|\geq d(z,\sigma(T))^{-1},\quad z\in\rho(T)\cap\mathbb{C}\,.$$

The resolvent $z\mapsto(T-z)^{-1}$ is an analytic function of $\rho(T)\cap\mathbb{C}$ into $\mathcal{B}(\mathcal{X})$. If $\infty\in\rho(T)$, then $\lim_{|z|\to\infty}z(T-z)^{-1}=-I$.

Proof. First a preparatory observation: Let $w,z\in\mathbb{C}$, and set $M:=\begin{pmatrix}1&0\\ w-z&1\end{pmatrix}$. By the second part of Lemma 7.2.1 and the Closed Graph Theorem,

$$\phi_M\Big(\big\{T\in\mathcal{B}(\mathcal{X}):\,\|T\|<\frac{1}{|w-z|}\big\}\Big)\subseteq\mathcal{B}(\mathcal{X})\,.$$

Let $w \in \rho(T) \cap \mathbb{C}$ be given, then we have

$$(T-z)^{-1} = \phi_{\begin{pmatrix} 0 & 1 \\ 1 & -z \end{pmatrix}}(T) = \phi_{\begin{pmatrix} 1 & 0 \\ w-z & 1 \end{pmatrix}}\Big(\phi_{\begin{pmatrix} 0 & 1 \\ 1 & -w \end{pmatrix}}(T)\Big) = \phi_{\begin{pmatrix} 1 & 0 \\ w-z & 1 \end{pmatrix}}\big((T-w)^{-1}\big)$$

Hence, $|w-z| < \|(T-w)^{-1}\|^{-1}$ implies that $(T-z)^{-1} \in \mathcal{B}(\mathcal{X})$. In other words, $\rho(T)$ contains the disk centered at w with radius $\|(T-w)^{-1}\|^{-1}$. This shows that w is an inner point of $\rho(T)$, and that the distance of w to $\sigma(T)$ is at least equal to $\|(T-w)^{-1}\|^{-1}$.

The point $w = \infty$ belongs to $\rho(T)$ if and only if T is a bounded operator. In this case the exterior of the disk centered at 0 with radius $\|T\|$ entirely belongs to $\rho(T)$. We see that again w is an inner point of $\rho(T)$.

The resolvent of T depends analytically on z since it satisfies the resolvent identity. In fact,

$$\frac{d}{dz}(T-z)^{-1} = (T-z)^{-2}, \quad z \in \rho(T) \cap \mathbb{C}.$$

If $T \in \mathcal{B}(\mathcal{X})$, we have the *Neumann series*

$$(T-z)^{-1} = -\sum_{k=0}^{\infty} \frac{1}{z^{k+1}} T^k, \quad |z| > \|T\|,$$

and hence the stated limit relation follows. ❏

7.6 Linear relations in a Krein space

If T is a linear relation in a Krein space, we can say much more about the relation between T and T^*. First, a preparatory observation.

LEI21 **7.6.1 Lemma.** *Let $\mathcal{K}$ be a Krein space, and $T \in \mathrm{LR}(\mathcal{K})$. Then $T^{**} = \operatorname{Clos} T$.*

Proof. With $\mathcal{K}$ also the product space $\mathcal{K} \times \mathcal{K}$ endowed with the difference inner product $[\![.,.]\!]$ is a Krein space. Moreover, it carries the product topology. Hence,

$$T^{**} = T^{[\![\perp]\!][\![\perp]\!]} = \operatorname{Clos} T.$$

❏

We extend complex conjugation to an involution

$$\bar{.} : \mathbb{C}_\infty \to \mathbb{C}_\infty$$

by setting $\overline{\infty} := \infty$.

PRI18 **7.6.2 Proposition.** *Let $\mathcal{K}$ be a Krein space and $T \in \mathrm{CLR}(\mathcal{K})$. Then*

$$\sigma(T^*) = \big\{ z \in \mathbb{C}_\infty : \overline{z} \in \sigma(T) \big\}.$$

Proof. First of all note that, since $T^{**} = T$, it is enough to show $\{\overline{z} \in \mathbb{C}_\infty : z \in \rho(T)\} \subseteq \rho(T^*)$.

In the first step we settle the case $z = \infty$. Assume that $\infty \in \rho(T)$, i.e. $T \in \mathcal{B}(\mathcal{K})$. Let $\mathfrak{J}$ be a fundamental decomposition of $\mathcal{K}$, and set $S := JT^{(*)_{\mathfrak{J}}}J \in \mathcal{B}(\mathcal{K})$. Then

$$[Sx, y] = \big[JT^{(*)_{\mathfrak{J}}}Jx, y\big] = \big(T^{(*)_{\mathfrak{J}}}Jx, y\big)_{\mathfrak{J}} = \big(Jx, Ty\big)_{\mathfrak{J}} = [x, Ty], \quad x, y \in \mathcal{K},$$

and hence $S \subseteq T^{[*]}$. However, $\operatorname{dom} T = \mathcal{K}$, and hence T^* is an operator. Since $\operatorname{dom} S = \mathcal{K}$, it thus follows that $S = T^{[*]}$. This shows that $\infty \in \rho(T^{[*]})$.

Let $z \in \rho(T) \cap \mathbb{C}$, so that $(T - z)^{-1} \in \mathcal{B}(\mathcal{K})$. Since $[(T-z)^{-1}]^* = (T^* - \overline{z})^{-1}$, the first step gives $(T^* - \overline{z})^{-1} \in \mathcal{B}(\mathcal{K})$. ❑

M-selfadjointness of a relation has consequences for its spectrum.

PRI24 **7.6.3 Proposition.** *Let $\mathcal{K}$ be a Krein space, $M \in \mathrm{GL}(2, \mathbb{C})$, and let T be a closed M-selfadjoint relation in $\mathcal{K}$.*

(*i*) *We have*

$$z \in \sigma(T) \iff \overline{\phi_M(z)} \in \sigma(T).$$

(*ii*) ????????? *If $z \in \sigma_p(T)$ and $\ker(T - z)$ or $\operatorname{mul} T$, respectively, is nondegenerated, then $z = \phi_M^{-1}(\overline{z})$, and z is a semi-simple eigenvalue.*

(*iii*) ????????? *If $z \notin \phi_M(\mathbb{R}_\infty)$, then $\operatorname{span}(E_z(T) \cup E_{\phi_M(\overline{z})}(T))$ is nondegenerated.*

Proof. The first assertion follows immediately by combining Proposition 7.6.2 with the Spectral Mapping Theorem for fractional linear transformations: We have $z \in \sigma(T)$ if and only if $\overline{z} \in \sigma(T^*)$. However, $\sigma(T^*) = \sigma(\phi_M(T)) = \phi_M(\sigma(T))$.

insert proof ❑

normal eigenvalues

Chapter 8

The Riesz-Dunford functional calculus

In the previous chapter we had endowed $\mathbb{C}_\infty$ with the analytic structure $\{\phi_1, \phi_2\}$ where ϕ_j are the charts (7.2.5). A more natural way of considering $\mathbb{C}_\infty$ as an analytic manifold is to enrich this analytic structure by taking all fractional linear transformations as charts. Clearly, the analytic structure $\{\phi_1, \phi_2\}$ is isomorphic to the analytic structure given by the collection of charts

$$\Big\{\phi_M : \mathbb{C}_\infty \setminus \{\phi_M^{-1}(\infty)\} \to \mathbb{C} \text{ with } M \in \mathrm{GL}(2, \mathbb{C})\Big\}.$$

8.1 An algebra of functions

Let K be a nonempty closed (and hence compact) subset of $\mathbb{C}_\infty$, and consider the set $H(K)$ of all functions F which are defined and analytic on some open subset of $\mathbb{C}_\infty$ which contains K, i.e.

$$H(K) := \bigcup_{\substack{O \text{ open} \\ K \subseteq O}} \mathbb{H}(O). \tag{8.1.1}$$ J27

Note that the union (8.1.1) is a disjoint union, since equality of functions includes equality of domains. If $F \in H(K)$, we will generically denote the domain of F by O_F.

On $H(K)$ we define a relation '$\sim$' as

$$F \sim G \quad \Longleftrightarrow \quad \exists\, O \text{ open} : K \subseteq O \subseteq O_F \cap O_G \text{ and } F|_O = G|_O$$

It is obvious that '$\sim$' is an equivalence relation.

DEJ4 **8.1.1 Definition.** Let K be a closed subset of $\mathbb{C}_\infty$. Then we denote by $\mathbb{H}(K)$ the factor set of $H(K)$ with respect to $\sim$, i.e.

$$\mathbb{H}(K) := \Big(\bigcup_{\substack{O \text{ open} \\ K \subseteq O}} \mathbb{H}(O)\Big)\Big/_{\sim}$$

and refer to $\mathbb{H}(K)$ as the *algebra of germs of analytic functions on* K. If $F \in H(K)$, we denote the equivalence class which contains the element F by $\underline{F}$. Moreover, we let $\underline{\pi}$ denote the canonical projection of $H(K)$ onto $\mathbb{H}(K)$. //

NRJ6 *8.1.2. The set* $\mathbb{H}(K)$*. I. Algebraic structure:*

(a) *Algebra operations:* The pointwise algebraic operations on $\mathbb{H}(O)$ give rise to algebraic operations on $H(K)$; one only has to take care of the respective domains. Explicitly, we define ($\lambda \in \mathbb{C}$)

$$+ : \left\{ \begin{array}{rcl} H(K)\times H(K) & \to & H(K) \\ (F,G) & \mapsto & F|_{O_F\cap O_G} + G|_{O_F\cap O_G} \end{array} \right. , \qquad \lambda\cdot : \left\{ \begin{array}{rcl} H(K) & \to & H(K) \\ F & \mapsto & \lambda F \end{array} \right.$$

$$\cdot : \left\{ \begin{array}{rcl} H(K)\times H(K) & \to & H(K) \\ (F,G) & \mapsto & F|_{O_F\cap O_G} \cdot G|_{O_F\cap O_G} \end{array} \right.$$

Clearly,

$$F_1 \sim F_2, G_1 \sim G_2 \quad \Longrightarrow \quad \begin{array}{c} F_1 + F_2 \sim G_1 + G_2 \\ \lambda \cdot F_1 \sim \lambda \cdot F_2 \\ F_1 \cdot F_2 \sim G_1 \cdot G_2 \end{array}$$

and hence algebraic operations '+', '$\lambda\cdot$', and '$\cdot$' are well-defined on $\mathbb{H}(K)$ by

$$\underline{F} + \underline{G} := \underline{F+G}, \quad \lambda \cdot \underline{F} := \underline{\lambda \cdot F}, \quad \underline{F} \cdot \underline{G} := \underline{F \cdot G}, \quad F, G \in H(K), \lambda \in \mathbb{C}\,.$$

It is elementary to check that $\mathbb{H}(K)$ becomes a $\mathbb{C}$-algebra when endowed with these operations; we will not carry out the details.

(b) *The algebra* $\mathbb{H}(K)$ *as a direct limit:* Let us observe that $\mathbb{H}(K)$ is the direct limit of the algebras $\mathbb{H}(O)$: As an index set we take $\{O \subseteq \mathbb{C}_\infty : O \text{ open}, K \subseteq O\}$. This set is directed by set-theoretic inclusion, namely

$$O_1 \preceq O_2 \;:\Longleftrightarrow\; O_1 \supseteq O_2\,,$$

and for each pair $O_1 \preceq O_2$ we have the restriction map

$$\rho_{O_2}^{O_1} : \left\{ \begin{array}{rcl} \mathbb{H}(O_1) & \to & \mathbb{H}(O_2) \\ F & \mapsto & F|_{O_2} \end{array} \right.$$

Clearly, these maps are algebra homomorphisms and satisfy $\rho_{O_3}^{O_2} \circ \rho_{O_2}^{O_1} = \rho_{O_3}^{O_1}$ whenever $O_1 \preceq O_2 \preceq O_3$. Moreover, for each O, we have an algebra homomorphism of $\mathbb{H}(O)$ into $\mathbb{H}(K)$, namely the map $\underline{\pi} \circ \iota_O$ where $\iota_O : \mathbb{H}(O) \to H(K)$ is the set-theoretic inclusion map. Whenever $O_1 \preceq O_2$, these maps satisfy

$$\begin{tikzcd} \mathbb{H}(O_1) \arrow[rr, "\rho_{O_2}^{O_1}"] \arrow[dr, "\underline{\pi}\circ\iota_{O_1}"'] & & \mathbb{H}(O_2) \arrow[dl, "\underline{\pi}\circ\iota_{O_2}"] \\ & \mathbb{H}(K) & \end{tikzcd}$$

It is straightforward to check that for each $\mathbb{C}$-algebra $\mathfrak{A}$ together with algebra homomorphisms $\varphi_O : \mathbb{H}(O) \to \mathfrak{A}$ satisfying $\varphi_{O_2} \circ \rho_{O_2}^{O_1} = \varphi_{O_1}$, $O_1 \preceq O_2$, there exists a unique algebra homomorphism $\psi : \mathbb{H}(K) \to \mathfrak{A}$ with

$$\begin{tikzcd}
\mathbb{H}(O_1) \arrow[rr, "\rho^{O_1}_{O_2}"] \arrow[dr, "\underline{\pi}\circ\iota_{O_1}"] \arrow[ddr, bend right, "\varphi_{O_1}"'] & & \mathbb{H}(O_2) \arrow[dl, "\underline{\pi}\circ\iota_{O_2}"'] \arrow[ddl, bend left, "\varphi_{O_2}"] \\
& \mathbb{H}(K) \arrow[d, dashed, "\psi"] & \\
& \mathfrak{A} &
\end{tikzcd}$$

This, however, is just the defining property of a direct limit. Hence, $\mathbb{H}(K)$ together with the maps $\underline{\pi} \circ \iota_O$ is the direct limit $\varinjlim_O \mathbb{H}(O)$ in the category of $\mathbb{C}$-algebras. Let us note that, also if we consider $\mathbb{H}(O)$ and $\mathbb{H}(K)$ only as linear spaces or merely as sets, still $\mathbb{H}(K) = \varinjlim_O \mathbb{H}(O)$ in the respective category.

(c) *Composition:* Let K be a closed subset of $\mathbb{C}_\infty$, and $D \subseteq \mathbb{C}_\infty$ openwith $K \subseteq D$. Moreover, let $\phi \in \mathbb{H}(D, \mathbb{C}_\infty)$ be injective, and set $\tilde{K} := \phi(K)$, $\tilde{D} := \phi(D)$. Then $\tilde{K}$ is a closed subset of $\mathbb{C}_\infty$, $\tilde{D}$ is an open, and $\tilde{K} \subseteq \tilde{D}$. If $F \in H(\tilde{K})$, the composite $F \circ \phi$ belongs to $H(K)$. Clearly, $F_1 \sim F_2$ implies that $F_1 \circ \phi \sim F_2 \circ \phi$, and hence a map $\circ\phi : \mathbb{H}(\tilde{K}) \to \mathbb{H}(K)$ is well defined by

$$\circ\phi : \underline{F} \mapsto \underline{F \circ \phi}, \quad F \in H(\tilde{K})\,.$$

It is easy to check that $\circ\phi$ is an algebra homomorphism.

The map ϕ^{-1} belongs to $\mathbb{H}(\tilde{D}, D)$, and clearly $\circ\phi^{-1}$ is inverse to $\circ\phi$. Thus we have mutually inverse algebra isomorphisms

$$\begin{tikzcd}
\mathbb{H}(\tilde{K}) \arrow[r, bend left, "\circ\phi"] & \mathbb{H}(K) \arrow[l, bend left, "\circ\phi^{-1}"]
\end{tikzcd}$$

//

NRJ8 *8.1.3. The set* $\mathbb{H}(K)$*. II. Topologically:* As we observed above, $\mathbb{H}(K)$ is as a linear space the direct (or 'inductive') limit of the linear spaces $\mathbb{H}(O)$ coming together with the restriction maps $\rho^{O_1}_{O_2}$. The spaces $\mathbb{H}(O)$ carry a locally convex vector topology, namely the topology of locally uniform convergence. The restriction maps are clearly continuous. Hence, the linear space $\mathbb{H}(K)$ can be topologized naturally. Namely, there exists a finest locally convex vector topology on $\mathbb{H}(K)$ such that all maps $\underline{\pi} \circ \iota_O$ continuous. This topology has the property that a linear map φ of $\mathbb{H}(K)$ into some locally convex vector space X is continuous if and only if all compositions $\varphi \circ \underline{\pi} \circ \iota_O$ are continuous.

Let us remark that, if $O_1 \preceq O_2$ but $O_1 \neq O_2$, the initial topology on $\mathbb{H}(O_1)$ with respect to the map $\rho^{O_1}_{O_2}$ is strictly coarser than the topology of $\mathbb{H}(O_1)$. In the language of topological vector spaces this means that $\mathbb{H}(K)$ is not the strict inductive limit of the spaces $\mathbb{H}(O)$. //

With the topology introduced above, $\mathbb{H}(K)$ is a locally convex vector space. Hausdorff

Next, we show that multiplication with a fixed function and composition are compatible with this topology:

LEJ26 **8.1.4 Lemma.** *Let K be a closed subset of $\mathbb{C}_\infty$.*

(i) *For each fixed $\underline{G} \in \mathbb{H}(K)$ the map*

$$\cdot \underline{G} : \begin{cases} \mathbb{H}(K) & \to \quad \mathbb{H}(K) \\ \quad \underline{F} & \mapsto \quad \underline{F} \cdot \underline{G} \end{cases}$$

is continuous.

(ii) *Let $D \subseteq \mathbb{C}_\infty$ be open with $K \subseteq D$, and let $\phi \in \mathbb{H}(D, \mathbb{C}_\infty)$ be injective. Then the map $\circ\phi : \mathbb{H}(\phi(K)) \to \mathbb{H}(K)$ is a homeomorphism.*

Proof. Let $G \in \mathbb{H}(K)$ be fixed, and let an open set $O \subseteq \mathbb{C}_\infty$ with $K \subseteq O$ be given. Then we have

$$\begin{tikzcd} \mathbb{H}(O) \arrow[r, "\underline{\pi}\circ\iota_O"] \arrow[d, "\rho^O_{O\cap O_G}"'] & \mathbb{H}(K) \arrow[r, "\cdot\underline{G}"] & \mathbb{H}(K) \\ \mathbb{H}(O \cap O_G) \arrow[rr, "\cdot G"'] & & \mathbb{H}(O \cap O_G) \arrow[u, "\underline{\pi}\circ\iota_{O\cap O_G}"'] \end{tikzcd}$$

Thus $(\cdot\underline{G}) \circ (\underline{\pi} \circ \iota_O)$ is a composition of continuous maps, and hence itself continuous. This shows that $\cdot\underline{G}$ is continuous.

Let ϕ be given according to (ii), and let $\tilde{O}$ be an open sert with $\tilde{K} \subseteq \tilde{O}$. If $F_n \to F$ in $\mathbb{H}(\tilde{O})$, then also $F_n \circ \phi \to F \circ \phi$ in $\mathbb{H}(\phi^{-1}(\tilde{O}))$, and hence $\underline{F_n \circ \phi} \to \underline{F \circ \phi}$ in $\mathbb{H}(K)$. This shows that $(\circ\phi) \circ (\underline{\pi} \circ \iota_{\tilde{O}})$ is continuous. Since the same argument applies with $\circ\phi^{-1}$, it follows that $\circ\phi$ is a homeomorphism. ❑

NRJ10 *8.1.5. The set $\mathbb{H}(K)$. III. $\mathbb{H}(K)$ vs. $C(K), \mathbb{C}(z)$:*

(a) *Relation with $C(K)$:* If $F \in H(K)$, then the restriction $F|_K$ is a continuous function on K. Moreover, if $F_1 \sim F_2$, then $F_1|_K = F_2|_K$. Hence, the restriction map of $H(K)$ into $C(K)$ induces a map ρ_K of $\mathbb{H}(K)$ into $C(K)$.

It is clear that ρ_K is an algebra homomorphism. Moreover, it is continuous when $C(K)$ is endowed with the topology of uniform convergence. This follows since locally uniform convergence in $\mathbb{H}(O)$, $K \subseteq O$, implies uniform convergence on K, and hence each map $\rho_K \circ (\underline{\pi} \circ \iota_O)$ is continuous.

(b) *Relation with $\mathbb{C}(z)$:* Denote by $\mathbb{C}(z)$ the set of all rational functions with complex coefficients. If $p \in \mathbb{C}(z)$, then $p \in H(K)$ if and only if p has no poles in K. Since the maximal domain of analyticity of a rational function is always connected, we have $p_1 \sim p_2$ if and only if $p_1(z) = p_2(z)$, $z \in \mathbb{C}_\infty$. Hence, we have an injective embedding of $\{p \in \mathbb{C}(z) : p \text{ no poles on } K\}$ into $\mathbb{H}(K)$, and will via this embedding always consider this set as a subspace of $\mathbb{H}(K)$. Note that, using this abuse of language, we can also write $(\underline{\pi} \circ \iota_O)(\mathbb{C}(z) \cap \mathbb{H}(O)) \subseteq \mathbb{C}(z) \cap \mathbb{H}(K)$.

By Runge's Theorem, $\mathbb{C}(z) \cap \mathbb{H}(O)$ is dense in $\mathbb{H}(O)$. This fact transfers to $\mathbb{H}(K)$: $\mathbb{C}(z) \cap \mathbb{H}(K)$ is a dense subspace of $\mathbb{H}(K)$. To see this, let a nonempty open subset W of $\mathbb{H}(K)$ be given. Then, for each open set $O \subseteq \mathbb{C}_\infty$, $K \subseteq O$, the set $(\underline{\pi} \circ \iota_O)^{-1}(W)$ is open in $\mathbb{H}(O)$. Since $\mathbb{H}(K) = \bigcup_O (\underline{\pi} \circ \iota_O)(\mathbb{H}(O))$, there exists O with $(\underline{\pi} \circ \iota_O)^{-1}(W) \neq \emptyset$. Hence, also $W \cap (\mathbb{C}(z) \cap \mathbb{H}(K)) \neq \emptyset$.

Let us note that, ρ_K maps $\mathbb{C}(z) \cap \mathbb{H}(K)$ onto $\mathbb{C}(z) \cap C(K)$, and that this map is injective if and only if K is infinite. //

NRJ29 *8.1.6. The set* $\mathbb{H}(K)$*. IV. Symmetry:* For any subset V of $\mathbb{C}_\infty$, we define

$$V^\# := \{z \in \mathbb{C}_\infty : \overline{z} \in V\}.$$

Clearly, V being open or closed implies that $V^\#$ has the same property. For a function $F : V \to \mathbb{C}_\infty$, we define a function $F^\# : V^\# \to \mathbb{C}_\infty$ as

$$F^\#(z) := \overline{F(\overline{z})}.$$

If V is open, and F is analytic or meromorphic on V, then also $F^\#$ is analytic or meromorphic, respectively. Note that $(V^\#)^\# = V$ and $(F^\#)^\# = F$.

Let K be a closed subset of $\mathbb{C}_\infty$. Then the map $F \mapsto F^\#$ induces a conjugate linear map of $H(K)$ to $H(K^\#)$. Clearly, $F_1 \sim F_2$ implies that $F_1^\# \circ F_2^\#$, and hence we obtain a conjugate linear map $.^\# : \mathbb{H}(K) \to \mathbb{H}(K^\#)$. Using the explicit description of a neighbourhood of 0 in an inductive limit, it is straightforward to check that this map is continuous. Moreover, it is a homeomorphism; its inverse being given by $.^\# : \mathbb{H}(K^\#) \to \mathbb{H}(K)$.

Denote the closed real line by $\mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$. We will often meet the situation that K is *symmetric with respect to* $\mathbb{R}_\infty$, i.e. that $K = K^\#$. In this case, the map $.^\#$ becomes a homeomorphic conjugate linear involution of $\mathbb{H}(K)$ onto itself.

Using fractional linear transformations, we can also speak of other kinds of symmetry than with respect to $\mathbb{R}_\infty$. Let $M \in \mathrm{GL}(2, \mathbb{C})$ be given. For a set $V \subseteq \mathbb{C}_\infty$ and a function $F : V \to \mathbb{C}_\infty$, we set

$$V^\square := \{z \in \mathbb{C}_\infty : \overline{\phi_M(z)} \in V\} = \phi_M^{-1}(V^\#), \quad F^\square(z) := F^\# \circ \phi_M, \ z \in V^\square.$$

In this way we obtain a conjugate linear and homeomorphic map $.^\square$ of $\mathbb{H}(K)$ onto $\mathbb{H}(K^\square)$. The inverse of the map $.^\square$ constructed with M is given by the map $.^\square$ constructed with $\overline{M}^{-1}$. To see this, notice that $\phi_M^\# = \phi_{\overline{M}}$, and compute

$$\big(F^\# \circ \phi_M\big)^\# \circ \phi_{\overline{M}^{-1}} = (F^\#)^\# \circ \phi_M^\# \circ \phi_{\overline{M}^{-1}} = F \circ \phi_{\overline{M}} \circ \phi_{\overline{M}^{-1}} = F,$$

and, moreover remembering that $\phi_{\overline{M}^{-1}} = \phi_{\overline{M^{-1}}}$,

$$\big(F^\# \circ \phi_{\overline{M}^{-1}}\big)^\# \circ \phi_M = (F^\#)^\# \circ \phi_{M^{-1}} \circ \phi_M = F.$$

If K satisfies $K^\square = K$, we say that K is M*-symmetric*. In this case $.^\square$ becomes a map of $\mathbb{H}(K)$ onto itself. If $M^{-1} = \overline{M}$, then this map is involutory. //

The following result is useful to define elements of $\mathbb{H}(K)$ by pasting single parts.

PRJ9 **8.1.7 Proposition.** *Let K be a closed subset of $\mathbb{C}_\infty$, and assume that $K_1, \ldots, K_n$ are pairwise disjoint, nonempty, and relatively open subsets of K with $K = K_1 \cup \ldots \cup K_n$. Then the map*

$$\psi_0 : \begin{cases} H(K) & \to & H(K_1) \times \ldots \times H(K_n) \\ F & \mapsto & (F, \ldots, F) \end{cases}$$

induces an algebra isomorphism of $\mathbb{H}(K)$ onto $\mathbb{H}(K_1) \times \ldots \times \mathbb{H}(K_n)$. This isomorphism is also a homeomorphism.

Proof (of Proposition 8.1.7; part 1). It is clear that every element of $H(K)$ also belongs to $H(K_i)$, and that $F_1 \sim F_2$ in $H(K)$ implies that $F_1 \sim F_2$ in $H(K_i)$. Hence, if $\underline{\pi_i}$ denotes the projection of $H(K_i)$ onto $\mathbb{H}(K_i)$, there exists an algebra homomorphism $\psi : \mathbb{H}(K) \to \mathbb{H}(K_1) \times \mathbb{H}(K_n)$ with

$$\begin{tikzcd}
H(K) \arrow[r, "\psi_0"] \arrow[d, "\underline{\pi}"'] & H(K_1) \times \ldots \times H(K_n) \arrow[d, "\underline{\pi_1} \times \ldots \times \underline{\pi_n}"] \\
\mathbb{H}(K) \arrow[r, dashed, "\psi"'] & \mathbb{H}(K_1) \times \ldots \times \mathbb{H}(K_n)
\end{tikzcd}$$

Let $O \subseteq \mathbb{C}_\infty$ be open with $K \subseteq O$, and denote by $\iota_{i,O}$ the set-theoretic inclusion of $\mathbb{H}(O)$ into $\mathbb{H}(K_i)$. Moreover, let π_i generically denote the projection of a direct product onto its i-th component. Due to the definition of ψ_0 and ψ, we have

$$\begin{tikzcd}
\mathbb{H}(O) \arrow[r, "\underline{\pi} \circ \iota_O"] \arrow[dr, "\iota_O"'] \arrow[drrr, bend right, "\iota_{i,O}"'] & \mathbb{H}(K) \arrow[r, "\psi"] & \mathbb{H}(K_1) \times \ldots \times \mathbb{H}(K_n) \arrow[r, "\pi_i"] & \mathbb{H}(K_i) \\
& H(K) \arrow[u, "\underline{\pi}"] \arrow[r, "\psi_0"'] & H(K_1) \times \ldots \times H(K_n) \arrow[u, "\underline{\pi_1} \times \ldots \times \underline{\pi_n}"'] \arrow[r, "\pi_i"] & H(K_i) \arrow[u, "\underline{\pi_i}"']
\end{tikzcd}$$

i.e. $\pi_i \circ \psi \circ (\underline{\pi} \circ \iota_O) = \underline{\pi_i} \circ \iota_{i,O}$. It follows that ψ is continuous. ❑

To complete the proof, we need to construct an inverse to ψ. This is based on two general observations.

Observation 1: Let K be a closed subset of $\mathbb{C}_\infty$, and $K_1, \ldots, K_n$ pairwise disjoint, nonempty, and relatively open subsets of K whose union covers K. Then, for each $i \in \{1, \ldots, n\}$, K_i is a closed subset of $\mathbb{C}_\infty$. There exist pairwise disjoint open sets $O_i \subseteq \mathbb{C}_\infty$, $i = 1, \ldots, n$, such that

$$K_i = K \cap O_i, \quad i = 1, \ldots, n\,.$$

To see this, choose open sets $\tilde{O}_i \subseteq \mathbb{C}_\infty$ with $K_i = \tilde{O}_i \cap K$. We have

$$\Big(\bigcup_{\substack{j=1 \\ j \neq i}}^{n} O_j \Big)^c \cap K = K_i, \quad i = 1, \ldots, n\,, \tag{8.1.2}$$ J32

and hence the sets K_i are closed in $\mathbb{C}_\infty$. Since $\mathbb{C}_\infty$ is normal, we can therefore choose pairwise disjoint open sets O_i with the required property.

Observation 2: Let $V_1, \ldots, V_m$ be pairwise disjoint, nonempty, and open subsets of $\mathbb{C}_\infty$, and set $V := \bigcup_{i=1}^m V_i$. Then the map

$$\left\{ \begin{array}{rcl} \mathbb{H}(V) & \to & \mathbb{H}(V_1) \times \ldots \times \mathbb{H}(V_m) \\ F & \mapsto & \big(F|_{V_1}, \ldots, F|_{V_m}\big) \end{array} \right.$$

is invertible. Its inverse is namely given by assigning to a tuple $(F_1, \dots, F_m) \in \mathbb{H}(V_1) \times \dots \times \mathbb{H}(V_m)$ the pasted function

$$F(z) := \begin{cases} F_1(z)\ , & z \in V_1 \\ \vdots & \\ F_m(z)\,, & z \in V_m \end{cases} \tag{8.1.3}$$ J28

Proof (of Proposition 8.1.7; part 2). Let K and K_i be given according to the statement, choose pairwise disjoint open sets $O_i \subseteq \mathbb{C}_\infty$, $i = 1, \dots, n$, according to Observation 1, and set $O_0 := \bigcup_{i=1}^n O_i$. It is straightforward to check that a continuous algebra homomorphism $\lambda_i : \mathbb{H}(K_i) \to \mathbb{H}(K)$ is well-defined by the requirement that for every open set $O \subseteq \mathbb{C}_\infty$ with $K \subseteq O$

$$\begin{tikzcd}
\mathbb{H}(O) \arrow[r, "F \mapsto F|_{O \cap O_i}"] \arrow[d] & \mathbb{H}(O \cap O_i) \arrow[r, "F \mapsto (F, 0, \dots, 0)"] & \mathbb{H}(O \cap O_i) \times \prod_{\substack{j=1 \\ j \neq i}}^{n} \mathbb{H}(O_j) \arrow[d, "\text{pasting map (8.1.3)}"] \\
H(K_i) \arrow[r, dashed] \arrow[d, "\pi_i"'] & H(K) \arrow[d, "\pi"] & \mathbb{H}\Big((O \cap O_i) \cup \bigcup_{\substack{j=1 \\ j \neq i}}^{n} O_j\Big) \arrow[l] \\
\mathbb{H}(K_i) \arrow[r, dashed, "\lambda_i"'] & \mathbb{H}(K) &
\end{tikzcd}$$

This definition of λ_i ensures that the map λ defined as

$$\lambda : \begin{cases} \mathbb{H}(K_1) \times \dots \times \mathbb{H}(K_n) & \to & \mathbb{H}(K) \\ (\underline{F}_1, \dots, \underline{F}_n) & \mapsto & \lambda_1(\underline{F}_1) + \dots + \lambda_n(\underline{F}_n) \end{cases}$$

satisfies $\lambda \circ \psi = \mathrm{id}$ and $\psi \circ \lambda = \mathrm{id}$. ❑

Note that, due to compactness, each decomposition of K into disjoint relatively open subsets is necessarily finite.

Clearly, in the situation of Proposition 8.1.7, each connected component of K must be entirely contained in one of the subsets K_i. However, the connected components of K themselves will in general not be suitable since they need not be open. Also note here that K may have infinitely many components. However, the set K is connected if and only if it does not allow a decomposition of the form used in Proposition 8.1.7.

Let us give some more facts which emphasize the interplay between algebraic and topological structures.

PRJ5 **8.1.8 Proposition.** *Let K be a closed subset of $\mathbb{C}_\infty$.*

(*i*) *$\mathbb{H}(K)$ is an integral domain if and only if K is connected.*

(*ii*) *The restriction map $\rho_K : \mathbb{H}(K) \to C(K)$ is injective if and only if K contains no isolated points.*

Proof. If K is not connected, we can write $K = K_1 \cup K_2$ with nonempty, disjoint, and relatively open subsets K_1, K_2. By Proposition 8.1.7, we have $\mathbb{H}(K) \cong \mathbb{H}(K_1) \times \mathbb{H}(K_2)$, and hence $\mathbb{H}(K)$ contains zero divisors. Conversely, assume that $\underline{F}, \underline{G} \in \mathbb{H}(K) \setminus \{0\}$ and $\underline{F} \cdot \underline{G} = 0$. Then there exists an open set O, $K \subseteq O$, with $O \subseteq O_F \cap O_G$ and $F(z)G(z) = 0$, $z \in O$. We may assume without loss of generality that every connected component of O intersects K; simply by removing from O all those components which do not intersect K. Let O_1 be the union of all those components where the function F vanishes identically, and let O_2 be the union of all other components. Clearly, O_1 and O_2 are disjoint open sets and their union contains K. Since $\underline{F} \neq 0$, the set O_2 is not empty, and hence $O_2 \cap K \neq \emptyset$. On the other hand, on O_2 the function G must vanish identically. Since $\underline{G} \neq 0$, O_2 cannot cover all of K. Thus also $O_1 \cap K \neq \emptyset$. It follows that K is not connected.

Next, we turn to the proof of (ii). Assume first that w is an isolated point of K. Then K is the disjoint union of the two relatively open sets $\{w\}$ and $K \setminus \{w\}$. Thus $\mathbb{H}(K) \cong \mathbb{H}(\{w\}) \times \mathbb{H}(K \setminus \{w\})$. The set $H(\{w\})$ contains nonzero elements which vanish at w, e.g. take $F_w(z) := z - w$ if $w \neq \infty$, or $F_w(z) := \frac{1}{z}$ if $w = \infty$. The element $(F_w, 0) \in \mathbb{H}(\{w\}) \times \mathbb{H}(K \setminus \{w\})$ is nonzero, still, its restriction to K is identically zero. For the converse, assume that K contains no isolated points, and let $\underline{F} \in \ker \rho_K$. Let O_i, $I \in I$, be those connected components of O_F which intersect K. Each component O_i must intersect K in infinitely many points. Hence, $O_i \cap K$ has an accumulation point in K. Since $O_i \cap K$ is closed, remember (8.1.2), each such point lies inside O_i. Since F vanishes on K, it thus vanishes identically on O_i. The union of all O_i, $i \in I$, covers K, and we conclude that $\underline{F} = 0$. ❑

In view of our later needs, we will now investigate divisibility in $\mathbb{H}(K)$. Let us recall the notion of the *divisor* $\mathfrak{d}_f$ of a meromorphic function f. Let X be an analytic manifold, $f \in \mathbb{H}(X, \mathbb{C}_\infty)$, and $w \in X$. If f vanishes identically on some neighbourhood of w, we set $\mathfrak{d}_f(w) := +\infty$. Otherwise, choose a chart ϕ whose domain contains w, and let $\mathfrak{d}_f(w)$ be the unique integer such that the Laurent expansion of $f \circ \phi^{-1}$ at $\phi(w)$ is of the form

$$(f \circ \phi^{-1})(z) = \sum_{n=\mathfrak{d}_f(w)}^{\infty} a_n (z - \phi(w))^n \text{ with } a_{\mathfrak{d}_f(w)} \neq 0 .$$

In this way, $\mathfrak{d}_f$ is a well-defined function of X into $\mathbb{Z} \cup \{+\infty\}$. Note that, for each two functions $f, g \in \mathbb{H}(X, \mathbb{C}_\infty)$ also $f \cdot g \in \mathbb{H}(X, \mathbb{C}_\infty)$ and $\mathfrak{d}_{f \cdot g} = \mathfrak{d}_f + \mathfrak{d}_g$.

Now consider a closed subset K of $\mathbb{C}_\infty$, and let $F_1, F_2 \in H(K)$. If $F_1 \sim F_2$, then for each $w \in K$ we have $\mathfrak{d}_{F_1}(w) = \mathfrak{d}_{F_2}(w)$. Hence, for an element $\underline{F} \in \mathbb{H}(K)$, a function $\mathfrak{d}_{\underline{F}} : K \to \mathbb{N}_0 \cup \{+\infty\}$ is well-defined by

$$\mathfrak{d}_{\underline{F}}(w) := \mathfrak{d}_F(w), \quad \underline{F} \in \mathbb{H}(K) .$$

We will call $\mathfrak{d}_{\underline{F}}$ the *divisor* of $\underline{F}$.

PRJ7 **8.1.9 Proposition.** *Let K be a closed subset of $\mathbb{C}_\infty$, $K \neq \mathbb{C}_\infty$.*

(i) Let $\underline{F}, \underline{G} \in \mathbb{H}(K)$. Then $\underline{F} \,|\, \underline{G}$ in $\mathbb{H}(K)$ if and only $\mathfrak{d}_{\underline{F}} \leq \mathfrak{d}_{\underline{G}}$. An element $\underline{F} \in \mathbb{H}(K)$ is a unit if and only if $\mathfrak{d}_{\underline{F}} = 0$.

(*ii*) *For each function* $\mathfrak{d} : K \to \mathbb{N}_0$ *with finite support there exists an element* $\underline{F} \in \mathbb{H}(K)$ *with* $\mathfrak{d} = \mathfrak{d}_{\underline{F}}$. *The function* F *can be chosen to be rational.*

(*iii*) *Let* $\mathcal{M}$ *be a nonempty subset of* $\mathbb{H}(K)$, *and assume that for each point* $w \in K$ *there exists an element* $\underline{F} \in \mathcal{M}$ *with* $\mathfrak{d}_{\underline{F}}(w) < +\infty$. *Set*

$$\mathfrak{d}_{\mathcal{M}} := \min\big\{\mathfrak{d}_{\underline{F}} : \underline{F} \in \mathcal{M}\big\},$$

then $\mathfrak{d}_{\mathcal{M}}$ *maps* K *into* $\mathbb{N}_0$ *and* $\operatorname{supp}\mathfrak{d}_{\mathcal{M}}$ *is finite. There exist finitely many elements* $\underline{F_1}, \ldots, \underline{F_n} \in \mathcal{M}$, *such that*

$$\mathfrak{d}_{\mathcal{M}} := \min\big\{\mathfrak{d}_{\underline{F_1}}, \ldots, \mathfrak{d}_{\underline{F_n}}\big\}. \tag{8.1.4}$$ J31

An element $\underline{D} \in \mathbb{H}(K)$ *is a greatest common divisor of* $\mathcal{M}$ *in* $\mathbb{H}(K)$ *if and only if* $\mathfrak{d}_{\underline{D}} = \mathfrak{d}_{\mathcal{M}}$. *In particular,* $\mathcal{M}$ *has a greatest common divisor.*

(*iv*) *Let* $\underline{F}, \underline{G} \in \mathbb{H}(K)$ *and assume that*

$$\min\{\mathfrak{d}_{\underline{F}}, \mathfrak{d}_{\underline{G}}\} = 0. \tag{8.1.5}$$ J30

Then there exist $\underline{A}, \underline{B} \in \mathbb{H}(K)$ *such that* $\underline{A} \cdot \underline{F} + \underline{B} \cdot \underline{G} = 1$.

(*v*) *Let* $\mathcal{M}$ *be a nonempty subset of* $\mathbb{H}(K)$, *and assume that for each point* $w \in K$ *there exists an element* $\underline{F} \in M$ *with* $\mathfrak{d}_{\underline{F}}(w) < +\infty$. *Moreover, let* $\underline{D} \in \mathbb{H}(K)$ *be a greatest common divisor of* $\mathcal{M}$. *Then there exist finitely many elements* $\underline{F_1}, \ldots, \underline{F_n} \in \mathcal{M}$ *and* $\underline{B_1}, \ldots, \underline{B_n} \in \mathbb{H}(K)$ *such that*

$$\underline{D} = \sum_{i=1}^{n} \underline{B_i} \cdot \underline{F_i}.$$

Proof. The statement (*i*) is easy to see. Assume that $\underline{F}\,|\,\underline{G}$ in $\mathbb{H}(K)$, i.e. that there exists an element $\underline{H} \in \mathbb{H}(K)$ with $\underline{G} = \underline{H} \cdot \underline{F}$. Then, for each $w \in K$,

$$\mathfrak{d}_{\underline{G}}(w) = \mathfrak{d}_{\underline{H}\cdot\underline{F}}(w) = \mathfrak{d}_{HF}|_K(w) = \mathfrak{d}_H|_K(w) + \mathfrak{d}_F|_K(w) \geq \mathfrak{d}_F|_K(w) = \mathfrak{d}_{\underline{F}}(w).$$

Conversely, assume that $\mathfrak{d}_{\underline{F}} \leq \mathfrak{d}_{\underline{G}}$. Denote by $O_1, \ldots, O_n$ the connected components of $O_F \cap O_G$ which intersect K. Set

$$H_i(z) := \begin{cases} \frac{G(z)}{F(z)}, & z \in O_i,\ F \text{ does not vanish identically on } O_i \\ 0 \quad , & z \in O_i,\ F \text{ vanishes identically on } O_i \end{cases}$$

then $H_i \in H(O_i)$. If $\underline{H}$ denotes the element of $\mathbb{H}(K)$ obtained by pasting $\underline{H_1}, \ldots, \underline{H_n}$ by means of Proposition 8.1.7, then $\underline{G} = \underline{H} \cdot \underline{F}$. Note here that, if F vanishes identically on O_i, also G does.

If $\underline{F}$ is a unit in $\mathbb{H}(K)$, then $\mathfrak{d}_{\underline{F}} \leq \mathfrak{d}_{\underline{1}} = 0$. Conversely, if $\mathfrak{d}_{\underline{F}} = 0$, then the function $\frac{1}{F}$ is analytic on $O := \{z \in O_F : F(z) \neq 0\}$, and hence belongs to $H(K)$. Clearly, $\underline{(\frac{1}{F})} \cdot \underline{F} = \underline{1}$.

For the proof of (*ii*), let $\mathfrak{d} : K \to \mathbb{N}_0$ with finite support be given. Since $K \neq \mathbb{C}_\infty$, we can choose $M \in \mathrm{GL}(2, \mathbb{C})$ with $\phi_M^{-1}(\infty) \notin K$. Let p be the polynomial

$$p(z) := \prod_{w \in K} \big(z - \phi_M(w)\big)^{\mathfrak{d}(w)},$$

then $p \circ \phi_M$ is analytic on $\mathbb{C}_\infty \setminus \{\phi_M^{-1}(\infty)\}$ and hence belongs to $\mathbb{H}(K)$. The element $\underline{F} := \underline{p \circ \phi_M}$ has the required property.

We come to the proof of (iii). By assumption, for each $w \in K$ there exists an element $\underline{F_w} \in \mathcal{M}$ with $\mathfrak{d}_{\underline{F_w}}(w) \neq 0$. Let O_w be the connected component of O_{F_w} which contains w, then $\mathfrak{d}_{\underline{F_w}}(z) < +\infty$ for all $z \in O_w$, and $\mathfrak{d}_{\underline{F_w}}(z) \neq 0$ for at most finitely many values of $z \in O_w \cap K$. The open sets O_w, $w \in K$, cover K and by compactness we can extract finitely many points $w_1, \dots, w_m$ such that $O_{w_1} \cup \dots \cup O_{w_m} \supseteq K$. The function

$$\mathfrak{d}_0 := \min \big\{ \mathfrak{d}_{\underline{F_{w_1}}}, \dots, \mathfrak{d}_{\underline{F_{w_m}}} \big\}$$

maps K into $\mathbb{N}_0$ and has finite support. Since $\mathfrak{d}_\mathcal{M} \leq \mathfrak{d}_0$, it already follows that $\mathfrak{d}_\mathcal{M}$ has finite support. Moreover, since each descending chain of functions $\mathfrak{d} : K \to \mathbb{N}_0$ with finite support must remain constant from some index on, it follows that we can find elements $\underline{F_{m+1}}, \dots, \underline{F_n} \in \mathcal{M}$ such that (8.1.4) holds with $\underline{F_i} := \underline{F_{w_i}}$, $i = 1, \dots, m$, and $\underline{F_{m+1}}, \dots, \underline{F_n}$.

By (i), an element $\underline{D} \in \mathbb{H}(K)$ is a common divisor of $\mathcal{M}$ if and only if $\mathfrak{d}_{\underline{D}} \leq \mathfrak{d}_\mathcal{M}$. We already see that each element $\underline{D}$ with $\mathfrak{d}_{\underline{D}} = \mathfrak{d}_\mathcal{M}$ is a greatest common divisor of $\mathcal{M}$. Assume that $\mathfrak{d}_{\underline{D}} \leq \mathfrak{d}_\mathcal{M}$ and that there exists $w \in K$ with $\mathfrak{d}_{\underline{D}}(w) < \mathfrak{d}_\mathcal{M}(w)$. Choose $M \in \mathrm{GL}(2, \mathbb{C})$ with $\phi_M^{-1}(\infty) \notin K$, then the function

$$\underline{F} := \big(\phi_M(z) - \phi_M(w)\big) \cdot \underline{D}$$

is a common divisor of $\mathcal{M}$. However, apparently, it is not a divisor of $\underline{D}$. We have shown that $\underline{D}$ is a greatest common divisor of $\mathcal{M}$ if and only if $\mathfrak{d}_{\underline{D}} = \mathfrak{d}_\mathcal{M}$. It remains to note that, by (ii), such elements do exist.

For the proof of (iv), let $\underline{F}, \underline{G} \in \mathbb{H}(K)$ with (8.1.5) be given. Let $O_1, \dots, O_n$ be the connected components of $O_F \cap O_G$ which intersect K. We are going to define, for each $i \in \{1, \dots, n\}$, an open set $\tilde{O}_i$ with $O_i \cap K \subseteq \tilde{O}_i \subseteq O_i$ and functions $A_i, B_i \in \mathbb{H}(O_i)$.

Case 1; $F|_{O_i} = 0$: Set $\tilde{O}_i := O_i$ and $A_i := 0$. Due to (8.1.5), the function $G|_{O_i}$ is zerofree. Hence, we may set $B_i := (G|_{O_i})^{-1}$. Then, trivially, $A_i \cdot F + B_i \cdot G = 1$ in $\mathbb{H}(\tilde{O}_i)$.

Case 2; $G|_{O_i} = 0$: Set $\tilde{O}_i := O_i$ and $B_i := 0$. Due to (8.1.5), the function $F|_{O_i}$ is zerofree, and we thus may set $A_i := (F|_{O_i})^{-1}$. Again, $A_i \cdot F + B_i \cdot G = 1$ in $\mathbb{H}(\tilde{O}_i)$.

Case 3; neither $F|_{O_i} = 0$ nor $G|_{O_i} = 0$: The function $(F|_{O_i} \cdot G|_{O_i})^{-1}$ is meromorphic in O_i. From (8.1.5) it follows that the set of its poles in K is the disjoint union of the sets of zeros in K of F and of G. In fact, we have

$$\mathfrak{d}_{(FG)^{-1}}(w) = -\mathfrak{d}_F(w) - \mathfrak{d}_G(w) = \begin{cases} -\mathfrak{d}_F(w)\,, & \mathfrak{d}_F(w) > 0 \\ -\mathfrak{d}_G(w)\,, & \mathfrak{d}_G(w) > 0 \\ 0 \qquad\quad , & \text{otherwise} \end{cases}$$

Denote by H_w the principal part of the Laurent expansion of $(FG)^{-1}$ at $w \in K$, understanding $H_w = 0$ if w is not a pole of $(FG)^{-1}$. Then

$$FH_w \in \mathbb{H}(O_i),\ \mathfrak{d}_F(w) > 0 \quad \text{and} \quad GH_w \in \mathbb{H}(O_i),\ \mathfrak{d}_G(w) > 0\,.$$

The function $H := (FG)^{-1} - \sum_{w\in K} H_w$ is analytic on some open set $\tilde{O}_i$ with $K \cap O_i \subseteq \tilde{O}_i \subseteq O_i$. Note here that in this sum only finitely many summands are nonzero. Set

$$A_i := GH + \sum_{\substack{w\in K\\ \mathfrak{d}_G(w)>0}} GH_w, \quad B_i := \sum_{\substack{w\in K\\ \mathfrak{d}_F(w)>0}} FH_w\,,$$

then $A_i, B_i \in \mathbb{H}(\tilde{O}_i)$ and $A_i \cdot F + B_i \cdot G = 1$ in $\mathbb{H}(\tilde{O}_i)$.

As a consequence of (8.1.5), on no set O_i both $F|_{O_i}$ and $G|_{O_i}$ can vanish identically, and hence we have defined $\tilde{O}_i$, A_i, B_i, for all $i \in \{1,\dots,n\}$. The required elements $\underline{A}, \underline{B} \in \mathbb{H}(K)$ are now obtained by pasting the elements $\underline{A_1},\dots,\underline{A_n}$ and $\underline{B_1},\dots,\underline{B_n}$, respectively, by means of Proposition 8.1.7.

Finally, we turn to the proof of (v). Due to (iii), it suffices to show that some greatest common divisor of a finite set $\underline{F_1},\dots,\underline{F_n}$ can be represented as a sum $\sum_{i=1}^n \underline{B_i}\cdot\underline{F_i}$. To show this, we use induction on n. If $n=1$, then $\underline{D} = \underline{F_1}$ is a greatest common divisor of $\{\underline{F_1}\}$, and the desired representation is trivially present. Let $n>1$, let $\underline{D}$ be a greatest common divisor of $\{\underline{F_1},\dots,\underline{F_n}\}$, and $\underline{D_0}$ one of $\{\underline{F_1},\dots,\underline{F_{n-1}}\}$. By the inductive hypothesis, we find $\underline{B_1},\dots,\underline{B_{n-1}} \in \mathbb{H}(K)$ with $\underline{D_0} = \sum_{i=1}^{n-1} \underline{B_i}\cdot\underline{F_i}$. We have

$$\min\big\{\mathfrak{d}_{\underline{D}^{-1}\underline{D_0}}, \mathfrak{d}_{\underline{D}^{-1}\underline{F_n}}\big\} = 0\,,$$

and hence find $\underline{A}, \underline{B} \in \mathbb{H}(K)$ with $\underline{A}\cdot(\underline{D}^{-1}\underline{D_0}) + \underline{B}\cdot(\underline{D}^{-1}\underline{F_n}) = 1$. This gives

$$\underline{D} = \underline{A}\cdot\sum_{i=1}^{n-1}\underline{B_i}\cdot\underline{F_i} + \underline{B}\cdot\underline{F_n}\,.$$

❑

8.2 Definition of the functional calculus

For a closed rectifiable path $\gamma : [0,1] \to \mathbb{C}$ and $z \in \mathbb{C}$, we denote by $n(\gamma_j, z)$ the winding number of γ around z. Moreover, we agree that $n(\gamma,\infty) := 0$.

Let $\gamma_1,\dots,\gamma_n : [0,1] \to \mathbb{C}$ be closed and piecewise smooth paths, and let $K \subseteq \mathbb{C}$ be compact and $O \subseteq \mathbb{C}$ open with $K \subseteq O$. Then we say that the collection $\gamma_1,\dots,\gamma_n$ of paths satisfies (8.2.1) for O, K, if

$$\begin{aligned} &\gamma_j\big([0,1]\big) \subseteq O\setminus K, \quad j=1,\dots,n \\ &\sum_{j=1}^n n(\gamma_j,z) = \begin{cases} 0\,, & z\notin O \\ 1\,, & z\in K \end{cases} \end{aligned} \tag{8.2.1}$$ J11

DEJ12 **8.2.1 Definition.** Let $\mathcal{X}$ be a Banach space and $T \in \mathrm{CLR}(\mathcal{X})$ with $\rho(T) \neq \emptyset$. Then we define a map

$$\Phi^T_{\mathrm{RD}} : \mathbb{H}(\sigma(T)) \to \mathcal{B}(\mathcal{X})$$

by the following procedure: If $\underline{F} \in \mathbb{H}(\sigma(T))$ is given, choose $M \in \mathrm{GL}(2,\mathbb{C})$ such that $\phi_M^{-1}(\infty) \notin \sigma(T)$, choose finitely many closed piecewise smooth paths $\gamma_1,\dots,\gamma_n$ which satisfy (8.2.1) for $\phi_M(O_F)\cap\mathbb{C}, \phi_M(\sigma(T))$, and set

$$\Phi^T_{\mathrm{RD}}(\underline{F}) := \frac{1}{2\pi i}\sum_{j=1}^n \int_{\gamma_j} (F\circ\phi_M^{-1})(\zeta)\cdot(\zeta - \phi_M(T))^{-1}\,d\zeta\,. \tag{8.2.2}$$ J13

The map Φ^T_{RD} is called the *Riesz-Dunford functional calculus.* //

First of all, we have to justify this definition.

PRJ14 **8.2.2 Proposition.** *Let $\mathcal{X}$ be a Banach space, $T \in \mathrm{CLR}(\mathcal{X})$ with $\rho(T) \neq \emptyset$, and $\underline{F} \in \mathbb{H}(\sigma(T))$. Then there exists $M \in \mathrm{GL}(2,\mathbb{C})$ such that $\phi_M^{-1}(\infty) \notin \sigma(T)$, and there exist closed piecewise smooth paths $\gamma_1, \dots, \gamma_n$ which satisfy* (8.2.1) *for $\phi_M(O_F) \cap \mathbb{C}, \phi_M(\sigma(T))$. The operator on the right side of* (8.2.2) *neither depends on the choice of M and γ_j subject to these properties, nor on the choice of the representant $F \in H(K)$ of the element $\underline{F} \in \mathbb{H}(K)$.*

The proof of this proposition is split into several lemmata. First an elementary but elaborate fact.

LEJ15 **8.2.3 Lemma.** *Let $\delta > 0$ and a paraxial grid of squares with edge length δ be given. Moreover, let $\mathcal{Q}$ be a finite set of squares of this grid, and set*

$$K = \bigcup_{Q \in \mathcal{Q}} \overline{Q}\,.$$

Then the set K is compact and its boundary ∂K is the union of all (closed) edges with the property that exactly one of the two adjacent squares belongs to $\mathcal{Q}$.

There exist closed paths $\gamma_1, \dots, \gamma_n$, each of which consists of a finite number of edges of squares in $\mathcal{Q}$, such that

(*i*) $\partial K = \bigcup_{k=1}^n \gamma_k$,

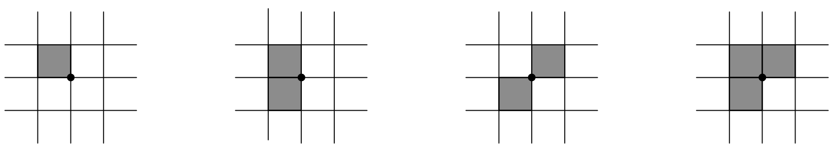
(*ii*) *each edge lying in γ_k is oriented such that the adjacent square in $\mathcal{Q}$ lies to the left,*

(*iii*) *no edge appears more than once in one path γ_j, or appears in two different paths,*

(*iv*) *we have*

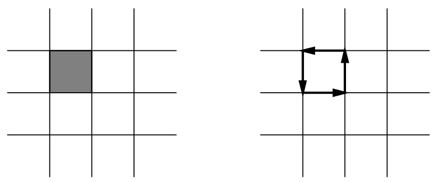
$$\sum_{k=1}^n n(\gamma_k, z) = \begin{cases} 0 & , \ z \in K^c \\ 1 & , \ z \in \mathring{K} \end{cases}$$

Proof. Let $w \in \mathbb{C}$. Assume first that w lies on some (closed) edge E with the stated property. Then, clearly, $w \in \partial K$. Conversely, assume that $w \in \partial K$. Then w cannot be an inner point of any square of the grid, since the interior of each square either belongs entirely to K or entirely to K^c. Thus w must be located on an edge of the grid. If w is not a vertex, then exactly one of the two adjacent squares belongs to $\mathcal{Q}$. If w is a vertex, then at least one of the four adjacent squares must belong to $\mathcal{Q}$ and at least one of them must not belong to $\mathcal{Q}$.



We see that in each case the point w lies on a closed edge E with the required property that one adjacent square belongs to $\mathcal{Q}$ and the other does not. This shows the required representation of ∂K.

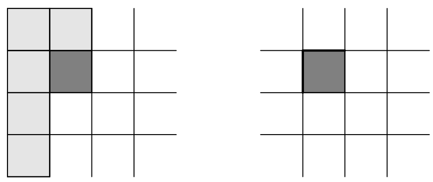
In order to show the existence of paths $\gamma_1, \ldots, \gamma_n$ with the properties $(i) - (iv)$, we use induction on the number of squares contained in $\mathcal{Q}$. If $\mathcal{Q}$ consists of only one square, then the assertion is obvious:



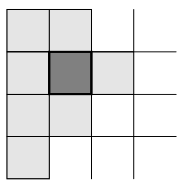
$\mathcal{Q}$ γ

Assume that a set $\mathcal{Q}$ which contains more than one square is given, and that the assertion of the lemma has already been proved for all sets $\mathcal{Q}'$ with less elements than $\mathcal{Q}$.

Let Q be the square in $\mathcal{Q}$ which has the maximal y-coordinate under all squares with minimal x-coordinate. Then at least the left and the upper edge of Q belongs to ∂K:



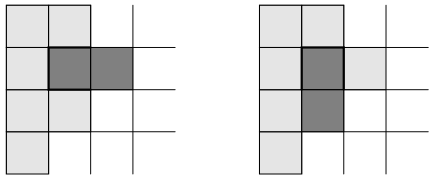
Case 1; all edges of Q belong to ∂K: Then we are in the situation



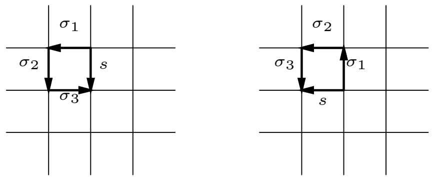
Set $\mathcal{Q}' := \mathcal{Q} \setminus \{Q\}$ and define K' correspondingly. Then $\partial K = \partial K' \cup \partial Q$ and $\partial K'$ and ∂Q have no edge in common. Applying the inductive hypothesis to $\mathcal{Q}'$ gives paths $\gamma_1', \ldots, \gamma_n'$. Let γ be the (positively oriented) boundary of Q. We are going to show that $\{\gamma_1', \ldots, \gamma_n', \gamma\}$ are paths with the required properties $(i) - (iv)$ for the set $\mathcal{Q}$. We already saw that (i) holds, the properties (ii) and (iii) are obvious. In order to see (iv), it is enough to note that $\mathring{K} = \mathring{K}' \dot{\cup} \mathring{Q}$ and that the path γ satisfies

$$n(\gamma, z) = \begin{cases} 0\,, & z \notin \overline{Q} \\ 1\,, & z \in \mathring{Q} \end{cases}$$

Case 2; three edges of Q belong to ∂K: Then we are in one of the two situations



Again set $\mathcal{Q}' := \mathcal{Q} \setminus \{Q\}$, denote by s the edge of Q which does not belong to ∂K liegt, and let $\sigma_1, \sigma_2, \sigma_3$ be the other edges oriented as



Removing the square Q from $\mathcal{Q}$ can change ∂K only in the edges of Q. It follows that $\partial K' = (\partial K \setminus \{\sigma_1, \sigma_2, \sigma_3\}) \cup \{s\}$. Let again $\gamma_1', \ldots, \gamma_n'$ be the paths the inductive hypothesis gives us for $\mathcal{Q}'$. Then the edge s appears in exactly one of $\gamma_1', \ldots, \gamma_n'$ and oriented such that Q lies to the right. Without loss of generality, assume that $s \in \gamma_1'$, and write γ_1' as the sequence of the oriented line segments $s, s_1, s_2, \ldots, s_m$.

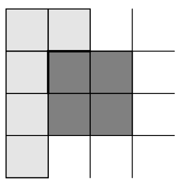
Define γ as the sequence of the oriented line segments $\sigma_1, \sigma_2, \sigma_3, s_1, s_2, \ldots, s_m$. Remember that $\sigma_1, \sigma_2, \sigma_3$ are oriented such that Q lies to the left. We are going to show that $\gamma, \gamma_2', \ldots, \gamma_n'$ are required paths for $\mathcal{Q}$. Again, $(i), (ii)$, and (iii) is obvious. Moreover, we have

$$n(\gamma, z) = \frac{1}{2\pi i} \int\limits_{\gamma} \frac{1}{\zeta - z}\, d\zeta = \frac{1}{2\pi i} \int\limits_{\gamma_1} \frac{1}{\zeta - z}\, d\zeta + \frac{1}{2\pi i} \oint\limits_{\partial Q} \frac{1}{\zeta - z}\, d\zeta =$$

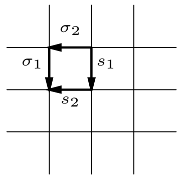
$$= \begin{cases} 0\,, & z \notin K' \cup \overline{Q} = K \\ 1\,, & z \in \mathring{K}' \cup \mathring{Q} \end{cases}$$

Due to continuity of winding numbers, this also implies that $n(\gamma, z) = 1$, $z \in s$. Altogether, (iv) holds.

Case 3; two sides of Q belong to ∂K and the square right and below of Q belongs to $\mathcal{Q}$: We are thus in the situation



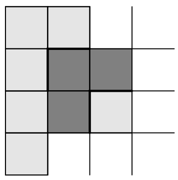
Let s_1, s_2, σ_1, and σ_2 be the right, lower, upper, and left, respectively, edge of Q. Thereby let s_1, s_2 be oriented such that Q lies to the right, and let σ_1, σ_2 be oriented such that Q lies to the left.

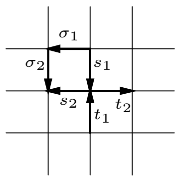


Again let $\mathcal{Q}' := \mathcal{Q} \setminus \{Q\}$, then $\partial K' = (\partial K \setminus \{\sigma_1, \sigma_2\}) \cup \{s_1, s_2\}$. Let $\gamma_1', \dots, \gamma_n'$ paths for $\mathcal{Q}'$, and assume that s_1 appears in γ_1'. Since the square right and below of Q belongs to $\mathcal{Q}'$, the edge which appears in γ_1' after s_1 must be s_2. Thus we can write γ_1' as the sequence of oriented lie segments $s_1, s_2, t_1, \dots, t_m$.

Let γ be defined as the sequence of oriented line segments $\sigma_1, \sigma_2, t_1, \dots, t_m$. Then one shows similar as in 'Case 2' that $\gamma, \gamma_2', \dots, \gamma_n'$ are appropriate paths for $\mathcal{Q}$.

Fall 4; two sides of Q belong to ∂K and the square right and below of Q does not belong to $\mathcal{Q}$: We are in the situation





Set $\mathcal{Q}' := \mathcal{Q} \setminus \{Q\}$, then $\partial K' = (\partial K \setminus \{\sigma_1, \sigma_2\}) \cup \{s_1, s_2\}$. Again let $\gamma_1', \dots, \gamma_n'$ be paths for $\mathcal{Q}'$.

Assume that all four edges s_1, s_2, t_1, t_2 lie on one path, say on γ_1'. Then either

(*a*) $\gamma_1' = s_1, s_2, u_1, \dots, u_n, t_1, t_2, u_{n+1}, \dots, u_m$,

or

(*b*) $\gamma_1' = s_1, t_2, u_1, \dots, u_n, t_1, s_2, u_{n+1} u_m$.

In the first case, set

$$\gamma := \sigma_1, \sigma_2, u_1, \dots, u_n, t_1, t_2 u_{n+1}, \dots, u_m \, ,$$

in the second

$$\gamma := \sigma_1, \sigma_2, u_{n+1}, \dots, u_m, \ \tilde{\gamma} := t_2, u_1, \dots, u_n, t_1 \, .$$

Assume that the four edges s_1, s_2, t_1, t_2 are distributed over two different paths, say $s_1 \in \gamma_1'$ and $t_1 \in \gamma_2'$. Then either

(*c*) $\gamma_1' = s_1, s_2, u_1, \dots, u_n$, $\gamma_2' = t_1, t_2, v_1, \dots, v_m$,

or

(*d*) $\gamma_1' = s_1, t_2, u_1, \dots, u_n$, $\gamma_2' = t_1, s_2, v_1, \dots, v_m$.

In the first case, set

$$\gamma := \sigma_1, \sigma_2, u_1, \dots, u_n \, ,$$

in the second

$$\gamma := \sigma_1, \sigma_2, v_1, \dots, v_m, t_1, t_2, u_1, \dots, u_m \, .$$

Corresponding to which case of (*a*), (*b*), (*c*) or (*d*) applies, consider the sets of paths

(a) $\gamma, \gamma_2', \ldots, \gamma_n'$,

(b) $\gamma, \tilde{\gamma}, \gamma_2', \ldots, \gamma_n'$,

(c) $\gamma, \gamma_2', \ldots, \gamma_n'$,

(d) $\gamma, \gamma_3', \ldots, \gamma_n'$.

These sets of paths satisfy (i)–(iii). In the cases (a) and (c) one shows in the same way as above that also (iv) holds. In case (b) we have

$$n(\gamma_1, z) = n(\gamma', z) + n(\gamma, z) - \oint_{\partial Q} \frac{1}{\zeta - z}\, d\zeta, \quad z \in \mathring{K}' \cap \mathring{Q}\,,$$

and hence

$$n(\gamma, z) + n(\gamma', z) + \sum_{k=2}^{n} n(\gamma_k, z) = \sum_{k=1}^{n} n(\gamma_n, z) + \oint_{\partial Q} \frac{1}{\zeta - z}\, d\zeta, \quad z \in \mathring{K}' \cap \mathring{Q}\,,$$

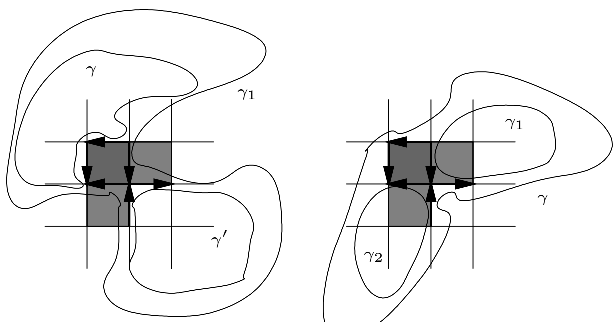
In case (d) we have

$$n(\gamma, z) = n(\gamma_1, z) + n(\gamma_2, z) + \oint_{\partial Q} \frac{1}{\zeta - z}\, d\zeta, \quad z \in \mathring{K}' \cap \mathring{Q}\,,$$

and hence

$$n(\gamma, z) + \sum_{k=3}^{n} n(\gamma_k, z) = \sum_{k=1}^{n} + \oint_{\partial Q} \frac{1}{\zeta - z}\, d\zeta, \quad z \in \mathring{K}' \cap \mathring{Q}\,.$$

In both cases (iv) follows again:



(b) (d)

❑

COJ16 **8.2.4 Corollary.** *Let $K \subseteq \mathbb{C}$ be compact, and $O \subseteq \mathbb{C}$ open with $K \subseteq O$. Then there exist closed piecewise smooth paths $\gamma_1, \ldots, \gamma_n : [0,1] \to \mathbb{C}$ which satisfy* (8.2.1) *for O, K.*

Proof. We apply Lemma 8.2.3 with $\delta := \frac{1}{2}d(O^c, K)$ and the set $\mathcal{Q}$ of all squares whose closure intersects K. This furnishes us with paths $\gamma_1, \dots, \gamma_n$. Since

$$K \subseteq \operatorname{Int}\Big(\bigcup_{Q\in\mathcal{Q}} \overline{Q}\Big) \subseteq \bigcup_{Q\in\mathcal{Q}} \overline{Q} \subseteq O\,,$$

these paths satisfy (8.2.1). ❑

Proof (of Proposition 8.2.2; Part 1). In this part of the proof we show existence of M, γ_j, and the integral in (8.2.2), and independence of this integral from the choice of γ_j when M is fixed.

For each point $w \in \mathbb{C}_\infty$, we can find a fractional linear transformation which maps this point to ∞. Since $\rho(T) \neq \emptyset$, we can thus choose ϕ_M with $\phi_M^{-1}(\infty) \in \rho(T)$. Corollary 8.2.4 applied with the compact set $\phi_M(\sigma(T))$ and the open set $\phi_M(O_F) \cap \mathbb{C}$ gives paths with the required properties.

By the Spectral Mapping Theorem for ϕ_M, the integrand in (8.2.2) is an analytic function on

$$O := (\phi_M(O_F) \cap \mathbb{C}) \setminus \phi_M(\sigma(T))\,.$$

In particular, the integral exists.

Let $M \in \mathrm{GL}(2,\mathbb{C})$ with $\phi_M^{-1}(\infty) \in \rho(T)$ be fixed, and assume that $\gamma_1, \dots, \gamma_n$ and $\gamma_1', \dots, \gamma_m'$ are two collections of piecewise smooth paths which satisfy (8.2.1) for $\phi_M(O_F) \cap \mathbb{C}, \phi_M(\sigma(T))$. Then

$$\sum_{k=1}^n n(\gamma_k, z) - \sum_{k=1}^m n(\gamma_k', z) = 0, \quad z \notin O\,,$$

and we obtain from the Cauchy Integral Theorem that

$$\sum_{k=1}^n \int_{\gamma_k} (F\circ\phi_M^{-1})(\zeta)\cdot(\zeta - \phi_M(T))^{-1}\,d\zeta - \sum_{k=1}^m \int_{\gamma_k'} (F\circ\phi_M^{-1})(\zeta)\cdot(\zeta - \phi_M(T))^{-1}\,d\zeta = 0\,.$$

❑

The fact that the right side of (8.2.2) does not depend on the choice of the chart ϕ_M will be deduced from its below, more explicit, representation.

LEJ17 **8.2.5 Lemma.** *Let $F \in \mathbb{H}(\sigma(T))$, and let $M \in \mathrm{GL}(2,\mathbb{C})$ be such that $\phi_M^{-1}(\infty) \notin \sigma(T)$. Assume that $\alpha_1, \dots, \alpha_n : [0,1] \to \mathbb{C}_\infty$ are closed piecewise smooth paths with*

$$\alpha_j\big([0,1]\big) \subseteq O_F \setminus \big(\sigma(T) \cup \{\infty, \phi_M^{-1}(\infty)\}\big), \quad j = 1, \dots, n\,,$$

and

$$\sum_{j=1}^n n(\phi_M \circ \alpha_j, z) = \begin{cases} 0\,, & z \notin \phi_M(O_F) \\ 1\,, & z \in \phi_M(\sigma(T)) \\ 0\,, & z = \phi_M(\infty) \text{ and } \infty \notin \sigma(T) \end{cases} \tag{8.2.3}$$ J18

Then the paths $\phi_M \circ \alpha_j$, $j = 1, \dots, n$, *satisfy* (8.2.1) *for* $\phi_M(O_F) \cap \mathbb{C}, \phi_M(\sigma(T))$, *and*

$$\frac{1}{2\pi i} \sum_{j=1}^{n} \int\limits_{\phi_M \circ \alpha_j} (F \circ \phi_M^{-1})(\zeta) \cdot (\zeta - \phi_M(T))^{-1} \, d\zeta =$$

$$= \begin{cases} \frac{1}{2\pi i} \sum\limits_{j=1}^{n} \int\limits_{\alpha_j} F(z) \cdot (z - T)^{-1} \, dz & , \quad \infty \notin \sigma(T) \\ F(\infty) I + \frac{1}{2\pi i} \sum\limits_{j=1}^{n} \int\limits_{\alpha_j} F(z) \cdot (z - T)^{-1} \, dz \, , & \infty \in \sigma(T) \end{cases} \tag{8.2.4}$$

J19

Proof. As a preliminary observation, let us compute the derivative of a fractional linear transformation:

$$\phi'_{\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)}(z) = \frac{d}{dz}\Big(\frac{\alpha z + \beta}{\gamma z + \delta}\Big) = \frac{\alpha(\gamma z + \delta) - (\alpha z + \beta)\gamma}{(\gamma z + \delta)^2} = \frac{\det M}{(\gamma z + \delta)^2} \, .$$

Let now F, M, and α_j be given according to the assumptions of the present lemma. The fact that $\gamma_j := \phi_M \circ \alpha_j$, $j = 1, \dots, n$, are closed piecewise smooth paths and satisfy (8.2.1) for $\phi_M(O_F) \cap \mathbb{C}, \phi_M(\sigma(T))$ is clear.

Consider a point $z \in \mathbb{C}_\infty \setminus \{\infty, \phi_M^{-1}(\infty)\}$. Then

$$z = \phi_{M^{-1}}(\phi_M(z)) = \frac{\delta \phi_M(z) - \beta}{-\gamma \phi_M(z) + \alpha} \, ,$$

and $-\gamma \phi_M(z) + \alpha \neq 0$. Using Lemma 7.2.1, we can compute

$$\big(\phi_M(T) - \phi_M(z)\big)^{-1} = \phi_{\left(\begin{smallmatrix} 0 & 1 \\ 1 & -\phi_M(z) \end{smallmatrix}\right)}\big(\phi_M(T)\big) = \phi_{\left(\begin{smallmatrix} 0 & 1 \\ 1 & -\phi_M(z) \end{smallmatrix}\right) M}(T) =$$

$$= \phi_{\left(\begin{smallmatrix} \gamma & \delta \\ \alpha - \phi_M(z)\gamma & \beta - \phi_M(z)\delta \end{smallmatrix}\right)}(T) =$$

$$= \frac{\gamma}{\alpha - \phi_M(z)\gamma} + \frac{\det M}{(\alpha - \phi_M(z)\gamma)^2} \cdot \Big(T + \underbrace{\frac{\beta - \phi_M(z)\delta}{\alpha - \phi_M(z)\gamma}}_{= -\phi_{M^{-1}}(\phi_M(z)) = -z}\Big)^{-1} =$$

$$= \frac{\gamma}{\alpha - \phi_M(z)\gamma} + \frac{\det M}{(\alpha - \phi_M(z)\gamma)^2} (T - z)^{-1} \, .$$

Since $\phi_M^{-1} = \phi_{\left(\begin{smallmatrix} \delta & -\beta \\ -\gamma & \alpha \end{smallmatrix}\right)}$, we have

$$\frac{\det M}{(\alpha - \gamma \phi_M(z))^2} = (\phi_M^{-1})'\big(\phi_M(z)\big) \, ,$$

and obtain

$$\int\limits_{\gamma_j} (F\circ\phi_M^{-1})(\zeta)\cdot(\zeta-\phi_M(T))^{-1}\,d\zeta =$$

$$= \int_0^1 F(\alpha_j(t))\cdot\big(\phi_M(\alpha_j(t))-\phi_M(T)\big)^{-1}\cdot\phi_M'(\alpha_j(t))\alpha_j'(t)\,dt =$$

$$= \int_0^1 F(\alpha_j(t))\cdot\frac{\gamma}{\gamma\phi_M(\alpha_j(t))-\alpha}\cdot\phi_M'(\alpha_j(t))\alpha_j'(t)\,dt +$$

$$+ \int_0^1 F(\alpha_j(t))\cdot(\phi_M^{-1})'(\phi_M(\alpha_j(t)))(\alpha_j(t)-T)^{-1}\cdot\phi_M'(\alpha_j(t))\alpha_j'(t)\,dt =$$

$$= \int\limits_{\gamma_j}(F\circ\phi_M^{-1})(\zeta)\cdot\frac{\gamma}{\gamma\zeta-\alpha}\,d\zeta + \int\limits_{\alpha_j} F(z)\cdot(z-T)^{-1}\,dz\,. \qquad (8.2.5)$$ J25

The second summand in this relation leads to the sum on the right side of (8.2.4). We need to have a closer look at the first summand in (8.2.5). If $\gamma = 0$, it vanishes. However, in this case, we have $\phi_M(\infty) = \infty$ and hence $\infty \notin \sigma(T)$. Thus (8.2.4) holds.

Assume that $\gamma \neq 0$ and $\frac{\alpha}{\gamma} \notin \phi_M(O_F)$. Then the integrand in the first summand in (8.2.5) is analytic in $\phi_M(O_F)\cap\mathbb{C}$. By (8.2.3) the Cauchy Integral Theorem applies, and it follows that the integral vanishes. However, $\frac{\alpha}{\gamma} \notin \phi_M(O_F)$ just means that $\infty \notin O_F$, and hence in particular $\infty \notin \sigma(T)$. Again (8.2.4) follows.

Assume that $\gamma \neq 0$ and $\frac{\alpha}{\gamma} \in \phi_M(O_F)$, i.e. $\infty \in O_F$. Then we use the Cauchy Integral Formula and (8.2.3) to evaluate

$$\sum_{j=1}^n \frac{1}{2\pi i}\int\limits_{\gamma_j}(F\circ\phi_M^{-1})(\zeta)\cdot\frac{1}{\zeta-\frac{\alpha}{\gamma}}\,d\zeta = \sum_{j=1}^n n(\gamma_j,\frac{\alpha}{\gamma})\cdot F(\phi_M^{-1}(\frac{\alpha}{\gamma}))$$

$$= \begin{cases} 0 & , \quad \infty\notin\sigma(T) \\ F(\infty)I\,, & \quad \infty\in\sigma(T)\end{cases}$$

The desired equality (8.2.4) follows also in this case. ❑

LEJ20 **8.2.6 Lemma.** *Let $F \in \mathbb{H}(\sigma(T))$ be given, and let $M, N \in \mathrm{GL}(2,\mathbb{C})$ be such that $\phi_M^{-1}(\infty), \phi_N^{-1}(\infty) \notin \sigma(T)$. Then there exist piecewise smooth paths $\alpha_1,\dots,\alpha_n : [0,1] \to \mathbb{C}_\infty$ which have the properties required in Lemma 8.2.5 for both matrices M and N in the same time.*

Proof. First we show a preliminary observation on winding numbers, namely: Let $G_1, G_2 \subseteq \mathbb{C}$ be open, $\psi : G_1 \to G_2$ analytic and bijective, and $\gamma_1,\dots,\gamma_n : [0,1] \to G_1$ closed piecewise smooth paths in G_1 with

$$\sum_{j=1}^n n(\gamma_j, z) = 0, \quad z \notin G_1\,.$$

Then $\sum_{j=1}^n n(\psi\circ\gamma_j,\psi(z)) = \sum_{j=1}^n n(\gamma_j,z)$, $z \in G_1$. To see this, compute

$$\int\limits_{\psi\circ\gamma_j}\frac{1}{\zeta-\psi(z)}\,d\zeta = \int_0^1\frac{1}{\psi(\gamma_j(t))-\psi(z)}\psi'(\gamma_j(t))\gamma_j'(t)\,dt = \int\limits_{\gamma_j}\frac{\psi'(\zeta)}{\psi(\zeta)-\psi(z)}\,d\zeta\,.$$

By the Theorem of Logarithmic Residues, thus

$$\sum_{j=1}^{n} n(\psi\circ\gamma_j,\psi(z)) = \sum_{j=1}^{n}\frac{1}{2\pi i}\int\limits_{\gamma_j}\frac{\psi'(\zeta)}{\psi(\zeta)-\psi(z)}\,d\zeta = \sum_{j=1}^{n} n(\gamma_j,z)\,.$$

Now we turn to the situation of the present lemma, i.e. assume that F, M, and N, are given. We apply Corollary 8.2.4 with

$$O := \begin{cases}\phi_M(O_F)\setminus\{\infty,\phi_M(\infty),\phi_M(\phi_N^{-1}(\infty))\}\,, & \infty\notin\sigma(T)\\ \phi_M(O_F)\setminus\{\infty,\phi_M(\phi_N^{-1}(\infty))\} & ,\ \ \infty\in\sigma(T)\end{cases},\quad K := \phi_M(\sigma(T))\,.$$

This gives paths $\gamma_1,\ldots,\gamma_n$. Set $\alpha_j := \phi_M^{-1}\circ\gamma_j$, then

$$\alpha_j\big([0,1]\big)\subseteq O_F\setminus\big(\sigma(T)\cup\{\phi_M^{-1}(\infty),\infty,\phi_N^{-1}(\infty)\}\big)\,,$$

$$\sum_{j=1}^{n} n(\phi_M\circ\alpha_j,z) = \sum_{j=1}^{n} n(\gamma_j,z) = \begin{cases}0\,, & z\notin\phi_M(O_F)\\ 1\,, & z\in\phi_M(\sigma(T))\\ 0\,, & z=\phi_M(\infty)\text{ and }\infty\notin\sigma(T)\\ 0\,, & z=\phi_M(\phi_N^{-1}(\infty))\end{cases}$$

In particular, the paths α_j satisfy all requirements of Lemma 8.2.5 for the matrix M.

The map $\phi_N\circ\phi_M^{-1}$ is an analytic bijection of $\mathbb{C}\setminus\{\phi_M(\phi_N^{-1}(\infty))\}$ onto $\mathbb{C}\setminus\{\phi_N(\phi_M^{-1}(\infty))\}$, and we have $\phi_N\circ\alpha_j = (\phi_N\circ\phi_M^{-1})\circ\gamma_j$. The above preliminary observation gives

$$\sum_{j=1}^{n} n\big(\phi_N\circ\alpha_j,\phi_N(\phi_M^{-1}(z))\big) = \sum_{j=1}^{n} n(\gamma_j,z)\,,$$

and in turn

$$\sum_{j=1}^{n} n\big(\phi_N\circ\alpha_j,w) = \begin{cases}0\,, & w\notin\phi_N(O_F)\\ 1\,, & w\in\phi_N(\sigma(T))\\ 0\,, & w=\phi_N(\infty)\text{ and }\infty\notin\sigma(T)\\ 0\,, & w=\phi_N(\phi_M^{-1}(\infty))\end{cases}$$

The last line thereby follows since $\sum_{j=1}^{n} n\big(\phi_N\circ\alpha_j,w)$ is a continuous function of w, and since $n(\gamma_j,z)=0$ when z lies in the unbounded component of $\mathbb{C}\setminus\bigcup_{j=1}^{n}\gamma_j([0,1])$. We see that the paths α_j also satisfy all requirements of Lemma 8.2.5 for the matrix N. ❑

Proof (of Proposition 8.2.2; Part 2). Next we show that the right side of (8.2.2) does not depend on the choice of M. Let M and N be given with $\phi_M^{-1}(\infty),\phi_N^{-1}(\infty)\notin\sigma(T)$. Choose paths α_j according to Lemma 8.2.6. Then, by Lemma 8.2.5, the right sides of (8.2.2) with M and $\phi_M\circ\alpha_j$ on the one hand, and with N and $\phi_N\circ\alpha_j$ on the other coincide.

In order to finish the proof of Proposition 8.2.2, it remains to show that the right side of (8.2.2) does not depend on the choice of the representant. Assume that $F_1\sim F_2$, i.e. $F_1|_O = F_2|_O$ for some open set O with $\sigma(T)\subseteq O$. Choose $M\in\mathrm{GL}(2,\mathbb{C})$ with $\phi_M^{-1}(\infty)\notin\sigma(T)$ and choose paths which satisfy (8.2.1) for $\phi_M(O)\cap\mathbb{C},\phi_M(\sigma(T))$. Then these paths may be used in (8.2.2) for both F_1 and F_2. It follows that the right sides of (8.2.2) are the same when buildt with F_1 or F_2. ❑

8.3 Properties of Φ_{RD}^{T}

In the below theorem we collect the main properties of the Riesz-Dunford functional calculus.

From now on we will often drop explicit distinction between functions $F \in H(K)$ and equivalence classes $\underline{F} \in \mathbb{H}(K)$. For example, we will often write $\Phi_{\mathrm{RD}}^{T}(F)$ when $\underline{F} \in H(K)$.

THJ21 **8.3.1 Theorem.** *Let $\mathcal{X}$ be a Banach space and $T \in \mathrm{CLR}(\mathcal{X})$ with $\rho(T) \neq \emptyset$.*

(i) The map Φ_{RD}^{T} is a continuous algebra homomorphism of $\mathbb{H}(\sigma(T))$ into $\mathcal{B}(\mathcal{X})$.

(ii) Let $F \in \mathbb{C}(z) \cap \mathbb{H}(\sigma(T))$, and write

$$F(z) = a + \sum_{k=1}^{\eth_F(\infty)} b_k z^k + \sum_{w\in\mathbb{C}} \sum_{k=1}^{\eth_F(w)} \frac{c_{w,k}}{(z-w)^k}$$

with $a, b_k, c_{w,k} \in \mathbb{C}$. Then

$$\Phi_{\mathrm{RD}}^{T}(F) = aI + \sum_{k=1}^{\eth_F(\infty)} b_k T^k + \sum_{w\in\mathbb{C}} \sum_{k=1}^{\eth_F(w)} c_{w,k}(T-w)^{-k}\,.$$

(iii) Let $M \in \mathrm{GL}(2,\mathbb{C})$ and $G \in \mathbb{H}(\sigma(\phi_M(T)))$. Then $G \circ \phi_M \in \mathbb{H}(\sigma(T))$ and we have

$$\Phi_{\mathrm{RD}}^{T}(G \circ \phi_M) = \Phi_{\mathrm{RD}}^{\phi_M(T)}(G)\,.$$

In particular, if $\phi_M^{-1}(\infty) \notin \sigma(T)$, then $\Phi_{\mathrm{RD}}^{T}(\phi_M) = \phi_M(T)$.

(iv) The Spectral Mapping Theorem*: Whenever $F \in \mathbb{H}(\sigma(T))$, we have*

$$\sigma(\Phi_{\mathrm{RD}}^{T}(F)) = F(\sigma(U))\,.$$

(v) Let $F \in \mathbb{H}(\sigma(T))$ and $G \in \mathbb{H}(\sigma(\Phi_{\mathrm{RD}}^{T}(F)))$. Then $G \circ F \in \mathbb{H}(\sigma(T))$, and we have

$$\Phi_{\mathrm{RD}}^{T}(G \circ F) = \Phi_{\mathrm{RD}}^{\Phi_{\mathrm{RD}}^{T}(F)}(G)\,.$$

If $G(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ is a Laurent series whose domain of convergence contains $\sigma(\Phi_{\mathrm{RD}}^{T}(F))$, then the series $\sum_{n=-\infty}^{\infty} a_n(\Phi_{\mathrm{RD}}^{T}(F)-z_0)^n$ converges in the norm of $\mathcal{B}(\mathcal{X})$, and its sum is equal to $\Phi_{\mathrm{RD}}^{T}(G \circ F)$.

(vi) Let $S \in \mathcal{B}(\mathcal{X})$, and let $O \subseteq \mathbb{C}_\infty$ be open with $\sigma(T) \subseteq O$. Then we have $S(T-w)^{-1} = (T-w)^{-1}S$, $w \in O \cap \rho(T) \cap \mathbb{C}$, if and only if $S\Phi_{\mathrm{RD}}^{T}(F) = \Phi_{\mathrm{RD}}^{T}(F)S$, $F \in \mathbb{H}(\sigma(T))$.

Before we start the proof of this theorem, let us note that the assertion in item (iii) is not a particular case of item (v): in (iii) we do not require that $\phi_M \in \mathbb{H}(\sigma(T))$.

Proof.
Step 1; Compatibility with '$+$', '$\lambda\cdot$', '$\cdot$': Fix $M \in \mathrm{GL}(2,\mathbb{C})$ with $\phi_M^{-1}(\infty) \notin \sigma(T)$.

The fact that $\Phi^T_{\mathrm{RD}}(\lambda F) = \lambda\Phi^T_{\mathrm{RD}}(F)$ is obvious. Let $F, G \in \mathbb{H}(\sigma(T))$ be given. Choose paths which satisfy (8.2.1) for

$$\phi_M(O_F) \cap \phi_M(O_G) \cap \mathbb{C}, \phi_M(\sigma(T)),$$

then these paths may be used in the definition of both $\Phi^T_{\mathrm{RD}}(F)$ and $\Phi^T_{\mathrm{RD}}(G)$. It follows that

$$\Phi^T_{\mathrm{RD}}(F+G) = \Phi^T_{\mathrm{RD}}(F) + \Phi^T_{\mathrm{RD}}(G).$$

Multiplicativity is not so straightforward. First choose paths $\gamma_1, \ldots, \gamma_n$ which satisfy (8.2.1) for $\phi_M(O_F) \cap \phi_M(O_G) \cap \mathbb{C}, \phi_M(\sigma(T))$. Once this is done, choose paths $\gamma_1', \ldots, \gamma_m'$ which satisfy (8.2.1) for

$$O' := \bigcup \left\{ \begin{array}{l} \text{connected components of } \phi_M(O_F) \cap \phi_M(O_G) \cap \mathbb{C} \\ \text{which intersect } \phi_M(\sigma(T)) \end{array} \right\}, \quad \phi_M(\sigma(T)).$$

We use γ_j to compute $\Phi^T_{\mathrm{RD}}(F)$, and γ_l' to compute $\Phi^T_{\mathrm{RD}}(G)$. Doing so, gives (for abbreviation set $f(z) := (F \circ \phi_M^{-1})(z)$ and $g(z) := (G \circ \phi_M^{-1})(z))$

$$\begin{aligned}
&\Phi^T_{\mathrm{RD}}(F) \cdot \Phi^T_{\mathrm{RD}}(G) = \\
&\quad = \frac{1}{2\pi i} \sum_{j=1}^n \int_{\gamma_j} f(\zeta)(\zeta - \phi_M(T))^{-1}\, d\zeta \cdot \frac{1}{2\pi i} \sum_{l=1}^m \int_{\gamma_l'} g(\lambda)(\lambda - \phi_M(T))^{-1}\, d\lambda = \\
&\quad = \Big(\frac{1}{2\pi i}\Big)^2 \sum_{j=1}^n \sum_{l=1}^m \int_{\gamma_j} \int_{\gamma_l'} f(\zeta) g(\lambda) \cdot (\zeta - \phi_M(T))^{-1}(\lambda - \phi_M(T))^{-1}\, d\lambda\, d\zeta = \\
&\quad = \Big(\frac{1}{2\pi i}\Big)^2 \sum_{j=1}^n \sum_{l=1}^m \int_{\gamma_j} \int_{\gamma_l'} f(\zeta) g(\lambda) \frac{(\zeta - \phi_M(T))^{-1} - (\lambda - \phi_M(T))^{-1}}{\lambda - \zeta}\, d\lambda\, d\zeta = \\
&\quad = \frac{1}{2\pi i} \sum_{j=1}^n \int_{\gamma_j} f(\zeta)(\zeta - \phi_M(T))^{-1} \cdot \Big(\frac{1}{2\pi i} \sum_{l=1}^m \int_{\gamma_l'} \frac{g(\lambda)}{\lambda - \zeta}\, d\lambda \Big)\, d\zeta + \\
&\qquad + \frac{1}{2\pi i} \sum_{l=1}^m \int_{\gamma_l'} g(\lambda)(\lambda - \phi_M(T))^{-1} \cdot \Big(\frac{1}{2\pi i} \sum_{j=1}^n \int_{\gamma_j} \frac{f(\zeta)}{\zeta - \lambda}\, d\zeta \Big)\, d\lambda.
\end{aligned}$$

If $\zeta \in \bigcup_{j=1}^n \gamma_j([0,1])$, then $\zeta \notin O'$. Thus, the inner integral in the first summand evaluates as

$$\frac{1}{2\pi i} \sum_{l=1}^m \int_{\gamma_l'} \frac{g(\lambda)}{\lambda - \zeta}\, d\lambda = g(\zeta) \sum_{l=1}^m n(\gamma_l', \zeta) = 0.$$

If $\lambda \in \bigcup_{l=1}^m \gamma_l([0,1])$, then λ lies in the same connected component of O' as some point of $\phi_M(\sigma(T))$, and hence $\sum_{j=1}^n n(\gamma_j, \lambda) = 1$. The inner integral in the second summand thus evaluates as

$$\frac{1}{2\pi i} \sum_{k=1}^n \int_{\gamma_k} \frac{f(\zeta)}{\zeta - \lambda}\, d\zeta = f(\lambda) \sum_{j=1}^n n(\gamma_j, \lambda) = f(\lambda).$$

Together, we obtain $\Phi^T_{\mathrm{RD}}(F) \cdot \Phi^T_{\mathrm{RD}}(G) = \Phi^T_{\mathrm{RD}}(FG)$.

Step 2; Computation of $\Phi^T_{\mathrm{RD}}(z^n \circ \phi_M)$: Let $M \in \mathrm{GL}(2,\mathbb{C})$ be such that $\phi_M^{-1}(\infty) \notin \sigma(T)$, and let $n \in \mathbb{N}_0$. In order to compute $\Phi^T_{\mathrm{RD}}(z^n \circ \phi_M)$, we use the chart ϕ_M and the path $\gamma(t) := 2\|\phi_M(T)\|e^{it}$, $t \in [0, 2\pi]$. This gives

$$\Phi^T_{\mathrm{RD}}(z^n \circ \phi_M) = \frac{1}{2\pi i}\int\limits_\gamma \zeta^n \cdot (\zeta - \phi_M(T))^{-1}\, d\zeta \,.$$

The Neumann series $(\zeta - \phi_M(T))^{-1} = \sum_{k=0}^\infty \frac{1}{z^{k+1}}\phi_M(T)^k$ converges uniformly on γ, and hence

$$\frac{1}{2\pi i}\int\limits_\gamma \zeta^n(\zeta - \phi_M(T))^{-1}\, d\zeta = \frac{1}{2\pi i}\int\limits_\gamma \sum_{k=0}^\infty \zeta^{n-k-1}\phi_M(T)^k\, d\zeta =$$

$$= \sum_{k=0}^\infty \Big(\frac{1}{2\pi i}\int\limits_\gamma \zeta^{n-k-1}\, d\zeta\Big)\phi_M(T)^k = \phi_M(T)^n \,.$$

We see that $\Phi^T_{\mathrm{RD}}(z^n \circ \phi_M) = \phi_M(T)^n$. Note that, in particular, $\Phi^T_{\mathrm{RD}}(1) = I$.

Step 3; Rational functions: Let $F \in \mathbb{C}(z) \cap \mathbb{H}(\sigma(T))$ be given, and write F as in the statement of (ii). We already know that $\Phi^T_{\mathrm{RD}}(a) = aI$. If the first sum appears, i.e. if $\mathfrak{d}_F(\infty) > 0$, we must have $\infty \in \rho(T)$. Hence, we obtain from the computation in Step 2 that

$$\Phi^T_{\mathrm{RD}}(z^k) = \Phi^T_{\mathrm{RD}}(z^k \circ \phi_I) = T^k \,.$$

Consider a summand $\frac{1}{(z-w)^k}$. Again, occurance of such a term in the representation of F implies that $w \in \rho(T)$, and hence we may apply Step 2 with $M := \begin{pmatrix} 0 & 1 \\ 1 & -w \end{pmatrix}$. This gives

$$\Phi^T_{\mathrm{RD}}\Big(\frac{1}{(z-w)^k}\Big) = \Phi^T_{\mathrm{RD}}\Big(z^k \circ \phi_{\begin{pmatrix} 0 & 1 \\ 1 & -w \end{pmatrix}}\Big) = \phi_{\begin{pmatrix} 0 & 1 \\ 1 & -w \end{pmatrix}}(T)^k = (T-w)^{-k} \,.$$

Step 4; Continuity: In order to see continuity of Φ^T_{RD}, let an open set O with $\sigma(T) \subseteq O$ be given, and consider the map $\Phi^T_{\mathrm{RD}} \circ (\underline{\pi} \circ \iota_O) : \mathbb{H}(O) \to \mathcal{B}(\mathcal{X})$. Let $F_n, F \in \mathbb{H}(O)$ with $F_n \to F$ locally uniformly. If $M \in \mathrm{GL}(2,\mathbb{C})$ and paths γ_j are paths which satisfy (8.2.1) for $\phi_M(O) \cap \mathbb{C}$, $\phi_M(\sigma(T))$, then these paths are suitable forcomputing all operators $\Phi^T_{\mathrm{RD}}(F_n)$, $n \in \mathbb{N}$, and $\Phi^T_{\mathrm{RD}}(F)$. However, $F_n \circ \phi_M^{-1}$ converges to $F \circ \phi_M^{-1}$ uniformly on $\bigcup_{j=1}^n \gamma_j([0,1])$, and hence

$$(F_n \circ \phi_M^{-1})(\zeta) \cdot (\zeta - \phi_M(T))^{-1} \to (F \circ \phi_M^{-1})(\zeta) \cdot (\zeta - \phi_M(T))^{-1}$$

in $\mathcal{B}(\mathcal{X})$ uniformly on $\bigcup_{j=1}^n \gamma_j([0,1])$. Thus, $\Phi^T_{\mathrm{RD}}(F_n) \to \Phi^T_{\mathrm{RD}}(F)$ in $\mathcal{B}(\mathcal{X})$.

Step 5; Proof of (iii): Assume that M and G are given according to (iii). We have $\sigma(\phi_M(T)) = \phi_M(\sigma(T))$ and $O_{G\circ\phi_M} = \phi_M^{-1}(O_G)$. Hence indeed $G \circ \phi_M \in \mathbb{H}(\sigma(T))$. Choose $N \in \mathrm{GL}(2,\mathbb{C})$ and paths γ_j with

$$\phi_N^{-1}(\infty) \notin \sigma(T), \qquad \gamma_j \text{ satisfy (8.2.1) for } \phi_N(O_{G\circ\phi_M}) \cap \mathbb{C}, \phi_N(\sigma(T))\,,$$

i.e. N and γ_j suitable for the computation of $\Phi_{\mathrm{RD}}^T(G \circ \phi_M)$. We have

$$\phi_{NM^{-1}}^{-1}(\infty) = \phi_M(\phi_N^{-1}(\infty)) \notin \phi_M(\sigma(T)) = \sigma(\phi_M(T)),$$

and

$$\phi_{NM^{-1}}(O_G) = \phi_N(\phi_M^{-1}(O_G)) = \phi_N(O_{G\circ\phi_M}),$$
$$\phi_{NM^{-1}}\big(\sigma(\phi_M(T))\big) = (\phi_N \circ \phi_{M^{-1}})\big(\phi_M\sigma((T))\big) = \phi_N(\sigma(T)).$$

Hence, the matrix NM^{-1} and the paths γ_j are suitable to compute $\Phi_{\mathrm{RD}}^{\phi_M(T)}(G)$. Doing so gives

$$\Phi_{\mathrm{RD}}^T(G \circ \phi_M) = \frac{1}{2\pi i}\sum_{j=1}^n \int_{\gamma_j} \big((G\circ\phi_M)\circ\phi_N^{-1}\big)(\zeta)\cdot(\zeta - \phi_N(T))^{-1}\,d\zeta =$$

$$= \frac{1}{2\pi i}\sum_{j=1}^n \int_{\gamma_j} (G\circ\phi_{NM^{-1}})(\zeta)\cdot\big(\zeta - \phi_{NM^{-1}}(\phi_M(T))\big)^{-1}\,d\zeta = \Phi_{\mathrm{RD}}^{\phi_M(T)}(G).$$

Step 6; The Spectral Mapping Theorem: We start with showing the inclusion '$\sigma(\Phi_{\mathrm{RD}}^T(F)) \subseteq F(\sigma(T))$'. Since $\infty \notin \sigma(\Phi_{\mathrm{RD}}^T(F))$, it suffices to show that

$$\mathbb{C} \setminus F(\sigma(T)) \subseteq \rho(\Phi_{\mathrm{RD}}^T(F)).$$

Let $w \in \mathbb{C} \setminus F(\sigma(T))$, then the function $G(z) := (F(z) - w)^{-1}$ belongs to $\mathbb{H}(\sigma(T))$, and we have

$$\Phi_{\mathrm{RD}}^T(G)\big(\Phi_{\mathrm{RD}}^T(F) - w\big) = \Phi_{\mathrm{RD}}^T(G)\Phi_{\mathrm{RD}}^T(F - w) = \Phi_{\mathrm{RD}}^T(1) = I.$$

This shows that $w \in \rho(\Phi_{\mathrm{RD}}^T(F))$, in fact

$$\big(\Phi_{\mathrm{RD}}^T(F) - w\big)^{-1} = \Phi_{\mathrm{RD}}^T\Big(\frac{1}{F(z) - w}\Big), \quad w \notin F(\sigma(T)). \qquad (8.3.1)$$ J24

For the reverse inclusion, assume first that $\infty \notin \sigma(T)$. Let $w \in \sigma(T)$ be given, then the function $G(z) := \frac{F(z)-F(w)}{z-w}$ belongs to $\mathbb{H}(\sigma(T))$, and we have

$$(T - w)\Phi_{\mathrm{RD}}^T(G) = \Phi_{\mathrm{RD}}^T(z - w)\Phi_{\mathrm{RD}}^T(G) = \Phi_{\mathrm{RD}}^T(F) - F(w).$$

If we had $F(w) \in \rho(\Phi_{\mathrm{RD}}^T(F))$, then also $(T - w)$ had a bounded inverse. It follows that $F(w) \in \sigma(\Phi_{\mathrm{RD}}^T(F))$.

The general case now follows easily. Choose $M \in \mathrm{GL}(2,\mathbb{C})$ with $\phi_M^{-1}(\infty) \notin \sigma(T)$. Then, by the above paragraph, the already proved item (*iii*), and the Spectral Mapping Theorem for fractional linear transformations, we obtain

$$\sigma\big(\Phi_{\mathrm{RD}}^T(F)\big) = \sigma\big(\Phi_{\mathrm{RD}}^{\phi_M(T)}(F\circ\phi_M^{-1})\big) = (F\circ\phi_M^{-1})\big(\sigma(\phi_M(T))\big) = F\big(\sigma(T)\big).$$

Step 7; Proof of (v)*:* Let F and G be given as in the statement of (*iv*). By the Spectral Mapping Theorem, indeed $G \circ F \in \mathbb{H}(\sigma(T))$.

The relation $\Phi_{\mathrm{RD}}^T(F)$ is a bounded operator, and hence we may compute $\Phi_{\mathrm{RD}}^{\Phi_{\mathrm{RD}}^T(F)}(G)$ using the chart Φ_I. Moreover, we choose paths $\gamma_1, \dots, \gamma_n$ which satisfy (8.2.1) for $O_G \cap \mathbb{C}, \sigma(\Phi_{\mathrm{RD}}^T(F))$. Then

$$\Phi_{\mathrm{RD}}^{\Phi_{\mathrm{RD}}^T(F)}(G) = \frac{1}{2\pi i}\sum_{j=1}^n \int_{\gamma_j} G(\zeta)\cdot\big(\zeta - \Phi_{\mathrm{RD}}^T(F)\big)^{-1}\,d\zeta. \qquad (8.3.2)$$ J22

Next, we choose paths appropriate for the definition of $\Phi^T_{\mathrm{RD}}(G\circ F)$. Set

$$O' := \bigcup \left\{ \begin{array}{l} \text{connected components of } (O_G\cap\mathbb{C})\setminus\bigcup_{j=1}^n \gamma_j([0,1]) \\ \text{which intersect } \sigma(\Phi^T_{\mathrm{RD}}(F)) = F(\sigma(T)) \end{array} \right\}$$

and

$$O'' := O_{G\circ F}\cap\big(F^{-1}(O')\setminus\{\phi_M^{-1}(\infty)\}\big)\,.$$

Then O'' is an open subset of $\mathbb{C}_\infty$ which contains $\sigma(T)$, and the function $G\circ F$ is analytic on O''. The set $\phi_M(O'')$ is an open subset of $\mathbb{C}$ which contains $\phi_M(\sigma(T))$. Choose paths $\gamma_1',\dots,\gamma_m'$ which satisfy (8.2.1) for $\phi_M(O''),\phi_M(\sigma(T))$. Then

$$\Phi^T_{\mathrm{RD}}(G\circ F) = \frac{1}{2\pi i}\sum_{l=1}^m \int_{\gamma_l'} (G\circ F\circ\phi_M^{-1})(\lambda)\cdot\big(\lambda-\phi_M(T)\big)^{-1}\,d\lambda\,. \tag{8.3.3}$$

J23

If $z\in O''$ then $F(z)\in O'$, and hence $F(z)\notin\bigcup_{j=1}^n\gamma_j([0,1])$. In other words, we have

$$O_{(\zeta-F(z))^{-1}}\supseteq O'',\quad \zeta\in\bigcup_{j=1}^n\gamma_j([0,1])\,.$$

The paths γ_l' satisfy (8.2.1) for $\phi_M(O_{(\zeta-F(z))^{-1}})$, and hence

$$\Phi^T_{\mathrm{RD}}\Big(\frac{1}{\zeta-F(z)}\Big) = \frac{1}{2\pi i}\sum_{l=1}^m\int_{\gamma_l'}\frac{1}{\zeta-(F\circ\phi_M^{-1})(\lambda)}\cdot\big(\lambda-\phi_M(T)\big)^{-1}\,d\lambda\,.$$

However, as we saw in (8.3.1), $\Phi^T_{\mathrm{RD}}\big((\zeta-F(z))^{-1}\big)=(\zeta-\Phi^T_{\mathrm{RD}}(F))^{-1}$. Substituting this into (8.3.2) gives

$$\begin{aligned}
\Phi_{\mathrm{RD}}^{\Phi^T_{\mathrm{RD}}(F)}(G) &= \\
&=\Big(\frac{1}{2\pi i}\Big)^2\sum_{j=1}^n\int_{\gamma_j}G(\zeta)\Big(\sum_{l=1}^m\int_{\gamma_l'}\frac{1}{\zeta-(F\circ\phi_M^{-1})(\lambda)}\big(\lambda-\phi_M(T)\big)^{-1}\,d\lambda\Big)\,d\zeta = \\
&=\frac{1}{2\pi i}\sum_{l=1}^m\int_{\gamma_l'}\Big(\frac{1}{2\pi i}\sum_{j=1}^n\int_{\gamma_j}\frac{G(\zeta)}{\zeta-(F\circ\phi_M^{-1})(\lambda)}\,d\zeta\Big)\cdot\big(\lambda-\phi_M(T)\big)^{-1}\,d\lambda\,.
\end{aligned}$$

The interchange of integrals is thereby justified since the integrand is analytic on the compact domain of integration.

If $\lambda\in\bigcup_{l=1}^m\gamma_l'([0,1])$ then $(F\circ\phi_M^{-1})(\lambda)\in O'$, and hence belongs to the same connected component of $(O_G\cap\mathbb{C})\setminus\bigcup_{j=1}^n\gamma_j([0,1])$ as some point of $\sigma(\Phi^T_{\mathrm{RD}}(F))$. Thus the inner integral evalutes as

$$\frac{1}{2\pi i}\sum_{j=1}^n\int_{\gamma_j}\frac{G(\zeta)}{\zeta-(F\circ\phi_M^{-1})(\lambda)}\,d\zeta = G\big((F\circ\phi_M^{-1})(\lambda)\big)\underbrace{\sum_{j=1}^n n\big(\gamma_j,(F\circ\phi_M^{-1})(\lambda)\big)}_{=1}\,.$$

Comparing with (8.3.3) shows that $\Phi_{\mathrm{RD}}^{\Phi^T_{\mathrm{RD}}(F)}(G)=\Phi^T_{\mathrm{RD}}(G\circ F)$.

Assume now that G is a Laurent series as in the statement of (v). For each $N \in \mathbb{N}$ we have

$$\Phi_{\mathrm{RD}}^{T}\Big(\sum_{n=-N}^{N} a_n (z-z_0)^n\Big) = \sum_{n=-N}^{N} a_n (\Phi_{\mathrm{RD}}^{T}(F) - z_0)^n \,.$$

Continuity of Φ_{RD}^{T} implies the desired assertion.

Step 8; Proof of (vi): Assume first that S commutes with all operators $\Phi_{\mathrm{RD}}^{T}(F)$. If $w \in \rho(T) \cap \mathbb{C}$, then $\frac{1}{z-w} \in \mathbb{H}(\sigma(T))$, and hence $(T-w)^{-1} = \Phi_{\mathrm{RD}}^{T}(\frac{1}{z-w})$. Thus S commutes with all resolvents $(T-w)^{-1}$, $w \in \rho(T) \cap \mathbb{C}$.

For the converse, assume that S commutes with $(T-w)^{-1}$ for all $w \in O \cap \rho(T) \cap \mathbb{C}$, and let $F \in H(\sigma(T))$ be given. Without loss of generality, we may assume that $O_F \subseteq O$. Choose $M \in \mathrm{GL}(2,\mathbb{C})$ with $\phi_M^{-1}(\infty) \notin \sigma(T)$, and choose paths α_j which satisfy the hypothesis of Lemma 8.2.5. By Lemma 8.2.6 this choice of paths is certainly possible. Then the relation (8.2.4) proved in Lemma 8.2.5 implies that S commutes with $\Phi_{\mathrm{RD}}^{T}(F)$. ❑

If the spectrum of T splits into a disjoint union of finitely many relatively open subsets, the Riesz-Dunford functional calculus together with the algebraic decomposition result Proposition 8.1.7 can be used to obtain a decomposition of the space $\mathcal{X}$ into T-invariant subspaces.

DEJ2 **8.3.2 Definition.** Let $\mathcal{X}$ be a Banach space, $T \in \mathrm{CLR}(\mathcal{X})$ with $\rho(T) \neq \emptyset$, and assume that $\sigma(T)$ is the disjoint union of nonempty and relatively open subsets $\sigma_1, \dots, \sigma_n$. Let $\lambda_i : \mathbb{H}(\sigma_i) \to \mathbb{H}(\sigma(T))$ be the map constructed in the proof of Proposition 8.1.7. Then we denote

$$\Phi_{\mathrm{RD}}^{T,\sigma_i} := \Phi_{\mathrm{RD}}^{T} \circ \lambda_i : \mathbb{H}(\sigma_i) \to \mathcal{B}(\mathcal{X})\,.$$

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As a composition of continuous algebra homomorphisms, the maps $\Phi_{\mathrm{RD}}^{T,\sigma_i}$ are themselves continuous algebra homomorphisms.

PRJ3 **8.3.3 Proposition.** *Let $\mathcal{X}$ be a Banach space, $T \in \mathrm{CLR}(\mathcal{X})$ with $\rho(T) \neq \emptyset$, and assume that $\sigma(T)$ splits into the disjoint union of relatively open subsets $\sigma_1, \dots, \sigma_n$. Set*

$$P_i := \Phi_{\mathrm{RD}}^{T,\sigma_i}(1), \quad i = 1, \dots, n\,.$$

Then the operators P_i are continuous projections, $P_iP_j = P_jP_i = 0$, $i \neq j$, and $P_1 + \ldots + P_n = I$.

Set $\mathcal{X}_i := \mathrm{ran}\, P_i$, $i = 1, \dots, n$. Then each space $\mathcal{X}_i$ is closed and $\mathcal{X} = \mathcal{X}_1 \dot{+} \ldots \dot{+} \mathcal{X}_n$. This decomposition of $\mathcal{X}$ reduces T, and we have

$$\sigma\big(T \cap \mathcal{X}_i^2\big) = \sigma_i\,. \qquad (8.3.4)$$ J35

Proof. Consider the elements

$$x_i := (0, \dots, 0, \underset{\substack{\uparrow \\ i\text{-th place}}}{1}, 0, \dots, 0) \in \mathbb{H}(\sigma_1) \times \ldots \times \mathbb{H}(\sigma_n), \quad i = 1, \dots, n\,.$$

Clearly,

$$x_i x_j = \begin{cases} x_i\,, & i = j \\ 0\;, & i \neq j \end{cases}\,, \qquad \sum_{i=1}^n x_1 = 1\,.$$

Let $\lambda : \mathbb{H}(\sigma_1) \times \ldots \times \mathbb{H}(\sigma_n) \to \mathbb{H}(\sigma(T))$ be the algebra homomorphism constructed in Proposition 8.1.7. Applying $\Phi_{\mathrm{RD}}^T \circ \lambda$, gives

$$(\Phi_{\mathrm{RD}}^T \circ \lambda)(x_i)(\Phi_{\mathrm{RD}}^T \circ \lambda)(x_j) = \begin{cases} (\Phi_{\mathrm{RD}}^T \circ \lambda)(x_i)\,, & i = j \\ 0 \qquad\qquad\qquad , & i \neq j \end{cases}\,, \qquad \sum_{i=1}^n (\Phi_{\mathrm{RD}}^T \circ \lambda)(x_1) = I\,.$$

However, by definition, $(\Phi_{\mathrm{RD}}^T \circ \lambda)(x_i) = \Phi_{\mathrm{RD}}^{T,\sigma_i}(1)$. This shows that the operators P_i are projections, that $P_i P_j = P_j P_i = 0$, $i \neq j$, and that $P_1 + \ldots + P_n = I$.

It remains to show that each space $\operatorname{ran} P_i$ is T-invariant. However, $\Phi_{\mathrm{RD}}^T(\mathbb{H}(\sigma(T)))$ is a commutative subalgebra of $\mathcal{B}(\mathcal{X})$ which contains all resolvents $(T-w)^{-1}$, $w \in \rho(T) \cap \mathbb{C}$ and T in case $\infty \in \rho(T)$. Hence, P_i commutes with all resolvents, and this just says that

$$(T-z)^{-1}(\operatorname{ran} P_i) \subseteq \operatorname{ran} P_i, \quad i = 1, \ldots, n\,.$$

Proposition 7.3.8 implies that $\mathcal{X} = \mathcal{X}_1 \dot{+} \ldots \dot{+} \mathcal{X}_n$ reduces T.

By Proposition 7.3.8 we have $\sigma(T) = \bigcup_{i=1}^n \sigma(T \cap \mathcal{X}_i^2)$. Hence, in order to show (8.3.4), it is enough to show that $\sigma(T \cap \mathcal{X}_i^2) \subseteq \sigma_i$.

We consider first the case that $\infty \in \rho(T)$. Let $w \in \mathbb{C}$, then $w \in \rho(T \cap \mathcal{X}_i^2)$ if and only if there exists an operator $S \in \mathcal{B}(\mathcal{X})$ with

$$S(T-w) = (T-w)S = P_i\,.$$

However, if $w \notin \sigma_i$, we have

$$\Phi_{\mathrm{RD}}^{T,\sigma_i}\Big(\frac{1}{z-w}\Big) \cdot (T-w) = \Phi_{\mathrm{RD}}^{T,\sigma_i}\Big(\frac{1}{z-w}\Big) \cdot \Phi_{\mathrm{RD}}^T(z-w) = \Phi_{\mathrm{RD}}^{T,\sigma_i}(1) = P_i\,,$$

and hence have found an operator with this property. Thus the required inclusion $\sigma(T \cap \mathcal{X}_i^2) \subseteq \sigma_i$ holds.

Next, reduce the general case to the already treated on with help of fractional linear transformations. Let $M \in \mathrm{GL}(2,\mathbb{C})$, and consider the relation $\phi_M(T)$. Then $\sigma(\phi_M(T)) = \phi_M(\sigma(T))$. Hence, $\sigma(\phi_M(T))$ is the disjoint union of the nonempty and relatively open subsets $\phi_M(\sigma_i)$, $i = 1, \ldots, n$. We have the diagram

$$\begin{tikzcd}
\mathbb{H}\big(\phi_M(\sigma_i)\big) \arrow[r, "\lambda_i"] \arrow[d, "\circ\phi_M"'] \arrow[rrd, bend left, "\Phi_{\mathrm{RD}}^{\phi_M(T),\phi_M(\sigma_i)}"] & \mathbb{H}\big(\phi_M(\sigma(T))\big) \arrow[rd, "\Phi_{\mathrm{RD}}^{\phi_M(T)}"] \arrow[d, "\circ\phi_M"'] & \\
\mathbb{H}(\sigma_i) \arrow[r, "\lambda_i"'] \arrow[rr, bend right, "\Phi_{\mathrm{RD}}^{T,\sigma_i}"'] & \mathbb{H}(\sigma(T)) \arrow[r, "\Phi_{\mathrm{RD}}^T"'] & \mathcal{B}(\mathcal{X})
\end{tikzcd} \tag{8.3.5}$$

J33

It follows that $\Phi_{\mathrm{RD}}^{\phi_M(T),\phi_M(\sigma_i)}(1) = \Phi_{\mathrm{RD}}^{T,\sigma_i}(1)$, i.e. the reducing decomposition obtained for $\phi_M(T)$ is the same as for T, namely $\mathcal{X} = \mathcal{X}_1 \dot{+} \ldots \dot{+} \mathcal{X}_n$. It is immediate from the definition of ϕ_M that $\phi_M(T) \cap \mathcal{X}_i^2 = \phi_M(T \cap \mathcal{X}_i^2)$.

Making the choice of $M \in \mathrm{GL}(2,\mathbb{C})$ in the above paragraph such that $\phi_M^{-1}(\infty) \notin \sigma(T)$, we have $\phi_M(T) \in \mathcal{B}(\mathcal{X})$. Applying the already proved case to the bounded operator $\phi_M(T)$ gives $\sigma(\phi_M(T) \cap \mathcal{X}_i^2) \subseteq \phi_M(\sigma_i)$. Again using the Spectral Mapping Theorem for ϕ_M, we see that

$$\phi_M(\sigma(T \cap \mathcal{X}_i^2)) = \sigma(\phi_M(T) \cap \mathcal{X}_i^2) \subseteq \phi_M(\sigma_i)\,,$$

and hence $\sigma(T \cap \mathcal{X}_i^2) \subseteq \sigma_i$. ❑

If $\mathcal{X}$ is not only a Banach space but in fact a Krein space, we have an additional operation on $\mathcal{B}(\mathcal{K})$, namely conjugation. Let us show that the Riesz-Dunford functional calculus is compatible with conjugation in a most natural sense.

PRJ1 **8.3.4 Proposition.** *Let $\mathcal{K}$ be a Krein space, and $T \in \mathrm{CLR}(\mathcal{K})$ with $\rho(T) \neq \emptyset$. Since $\sigma(T^*) = \sigma(T)^\#$, the map $.^\#$ is a conjugate linear homeomorphism of $\mathbb{H}(\sigma(T))$ onto $\mathbb{H}(\sigma(T)^\#)$. We have*

$$\Phi_{\mathrm{RD}}^T(F)^* = \Phi_{\mathrm{RD}}^{T^*}(F^\#), \quad F \in \mathbb{H}(\sigma(T))\,.$$

Proof. Let $F \in H(\sigma(T))$ be given. In order to compute $\Phi_{\mathrm{RD}}^T(F)$, choose $M \in \mathrm{GL}(2,\mathbb{C})$ with $\phi_M^{-1}(\infty) \notin \sigma(T)$ and paths γ_j which satisfy (8.2.1) for $\phi_M(O_F) \cap \mathbb{C}, \phi_M(\sigma(T))$. Then we have

$$\Phi_{\mathrm{RD}}^T(F)^* = \Big(\frac{1}{2\pi i}\sum_{j=1}^n \int_{\gamma_j} (F \circ \phi_M^{-1})(\zeta)\cdot(\zeta - \phi_M(T))^{-1}\,d\zeta\Big)^* =$$

$$= \frac{1}{2\pi i}\sum_{j=1}^n \int_{\overline{\gamma_j}} (F \circ \phi_M^{-1})^\#(\zeta)\cdot(\zeta - \phi_M(T)^*)^{-1}\,d\zeta =$$

$$= \frac{1}{2\pi i}\sum_{j=1}^n \int_{\overline{\gamma_j}} (F^\# \circ \phi_{\overline{M}}^{-1})(\zeta)\cdot(\zeta - \phi_{\overline{M}}(T^*))^{-1}\,d\zeta\,.$$

Since $\phi_{\overline{M}}^{-1}(\infty) = \overline{\phi_M^{-1}(\infty)}$, we have $\phi_{\overline{M}}^{-1}(\infty) \notin \sigma(T^*)$. If we can show that the paths $\overline{\gamma_j}$ satisfy (8.2.1) for $\phi_{\overline{M}}(O_{F^\#}) \cap \mathbb{C}, \phi_{\overline{M}}(\sigma(T^*))$, we are done, since then the last sum of integrals in the above computation equals $\Phi_{\mathrm{RD}}^{T^*}(F^\#)$. In order to see this, however, it is enough to note that

$$\phi_{\overline{M}}(O_{F^\#}) \cap \mathbb{C} = (\phi_M(O_F) \cap \mathbb{C})^\#, \quad \phi_{\overline{M}}(\sigma(T^*)) = (\phi_M(\sigma(T)))^\#\,,$$

and that $n(\overline{\gamma_j}, \overline{w}) = n(\gamma_j, w)$. ❑

COJ34 **8.3.5 Corollary.** *Let $\mathcal{K}$ be a Krein space, $M \in \mathrm{GL}(2,\mathbb{C})$, and T a closed M-selfadjoint relation in $\mathcal{K}$ with $\rho(T) \neq \emptyset$. Then the spectrum of T is M-symmetric, cf. Proposition 7.6.3, and hence we have the conjugate linear homeomorphism $.^\square$ of $\mathbb{H}(\sigma(T))$ onto itself, cf. 8.1.6. With these notations it holds that*

$$\Phi_{\mathrm{RD}}^T(F)^* = \Phi_{\mathrm{RD}}^T(F^\square), \quad F \in \mathbb{H}(\sigma(T))\,.$$

Proof. Applying Proposition 8.3.4 and Theorem 8.3.1, (*iii*), gives

$$\Phi_{\mathrm{RD}}^T(F)^* = \Phi_{\mathrm{RD}}^{T^*}(F^\#) = \Phi_{\mathrm{RD}}^{\phi_M(T)}(F^\#) = \Phi_{\mathrm{RD}}^T(F^\# \circ \phi_M) = \Phi_{\mathrm{RD}}^T(F^\square)\,.$$

❑

Chapter 9

The Langer-Jonas functional calculus

If $\mathcal{H}$ is a Hilbert space, and $A \in \mathcal{B}(\mathcal{H})$ is selfadjoint, then the Riesz-Dunford functional calculus can be extended to all bounded and measurable functions defined on the spectrum of A. Existence of this extension can be shown in different ways, for example

(a) via extending the polynomial functional calculus $p \mapsto p(A)$, $p \in \mathbb{C}[z]$, by continuity.

(b) via the Gelfand-transform; an approach which works even for normal operators.

Both approaches are bound to the fact that $\{A, A^*\}$ generates a commutative B^*-algebra. If $\mathcal{K}$ is a Krein space, the algebra $\mathcal{B}(\mathcal{K})$ endowed with the operator norm induced by some fundamental decomposition is not anymore a B^*-algebra; the law $\|xx^*\| = \|x\|^2$ fails. Hence, neither of these approaches works if we move to the indefinite situation. Also considering A as an operator in the Hilbert space $\langle \mathcal{K}, (.,.)_{\mathfrak{J}} \rangle$ does not help; A is in general not even normal in this Hilbert space, remember that $A^{(*)_{\mathfrak{J}}} = JA^{[*]}J$ and hence $A = A^{[*]}$ gives

$$A^{(*)_{\mathfrak{J}}}A = JA^{[*]}JA = JAJA \quad \text{whereas} \quad AA^{(*)_{\mathfrak{J}}} = AJA^{[*]}J = AJAJ\,.$$

However, if we restrict to a certain subclass of selfadjoint operators, then we can mimic the above approach (a). Continuity will arise from another source than in the Hilbert space case, namely from

(c) imposing and exploiting a positivity condition on A.

9.1 $\mathcal{B}(\mathcal{K})$-valued measures

An object of major importance in spectral theory is the space of bounded measureable functions on the spectrum of an operator. We denote in general, for a set Ω and a σ-algebra Σ of subsets of Ω, by $\mathrm{BM}(\Omega, \Sigma)$ the linear space of all complex-valued, bounded, and Σ-measureable functions on Ω. Moreover, we set

$$\|f\|_\infty := \sup_{x\in\Omega} |f(x)|, \quad f \in \mathrm{BM}(\Omega, \Sigma)\,.$$

The space $\langle \mathrm{BM}(\Omega,\Sigma), \|.\|_\infty\rangle$ is a Banach space. On $\mathrm{BM}(\Omega,\Sigma)$ we have the pointwise defined operations of multiplication $((f\cdot g)(x) := f(x)g(x))$ and conjugation $(\overline{f})(x) := \overline{f(x)}$. Endowed with this additional algebraic structure $\mathrm{BM}(\Omega,\Sigma)$ becomes a commutative B^*-algebra.

We will frequently make use of another notion of convergence on $\mathrm{BM}(\Omega,\Sigma)$ than norm-convergence. This notion is closely related to the Banach space $L^\infty(\mu)$ when $\mu : \Sigma \to [0,\infty]$ is some (additionally given) measure on Ω. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in $\mathrm{BM}(\Omega,\Sigma)$ and let $f \in \mathrm{BM}(\Omega,\Sigma)$. Then we say that $(f_n)_{n\in\mathbb{N}}$ converges *μ-boundedly pointwise* to f, if

$$\sup_{n\in\mathbb{N}} \|f_n\|_\infty < \infty \quad \text{and} \quad \lim_{n\to\infty} f_n(x) = f(x),\ \mu\text{-a.e.}$$

REH16 *9.1.1 Remark.* The following facts are immediate from the definition of μ-boundedly pointwise convergence.

(*i*) Let $(f_n)_{n\in\mathbb{N}}$ be a uniformly bounded sequence in $\mathrm{BM}(\Omega,\Sigma)$ which converges pointwise to a function f. Then f belongs to $\mathrm{BM}(\Omega,\Sigma)$, and $(f_n)_{n\in\mathbb{N}}$ converges to f μ-boundedly pointwise for each measure μ.

(*ii*) The above item applies in particular to every uniformly convergent sequence.

(*iii*) Provided μ is a finite measure, μ-boundedly pointwise convergence implies convergence in the norm of $L^1(\mu)$.

(*iv*) The algebraic operations of the *-algebra $\mathrm{BM}(\Omega,\Sigma)$ are *μ-boundedly pointwise continuous*. Explicitly, we mean by this that $f_n \to f$, $g_n \to g$, μ-boundedly pointwise implies that $f_n + g_n \to f + g$ μ-boundedly pointwise, $f_n g_n \to fg$ μ-boundedly pointwise, etc.

//

Thinking of spectral theory, we will be most interested in the case that $\Omega = K$ is a compact subset of $\mathbb{C}_\infty$ and Σ is the σ-algebra of all Borel sets on K. More generally, if K is a compact Hausdorff space and $\mathrm{Bor}(K)$ is the σ-algebra of Borel sets on K, we will write abbreviatory $\mathrm{BM}(K)$ instead of $\mathrm{BM}(K,\mathrm{Bor}(K))$.

As usual we denote by $C(K)$ the Banach space of all continuous functions on K endowed with the maximum norm. Clearly, $C(K) \subseteq \mathrm{BM}(K)$. Moreover, if μ is a Borel measure on K, then $\mathrm{BM}(K) \subseteq L^1(\mu)$. In this place we should say explicitly that we understand the term *Borel measure* as including that the measure of compact sets is finite. Thus, if μ is a Borel measure on a compact space K, then μ is a finite measure.

The following density properties are often of good use.

REH17 *9.1.2 Remark.* Let Ω be a set and Σ a σ-algebra on Ω.

(*i*) For each function $f \in \mathrm{BM}(\Omega,\Sigma)$ there exists a uniformly bounded sequence $(f_n)_{n\in\mathbb{N}}$ of measureable simple functions with which converges pointwise to f.

This is immediate if f is nonnegative, cf. [R2, Theorem 1.17], and transfers by linearity to arbitrary complex valued functions f.

Let K be a compact Hausdorff space, and let μ be a regular Borel measure on K.

(*ii*) $C(K)$ is dense in $L^1(\mu)$ with respect to the L^1-norm.

(*iii*) For each function $f \in \mathrm{BM}(K)$ there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of continuous functions which converges μ-boundedly pointwise to f.

These are classical facts. See e.g. [R2, Theorem 3.14] for (*i*), and the Corollary to Lusin's Theorem [R2, p.56] for (*ii*).

Assume in addition that K is a compact subset of $\mathbb{C}_\infty$, where $\mathbb{C}_\infty$ is the one-point compactification of the complex numbers.

(*iv*) For each $f \in \mathrm{BM}(K)$ there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of rational functions being continuous on K which converges μ-boundedly pointwise to f.

This follows by combining the above item (*iii*) with the Stone-Weierstraß Theorem, and remembering that uniform convergence implies μ-boundedly pointwise convergence. //

DEH1 **9.1.3 Definition.** Let Ω be a set, Σ a σ-algebra on Ω, and $\langle\mathcal{K},[.,.]\rangle$ a Krein space. We call a map $E:\Sigma\to\mathcal{B}(\mathcal{K})$ a *weak $\mathcal{B}(\mathcal{K})$-valued measure*, if for each countable family Δ_n, $n\in\mathbb{N}$, of disjoint elements of Σ the series $\sum_{n=1}^\infty E(\Delta_n)$ converges in the weak operator topology and its sum is equal to $E(\bigcup_{n=1}^\infty \Delta_n)$. //

If E is a weak $\mathcal{B}(\mathcal{K})$-valued measure, then for each fixed $x,y\in\mathcal{K}$ the function

$$E_{x,y}:\left\{\begin{array}{lll}\Sigma & \to & \mathbb{C}\\ \Delta & \mapsto & [E(\Delta)x,y]\end{array}\right.$$

is a complex measure.

For any complex measure μ defined on some σ-algebra of subsets of a set Ω, we denote by $|\mu|$ the *total variation measure* of μ, and by $\|\mu\|$ the *total variation* of μ, that is $\|\mu\| = |\mu|(\Omega)$.

DEH2 **9.1.4 Definition.** Let $\langle\mathcal{K},[.,.]\rangle$ be a Krein space, and choose a norm $\|.\|_\mathcal{K}$ on $\mathcal{K}$ which induces the Krein space topology of $\mathcal{K}$. We call a weak $\mathcal{B}(\mathcal{K})$-valued measure E *uniformly bounded*, if

$$\sup_{\|x\|_\mathcal{K}\leq 1}\|E_{x,x}\| < \infty\,. \tag{9.1.1}$$ H3

//

Note that finiteness of the supremum in (9.1.1) does not depend on the choice of the norm $\|.\|_\mathcal{K}$; the actual value of the supremum of course does. Moreover, by the parallelogram rule for the inner product $[.,.]$, validity of (9.1.1) implies that

$$\|E\| := \sup_{\|x\|_\mathcal{K},\|y\|_\mathcal{K}\leq 1}\|E_{x,y}\| < \infty\,.$$

In turn, by linearity, this gives

$$\|E_{x,y}\| \leq \|E\|\cdot\|x\|_\mathcal{K}\|y\|_\mathcal{K},\quad x,y\in\mathcal{K}\,. \tag{9.1.2}$$ H5

If Ω is a set, and $\Delta\subseteq\Omega$, we denote by χ_Δ the *characteristic function* of the set Δ. That is

$$\chi_\Delta(x) := \begin{cases}1\,, & x\in\Delta\\ 0\,, & x\in\Omega\setminus\Delta\end{cases}$$

PRH4

9.1.5 Proposition. *Let $\langle\mathcal{K},[.,.]\rangle$ be a Krein space, and fix a norm $\|.\|_{\mathcal{K}}$ on $\mathcal{K}$ which induces the Krein space topology of $\mathcal{K}$. Let Ω be a set, Σ a σ-algebra on Ω, and E a uniformly bounded weak $\mathcal{B}(\mathcal{K})$-valued measure defined on Σ. Then there exists a linear operator $\Psi_E : \mathrm{BM}(\Omega,\Sigma) \to \mathcal{B}(\mathcal{K})$ such that*

(i) $\Psi_E(\chi_\Delta) = E(\Delta)$ *for all* $\Delta \in \Sigma$.

(ii) Ψ_E *is bounded. In fact the* $\|.\|_{\mathcal{K}}$*-to-*$\|.\|_{\mathcal{K}}$*–operator norm* $\|\Psi_E\|$ *of* Ψ_E *is equal to* $\|E\|$ *defined using the norm* $\|.\|_{\mathcal{K}}$.

(iii) Ψ_E *has the following additional continuity property:*

$$\begin{aligned}&\forall\, x,y \in \mathcal{K}\ \ \exists\,\mu \text{ finite positive measure}:\\ &\qquad f_n \to f\ \mu\text{-boundedly pointwise} \ \Rightarrow\ [\Psi_E(f_n)x,y] \to [\Psi_E(f)x,y]\end{aligned}$$

(iv) Let $T \in \mathcal{B}(\mathcal{K})$*. Then* T *commutes with all operators* $E(\Delta)$, $\Delta \in \Sigma$, *if and only if* T *commutes with all operators* $\Psi_E(f)$, $f \in \mathrm{BM}(\Omega,\Sigma)$.

Proof. Let $f \in \mathrm{BM}(\Omega,\Sigma)$ be given. Consider the map

$$[.,.]_f : \begin{cases} \mathcal{K}\times\mathcal{K} & \to & \mathbb{C} \\ (x,y) & \mapsto & \int\limits_\Omega f\, dE_{x,y} \end{cases}$$

We have

$$E_{x_1+x_2,y}(\Delta) = [E(\Delta)(x_1+x_2),y] = [E(\Delta)x_1,y] + [E(\Delta)x_2,y] =$$

$$= E_{x_1,y}(\Delta) + E_{x_2,y}(\Delta), \quad \Delta \in \Sigma\,,$$

and similarly

$$E_{x,y_1+y_2}(\Delta) = E_{x,y_1}(\Delta) + E_{x,y_2}(\Delta)\,,$$

$$E_{\lambda x,y}(\Delta) = \lambda E_{x,y}(\Delta),\ E_{x,\lambda y}(\Delta) = \overline{\lambda} E_{x,y}(\Delta)\,.$$

Hence $[.,.]_f$ is a sesquilinear form on $\mathcal{K}$. Using (9.1.2), we obtain

$$|[x,y]_f| \leq \|f\|_\infty \|E\| \cdot \|x\|_{\mathcal{K}} \|y\|_{\mathcal{K}}, \quad x,y \in \mathcal{K}\,,$$

i.e. $[.,.]_f$ is a bounded sesquilinear form.

By **??** there exists an operator $B_f \in \mathcal{B}(\mathcal{K})$ with $\|B_f\| \leq \|E\| \cdot \|f\|_\infty$ such that

$$[x,y]_f = [B_f x, y], \quad x,y \in \mathcal{K}\,.$$

Define $\Psi_E : \mathrm{BM}(\Omega,\Sigma) \to \mathcal{B}(\mathcal{K})$ as $\Psi_E(f) := B_f$. By linearity of the integral $\int_\Omega f\, dE_{x,y}$ in the argument f, the map Ψ_E is linear. As we have noted above $\|\Psi_E(f)\| \leq \|E\| \cdot \|f\|_\infty$, i.e. Ψ_E is bounded and $\|\Psi_E\| \leq \|E\|$. Moreover, by its definition,

$$[\Psi_E(\chi_\Delta)x,y] = \int\limits_\Omega \chi_\Delta\, dE_{x,y} = E_{x,y}(\Delta) = [E(\Delta)x,y], \quad x,y \in \mathcal{K}\,,$$

and this says that $\Psi_E(\chi_\Delta) = E(\Delta)$.

To show the continuity property (iii), let $x, y \in \mathcal{K}$ be given, and consider the positive and finite measure $\mu := |E_{x,y}|$. If $(f_n)_{n\in\mathbb{N}}$ converges μ-boundedly pointwise to f, then by the dominated convergence theorem

$$\big|[\Psi_E(f_n)x, y] - [\Psi_E(f)x, y]\big| = \Big|\int\limits_\Omega (f_n - f)\, dE_{x,y}\Big| \leq \int\limits_\Omega |f_n - f|\, d\mu \to 0\,.$$

Next, we show the inequality $\|E\| \leq \|\Psi_E\|$. Let $x, y \in \mathcal{K}$, and let Δ_n, $n \in \mathbb{N}$, be a disjoint family of elements of Σ with $\bigcup_{n=1}^\infty \Delta_n = \Omega$. Choose $\epsilon_n \in \mathbb{C}$, $|\epsilon_n| = 1$, such that $\epsilon_n[\Psi_E(\chi_{\Delta_n})x, y] \geq 0$, then

$$\sum_{n=1}^\infty |E_{x,y}(\Delta_n)| = \sum_{n=1}^\infty \big|[\Psi_E(\chi_{\Delta_n})x, y]\big| = \sum_{n=1}^\infty [\Psi_E(\chi_{\epsilon_n\Delta_n})x, y]\,.$$

Since the sets Δ_n are disjoint, we have

$$\Big\|\sum_{n=1}^N \epsilon_n\chi_{\Delta_n}\Big\|_\infty \leq 1, \quad N \in \mathbb{N}\,,$$

and, for each $\zeta \in \Omega$, the series $\sum_{n=1}^\infty \epsilon_n\chi_{\Delta_n}(\zeta)$ converges. Thus, by the already proved continuity property (iii), we obtain that

$$\sum_{n=1}^\infty [\Psi_E(\chi_{\epsilon_n\Delta_n})x, y] = \Big[\Psi_E\Big(\sum_{n=1}^\infty \chi_{\epsilon_n\Delta_n}\Big)x, y\Big] \leq \|\Psi_E\| \cdot \|x\|_\mathcal{K}\|y\|_\mathcal{K}\,.$$

It follows that $\|E_{x,y}\| \leq \|\Psi_E\| \cdot \|x\|_\mathcal{K}\|y\|_\mathcal{K}$.

Finally, let $T \in \mathcal{B}(\mathcal{K})$ be given. If T commutes with all operators $\Psi_E(f)$, $f \in \mathrm{BM}(\Omega, \Sigma)$, then it commutes in particular with $\Psi_E(\chi_\Delta) = E(\Delta)$, $\Delta \in \Sigma$. Conversely, assume that $TE(\Delta) = E(\Delta)T$, $\Delta \in \Sigma$. Then, for $x, y \in \mathcal{K}$,

$$E_{Tx,y}(\Delta) = [E(\Delta)Tx, y] = [TE(\Delta)x, y] = [E(\Delta)x, T^*y] = E_{x,T^*y}(\Delta), \quad \Delta \in \Sigma\,,$$

i.e. $E_{Tx,y} = E_{x,T^*y}$. It follows that

$$[\Psi_E(f)Tx, y] = \int\limits_\Omega f\, dE_{Tx,y} = \int\limits_\Omega f\, dE_{x,T^*y} = [\Psi_E(f)x, T^*y] = [T\Psi_E(f)x, y]\,.$$

❑

The continuity property which appeared in Proposition 9.1.5, (iii), plays an important role.

DEH46 **9.1.6 Definition.** Let Ω be a set, Σ a σ-algebra on Ω, and $\mathcal{K}$ a Krein space. We will say that a map Ψ defined on a subset of $\mathrm{BM}(\Omega, \Sigma)$ and taking values in $\mathcal{B}(\mathcal{K})$ is (9.1.3)-*continuous*, if

$$\begin{aligned} &\forall\, x, y \in \mathcal{K}\ \ \exists\, \mu \text{ finite positive measure}: \\ &f_n \to f\ \mu\text{-boundedly pointwise} \ \Rightarrow\ [\Psi(f_n)x, y] \to [\Psi(f)x, y] \end{aligned} \qquad (9.1.3)$$ H7

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The following notices are sometimes practical.

LEH15 **9.1.7 Lemma.** *Let* $D \subseteq \mathrm{BM}(\Omega, \Sigma)$.

(*i*) *Let* $\Psi_j : D \to \mathcal{B}(\mathcal{K})$, $j = 1, \dots, n$, *be* (9.1.3)*-continuous, let* $\lambda_j \in \mathbb{C}$, $j = 1, \dots, n$, *and consider the map* $\sum_{j=1}^n \lambda_j \Psi_j$ *given by*

$$(\sum_{j=1}^n \lambda_j \Psi_j)(f) := \sum_{j=1}^n \lambda_j \Psi_j(f), \quad f \in D.$$

Then $\sum_{j=1}^n \lambda_j \Psi_j$ *is* (9.1.3)*-continuous.*

(*ii*) *Let* $\Psi : D \to \mathcal{B}(\mathcal{K})$ *be* (9.1.3)*-continuous, and let* $T \in \mathcal{B}(\mathcal{K})$*. Consider the maps* $T\Psi, \Psi T : D \to \mathcal{B}(\mathcal{K})$ *given by*

$$(T\Psi)(f) := T\Psi(f), \ (\Psi T)(f) := \Psi(f)T, \quad f \in D.$$

Then $T\Psi$ *and* ΨT *are* (9.1.3)*-continuous.*

(*iii*) *Let* $\Psi : D \to \mathcal{B}(\mathcal{K})$ *be* (9.1.3)*-continuous, and consider the map* $\Psi^* : D \to \mathcal{B}(\mathcal{K})$ *given by*

$$\Psi^* : f \mapsto \Psi(f)^*, \quad f \in D.$$

Then Ψ^* *is* (9.1.3)*-continuous.*

Proof. Throughout the proof let $x, y \in \mathcal{K}$ be fixed.

Let Ψ_j and λ_j, $j = 1, \dots, n$, be given. Then, for each $j \in \{1, \dots, n\}$, there exists a positive finite measure μ_j such that $f_n \to f$ μ_j-boundedly pointwise implies $[\Psi_j(f_n)(\lambda_j x), y] \to [\Psi_j(f)(\lambda_j x), y]$. Set $\mu := \sum_{j=1}^n \mu_j$. Then, clearly, μ-boundedly pointwise convergence implies μ_j-boundedly pointwise for all j. We conclude that $f_n \to f$ μ-boundedly pointwise implies $[\sum_{j=1}^n \lambda_j \Psi_j(f_n)x, y] \to [\sum_{j=1}^n \lambda_j \Psi_j(f)x, y]$.

Next, let Ψ and T be given. Then there exists a positive finite measure μ such that $f_n \to f$ μ-boundedly pointwise implies $[\Psi(f_n)(Tx), y] \to [\Psi(f)(Tx), y]$, i.e. $[(\Psi T)(f_n)x, y] \to [(\Psi T)(f)x, y]$. Also there exists a positive finite measure μ such that $f_n \to f$ μ-boundedly pointwise implies $[\Psi(f_n)x, T^*y] \to [\Psi(f)x, T^*y]$, and this gives $[(T\Psi)(f_n)x, y] \to [(T\Psi)(f)x, y]$.

Finally, for (*iii*), choose a positive finite measure μ such that $f_n \to f$ μ-boundedly pointwise implies that $[\Psi(f_n)y, x] \to [\Psi(f)y, x]$. It follows that $f_n \to f$ μ-boundedly pointwise implies that $[\Psi^*(f_n)x, y] \to [\Psi^*(f)x, y]$. ❑

Also a converse to Proposition 9.1.5 holds.

PRH8 **9.1.8 Proposition.** *Let* $\Psi : \mathrm{BM}(\Omega, \Sigma) \to \mathcal{B}(\mathcal{K})$ *be a bounded and* (9.1.3)*-continuous linear map. Then there exists a uniformly bounded weak* $\mathcal{B}(\mathcal{K})$*-valued measure* E_Ψ *such that* $\Psi = \Psi_{E_\Psi}$*. The assignments* $E \mapsto \Psi_E$ *and* $\Psi \mapsto E_\Psi$ *are mutually inverse.*

Proof. Define $E_\Psi : \Sigma \to \mathcal{B}(\mathcal{K})$ as

$$E_\Psi(\Delta) := \Psi(\chi_\Delta), \quad \Delta \in \Sigma.$$

Let Δ_n, $n \in \mathbb{N}$, be a family of disjoint elements of Σ, and set $\Delta := \bigcup_{n=1}^{\infty} \Delta_n$. Then the sequence $\big(\sum_{n=1}^{N} \chi_{\Delta_n}\big)_{N\in\mathbb{N}}$ is uniformly bounded and pointwise convergent to χ_Δ. Let $x, y \in \mathcal{K}$ be given, and choose μ as in (9.1.3). Since $\sum_{n=1}^{N} \chi_{\Delta_n} \to \chi_\Delta$ μ-boundedly pointwise, it follows that

$$\Big[\Psi\Big(\sum_{n=1}^{N} \chi_{\Delta_n}\Big)x, y\Big] \to [\Psi(\chi_\Delta)x, y]\,.$$

The elements $x, y \in \mathcal{K}$ were arbitrary, and thus $\Psi\big(\sum_{n=1}^{N} \chi_{\Delta_n}\big) \to \Psi(\chi_\Delta)$ weakly. However, $\Psi\big(\sum_{n=1}^{N} \chi_{\Delta_n}\big) = \sum_{n=1}^{N} E_\Psi(\Delta_n)$ and $\Psi(\chi_\Delta) = E_\Psi(\Delta)$. We conclude that E_Ψ is a weak $\mathcal{B}(\mathcal{K})$-valued measure.

To show that E_Ψ is uniformly bounded, we use the same argument as in the proof of Proposition 9.1.5. Let $x, y \in \mathcal{K}$, $\Delta_n \in \Sigma$, $n \in \mathbb{N}$, be disjoint with $\bigcup_{n=1}^{\infty} \Delta_n = \Omega$, and $|\epsilon_n| = 1$ with $\epsilon_n[\Psi(\Delta_n)x, y] \geq 0$. Then $\big(\sum_{n=1}^{N} \epsilon_n \chi_{\Delta_n}\big)_{n\in\mathbb{N}}$ is uniformly bounded by 1 and converges pointwise. Thus, using again (9.1.3)-continuity of Ψ, we obtain

$$\sum_{n=1}^{\infty} |(E_\Psi)_{x,y}(\Delta_n)| = \sum_{n=1}^{\infty} [\Psi(\epsilon_n \chi_{\Delta_n})x, y] =$$

$$\Big[\Psi\Big(\sum_{n=1}^{\infty} \epsilon_n \chi_{\Delta_n}\Big)x, y\Big] \leq \|\Psi\| \cdot \|x\|_\mathcal{K} \|y\|_\mathcal{K}\,.$$

It follows that $\|(E_\Psi)_{x,y}\| \leq \|\Psi\| \cdot \|x\|_\mathcal{K} \|y\|_\mathcal{K}$, and thus that E_Ψ is uniformly bounded; in fact $\|E_\Psi\| \leq \|\Psi\|$.

Let us consider the operator Ψ_{E_Ψ}. Then we have

$$\Psi_{E_\Psi}(\chi_\Delta) = E_\Psi(\chi_\Delta) = \Psi(\chi_\Delta), \quad \Delta \in \Sigma\,,$$

and hence $\Psi_{E_\Psi}(f) = \Psi(f)$ for all measurable simple functions. Let $x, y \in \mathcal{K}$, and choose μ_1 according to (9.1.3)-continuity of Ψ and μ_2 according to (9.1.3)-continuity of Ψ_{E_Ψ}. If $f \in \mathrm{BM}(\Omega, \Sigma)$, we can find a sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions which is uniformly bounded and converges pointwise to f, cf. Remark 9.1.2. This sequence thus also converges μ_j-boundedly pointwise to f, $j = 1, 2$, and we obtain

$$[\Psi_{E_\Psi}(f)x, y] = \lim_{n\to\infty} [\Psi_{E_\Psi}(f_n)x, y] = \lim_{n\to\infty} [\Psi(f_n)x, y] = [\Psi(f)x, y]\,.$$

Finally, if E is any weak $\mathcal{B}(\mathcal{K})$-valued measure, then by the respective definitions we have

$$E_{\Psi_E}(\Delta) = \Psi_E(\Delta) = E(\Delta), \quad \Delta \in \Sigma\,.$$

❑

The next statement is a variant of the Riesz Representation Theorem. Let $\mathcal{K}$ be a Krein space. An operator $T \in \mathcal{B}(\mathcal{K})$ is called *positive*, if

$$[Tx, x] \geq 0, \quad x \in \mathbb{K}\,.$$

In this case, we write $T \geq 0$.

PRH6 **9.1.9 Proposition.** *Let K be a compact Hausdorff space, $\mathcal{K}$ a Krein space, and $\Psi : C(K) \to \mathcal{B}(\mathcal{K})$ a bounded linear map. Then there exists a uniformly bounded weak $\mathcal{B}(\mathcal{K})$-valued measure E defined on* $\mathrm{Bor}(K)$*, such that*

(*i*) *$\Psi(f) = \Psi_E(f)$ for all $f \in C(K)$.*

(*ii*) *For each $x, y \in \mathcal{K}$, the complex Borel measure $E_{x,y}$ is regular.*

(*iii*) *$\|E\| = \|\Psi\|$ where these norms understand with respect to a norm on $\mathcal{K}$ which is induced by some fundamental decomposition.*

(*iv*) *Let $T \in \mathcal{B}(\mathcal{K})$. Then T commutes with all operators $E(\Delta)$, $\Delta \in \mathrm{Bor}(K)$, if and only if T commutes with all operators $\Psi(f)$, $f \in C(K)$.*

(*v*) *If Ψ has the property that*

$$\Psi(f) \geq 0 \text{ for all } f \in C(K) \text{ with } f \geq 0\,, \tag{9.1.4}$$ H44

so does Ψ_E.

The measure E is uniquely determined by its properties (*i*) *and* (*ii*).

Proof. Let $\|.\|_{\mathcal{K}}$ be a norm on $\mathcal{K}$ which is induced by some fundamental decomposition. For $x, y \in \mathcal{K}$ consider the linear functional

$$\psi_{x,y} : \begin{cases} C(K) & \to & \mathbb{C} \\ f & \mapsto & [\Psi(f)x, y] \end{cases}$$

We have

$$|\psi_{x,y}(f)| \leq \|\Psi\| \cdot \|f\|_\infty \cdot \|x\|_{\mathcal{K}} \|y\|_{\mathcal{K}}\,,$$

i.e. $\psi_{x,y}$ is bounded and $\|\psi_{x,y}\| \leq \|\Psi\| \cdot \|x\|_{\mathcal{K}} \|y\|_{\mathcal{K}}$. By the Riesz Representation Theorem there exists a regular complex Borel measure $\mu_{x,y}$ with $\|\mu_{x,y}\| = \|\psi_{x,y}\|$, such that

$$\psi_{x,y}(f) = \int_K f \, d\mu_{x,y}, \quad f \in C(K)\,.$$

For $\Delta \in \mathrm{Bor}(K)$ set

$$[.,.]_\Delta : \begin{cases} \mathcal{K} \times \mathcal{K} & \to & \mathbb{C} \\ (x, y) & \mapsto & \mu_{x,y}(\Delta) \end{cases}$$

We have

$$\int_K f \, d\mu_{x_1+x_2,y} = [\Psi(f)(x_1 + x_2), y] = [\Psi(f)x_1, y] + [\Psi(f)x_2, y] =$$

$$= \int_K f \, d\mu_{x_1,y} + \int_K f \, d\mu_{x_2,y}, \quad f \in C(K)\,,$$

and hence $\mu_{x_1+x_2,y} = \mu_{x_1,y} + \mu_{x_2,y}$. Similarly, $\mu_{x,y_1+y_2} = \mu_{x,y_1} + \mu_{x_2,y}$, $\mu_{\lambda x,y} = \lambda\mu_{x,y}$, and $\mu_{x,\lambda y} = \overline{\lambda}\mu_{x,y}$. Thus $[.,.]_\Delta$ is a sesquilinear form. Moreover,

$$|[x, y]_\Delta| = |\mu_{x,y}(\Delta)| \leq \|\mu_{x,y}\| = \|\psi_{x,y}\| \leq \|\Psi\| \cdot \|x\|_{\mathcal{K}} \|y\|_{\mathcal{K}}\,.$$

By **??** there exists $B_\Delta \in \mathcal{B}(\mathcal{K})$ with $\|B_\Delta\| \leq \|\Psi\|$ such that

$$[x,y]_\Delta = [B_\Delta x, y], \quad x,y \in \mathcal{K}\,.$$

Define a function $E : \mathrm{Bor}(K) \to \mathcal{B}(\mathcal{K})$ by

$$E(\Delta) := B_\Delta, \quad \Delta \in \mathrm{Bor}(K)\,.$$

Let $\Delta_n \in \mathrm{Bor}(K)$, $n \in \mathbb{N}$, be disjoint, and set $\Delta := \bigcup_{n=1}^\infty \Delta_n$. Then

$$[B_\Delta x, y] = \mu_{x,y}(\Delta) = \sum_{n=1}^\infty \mu_{x,y}(\Delta_n) = \sum_{n=1}^\infty [B_{\Delta_n} x, y]\,,$$

i.e. $\sum_{n=1}^\infty E(\Delta_n) = E(\Delta)$ where the series converges weakly. This says that E is a weak $\mathcal{B}(\mathcal{K})$-valued measure. By the definition of E we have

$$E_{x,y}(\Delta) = [E(\Delta)x, y] = [x,y]_\Delta = \mu_{x,y}(\Delta), \quad \Delta \in \mathrm{Bor}(K)\,,$$

i.e. $E_{x,y} = \mu_{x,y}$. We see that

$$\|E_{x,y}\| = \|\mu_{x,y}\| \leq \|\Psi\| \cdot \|x\|_{\mathcal{K}} \|y\|_{\mathcal{K}}\,,$$

and hence E is uniformly bounded; in fact $\|E\| \leq \|\Psi\|$.

For $f \in C(K)$ we compute

$$[\Psi_E(f)x, y] = [x,y]_f = \int_K f\, dE_{x,y} = \int_K f\, d\mu_{x,y} = \psi_{x,y}(f) = [\Psi(f)x, y]\,.$$

This shows that $\Psi_E(f) = \Psi(f)$ for all $f \in C(K)$. It also follows that $\|E\| = \|\Psi_E\| \geq \|\Psi\|$. We have thus established existence of a uniformly bounded weak $\mathcal{B}(\mathcal{K})$-value measure with (i)–(iii).

To show (iv), let $T \in \mathcal{B}(\mathcal{K})$ be given. If $TE(\Delta) = E(\Delta)T$, $\Delta \in \mathrm{Bor}(K)$, then $T\Psi_E(f) = \Psi_E(f)T$, $f \in \mathrm{BM}(K)$. In particular, T commutes with all operators $\Psi(f)$, $f \in C(K)$. Conversely, assume that $T\Psi(f) = \Psi(f)T$, $f \in C(K)$. Then

$$\psi_{Tx,y}(f) = [\Psi(f)Tx, y] = [T\Psi(f)x, y] = [\Psi(f)x, T^*y] = \psi_{x,T^*y}(f)\,,$$

and hence $\mu_{Tx,y} = \mu_{x,T^*y}$. This implies that $[B_\Delta Tx, y] = [B_\Delta x, T^*y]$, $\Delta \in \mathrm{Bor}(K)$, and hence that

$$E(\Delta)T = B_\Delta T = TB_\Delta = TE(\Delta)\,.$$

Next, assume that Ψ has the positivity property (9.1.4). Then, for each $x \in \mathcal{K}$, the functional $\psi_{x,x}$ maps nonnegative functions to nonnegative numbers. The Riesz Representations Theorem thus tells us that the measure $\mu_{x,x}$ is positive. Since $[\Psi_E(f)x, x] = \int_K f\, d\mu_{x,x}$, this implies that Ψ_E satisfies (9.1.4).

In order to show the desired uniqueness assertion, assume that E_1 and E_2 are uniformly bounded weak $\mathcal{B}(\mathcal{K})$-valued measures which satisfy (i) and (ii). Then $\Psi_{E_1}(f) = \Psi(f) = \Psi_{E_2}(f)$, $f \in C(K)$, and hence

$$\int_K f\, d(E_1)_{x,y} = [\Psi_{E_1}(f)x, y] = [\Psi_{E_2}(f)x, y] = \int_K f\, d(E_2)_{x,y}, \quad f \in C(K)\,.$$

By the uniqueness part of the Riesz Representation Theorem, this implies that $(E_1)_{x,y} = (E_2)_{x,y}$. Since $x, y \in \mathcal{K}$ were arbitrary, it follows that $E_1 = E_2$. ❑

Let us explicitly state the following consequence of Proposition 9.1.9.

COH9 **9.1.10 Corollary.** *Let K be a compact Hausdorff space, $\mathcal{K}$ a Krein space, and Ψ a bounded linear map of $C(K)$ into $\mathcal{B}(\mathcal{K})$. Then there exists a continuation $\tilde{\Psi} : \mathrm{BM}(K) \to \mathcal{B}(\mathcal{K})$ of Ψ which has the same norm as Ψ, and is* (9.1.3)*-continuous. In particular, Ψ itself is* (9.1.3)*-continuous. Moreover, we have*

(i) *If Ψ satisfies* (9.1.4)*, then $\tilde{\Psi}$ has the corresponding property, i.e.*

$$\tilde{\Psi}(f) \geq 0 \textit{ for all } f \in \mathrm{BM}(K) \textit{ with } f \geq 0\,.$$

(ii) *Let $T \in \mathcal{B}(\mathcal{K})$. Then T commutes with all operators $\Psi(f)$, $f \in C(K)$, if and only if T commutes with all operators $\tilde{\Psi}(f)$, $f \in \mathrm{BM}(K)$.*

❑

REH10 *9.1.11 Remark.* The continuation $\tilde{\Psi}$ of Ψ to $\mathrm{BM}(K)$ with the properties stated in Corollary 9.1.10 need not be unique. However, among all such continuations there is exactly one with the property that for each $x, y \in \mathcal{K}$ the function $\Delta \mapsto [\tilde{\Psi}(\chi_\Delta)x, y]$ is a regular complex Borel measure. Hence, if K has the property that all Borel measures are regular, then the continuation in Corollary 9.1.10 is unique. This applies, in particular, if K is metrizable. //

9.2 An algebra of functions

Denote by $\mathrm{GL}(2,\mathbb{R})$ the subgroup of $\mathrm{GL}(2,\mathbb{C})$ of all matrices with real entries.

DEH18 **9.2.1 Definition.** Let $\mathfrak{d} : \mathbb{R}_\infty \to \mathbb{N}_0$ be a function with finite support. Then we denote by $\mathfrak{A}(\mathfrak{d})$ the set of all functions $f \in \mathrm{BM}(\mathbb{R}_\infty)$ with the following property: For each $w \in \operatorname{supp}\mathfrak{d}$ and $M \in \mathrm{GL}(2,\mathbb{R})$ with $\phi_M(w) = 0$, there exist $a_0(w), \dots, a_{\mathfrak{d}(w)-1}(w) \in \mathbb{C}$, $\epsilon_w > 0$, and $f_w \in \mathrm{BM}([-\epsilon_w, \epsilon_w])$, such that

$$(f \circ \phi_M^{-1})(x) = \sum_{j=0}^{\mathfrak{d}(w)-1} a_j(w)x^j + x^{\mathfrak{d}(w)} f_w(z), \quad x \in [-\epsilon_w, \epsilon_w]\,. \qquad (9.2.1)$$ H42

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Obviously, $\mathfrak{A}(\mathfrak{d})$ is a *-subalgebra of $\mathrm{BM}(\mathbb{R}_\infty)$. However, it is not closed with respect to $\|.\|_\infty$ unless $\mathfrak{d} = 0$ in which case $\mathfrak{A}(\mathfrak{d}) = \mathrm{BM}(\mathbb{R}_\infty)$. Moreover, clearly, $\mathfrak{d}_1 \leq \mathfrak{d}_2$ implies that $\mathfrak{A}(\mathfrak{d}_1) \supseteq \mathfrak{A}(\mathfrak{d}_2)$.

REH11 *9.2.2 Remark.* Let $\mathfrak{d} : \mathbb{R}_\infty \to \mathbb{N}_0$ be a function with finite support, and $f \in \mathfrak{A}(\mathfrak{d})$. Then $f \in \mathfrak{A}(\mathfrak{d})$ if and only if for each $w \in \operatorname{supp}\mathfrak{d}$ there exist $M \in \mathrm{GL}(2,\mathbb{R})$ with $\phi_M(w) = 0$, $a_0(w), \dots, a_{\mathfrak{d}(w)-1}(w) \in \mathbb{C}$, $\epsilon_w > 0$, and $f_w \in \mathrm{BM}([-\epsilon_w, \epsilon_w])$, such that (9.2.1) holds.

For example, it is enough to check (9.2.1) with the matrices M_w, $w \in \mathbb{C}_\infty$, defined as

$$M_w := \begin{cases} \begin{pmatrix} 1 & -w \\ 0 & 1 \end{pmatrix}, & w \in \mathbb{C} \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & w = \infty \end{cases}$$

//

We start with showing a basic representation of $\mathfrak{A}(\mathfrak{d})$. For $\delta \in \mathbb{N}$ denote by $\mathbb{C}[z]_\delta$ the set of all polynomials with degree at most $\delta - 1$. Note that $\dim \mathbb{C}[z]_\delta = \delta$.

PRH19 **9.2.3 Proposition.** *Let $\mathfrak{d} : \mathbb{R}_\infty \to \mathbb{N}_0$ be a function with finite support, and set $\delta := \sum_{w \in \mathbb{R}_\infty} \mathfrak{d}(w)$. Choose $d \in \mathbb{H}(\mathbb{R}_\infty)$ with $\mathfrak{d}_d = \mathfrak{d}$. Then*

$$\mathfrak{A}(\mathfrak{d}) = \mathbb{H}(\mathbb{R}_\infty) + d \cdot \mathrm{BM}(\mathbb{R}_\infty) = \frac{1}{(z+i)^{\delta-1}} \mathbb{C}[z]_\delta \dot{+} d \cdot \mathrm{BM}(\mathbb{R}_\infty), \tag{9.2.2}$$ H23

where we consider $\mathbb{H}(\mathbb{R}_\infty)$ as a subspace of $\mathrm{BM}(\mathbb{R}_\infty)$ via the map $\rho_{\mathbb{R}_\infty}$, cf. 8.1.5.

Proof. Let $p \in \mathbb{H}(\mathbb{R}_\infty)$, $g \in \mathrm{BM}(\mathbb{R}_\infty)$, $w \in \operatorname{supp} \mathfrak{d}$, and $M \in \mathrm{GL}(2, \mathbb{R})$ with $\phi_M(w) = 0$. Then $p \circ \phi_M \in \mathbb{H}(\mathbb{R}_\infty)$, and hence on some neighbourhood $[-\epsilon_w, \epsilon_w]$ of 0 we have the expansion

$$(p \circ \phi_M^{-1})(x) = \sum_{j=0}^{\mathfrak{d}(w)-1} \frac{1}{j!} [p \circ \phi_M^{-1}]^{(j)}(0) x^j + x^{\mathfrak{d}(w)} \sum_{j=0}^{\infty} \frac{1}{j!} [p \circ \phi_M^{-1}]^{(j)}(0) x^{j-\mathfrak{d}(w)}, \quad x \in [-\epsilon_w, \epsilon_w].$$

Moreover, since $\mathfrak{d}_{d \circ \phi_M}(0) = \mathfrak{d}_d(w) = \mathfrak{d}(w)$,

$$\big[(dg) \circ \phi_M\big](x) = x^{\mathfrak{d}(w)} \cdot \Big(\frac{d(x)}{x^{\mathfrak{d}(w)}} g(x)\Big).$$

We see that $p + dg \in \mathfrak{A}(\mathfrak{d})$. The second inclusion '$\supseteq$' in (9.2.2) is trivial.

For the converse, we start with a preliminary observation. Denote $\mathcal{R} := (z+i)^{-(\delta-1)} \mathbb{C}[z]_\delta$, then $\mathcal{R}$ is a linear subspace of $\mathbb{H}(\mathbb{R}_\infty)$ with dimension δ. The map

$$\nu : \begin{cases} \mathcal{R} & \to \quad \prod_{w \in \operatorname{supp} \mathfrak{d}} \mathbb{C}^{\mathfrak{d}(w)} \\ r & \mapsto \quad \big(([r \circ \Phi_{M_w}^{-1}]^{(j)})_{j=0}^{\mathfrak{d}(w)-1}\big)_{w \in \operatorname{supp} \mathfrak{d}} \end{cases} \tag{9.2.3}$$ H63

is linear. If $r \in \ker \nu$, then $(z+i)^{\delta-1} r(z)$ is a polynomial whose degree does not exceed $\delta - 1 - \mathfrak{d}(\infty)$, and which has zeros on $\mathbb{R}$ of total multiplicity at least $\sum_{w \in \operatorname{supp} \mathfrak{d} \cap \mathbb{R}} \mathfrak{d}(w)$. Since $\mathfrak{d}(\infty) + \sum_{w \in \operatorname{supp} \mathfrak{d} \cap \mathbb{R}} \mathfrak{d}(w) = \delta$, this implies that $(z+i)^{\delta-1} r(z) = 0$, and hence that $r = 0$. Thus ν is injective, and by equality of dimensions hence bijective.

We show that $\mathfrak{A}(\mathfrak{d})$ is contained in the rightmost sum. Let $f \in \mathfrak{A}(\mathfrak{d})$ be given, and let $a_j(w)$ be as in (9.2.1). Define

$$p := \nu^{-1}\big((a_j(w))_{j=0}^{\mathfrak{d}(w)-1}\big)_{w \in \operatorname{supp} \mathfrak{d}},$$

and a function $g_0 : \mathbb{R}_\infty \setminus \operatorname{supp} \mathfrak{d} \to \mathbb{C}$ as

$$g_0(x) := \frac{f(x) - p(x)}{d(x)}, \quad x \in \mathbb{R}_\infty \setminus \operatorname{supp} \mathfrak{d}.$$

The certainly g_0 is measurable. Our aim is to show that g_0 is bounded. On each compact set which does not intersect $\operatorname{supp} \mathfrak{d}$ this is clear. Hence, it suffices to find neighbourhoods of each point $w \in \operatorname{supp} \mathfrak{d}$ such that g_0 is bounded on the respective neighbourhood. To this end, let ϵ_w and f_w be as in (9.2.1), and choose $\epsilon'_w \in (0, \epsilon_w)$ such that the functions $p \circ \phi_{M_w}^{-1}$ and $d \circ \phi_{M_w}^{-1}$ are analytic on some open neighbourhood of the closed disk $\{|z| \le \epsilon'_w\}$ and $d \circ \phi_{M_w}^{-1}$ has no other zero than 0 in this disk. Then we have, for $x \in [-\epsilon'_w, \epsilon'_w] \setminus \{0\}$,

$$(d \circ \phi_{M_w}^{-1})(x) \cdot (g_0 \circ \phi_{M_w}^{-1})(x) = (f \circ \phi_{M_w}^{-1})(x) - (p \circ \phi_{M_w}^{-1})(x) =$$

$$= x^{\mathfrak{d}(w)} \Big[f_w(x) - \sum_{j=\mathfrak{d}(w)}^{\infty} \frac{1}{j!} [p \circ \phi_{M_w}^{-1}]^{(j)}(0) x^{j-\mathfrak{d}(w)} \Big] ,$$

and hence

$$(g_0 \circ \phi_{M_w}^{-1})(x) = \Big[\frac{(d \circ \phi_{M_w}^{-1})(x)}{x^{\mathfrak{d}(w)}} \Big]^{-1} \Big[f_w(x) - \sum_{j=\mathfrak{d}(w)}^{\infty} \frac{1}{j!} [p \circ \phi_{M_w}^{-1}]^{(j)}(0) x^{j-\mathfrak{d}(w)} \Big] . \tag{9.2.4}$$ H64

Since $\phi_{M_w}^{-1}$ is a bijective analytic map of $\mathbb{C}_\infty$ onto itself, composition with $\phi_{M_w}^{-1}$ preserves zero-order. Thus the first factor is an analytic function on some neighbourhood of $[-\epsilon'_w, \epsilon'_w]$. The second factor is, as the sum of a bounded function and an analytic function, certainly bounded on this interval. Altogether, we see that g_0 is bounded on $\phi_{M_w}^{-1}([-\epsilon'_w, \epsilon'_w] \setminus \{0\})$.

Extending g_0 arbitrarily to all of $\mathbb{R}_\infty$, e.g. by setting $g(w) := 0$, $w \in \operatorname{supp} \mathfrak{d}(w)$, yields a bounded and measureable function $g : \mathbb{R}_\infty \to \mathbb{C}$. By construction, the equality $f = p + dg$ holds. We have established that $\mathfrak{A}(\mathfrak{d}) \subseteq \mathcal{R} + d \cdot \mathrm{BM}(\mathbb{R}_\infty)$. To show that this sum is indeed direct, assume that $r \in \mathcal{R} \cap d \cdot \mathrm{BM}(\mathbb{R}_\infty)$. Then $\mathfrak{d}_r(w) \ge \mathfrak{d}(w)$, $w \in \mathbb{R}_\infty$, and hence $r \in \ker \nu = \{0\}$. ❑

Let $\mathfrak{d} : \mathbb{R}_\infty \to \mathbb{N}_0$ be a function with finite support, and choose $d \in \mathbb{H}(\mathbb{R}_\infty)$ with $\mathfrak{d}_d = \mathfrak{d}$. By means of (9.2.2) we have a surjective map

$$\pi_d : \begin{cases} \mathbb{H}(\mathbb{R}_\infty) \times \mathrm{BM}(\mathbb{R}_\infty) & \to \quad \mathfrak{A}(\mathfrak{d}) \\ (p, g) & \mapsto \quad p + dg \end{cases} \tag{9.2.5}$$ H53

This map can be used to transfer properties of $\mathbb{H}(\mathbb{R}_\infty) \times \mathrm{BM}(\mathbb{R}_\infty)$ to $\mathfrak{A}(\mathfrak{d})$. First let us introduce more algebraic operations on $\mathbb{H}(\mathbb{R}_\infty) \times \mathrm{BM}(\mathbb{R}_\infty)$.

NRH21 *9.2.4.* $\mathbb{H}(\mathbb{R}_\infty) \times \mathrm{BM}(\mathbb{R}_\infty)$ *as an algebra:* Let $d \in \mathbb{H}(\mathbb{R}_\infty)$ be fixed. Then we define a multiplication '$\diamond_d$' on $\mathbb{H}(\mathbb{R}_\infty) \times \mathrm{BM}(\mathbb{R}_\infty)$ as

$$(p_1, g_1) \diamond_d (p_2, g_2) := (p_1 p_2, p_1 g_2 + p_2 g_1 + d g_1 g_2) ,$$

and a conjugation

$$.^{\#} : \begin{cases} \mathbb{H}(\mathbb{R}_\infty) \times \mathrm{BM}(\mathbb{R}_\infty) & \to \quad \mathbb{H}(\mathbb{R}_\infty) \times \mathrm{BM}(\mathbb{R}_\infty) \\ (p, g) & \mapsto \quad (p^{\#}, \overline{g}) \end{cases}$$

One can show by elementary computation that $\mathbb{H}(\mathbb{R}_\infty) \times \mathrm{BM}(\mathbb{R}_\infty)$ becomes a commutative algebra if endowed with the usual vector space operations and the multiplication '$\diamond_d$'. For example, let us check distributivity and associativity:

$$\begin{aligned} \big((p_1, g_1) + (p_2, g_2)\big) \diamond_d (p_3, g_3) &= (p_1 + p_2, g_1 + g_2) \diamond_d (p_3, g_3) = \\ &= \big((p_1 + p_2) p_3, (p_1 + p_2) g_3 + p_3 (g_1 + g_2) + d (g_1 + g_2) g_3\big) = \\ &= (p_1 p_3, p_1 g_3 + p_3 g_1 + d g_1 g_3) + (p_2 p_3, p_2 g_3 + p_3 g_2 + d g_2 g_3) = \\ &= (p_1, g_1) \diamond_d (p_3, g_3) + (p_2, g_2) \diamond_d (p_3, g_3) \end{aligned}$$

$$\begin{aligned}
\big((p_1,g_1)\diamond_d(p_2,g_2)\big)\diamond_d(p_3,g_3) &= (p_1p_2, p_1g_2+p_2g_1+dg_1g_2)\diamond_d(p_3,g_3) = \\
&= \big((p_1p_2)p_3, (p_1p_2)g_3+p_3(p_1g_2+p_2g_1+dg_1g_2)+ \\
&\qquad\qquad +d(p_1g_2+p_2g_1+dg_1g_2)g_3\big) = \\
&= \big(p_1p_2p_3, p_1p_2g_3+p_1g_2p_3+g_1p_2p_3+ \\
&\qquad\qquad +d(g_1g_2p_3+g_1p_2g_3+p_1g_2g_3)+d^2g_1g_2g_3\big) = \\
&= (p_1,g_1)\diamond_d\big((p_2,g_2)\diamond_d(p_3,g_3)\big)
\end{aligned}$$

The conjugation '$.^\#$' is obviously a conjugate linear involution. However, it is compatible with $\diamond_d$ only if $d=d^\#$; in general we have

$$\big((p_1,g_1)\diamond_d(p_2,g_2)\big)^\# = (p_1,g_1)^\#\diamond_{(d^\#)}(p_2,g_2)^\# .$$

The maps $\diamond_d:[\mathbb{H}(\mathbb{R}_\infty)\times\mathrm{BM}(\mathbb{R}_\infty)]^2\to\mathbb{H}(\mathbb{R}_\infty)\times\mathrm{BM}(\mathbb{R}_\infty)$ and $.^\#:\mathbb{H}(\mathbb{R}_\infty)\times\mathrm{BM}(\mathbb{R}_\infty)\to\mathbb{H}(\mathbb{R}_\infty)\times\mathrm{BM}(\mathbb{R}_\infty)$ are continuous with respect to this topology. For $.^\#$ this is immediate, for $\diamond_d$ remember that convergence in $\mathbb{H}(\mathbb{R}_\infty)$ implies uniform convergence on $\mathbb{R}_\infty$.

The space $\mathbb{H}(\mathcal{R}_\infty)\times\mathrm{BM}(\mathbb{R}_\infty)$ is naturally topologized with the product topology of the inverse limit topology on $\mathbb{H}(K)$ and the norm topology on $\mathrm{BM}(\mathbb{R}_\infty)$. //

NRH39 *9.2.5. $\mathfrak{A}(\mathfrak{d})$ as a quotient:* Let $\mathfrak{d}:\mathbb{R}_\infty\to\mathbb{N}_0$ be a function with finite support, and $d\in\mathbb{H}(\mathbb{R}_\infty)$ with $\mathfrak{d}_d=\mathfrak{d}$. Then the map $\pi_d:\mathbb{H}(\mathbb{R}_\infty)\times\mathrm{BM}(\mathbb{R}_\infty)\to\mathfrak{A}(\mathfrak{d})$ is homomorphic with respect to multiplication. With the obvious modification, it is also compatible with conjugation. To be precise, we have

$$\begin{tikzcd}
{[\mathbb{H}(\mathbb{R}_\infty)\times\mathrm{BM}(\mathbb{R}_\infty)]^2} \arrow[r, "\pi_d\times\pi_d"] \arrow[d, "\diamond_d"'] & \mathfrak{A}(\mathfrak{d})^2 \arrow[d, "\cdot"] \\
\mathbb{H}(\mathbb{R}_\infty)\times\mathrm{BM}(\mathbb{R}_\infty) \arrow[r, "\pi_d"'] & \mathfrak{A}(\mathfrak{d})
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{H}(\mathbb{R}_\infty)\times\mathrm{BM}(\mathbb{R}_\infty) \arrow[r, "\pi_d"] \arrow[d, ".^\#"'] & \mathfrak{A}(\mathfrak{d}) \arrow[d, "\overline{\cdot}"] \\
\mathbb{H}(\mathbb{R}_\infty)\times\mathrm{BM}(\mathbb{R}_\infty) \arrow[r, "\pi_{d^\#}"'] & \mathfrak{A}(\mathfrak{d})
\end{tikzcd}$$

Let us remark that $\pi_d(p,g)\in\mathbb{H}(\mathbb{R}_\infty)$ if and only if there exists $\tilde g\in\mathbb{H}(\mathbb{R}_\infty)$ with $g|_{\mathbb{R}_\infty\setminus\operatorname{supp}\mathfrak{d}}=\tilde g|_{\mathbb{R}_\infty\setminus\operatorname{supp}\mathfrak{d}}$.

In view of (9.2.5), $\mathfrak{A}(\mathfrak{d})$ is naturally topologized, namely with the quotient topology with respect to the map π_d. We will denote this topology on $\mathfrak{A}(\mathfrak{d})$ by $\mathcal{T}_{\mathfrak{A}}$. Let us show that $\mathcal{T}_{\mathfrak{A}}$ does not depend on the choice of d. If $d'\in\mathbb{H}(\mathbb{R}_\infty)$ is another element with $\mathfrak{d}_{d'}=\mathfrak{d}$, then $\frac{d}{d'}$ is a unit in $\mathbb{H}(\mathbb{R}_\infty)$. In particular, $\frac{d}{d'}$ is bounded above and away from zero on $\mathbb{R}_\infty$. Hence, multiplication with $\frac{d}{d'}$ is a homeomorphism of $\mathrm{BM}(\mathbb{R}_\infty)$ onto itself. We have the diagram

$$\begin{tikzcd}
\mathbb{H}(\mathbb{R}_\infty)\times\mathrm{BM}(\mathbb{R}_\infty) \arrow[r, "\pi_d"] \arrow[d, "\mathrm{id}\times(\cdot\frac{d}{d'})"'] & \mathfrak{A}(\mathfrak{d}) \\
\mathbb{H}(\mathbb{R}_\infty)\times\mathrm{BM}(\mathbb{R}_\infty) \arrow[ur, "\pi_{d'}"'] &
\end{tikzcd}$$

and this shows that the quotient topologies induced on $\mathfrak{A}(\mathfrak{d})$ by π_d and $\pi_{d'}$, respectively, coincide.

Let us remark that the topology $\mathcal{T}_{\mathfrak{A}}$ is finer than the topology which $\mathfrak{A}(\mathfrak{d})$ carries as a subspace of $\mathrm{BM}(\mathbb{R}_\infty)$, i.e. the norm topology induced by $\|.\|_\infty$. This follows since the map

$$\pi_d:\mathbb{H}(\mathbb{R}_\infty)\times\mathrm{BM}(\mathbb{R}_\infty)\to\langle\mathfrak{A}(\mathfrak{d}),\|.\|_\infty\rangle$$

is continuous.

Unless the contrary is explicitly stated, all topological terms refer to the topology $\mathfrak{T}_{\mathfrak{A}}$. //

LEH27 **9.2.6 Lemma.** *For each fixed $f_0 \in \mathfrak{A}(\mathfrak{d})$ the map*

$$\cdot f_0 : \begin{cases} \mathfrak{A}(\mathfrak{d}) & \to & \mathfrak{A}(\mathfrak{d}) \\ f & \mapsto & f \cdot f_0 \end{cases}$$

is continuous. Moreover, the map $\bar{.} : \mathfrak{A}(\mathfrak{d}) \to \mathfrak{A}(\mathfrak{d})$ is continuous.

Proof. Choose d as in Proposition 9.2.3, and write $f_0 = \pi_d(p_0, g_0)$. Then we have the diagram

$$\begin{tikzcd} \mathbb{H}(\mathbb{R}_\infty) \times \mathrm{BM}(\mathbb{R}_\infty) \arrow[r, "\pi_d"] \arrow[d, "\diamond_d(p_0,g_0)"'] & \mathfrak{A}(\mathfrak{d}) \arrow[d, "\cdot f_0"] \\ \mathbb{H}(\mathbb{R}_\infty) \times \mathrm{BM}(\mathbb{R}_\infty) \arrow[r, "\pi_d"'] & \mathfrak{A}(\mathfrak{d}) \end{tikzcd}$$

It follows that '$\cdot f_0$' is continuous. The (conjugate linear) map '$\bar{.}$' is treated similarly. ❑

NRH40 *9.2.7.* $\mathfrak{A}(\mathfrak{d})$ *as a Banach space:* In the quotient construction 9.2.5 we have used the first equality in (9.2.2). The second equality in this relation can be used to endow $\mathfrak{A}(\mathfrak{d})$ with a Banach space topology.

Let $\mathfrak{d} : \mathbb{R}_\infty \to \mathbb{N}_0$ be a function with finite support. Choose $d \in \mathbb{H}(\mathbb{R}_\infty)$ and a finite-dimensional subspace $\mathcal{R}$ of $\mathbb{H}(\mathbb{R}_\infty)$, such that

$$\mathfrak{d}_d = \mathfrak{d}, \quad \pi_d\big(\mathcal{R} \times \mathrm{BM}(\mathbb{R}_\infty)\big) = \mathfrak{A}(\mathfrak{d})\,. \tag{9.2.6}$$ H54

The space $\mathcal{R} \times \mathrm{BM}(\mathbb{R}_\infty)$ is a Banach space, when endowed with the sum norm ($\|p\|_\infty := \sup_{x \in \mathbb{R}_\infty} |p(x)|$)

$$\|(p, g)\|_+ := \|p\|_\infty + \|g\|_\infty, \quad (p, g) \in \mathcal{R} \times \mathrm{BM}(\mathbb{R}_\infty)\,,$$

Clearly, $\ker \pi_d$ is $\|.\|_+$-closed, and hence $\mathfrak{A}(\mathfrak{d})$ becomes a Banach space when endowed with the quotient norm of $\|.\|_+$ with respect to π_d. We will denote this norm as $\|.\|_{\mathcal{R},d}$ and the topology it induces on $\mathfrak{A}(\mathfrak{d})$ by $\mathcal{T}_{\mathfrak{A},\|.\|}$.

Let us show that two norms obtained in this way are equivalent. The same argument as in 9.2.5 shows that always $\|.\|_{\mathcal{R},d}$ is equivalent to $\|.\|_{\mathcal{R},d'}$ when $\mathcal{R}, d, d'$ are subject to (9.2.6). It remains to check equivalence of norms when d is fixed. Since each two finite-dimensional subspaces of $\mathbb{H}(\mathcal{R}_\infty)$ satisfying (9.2.6) are contained in one common finite-dimensional subspace with (9.2.6), namely in their linear span, it is enough to prove that $\|.\|_{\mathcal{R},d}$ is equivalent to $\|.\|_{\mathcal{R}',d}$ whenever $\mathcal{R} \subseteq \mathcal{R}'$. However, if $\mathcal{R} \subseteq \mathcal{R}'$, we have the diagram

$$\begin{tikzcd} \mathcal{R} \times \mathrm{BM}(\mathbb{R}_\infty) \arrow[r, "\pi_d"] \arrow[d, "\subseteq"'] & \langle \mathfrak{A}(\mathfrak{d}), \|.\|_{\mathcal{R},d} \rangle \arrow[d, "\mathrm{id}"] \\ \mathcal{R}' \times \mathrm{BM}(\mathbb{R}_\infty) \arrow[r, "\pi_d"] & \langle \mathfrak{A}(\mathfrak{d}), \|.\|_{\mathcal{R}',d} \rangle \end{tikzcd}$$

and hence the identity map is $\|.\|_{\mathcal{R},d}$-to-$\|.\|_{\mathcal{R}',d}$–continuous. By the Open Mapping Theorem, it is thus bicontinuous.

Note that the topology $\mathcal{T}_{\mathfrak{A},\|.\|}$ is finer than $\mathcal{T}_{\mathfrak{A}}$. This follows from continuity of the inclusion map $\subseteq: \mathcal{R} \times \mathrm{BM}(\mathbb{R}_\infty) \to \mathbb{H}(\mathbb{R}_\infty) \times \mathrm{BM}(\mathbb{R}_\infty)$ and the diagram

$$\begin{tikzcd}
\mathcal{R} \times \mathrm{BM}(\mathbb{R}_\infty) \arrow[r, "\pi_d"] \arrow[d, "\subseteq"'] & \mathfrak{A}(\mathfrak{d}) \\
\mathbb{H}(\mathbb{R}_\infty) \times \mathrm{BM}(\mathbb{R}_\infty) \arrow[ur, "\pi_d"'] &
\end{tikzcd}$$

//

It turns out that algebraic operations are continuous with respect to $\mathcal{T}_{\mathfrak{A}(\mathfrak{d}),\|.\|}$.

LEH22 **9.2.8 Lemma.** *The maps*

$$\cdot : \langle \mathfrak{A}(\mathfrak{d}), \mathcal{T}_{\mathfrak{A},\|.\|} \rangle^2 \to \langle \mathfrak{A}(\mathfrak{d}), \mathcal{T}_{\mathfrak{A},\|.\|} \rangle \quad \textit{and} \quad \bar{.} : \langle \mathfrak{A}(\mathfrak{d}), \mathcal{T}_{\mathfrak{A},\|.\|} \rangle \to \langle \mathfrak{A}(\mathfrak{d}), \mathcal{T}_{\mathfrak{A},\|.\|} \rangle$$

are continuous. In other words, whenever $\mathcal{R}$ and d are chosen with (9.2.6), *there exist constants $C_1, C_2 > 0$ such that*

$$\|f_1 f_2\|_{\mathcal{R},d} \leq C_1 \|f_1\|_{\mathcal{R},d} \|f_2\|_{\mathcal{R},d}, \ \|\overline{f}\|_{\mathcal{R},d} \leq C_2 \|f\|_{\mathcal{R},d}, \quad f_1, f_2, f \in \mathfrak{A}(\mathfrak{d})\,.$$

Proof. It is obvious from the definition that

$$\|(p_1, g_1) \diamond_d (p_2, g_2)\|_+ \leq (3 + \|d\|_\infty) \cdot \|(p_1, g_1)\|_+ \cdot \|(p_2, g_2)\|_+\,,$$

$$\|(p, g)^\#\|_+ = \|(p, g)\|_+\,,$$

whenever all occuring elements belong to $\mathcal{R} \times \mathrm{BM}(\mathbb{R}_\infty)$.

Set $\mathcal{R}_1 := (z+i)^{\delta-1}\mathbb{C}[z]_\delta$ and $\mathcal{R}_2 := (z+i)^{2\delta-2}\mathbb{C}[z]_{2\delta-1}$ Using distributivity and conjugate-linearity, respectively, it follows that

$$\diamond_d : [\mathcal{R}_1 \times \mathrm{BM}(\mathbb{R}_\infty)]^2 \to \mathcal{R}_2 \times \mathrm{BM}(\mathbb{R}_\infty)$$

$$.^\# : \mathcal{R}_1 \times \mathrm{BM}(\mathbb{R}_\infty) \to \mathcal{R}_1 \times \mathrm{BM}(\mathbb{R}_\infty)$$

are $\|.\|_+$-continuous. The assertion of the lemma now follows from the diagrams

$$\begin{tikzcd}
\langle \mathcal{R}_1 \times \mathrm{BM}(\mathbb{R}_\infty), \|.\|_+ \rangle^2 \arrow[r, "\diamond_d"] & \langle \mathcal{R}_2 \times \mathrm{BM}(\mathbb{R}_\infty), \|.\|_+ \rangle \arrow[d, "\pi_d"] \\
 & \langle \mathfrak{A}(\mathfrak{d}), \|.\|_{\mathcal{R}_2,d} \rangle \arrow[d, "\mathrm{id}"] \\
\langle \mathfrak{A}(\mathfrak{d}), \|.\|_{\mathcal{R}_1,d} \rangle^2 \arrow[uu, "(\pi_d \times \pi_d)^{-1}"] \arrow[r, "\cdot"'] & \langle \mathfrak{A}(\mathfrak{d}), \|.\|_{\mathcal{R}_1,d} \rangle
\end{tikzcd}
\qquad
\begin{tikzcd}
\langle \mathcal{R}_1 \times \mathrm{BM}(\mathbb{R}_\infty), \|.\|_+ \rangle \arrow[r, ".^\#"] & \langle \mathcal{R}_1 \times \mathrm{BM}(\mathbb{R}_\infty), \|.\|_+ \rangle \arrow[d, "\pi_{d^\#}"] \\
 & \langle \mathfrak{A}(\mathfrak{d}), \|.\|_{\mathcal{R}_1,d^\#} \rangle \arrow[d, "\mathrm{id}"] \\
\langle \mathfrak{A}(\mathfrak{d}), \|.\|_{\mathcal{R}_1,d} \rangle \arrow[uu, "\pi_d^{-1}"] \arrow[r, "\bar{.}"'] & \langle \mathfrak{A}(\mathfrak{d}), \|.\|_{\mathcal{R}_1,d} \rangle
\end{tikzcd}$$

❑

REH62 *9.2.9 Remark.* For later reference let us remark that, for each positive Borel measure μ and fixed $p_1, p_2, p \in \mathbb{H}(\mathbb{R}_\infty)$, the maps

$$(g_1, g_2) \mapsto (p_1, g_1) \diamond_d (p_2, g_2) \quad \text{and} \quad g \mapsto (p, g)^\#$$

are μ-boundedly pointwise continuous. //

Finally, let us show how the algebras $\mathfrak{A}(\mathfrak{d})$ transform when performing a fractional linear transformation.

LEH66 **9.2.10 Lemma.** *Let $\mathfrak{d} : \mathbb{R}_\infty \to \mathbb{N}_0$ be a function with finite support, let $N \in \mathrm{GL}(2,\mathbb{R})$, and set $\tilde{\mathfrak{d}} := \mathfrak{d} \circ \phi_N$. Then composition with ϕ_N is a homeomorphic and $\mathcal{T}_{\mathfrak{A},\|.\|}$-homeomorphic *-algebra isomorphism of $\mathfrak{A}(\mathfrak{d})$ onto $\mathfrak{A}(\tilde{\mathfrak{d}})$.*

Proof. Choose $d \in \mathbb{H}(\mathbb{R}_\infty)$ with $\mathfrak{d}_d = \mathfrak{d}$, and set $\tilde{d} := d \circ \phi_N$. Then $\mathfrak{d}_{\tilde{d}} = \mathfrak{d}_d \circ \phi_N = \tilde{\mathfrak{d}}$. We have the diagram

$$\begin{tikzcd}
\mathbb{H}(\mathbb{R}_\infty) \times \mathrm{BM}(\mathbb{R}_\infty) \arrow[r, "\pi_d"] \arrow[d, bend right, "\circ\phi_N \times \circ\phi_N"'] & \mathrm{BM}(\mathbb{R}_\infty) \arrow[d, bend right, "\circ\phi_N"'] \\
\mathbb{H}(\mathbb{R}_\infty) \times \mathrm{BM}(\mathbb{R}_\infty) \arrow[r, "\pi_{\tilde{d}}"'] \arrow[u, bend right, "\circ\phi_N^{-1} \times \circ\phi_N^{-1}"'] & \mathrm{BM}(\mathbb{R}_\infty) \arrow[u, bend right, "\circ\phi_N^{-1}"']
\end{tikzcd}$$

The maps $\circ\phi_N \times \circ\phi_N$ and $\circ\phi_N^{-1} \times \circ\phi_N^{-1}$ are mutually inverse bijections, and hence it follows that $\circ\phi_N$ maps $\mathfrak{A}(\mathfrak{d})$ bijectively onto $\mathfrak{A}(\tilde{\mathfrak{d}})$.

By Lemma 8.1.4, (ii), $\circ\phi_N : \mathbb{H}(\mathbb{R}_\infty) \to \mathbb{H}(\mathbb{R}_\infty)$ is homeomorphic. Clearly, $\circ\phi_N : \mathrm{BM}(\mathbb{R}_\infty) \to \mathrm{BM}(\mathbb{R}_\infty)$ is isometric. We conclude that $\circ\phi_N : \mathfrak{A}(\mathfrak{d}) \to \mathfrak{A}(\tilde{\mathfrak{d}})$ is a homeomorphism. Using the same argument, only restricting the left sides of the above diagram to $\mathcal{R} \times \mathrm{BM}(\mathbb{R}_\infty)$ where $\mathcal{R}$ satisfies (9.2.6), shows that $\circ\phi_N$ is also homeomorphic with respect to the topologies $\mathcal{T}_{\mathfrak{A},\|\cdot\|}$. The fact that composition with ϕ_N is homomorphic with respect to algebraic operations is immediate. ❑

9.3 The algebra $C^\infty(\mathbb{R}_\infty)$

The set $\mathbb{R}_\infty$ endowed with the restriction of the topology of $\mathbb{C}_\infty$ is nothing else but the one-point compactification of $\mathbb{R}$. Let V be an open subset of $\mathbb{R}_\infty$. Then V becomes a C^∞-manifold when endowed with the collection of charts

$$\Big\{\phi_M : V \setminus \{\phi_M^{-1}(\infty)\} \to \mathbb{C} \text{ with } M \in \mathrm{GL}(2,\mathbb{R})\Big\}. \tag{9.3.1}$$ H55

The linear space of all arbitrarily differentiable functions $f : V \to \mathbb{C}$ is denoted by $C^\infty(V)$. With the pointwise defined algebraic operations and conjugation $C^\infty(V)$ becomes a commutative *-algebra.

With help of the charts (9.3.1) we can also define a locally convex vector topology on $C^\infty(V)$. Namely, for $M \in \mathrm{GL}(2,\mathbb{R})$, $\epsilon > 0$ with $[-\epsilon,\epsilon] \subseteq \phi_M(V)$, and $k \in \mathbb{N}_0$, we consider the seminorms

$$p^k_{M,\epsilon}(f) := \sup_{x \in [-\epsilon,\epsilon]} \big|(f \circ \phi_M^{-1})^{(k)}(x)\big|, \quad f \in C^\infty(V)\,.$$

Then the family

$$\Big\{p^k_{M,\epsilon} :\ M \in \mathrm{GL}(2,\mathbb{R}), \epsilon > 0 \text{ with } [-\epsilon,\epsilon] \subseteq \phi_M(V), k \in \mathbb{N}_0\Big\}$$

is a separating family of seminorms, and hence defines a Hausdorff and locally convex topology on $C^\infty(V)$. We will refer to this topology as $\mathcal{T}_\infty$.

REH41 *9.3.1 Remark.*

(*i*) Let $M_i \in \mathrm{GL}(2,\mathbb{R})$, $\epsilon_i > 0$ with $[-\epsilon_i,\epsilon_i] \subseteq \phi_{M_i}(V)$, $i \in I$, be such that

$$\bigcup_{i \in I} \phi_{M_i}^{-1}\big((-\epsilon_i,\epsilon_i)\big) = V\,.$$

Then the family $\{p^k_{M_i,\epsilon_i} : i \in I, k \in \mathbb{N}_0\}$ is a separating family of seminorms on $C^\infty(V)$, and induces the topology $\mathcal{T}_\infty$.

(*ii*) Let $O \subseteq \mathbb{C}_\infty$ be open, and set $V := O \cap \mathbb{R}_\infty$. Then we have the restriction map $\rho_{O,V} : F \mapsto F|_V$. Clearly, $\rho_{O,V}$ maps $\mathbb{H}(O)$ into $C^\infty(V)$. Since the seminorms $p^k_{M,\epsilon}$ evaluate derivatives on compact intervals, $\rho_{O,V}$ is continuous.

//

It is an important fact that $C^\infty(\mathbb{R}_\infty)$ is contained in $\mathfrak{A}(\mathfrak{d})$.

PRH13 **9.3.2 Proposition.** *Let $\mathfrak{d} : \mathbb{R}_\infty \to \mathbb{N}_0$ be a function with finite support. Then $C^\infty(\mathbb{R}_\infty) \subseteq \mathfrak{A}(\mathfrak{d})$ and the set-theoretic inclusion map is $\mathcal{T}_\infty$-to-$\mathcal{T}_{\mathfrak{A},\|.\|}$-continuous.*

Proof.
Step 1; $C^\infty(\mathbb{R}_\infty) \subseteq \mathfrak{A}(\mathfrak{d})$: Let $f \in C^\infty(\mathbb{R}_\infty)$ be given. Then, for each $w \in \mathbb{R}_\infty$, the function $f \circ \phi^{-1}_{M_w}$ also belongs to $C^\infty(\mathbb{R}_\infty)$. Let $f_w : \mathbb{R} \setminus \{0\} \to \mathbb{C}$ be the function which is uniquely defined by the relation

$$(f \circ \phi^{-1}_{M_w})(x) = \sum_{j=0}^{\mathfrak{d}(w)-1} \frac{1}{j!}[f \circ \phi^{-1}_{M_w}]^{(j)}(0)x^j + x^{\mathfrak{d}(w)} f_w(x), \quad x \in \mathbb{R} \setminus \{0\}\,.$$

Then, by Taylor's Theorem, for each $x \in \mathbb{R} \setminus \{0\}$ there exists a point ξ_x on the line segment connecting x with 0 such that $f_w(x) = \frac{1}{\mathfrak{d}(w)!}[f \circ \phi^{-1}_{M_w}]^{(\mathfrak{d}(w))}(\xi_x)$. We see that, for each $\epsilon > 0$,

$$\sup_{x \in [-\epsilon,\epsilon]} |f_w(x)| \leq \frac{1}{\mathfrak{d}(w)!} \sup_{x \in [-\epsilon,\epsilon]} \left|[f \circ \phi^{-1}_{M_w}]^{(\mathfrak{d}(w))}(x)\right|\,. \tag{9.3.2}$$ H65

We see that $f \circ \phi^{-1}_{M_w}$ possesses a representation as required in (9.2.2), and conclude that $f \in \mathfrak{A}(\mathfrak{d})$.

Step 2; Continuity of g: Let $d \in \mathbb{H}(\mathbb{R}_\infty)$ be such that $\mathfrak{d}_d = \mathfrak{d}$, and set $\delta := \sum_{w \in \operatorname{supp} \mathfrak{d}} \mathfrak{d}(w)$. Let $f \in C^\infty(\mathbb{R}_\infty)$, then by means of Proposition 9.2.3 we find $p \in (z+i)^{-(\delta-1)}\mathbb{C}[z]_\delta$ and $g \in \mathrm{BM}(\mathbb{R}_\infty)$ such that $f = p + dg$. Then the function g can be chosen to be continuous. Continuity at a point $w \notin \operatorname{supp} \mathfrak{d}$ readily follows from the fact that $f = p + dg$. If $w \in \operatorname{supp} \mathfrak{d}$, then $g \circ \phi^{-1}_{M_w}$ coincides on some neighbourhood of 0 with the function f_w constructed above. However, if $x \to 0$ then also $\xi_x \to 0$, and hence $f_w(x) \to \frac{1}{\mathfrak{d}(w)!}[f \circ \phi^{-1}_{M_w}]^{(\mathfrak{d}(w))}(0)$. Thus redefining g on $\operatorname{supp} \mathfrak{d}$ as $g(w) := \frac{1}{\mathfrak{d}(w)!}[f \circ \phi^{-1}_{M_w}]^{(\mathfrak{d}(w))}(0)$ gives a continuous function.

Step 3; Continuity of the map $f \mapsto (p, g)$: By means of the previous steps a map of $C^\infty(\mathbb{R}_\infty) \to \mathcal{R} \times C(\mathbb{R}_\infty)$ is well-defined by mapping a function f to the pair (p, g) with $f = p + dg$. Let ν be the map defined in (9.2.3), and let ϵ'_w, $w \in \operatorname{supp} \mathfrak{d}$, be chosen as in the proof of Proposition 9.2.3. Let $\beta_w \in (0, \epsilon'_w)$ be such that the distance of the closed set $\phi^{-1}_{M_w}(\{|z| \leq \beta_w\})$ to the point $-i$ is at least $\frac{1}{2}$.

We consider the space $\mathcal{R}$ endowed with the supremum norm $\|r\| := \sup_{|z+i| \geq \frac{1}{2}} |r(z)|$, and the space $\prod_{w \in \operatorname{supp} \mathfrak{d}} \mathbb{C}^{\mathfrak{d}(w)}$ endowed with the maximum norm $\|((\alpha_{j,w})_{j=0}^{\mathfrak{d}(w)-1})_{w\in\operatorname{supp}\mathfrak{d}}\| := \max_{w,j}|a_{j,w}|$. Since ν is a bijection between these finite dimensional spaces, it is bicontinuous. Let $\|\nu^{-1}\|$ be the operator norm of ν^{-1} corresponding to these norms.

Choose finitely many points $w_1,\dots,w_n\in\mathbb{R}_\infty$ and numbers $\epsilon_1,\dots,\epsilon_n>0$, such that

$$\operatorname{supp}\mathfrak{d}\cap\bigcup_{i=1}^{n}\phi_{M_{w_i}}^{-1}\big([-\epsilon_i,\epsilon_i]\big)=\emptyset\,,$$

and

$$\bigcup_{w\in\operatorname{supp}\mathfrak{d}}\phi_{M_w}^{-1}\big((-\beta_w,\beta_w)\big)\cup\bigcup_{i=1}^{n}\phi_{M_{w_i}}^{-1}\big((-\epsilon_i,\epsilon_i)\big)=\mathbb{R}_\infty\,.$$

Since $x^{-\mathfrak{d}(w)}[(dg)\circ\phi_{M_w}^{-1}](x)$ remains bounded when x tends to 0, we must have

$$\nu(p)=\big((\frac{1}{j!}[f\circ\phi_{M_w}^{-1}]^{(j)}(0))_{j=0}^{\mathfrak{d}(w)-1}\big)_{w\in\operatorname{supp}\mathfrak{d}}$$

and it follows that

$$\|p\|\le\|\nu^{-1}\|\;\cdot\max_{\substack{w\in\operatorname{supp}\mathfrak{d}\\0\le j<\mathfrak{d}(w)}}\frac{1}{j!}\,p_{M_w,\beta_w}^{j}(f)\,.$$

Set $D_i:=(\min_{x\in\phi_{M_{w_i}}^{-1}([-\epsilon_i,\epsilon_i])}|d(x)|)^{-1}$. Then we have

$$|g(x)|=\Big|\frac{f(x)-p(x)}{d(x)}\Big|\le\Big(p_{M_{w_i},\epsilon_i}^{0}(f)+\|\nu^{-1}\|\;\cdot\max_{\substack{w\in\operatorname{supp}\mathfrak{d}\\0\le j<\mathfrak{d}(w)}}\frac{1}{j!}\,p_{M_w,\beta_w}^{j}(f)\Big)\;\cdot\;D_i,$$
$$x\in\phi_{M_{w_i}}^{-1}\big([-\epsilon_i,\epsilon_i]\big)\,.$$

To estimate $|g(x)|$ on the sets $\phi_{M_w}^{-1}\big((-\beta_w,\beta_w)\big)$, we use (9.2.4). Set $D_w:=\big(\min_{x\in[-\beta_w,\beta_w]}\frac{(d\circ\phi_{M_w}^{-1})(x)}{x^{\mathfrak{d}(w)}}\big)^{-1}$, and remember that (9.3.2) says

$$\sup_{x\in[-\beta_w,\beta_w]}|f_w(x)|\le\frac{1}{\mathfrak{d}(w)!}p_{M_w,\beta_w}^{\mathfrak{d}(w)}(f)\,.$$

To estimate the series in (9.2.4), we use the Maximum Principle. It gives

$$\Big|\sum_{j=\mathfrak{d}(w)}^{\infty}\frac{1}{j!}[p\circ\phi_{M_w}^{-1}]^{(j)}(0)x^{j-\mathfrak{d}(w)}\Big|=$$
$$=\Big|x^{-\mathfrak{d}(w)}\Big([p\circ\phi_{M_w}^{-1}](x)-\sum_{j=0}^{\mathfrak{d}(w)-1}\frac{1}{j!}[p\circ\phi_{M_w}^{-1}]^{(j)}(0)x^{j}\Big)\Big|\le$$
$$\le\Big(\frac{1}{\beta_w}\Big)^{\mathfrak{d}(w)}\Big(\|p\|+\sum_{j=0}^{\mathfrak{d}(w)-1}\frac{1}{j!}\,p_{M_w,\beta_w}^{j}(f)\beta_w^{j}\Big)\,.$$

Putting together these estimates gives

$$|g(x)|\le D_w\;\cdot\;\Big(\frac{1}{\mathfrak{d}(w)!}\,p_{M_w,\beta_w}^{\mathfrak{d}(w)}(f)+\Big(\frac{1}{\beta_w}\Big)^{\mathfrak{d}(w)}\|p\|+$$
$$+\Big(\frac{1}{\beta_w}\Big)^{\mathfrak{d}(w)}\sum_{j=0}^{\mathfrak{d}(w)-1}\frac{1}{j!}\,p_{M_w,\beta_w}^{j}(f)\beta_w^{j}\Big),\quad x\in\phi_{M_w}^{-1}\big((-\beta_w,\beta_w)\big)\,.$$

We see that

$$\|p\|_\infty = \sup_{x\in\mathbb{R}_\infty} |p(x)| \text{ and } \|g\|_\infty = \sup_{x\in\mathbb{R}_\infty} |g(x)|$$

are bounded by expressions which involve constants not depending on f and the seminorms

$$p^0_{M_{w_i},\epsilon_i},\ i=1,\ldots,n, \quad p^j_{M_w,\beta_w},\ w\in\operatorname{supp}\mathfrak{d}, 0\leq j\leq \mathfrak{d}(w)\,.$$

Step 4; Finish of proof: We have

$$\begin{tikzcd}
\mathcal{R}\times C(\mathbb{R}_\infty) \arrow[r, "\pi_d"] & \langle\mathfrak{A}(\mathfrak{d}), \mathcal{T}_{\mathfrak{A},\|.\|}\rangle \\
C^\infty(\mathbb{R}_\infty) \arrow[u, "f\mapsto(p,g)"] \arrow[ur, dashed, "\subseteq"'] &
\end{tikzcd}$$

and hence the inclusion map is the composition of two continuous maps. ❑

Let f be a function defined on an analytic manifold X and taking values in a locally convex vector space $\mathcal{X}$. Then we say that f is *strongly analytic* if for each chart $\varphi: U\to\mathbb{C}$ of X, then map $f\circ\varphi^{-1}:\varphi(U)\to\mathcal{X}$ is complex differentiable with respect to the topology of $\mathcal{X}$. We say that f is *weakly analytic*, if for each continuous linear functional $\lambda\in\mathcal{X}'$ the complex valued function $\lambda\circ f$ belongs to $\mathbb{H}(X)$.

Clearly, f being norm analytic implies that f is weakly analytic. If $\mathcal{X}$ is a Banach space, also the converse holds, and we will shortly speak of an analytic map.

In our later considerations, the following construction appears.

PRH56 **9.3.3 Proposition.** *Let $\mathfrak{d}:\mathbb{R}_\infty\to\mathbb{N}_0$ be a function with finite support, and let $f\in\mathfrak{A}(\mathfrak{d})$. For each fixed value of the parameter $z\in\mathbb{C}$ consider the function*

$$\xi_f : (\mathbb{C}_\infty\setminus\operatorname{supp} f)\times\mathbb{R}_\infty\to\mathbb{C}$$

defined as

$$\xi_f(w,x) := \begin{cases} \frac{w-z}{w-x}f(x)\,, & w\in\mathbb{C}\setminus\operatorname{supp} f, x\in\mathbb{R},\ w\neq x\\ 0 \quad, & w\in\mathbb{C}\setminus\operatorname{supp} f, x\in\mathbb{R},\ w=x\\ 0 \quad, & w\in\mathbb{C}\setminus\operatorname{supp} f, x=\infty\\ f(x) \quad, & w\in\{\infty\}\setminus\operatorname{supp} f, x\in\mathbb{R}_\infty\end{cases} \tag{9.3.3}$$ H58

Then, for each fixed $w\in\mathbb{C}_\infty\setminus\operatorname{supp} f$, the function $x\mapsto\xi_f(w,x)$ belongs to $\mathfrak{A}(\mathfrak{d})$. The function $w\mapsto\xi_f(w,.)$ is an analytic map of $\mathbb{C}_\infty\setminus\operatorname{supp} f$ into the Banach space $\langle\mathfrak{A}(\mathfrak{d}),\mathcal{T}_{\mathfrak{A},\|.\|}\rangle$.

The main argument is the following '$\mathbb{H}(O)$ version'.

LEH57 **9.3.4 Lemma.** *Let $O\subseteq\mathbb{C}_\infty$ be open, nonempty, with $\overline{O}\neq\mathbb{C}_\infty$, and consider the function*

$$g_O : (\mathbb{C}_\infty\setminus\overline{O})\times O\to\mathbb{C}$$

which is defined as

$$g_O(w,x) := \begin{cases} \frac{w-z}{w-x}\,, & w\in\mathbb{C}\setminus\overline{O},\ x\in O\setminus\{\infty\} \\ 0 \quad , & w\in\mathbb{C}\setminus\overline{O},\ x\in O\cap\{\infty\} \\ 1 \quad , & w\in\{\infty\}\setminus\overline{O},\ x\in O \end{cases}$$

Then, for each fixed $w\in\mathbb{C}_\infty\setminus\overline{O}$, the function $x\mapsto g_O(w,x)$ belongs to $\mathbb{H}(O)$. The function $w\mapsto g_O(w,.)$ is an analytic map of $\mathbb{C}_\infty\setminus\overline{O}$ into $\mathbb{H}(O)$.

Proof. We will use the charts

$$\varphi_1 := \phi_I : [\mathbb{C}_\infty\setminus\overline{O}]\setminus\{\infty\}\to\mathbb{C},\quad \varphi_2 := \phi_{\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}} : [\mathbb{C}_\infty\setminus\overline{O}]\setminus\{0\}\to\mathbb{C}$$

to describe the analytic manifold $\mathbb{C}_\infty\setminus\overline{O}$, and the charts

$$\psi_1 := \phi_I : O\setminus\{\infty\}\to\mathbb{C},\quad \psi_2 := \phi_{\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}} : O\setminus\{0\}\to\mathbb{C}$$

to describe the analytic manifold O. Moreover, we compute

$$\begin{aligned} g\circ\big(\varphi_1^{-1}(u)\times\psi_1^{-1}(t)\big) &= \tfrac{u-z}{u-t} &&, u\in\varphi_1([\mathbb{C}_\infty\setminus\overline{O}]\setminus\{\infty\}),\ t\in\psi_1(O\setminus\{\infty\}) \\ g\circ\big(\varphi_1^{-1}(u)\times\psi_2^{-1}(t)\big) &= \tfrac{t(u-z)}{tu-1} &&, u\in\varphi_1([\mathbb{C}_\infty\setminus\overline{O}]\setminus\{\infty\}),\ t\in\psi_2(O\setminus\{0\}) \\ g\circ\big(\varphi_2^{-1}(u)\times\psi_1^{-1}(t)\big) &= \tfrac{1-zu}{1-tu} &&, u\in\varphi_2([\mathbb{C}_\infty\setminus\overline{O}]\setminus\{0\}),\ t\in\psi_1(O\setminus\{\infty\}) \\ g\circ\big(\varphi_2^{-1}(u)\times\psi_2^{-1}(t)\big) &= \tfrac{t(1-zu)}{t-u} &&, u\in\varphi_2([\mathbb{C}_\infty\setminus\overline{O}]\setminus\{0\}),\ t\in\psi_2(O\setminus\{0\}) \end{aligned} \tag{9.3.4}$$ H59

Each of these functions depends, for u fixed, analytically on t. This already shows that $g_O(w,.)\in\mathbb{H}(O)$.

The function $w\mapsto g_O(w,.)$ being an analytic map of $\mathbb{C}_\infty\setminus\overline{O}$ into $\mathbb{H}(O)$, means that for both charts φ_j, $j=1,2$, and each $u_0\in\varphi_1([\mathbb{C}_\infty\setminus\overline{O}]\setminus\{\infty\})$ and $u_0\in\varphi_2([\mathbb{C}_\infty\setminus\overline{O}]\setminus\{0\})$, respectively, the limit

$$\lim_{u\to u_0}\frac{g_O(\varphi_j^{-1}(u),x)-g_O(\varphi_j^{-1}(u_0),x)}{u-u_0}$$

exists locally uniformly for $x\in O$. Since the sets $O\setminus\{\infty\}$ and $O\setminus\{0\}$ are an open cover of O, it is enough to show that, for each $j\in\{1,2\}$, the limits

$$g_{ji}(u_0,x) := \lim_{u\to u_0}\frac{g_O(\varphi_j^{-1}(u),x)-g_O(\varphi_j^{-1}(u_0),x)}{u-u_0},\ i=1,2\,,$$

exist locally uniformly on $O\setminus\{\infty\}$ and $O\setminus\{0\}$, respectively, and coincide on the intersection of these sets. This, however, follows from (9.3.4). It is straightforward that

$$g_{11}(u_0,\psi_1^{-1}(t)) = \lim_{u\to u_0}\frac{g_O(\varphi_1^{-1}(u),\psi_1^{-1}(t))-g_O(\varphi_1^{-1}(u_0),\psi_1^{-1}(t))}{u-u_0} = \frac{z-t}{(u_0-t)^2}$$

$$g_{12}(u_0,\psi_2^{-1}(t)) = \lim_{u\to u_0}\frac{g_O(\varphi_1^{-1}(u),\psi_2^{-1}(t))-g_O(\varphi_1^{-1}(u_0),\psi_2^{-1}(t))}{u-u_0} = \frac{zt-1}{(tu_0-1)^2}t$$

locally uniformly for $t\in\phi_1(O\setminus\{\infty\})$ and $t\in\phi_1(O\setminus\{0\})$, respectively, and that

$$g_{11}(u_0,x) = \frac{z-x}{(u_0-x)^2} = \frac{z\frac{1}{x}-1}{(\frac{1}{x}u_0-1)^2}\frac{1}{x} = g_{12}(u_0,x),\quad x\in O\setminus\{0,\infty\}\,.$$

Next,

$$g_{21}(u_0,\psi_1^{-1}(t)) = \lim_{u\to u_0} \frac{g_O(\varphi_2^{-1}(u),\psi_1^{-1}(t)) - g_O(\varphi_2^{-1}(u_0),\psi_1^{-1}(t))}{u-u_0} = \frac{z-t}{(1-tu_0)^2}$$

$$g_{22}(u_0,\psi_2^{-1}(t)) = \lim_{u\to u_0} \frac{g_O(\varphi_2^{-1}(u),\psi_2^{-1}(t)) - g_O(\varphi_2^{-1}(u_0),\psi_2^{-1}(t))}{u-u_0} = t\frac{1-zt}{(t-u_0)^2}$$

locally uniformly for $t\in\phi_1(O\setminus\{\infty\})$ and $t\in\phi_1(O\setminus\{0\})$, respectively, and

$$g_{21}(u_0,x) = \frac{z-x}{(1-xu_0)^2} = \frac{1}{x}\frac{1-z\frac{1}{x}}{(\frac{1}{x}-u_0)^2} = g_{22}(u_0,x), \quad x\in O\setminus\{0,\infty\}\,.$$

❑

Proof (of Proposition 9.3.3).
Step 1: Let $\chi\in C^\infty(\mathbb{R}_\infty)$, $O\subseteq\mathbb{C}_\infty$ open, and define a function

$$h_{\chi,O}:(\mathbb{C}_\infty\setminus\overline{O})\times\mathbb{R}_\infty\to\mathbb{C}$$

as

$$h_{\chi,O}(w,x):\begin{cases} g_O(w,x)\chi(x)\,, & x\in O\\ 0 & , \quad x\in\mathbb{R}_\infty\setminus\operatorname{supp}\chi\end{cases}$$

Note that, since $\operatorname{supp}\chi\subseteq O$, this function is well-defined on all of $\mathbb{R}_\infty$. Moreover, for each fixed $w\in\mathbb{C}_\infty\setminus\overline{O}$, we have $h_{\chi,O}(w,.)\in C^\infty(\mathbb{R}_\infty)$. Since multiplication with a fixed function is a continuous map of $C^\infty(O)$ into itself, the function $w\mapsto g_O(w,.)\chi(.)$ is analytic. The zero function of $\mathbb{C}_\infty\setminus\overline{O}$ into $C^\infty(\mathbb{C}_\infty\setminus\operatorname{supp}\chi)$ is trivially analytic. Again, since $\operatorname{supp}\chi\subseteq O$, the function $w\mapsto h_{\chi,O}(w,.)$ is thus an analytic map of $\mathbb{C}_\infty\setminus\overline{O}$ into $C^\infty(\mathbb{R}_\infty)$.

Step 2: Let $f\in\mathfrak{A}(\mathfrak{d})$ be given. Choose $O\subseteq\mathbb{C}_\infty$ open with $\operatorname{supp}f\subseteq O$, and choose a partition of unity $\chi_1,\chi_2\in C^\infty(\mathbb{R}_\infty)$ subordinate to the open cover $\{O\cap\mathbb{R}_\infty,\mathbb{R}_\infty\setminus\operatorname{supp}f\}$. Then $f=\chi_1 f$, and

$$\xi_f(w,x) = h_{\chi_1,O}(w,x)f(x), \quad w\in\mathbb{C}_\infty\setminus\overline{O}, x\in\mathbb{R}_\infty\,.$$

By Proposition 9.3.2, and since multiplication with a fixed function in $\mathfrak{A}(\mathfrak{d})$ is continuous, it follows that $\xi_f(w,.)|_{\mathbb{C}_\infty\setminus\overline{O}}$ is an analytic map of $\mathbb{C}_\infty\setminus\overline{O}$ into the Banach space $\mathfrak{A}(\mathfrak{d})$.

Since O was arbitrary, it follows that ξ_f is in fact analytic on all of $\mathbb{C}_\infty\setminus\operatorname{supp}f$. ❑

9.4 The functional calculus. I. Definitizability along $\mathbb{R}_\infty$

DEH52 **9.4.1 Definition.** Let $\mathcal{K}$ be a Krein space, and A be a selfadjoint linear relation in $\mathcal{K}$. Then we say that A is *definitizable along* $\mathbb{R}_\infty$, if

(i) The sets $\sigma(A)\cap\mathbb{R}_\infty$ and $\sigma(A)\setminus\mathbb{R}_\infty$ are relatively open in $\sigma(A)$.

(ii) Denote $\sigma_0:=\sigma(A)\cap\mathbb{R}_\infty$. There exists an element $d\in\mathbb{H}(\mathbb{R}_\infty)\setminus\{0\}$ such that $\Phi_{\mathrm{RD}}^{A,\sigma_0}(d)\geq 0$.

If $d \in \mathbb{H}(\sigma(A) \cap \mathbb{R}_\infty)$ is an element with the properties required in (ii), we call d *$\mathbb{R}_\infty$-definitizing for A.* //

In the next theorem we comprehensively formulate the properties of the functional calculus for selfadjoint relations which are definitizable along $\mathbb{R}_\infty$. Recall that $\rho_{\mathbb{R}_\infty}$ denotes the canonical map of $\mathbb{H}(\mathbb{R}_\infty)$ into $C(\mathbb{R}_\infty)$.

THH24 **9.4.2 Theorem.** *Let $\mathcal{K}$ be a Krein space, and A a selfadjoint relation in $\mathcal{K}$ which is definitizable along $\mathbb{R}_\infty$. Denote*

$$\mathfrak{d}_{\mathrm{crt}}^A := \min\big\{\mathfrak{d}_d :\ d\ \mathbb{R}_\infty\text{-definitizing for } A\big\},$$

and let $\mathfrak{A}_A$ be the commutative *-algebra $\mathfrak{A}_A := \mathfrak{A}(\mathfrak{d}_{\mathrm{crt}}^A)$. Then there exists a continuous *-algebra homomorphism $\Phi_A : \mathfrak{A}_A \to \mathcal{B}(\mathcal{K})$ which extends the Riesz-Dunford functional calculus in the sense that ($\sigma_0 := \sigma(A) \cap \mathbb{R}_\infty$)

$$\begin{tikzcd}
\mathbb{H}(\mathbb{R}_\infty) \arrow[r, "\Phi_{\mathrm{RD}}^{A,\sigma_0}"] \arrow[d, "\rho_{\mathbb{R}_\infty}"'] & \mathcal{B}(\mathcal{K}) \\
\mathfrak{A}_A \arrow[ur, "\Phi_A"'] &
\end{tikzcd}$$

The map Φ_A has the following additional properties:

(i) *Set $\mathcal{K}_1 := \operatorname{ran} \Phi_{\mathrm{RD}}^{A,\sigma_0}(1)$ and $\mathcal{K}_2 := \ker \Phi_{\mathrm{RD}}^{A,\sigma_0}(1)$. Then for all $f \in \mathfrak{A}_A$ the decomposition $\mathcal{K} = \mathcal{K}_1[\dot{+}]\mathcal{K}_2$ reduces $\Phi_A(f)$. Moreover, $\Phi_A(f)|_{\mathcal{K}_2} = 0$.*

Let $T \in \mathcal{B}(\mathcal{K}_1)$. Then we have $T \cdot (A-w)^{-1}|_{\mathcal{K}_1} = (A-w)^{-1}|_{\mathcal{K}_1} \cdot T$, $w \in \rho(A) \cap \mathbb{C}$, if and only if $T\Phi_A(f) = \Phi_A(f)T$, $f \in \mathfrak{A}_A$.

(ii) *Let d be $\mathbb{R}_\infty$-definitizing for A and $f \in \mathfrak{A}_A$. If $\frac{f}{d}$ is bounded and nonnegative, then $\Phi_A(f) \geq 0$.*

(iii) *The* Spectral Mapping Theorem*: Let $f \in \mathfrak{A}_A$, and assume that each function f_w in* (9.2.1) *is continuous at* 0*. Then*

$$\sigma\big(\Phi_A(f)|_{\mathcal{K}_1}\big) = f\big(\sigma(A) \cap \mathbb{R}_\infty\big).$$

(iv) *The set $\sigma(A) \cap \mathbb{R}_\infty$ is the smallest closed subset C of $\mathbb{R}_\infty$ with the property that*

$$\forall\, f \in \mathfrak{A}_A : \quad C \cap \operatorname{supp} f = \emptyset \ \Rightarrow\ \Phi_A(f) = 0 \tag{9.4.1}$$ H38

We refer to Φ_A as the Langer-Jonas functional calculus for *A.*

The rest of this section is devoted to the proof of this result. It is quite elaborate and will be carried out in several steps according to the following schedule:

Step 1: Positivity ensures continuity and therefore existence of continuous extensions.

Step 2: For each $\mathbb{R}_\infty$-definitizing element d a map $\Psi^d : \mathrm{BM}(\mathbb{R}_\infty) \to \mathcal{B}(\mathcal{K})$ is constructed.

Step 3: The maps Ψ^d give rise to continuous algebra homomorphisms $\Lambda_{b_j}^{d_j}$ of $\mathbb{H}(\mathbb{R}_\infty) \times \mathrm{BM}(\mathbb{R}_\infty)$ into $\mathcal{B}(\mathcal{K})$.

Step 4: We define the desired functional calculus Φ_A.

Step 5: We show properties (i) and (ii).

Step 6: We deduce the Spectral Mapping Theorem with help of a perturbation argument.

Step 7: We show that the support of the functional calculus is equal to the spectrum of A.

Step 1; Consequences of positivity

PRH20 **9.4.3 Proposition.** *Let $\mathcal{K}$ be a Krein space, and K a compact Hausdorff space. Moreover, let D be a dense linear subspace of $C(K)$ which contains the constant function 1 and is closed with respect to complex conjugation, and let $\Psi : D \to \mathcal{B}(\mathcal{K})$ be a linear map which fullfills the positivity property*

$$\Psi(f) \geq 0 \textit{ for all } f \in D \textit{ with } f \geq 0 . \tag{9.4.2}$$ H48

Then there exist a linear $\|.\|_\infty$-to-$\|.\|$–bounded and (9.1.3)-continuous extension $\tilde{\Psi}$ of Ψ to $\mathrm{BM}(K)$ with

$$\tilde{\Psi}(f) \geq 0 \textit{ for all } f \in \mathrm{BM}(K) \textit{ with } f \geq 0 . \tag{9.4.3}$$ H49

Moreover, an operator $T \in \mathcal{B}(\mathcal{K})$ commutes with all operators $\Psi(f)$, $f \in D$, if and only if it commutes with all operators $\tilde{\Psi}(f)$, $f \in \mathrm{BM}(K)$.

In the proof we use the following two statements which ensure norm-boundedness.

LEH25 **9.4.4 Lemma.** *Let K be a compact Hausdorff space. Moreover, let D be a linear subspace of $C(K)$ which contains the constant function 1 and is closed with respect to complex conjugation, and let $\varphi : D \to \mathbb{C}$ be a linear functional. If*

$$\varphi(f) \geq 0 \quad \textit{for all } f \in D \textit{ with } f \geq 0 ,$$

then φ is bounded.

Proof. First we consider a real-valued function $f \in D$. Then $\|f\|_\infty - f \geq 0$, and hence $\varphi(\|f\|_\infty - f) \geq 0$. The value $\varphi(\|f\|_\infty)$ is nonnegative. Since $\varphi(f) = \varphi(f - \|f\|_\infty) + \varphi(\|f\|_\infty)$, it follows that $\varphi(f)$ is real and satisfies the inequality

$$\varphi(f) \leq \varphi(\|f\|_\infty) = \|f\|_\infty \cdot \varphi(1) .$$

Since with f also $-f$ belongs to D and is real-valued, we find that also $-\varphi(f) = \varphi(-f) \leq \|f\|_\infty \cdot \varphi(1)$. In total, thus $|\varphi(f)| \leq \|f\|_\infty \cdot \varphi(1)$.

Let $f \in D$ be arbitrary, and write $f = f_1 + if_2$ with $f_1 = \frac{1}{2}(f + \overline{f})$ and $f_2 = \frac{1}{2i}(f - \overline{f})$. Since with f also $\overline{f}$ belongs to D, the functions f_1 and f_2 belong to D. Moreover, they are real-valued and $\|f_j\|_\infty \leq \|f\|_\infty$, $j = 1, 2$. From what we showed above, it follows that

$$|\varphi(f)| \leq |\varphi(f_1)| + |\varphi(f_2)| \leq 2\varphi(1) \cdot \|f\|_\infty .$$

❑

LEH26

9.4.5 Lemma. *Let $\mathcal{K}$ be a Krein space, and $T_j \in \mathcal{B}(\mathcal{K})$, $j \in J$, a family of bounded linear operators. If*

$$\sup_{j\in J} |[T_j x, x]| < \infty, \quad x \in \mathcal{K},$$

then $\sup_{j\in J} \|T_j\| < \infty$.

Proof. By the parallelogram rule we have

$$4[T_j x, y] = [T_j(x+y),(x+y)] - [T_j(x-y),(x-y)] + i[T_j(x+iy),(x+iy)] - \\ -i[T_j(x-iy),(x-iy)], \quad x, y \in \mathcal{K}.$$

Hence, the present hypothesis implies that

$$\sup_{j\in I} |[T_j x, y]| < \infty, \quad x, y \in \mathcal{K}.$$

Let $\mathfrak{J}$ be a fundamental decomposition of $\mathcal{K}$, and denote by J the corresponding fundamental symmetry. Then, for each $x \in \mathcal{K}$, the family

$$\big\{(., JT_j x)_{\mathfrak{J}} : j \in J\big\} \subseteq \langle \mathcal{K}, (.,.)_{\mathfrak{J}}\rangle'$$

is pointwise bounded. Using the Banach-Steinhaus Theorem twice, this implies that $\sup_{j\in J} \|JT_j x\| < \infty$, $x \in \mathcal{K}$, and in turn $\sup_{j\in J} \|JT_j\| < \infty$. Since $\|J^{-1}\| = 1$, thus also $\sup_{j\in J} \|T_j\| < \infty$. ❑

Proof (of Proposition 9.4.3). For each $x \in \mathcal{K}$, the map

$$\varphi_x : \begin{cases} D & \to & \mathbb{C} \\ f & \mapsto & [\Psi(f)x, x] \end{cases}$$

is a linear functional. By the assumption (9.4.2) of the proposition, it satisfies the hypothesis of Lemma 9.4.4, and hence is bounded. This means that

$$\sup_{\substack{f\in D\\ \|f\|_\infty \le 1}} |\varphi_x(f)| < \infty, \quad x \in \mathcal{K}.$$

Lemma 9.4.5, applied with the family $\{\Psi(f) : f \in D, \|f\|_\infty \le 1\}$, gives

$$\sup_{\substack{f\in D\\ \|f\|_\infty \le 1}} \|\Psi(f)\| < \infty.$$

This just says that $\Psi : D \to \mathcal{B}(\mathcal{K})$ is $\|.\|_\infty$-to-$\|.\|$–bounded.

Since Ψ is bounded, there exists a bounded linear continuation of Ψ to $C(K)$, say $\Psi_c : C(K) \to \mathcal{B}(\mathcal{K})$. By Corollary 9.1.10 Ψ_c possesses a $\|.\|_\infty$-to-$\|.\|$–bounded and (9.1.3)-continuous continuation

$$\tilde{\Psi} : \mathrm{BM}(D) \to \mathcal{B}(\mathcal{K}).$$

In order to show that $\tilde{\Psi}$ satisfies (9.4.3), it suffices to show that Ψ_c has the corresponding positivity property, cf. Corollary 9.1.10. Let $f \in C(K)$, $f \ge 0$, be given. Choose a sequence $f_n \in D$, $n \in \mathbb{N}$, with $f_n \to f$ in $C(K)$. Set

$$g_n := f_n - \min_{x\in K} f_n(x) + \min_{x\in K} f(x),$$

then $g_n \in D$, $g_n \geq 0$, and $g_n \to f$ in $C(K)$. Therefore

$$[\Psi_c(f)x,x] = \lim_{n\to\infty}[\Psi_c(g_n)x,x] = \lim_{n\to\infty}[\Psi(g_n)x,x] \geq 0\,.$$

In order to show the last assertion, assume that $T \in \mathcal{B}(\mathcal{K})$ and commutes with all operators $\Psi(f)$, $f \in D$. Since Ψ_c is the continuation of Ψ by continuity, it follows that T commutes with all operators $\Psi_c(f)$, $f \in C(K)$. Once again referring to Corollary 9.1.10, yields that T commutes with all operators $\tilde{\Psi}(f)$, $f \in \mathrm{BM}(K)$. ❑

Step 2; Construction of Ψ^d, d $\mathbb{R}_\infty$-definitizing

Let $d \in \mathbb{H}(\mathbb{R}_\infty)$ be $\mathbb{R}_\infty$-definitizing for A, and set $\sigma_0 := \sigma(A) \cap \mathbb{R}_\infty$. Since $\rho_{\mathbb{R}_\infty}$ maps $\mathbb{C}(z) \cap \mathbb{H}(\mathbb{R}_\infty)$ bijectively onto $\mathbb{C}(z) \cap C(\mathbb{R}_\infty)$, we may define

$$\Psi^d_{\mathrm{rat}} : \begin{cases} \mathbb{C}(z) \cap C(\mathbb{R}_\infty) & \to \quad \mathcal{B}(\mathcal{K}) \\ \qquad\qquad g & \mapsto \quad \Phi^{A,\sigma_0}_{\mathrm{RD}}\big(d \cdot \rho^{-1}_{\mathbb{R}_\infty}(g)\big) \end{cases}$$

Our aim is to apply Proposition 9.4.3 with $K = \mathbb{R}_\infty$. Clearly, Ψ^d_{rat} is linear. The set $\mathbb{C}(z) \cap C(\mathbb{R}_\infty)$ is a subalgebra of $C(\mathbb{R}_\infty)$ which contains the constant function 1. It is point separating since it contains the function $(z-i)^{-1}$, and it is closed with respect to complex conjugation since it contains with a function q also the function $q^{\#}$. By the Stone-Weierstraß Theorem, therefore, $\mathbb{C}(z) \cap C(\mathbb{R}_\infty)$ is dense in $C(\mathbb{R}_\infty)$.

The required positivity property (9.4.2) can be deduced from d being definitizing with help of the following lemma.

LEH14 **9.4.6 Lemma.** *Let $g \in \mathbb{C}(z) \cap C(\mathbb{R}_\infty)$, and assume that $g(z) \geq 0$, $z \in \mathbb{R}_\infty$. Then there exists $q \in \mathbb{C}(z) \cap C(\mathbb{R}_\infty)$ such that $g = q^{\#}q$.*

Proof. Since g takes real values along the real axis, we must have $g^{\#} = g$. In particular,

$$\mathfrak{d}_g(z) = \mathfrak{d}_{g^{\#}}(z) = \mathfrak{d}_g(\overline{z}), \quad z \in \mathbb{C}\,,$$

i.e. the zeros and poles of g are located symmetrically with respect to the real line. Moreover, since g takes only nonnegative values along the $\mathbb{R}$, each real zero must have even order. Set

$$s(z) := \prod_{\operatorname{Im} w > 0} (z-w)^{\mathfrak{d}_g(w)} \cdot \prod_{w \in \mathbb{R}} (z-w)^{\frac{\mathfrak{d}_g(w)}{2}}\,,$$

then the function $g \cdot (s^{\#}s)^{-1}$ is a rational function which has no poles and zeros in $\mathbb{C}$. Thus it is constant, say $g \cdot (s^{\#}s)^{-1} = \gamma$. Evaluating at a point $w \in \mathbb{R}$ which is no zero of g shows that $\gamma \geq 0$. We see that the function $q := \sqrt{\gamma} \cdot s$ satisfies the required identity $g = q^{\#}q$. ❑

Let $g \in \mathbb{C}(z) \cap C(\mathbb{R}_\infty)$ and assume that $g(z) \geq 0$, $z \in \mathbb{R}_\infty$. According to Lemma 9.4.6 we can choose $q \in \mathbb{C}(z) \cap C(\mathbb{R}_\infty)$ with $g = q^{\#}q$. It follows that

$$\begin{aligned}
\big[\Phi^d_{\mathrm{rat}}(g)x,x\big] &= \big[\Phi^{A,\sigma_0}_{\mathrm{RD}}\big(d \cdot \rho^{-1}_{\mathbb{R}_\infty}(q^{\#}q)\big)x,x\big] = \\
&= \big[\Phi^{A,\sigma_0}_{\mathrm{RD}}\big(\rho^{-1}_{\mathbb{R}_\infty}(q^{\#})\big) \cdot \Phi^{A,\sigma_0}_{\mathrm{RD}}(d) \cdot \Phi^{A,\sigma_0}_{\mathrm{RD}}\big(\rho^{-1}_{\mathbb{R}_\infty}(q)\big)x,x\big] = \\
&= \big[\Phi^{A,\sigma_0}_{\mathrm{RD}}(d) \cdot \Phi^{A,\sigma_0}_{\mathrm{RD}}\big(\rho^{-1}_{\mathbb{R}_\infty}(q)\big)x, \Phi^{A,\sigma_0}_{\mathrm{RD}}\big(\rho^{-1}_{\mathbb{R}_\infty}(q)\big)x\big] \geq 0\,.
\end{aligned}$$

All hypothesis of Proposition 9.4.3 are verfied, and we obtain a linear extension Ψ^d of Ψ^d_{rat} to $\mathrm{BM}(\mathbb{R}_\infty)$ which is $\|.\|_\infty$-to-$\|.\|$–bounded, (9.1.3)-continuous, and maps nonnegative functions to nonnegative operators.

LEH50 **9.4.7 Lemma.** *We have*

$$\Psi^d\big(\rho_{\mathbb{R}_\infty}(g)\big) = \Phi^{A,\sigma_0}_{\mathrm{RD}}(dg), \quad g \in \mathbb{H}(\mathbb{R}_\infty)\,.$$

Proof. By the definition of Ψ^d, we have

$$\Psi^d \circ \rho_{\mathbb{R}_\infty}\big|_{\mathbb{C}(z)\cap\mathbb{H}(\mathbb{R}_\infty)} = \Phi^{A,\sigma_0}_{\mathrm{RD}} \circ (d\cdot)\big|_{\mathbb{C}(z)\cap\mathbb{H}(\mathbb{R}_\infty)}\,.$$

Both functions $\Psi^d\circ\rho_{\mathbb{R}_\infty} : \mathbb{H}(\mathbb{R}_\infty) \to \mathcal{B}(\mathcal{K})$ and $\Phi^{A,\sigma_0}_{\mathrm{RD}}\circ(d\cdot) : \mathbb{H}(\mathbb{R}_\infty) \to \mathcal{B}(\mathcal{K})$ are continuous. Since $\mathbb{C}(z)\cap\mathbb{H}(\mathbb{R}_\infty)$ is dense in $\mathbb{H}(\mathbb{R}_\infty)$, cf. 8.1.5, these functions coincide on all of $\mathbb{H}(\mathbb{R}_\infty)$. ❑

Step 3; The maps $\Lambda^{d_j}_{b_j}$

From now on we will suppress explicit notation of $\rho_{\mathbb{R}_\infty}$, and consider $\mathbb{H}(K)$ as a 'subset' of $C(K)$. Note that, in places, this abuse of language has to handled with care since $\rho_{\mathbb{R}_\infty}$ need not be injective.

Let $d_1,\dots,d_n \in \mathbb{H}(\mathbb{R}_\infty)$ be $\mathbb{R}_\infty$-definitizing for A, and let $b_1,\dots,b_n \in \mathbb{H}(\mathbb{R}_\infty)$. Then we define a map

$$\Lambda^{d_j}_{b_j} : \begin{cases} \mathbb{H}(\mathbb{R}_\infty)\times\mathrm{BM}(\mathbb{R}_\infty) & \to \quad \mathcal{B}(\mathcal{K}) \\ \qquad (p,g) & \mapsto \quad \Phi^{A,\sigma_0}_{\mathrm{RD}}(p) + \sum\limits_{j=1}^n \Psi^{d_j}(b_j g) \end{cases}$$

LEH31 **9.4.8 Lemma.** *The map $\Lambda^{d_j}_{b_j}$ is linear and continuous. Set $d := \sum_{j=1}^n b_j d_j$, then*

$$\begin{tikzcd} \mathbb{H}(\mathbb{R}_\infty)\times\mathbb{H}(\mathbb{R}_\infty) \arrow[r, "\Lambda^{d_j}_{b_j}"] \arrow[d, "\pi_d"'] & \mathcal{B}(\mathcal{K}) \\ \mathbb{H}(\mathbb{R}_\infty) \arrow[ur, "\Phi^{A,\sigma_0}_{\mathrm{RD}}"'] & \end{tikzcd} \tag{9.4.4}$$ H51

For each fixed $p \in \mathbb{H}(\mathbb{R}_\infty)$ the function $g \mapsto \Lambda^{d_j}_{b_j}(p,g)$ is (9.1.3)-continuous.

Proof. The map $\Lambda^{d_j}_{b_j}$ is a composition of linear and continuous maps, namely of the projections, the maps $g \mapsto b_j g$, and the maps $\Phi^{A,\sigma_0}_{\mathrm{RD}}$, Ψ^{d_j}. Thus it is itself linear and continuous. In order to show (9.4.4), let $(p,g) \in \mathbb{H}(\mathbb{R}_\infty)\times\mathbb{H}(\mathbb{R}_\infty)$ be given. Using Lemma 9.4.7, we obtain

$$\Lambda^{d_j}_{b_j}(p,g) = \Phi^{A,\sigma_0}_{\mathrm{RD}}(p) + \sum_{j=1}^n \Psi^{d_j}(b_j g) = \Phi^{A,\sigma_0}_{\mathrm{RD}}(p) + \sum_{j=1}^n \Phi^{A,\sigma_0}_{\mathrm{RD}}(d_j b_j g) =$$

$$= \Phi^{A,\sigma_0}_{\mathrm{RD}}\big(p + \sum_{j=1}^n d_j b_j g\big) = \Phi^{A,\sigma_0}_{\mathrm{RD}}(p + dg) = \big(\Phi^{A,\sigma_0}_{\mathrm{RD}}\circ\pi_d\big)(p,g)\,.$$

Finally, let $p \in \mathbb{H}(\mathbb{R}_\infty)$ be fixed. For each measure μ, multiplication by b_j maps μ-boundedly pointwise convergent sequences to μ-boundedly pointwise convergent sequences. Hence, (9.1.3)-continuity of Ψ^{d_j} implies that the map $g \mapsto \Psi^{d_j}(b_j g)$ has the same property. Thus also $g \mapsto \Lambda^{d_j}_{b_j}(p,g)$ is (9.1.3)-continuous. ❑

We can now deduce the main properties of $\Lambda^{d_j}_{b_j}$.

PRH30 **9.4.9 Proposition.** *Let $d_1, \dots, d_n \in \mathbb{H}(\mathbb{R}_\infty)$ be definitizing functions for A, $b_1, \dots, b_n \in \mathbb{H}(\mathbb{R}_\infty)$, and set $d := \sum_{j=1}^n b_j d_j$.*

(i) The map $\Lambda^{d_j}_{b_j} : \mathbb{H}(\mathbb{R}_\infty) \times \mathrm{BM}(\mathbb{R}_\infty) \to \mathcal{B}(\mathcal{K})$ is a continuous algebra homomorphism, when $\mathbb{H}(\mathbb{R}_\infty) \times \mathrm{BM}(\mathbb{R}_\infty)$ is endowed with the multiplication '$\diamond_d$'.

(ii) We have

$$\Lambda^{d_j}_{b_j}(p^\#, \overline{g}) = \big[\Lambda^{d_j^\#}_{b_j^\#}(p,g)\big]^*, \quad (p,g) \in \mathbb{H}(\mathbb{R}_\infty) \times \mathrm{BM}(\mathbb{R}_\infty).$$

(iii) We have $\ker \pi_d \subseteq \ker \Lambda^{d_j}_{b_j}$.

(iv) Let in addition to d_j, b_j some $\mathbb{R}_\infty$-definitizing elements $d'_1, \dots, d'_m \in \mathbb{H}(\mathbb{R}_\infty)$ and elements $b'_1, \dots, b'_n \in \mathbb{H}(\mathbb{R}_\infty)$ be given, and set $d' := \sum_{j=1}^m b'_j d'_j$. If $d'|d$ in $\mathbb{H}(\mathbb{R}_\infty)$, then

$$\Lambda^{d_j}_{b_j}(p,g) = \Lambda^{d'_j}_{b'_j}\Big(p, \frac{d}{d'} g\Big), \quad (p,g) \in \mathbb{H}(\mathbb{R}_\infty) \times \mathrm{BM}(\mathbb{R}_\infty).$$

Proof. For (i) it remains to check compatibility with multiplication. To do so, we use (9.1.3)-continuity. For $p_1, p_2 \in \mathbb{H}(\mathbb{R}_\infty)$ and $g_1, g_2 \in \mathbb{C}(z) \cap \mathbb{H}(\mathbb{R}_\infty)$, we can compute

$$\begin{aligned} \Lambda^{d_j}_{b_j}\big((p_1,g_1) \diamond_d (p_2,g_2)\big) &= \Phi^{A,\sigma_0}_{\mathrm{RD}}\big(\pi_d\big((p_1,g_1) \diamond_d (p_2,g_2)\big)\big) = \\ = \Phi^{A,\sigma_0}_{\mathrm{RD}}\big(\pi_d(p_1,g_1)\big)\Phi^{A,\sigma_0}_{\mathrm{RD}}\big(\pi_d(p_2,g_2)\big) &= \Lambda^{d_j}_{b_j}(p_1,g_1)\Lambda^{d_j}_{b_j}(p_2,g_2), \end{aligned} \tag{9.4.5}$$ H28

Let $p_1, p_2 \in \mathbb{H}(\mathbb{R}_\infty)$ and $g_2 \in \mathbb{C}(z) \cap \mathbb{H}(\mathbb{R}_\infty)$ be fixed. Both functions

$$g_1 \mapsto \Lambda^{d_j}_{b_j}\big((p_1,g_1) \diamond_d (p_2,g_2)\big) \quad \text{and} \quad g_1 \mapsto \Lambda^{d_j}_{b_j}(p_1,g_1)\Lambda^{d_j}_{b_j}(p_2,g_2)$$

are (9.1.3)-continuous. Thus (9.4.5) implies that

$$\begin{aligned} \Lambda^{d_j}_{b_j}\big((p_1,g_1) \diamond_d (p_2,g_2)\big) = \Lambda^{d_j}_{b_j}(p_1,g_1)\Lambda^{d_j}_{b_j}(p_2,g_2), \quad & p_1, p_2 \in \mathbb{H}(\mathbb{R}_\infty), \\ & g_2 \in \mathbb{C}(z) \cap \mathbb{H}(\mathbb{R}_\infty),\ g_1 \in \mathrm{BM}(\mathbb{R}_\infty). \end{aligned} \tag{9.4.6}$$ H32

Next keep, besides $p_1, p_2 \in \mathbb{H}(\mathbb{R}_\infty)$, a function $g_1 \in \mathrm{BM}(\mathbb{R}_\infty)$ fixed. Both functions

$$g_2 \mapsto \Lambda^{d_j}_{b_j}\big((p_1,g_1) \diamond_d (p_2,g_2)\big) \quad \text{and} \quad g_2 \mapsto \Lambda^{d_j}_{b_j}(p_1,g_1)\Lambda^{d_j}_{b_j}(p_2,g_2)$$

are (9.1.3)-continuous. Thus (9.4.6) implies that

$$\Lambda_{b_j}^{d_j}\big((p_1,g_1)\diamond_d (p_2,g_2)\big) = \Lambda_{b_j}^{d_j}(p_1,g_1)\Lambda_{b_j}^{d_j}(p_2,g_2), \quad p_1,p_2 \in \mathbb{H}(\mathbb{R}_\infty), \\ g_1,g_2 \in \mathrm{BM}(\mathbb{R}_\infty)\,.$$

We come to the proof of (ii). We know that with d_j also $d_j^\#$ is $\mathbb{R}_\infty$-definitizing for A. Hence, the map $\Lambda_{b_j^\#}^{d_j^\#}$ is well-defined. Let $p \in \mathbb{H}(\mathbb{R}_\infty)$ and $g \in \mathbb{C}(z)\cap\mathbb{H}(\mathbb{R}_\infty)$. Then

$$\Lambda_{b_j}^{d_j}(p^\#,\overline{g}) = \Phi_{\mathrm{RD}}^{A,\sigma_0}\big(\pi_d(p^\#,\overline{g})\big) = \Phi_{\mathrm{RD}}^{A,\sigma_0}\big([\pi_{(d^\#)}(p,g)]^\#\big) = \\ = \Phi_{\mathrm{RD}}^{A,\sigma_0}\big(\pi_{(d^\#)}(p,g)\big)^* = \Big(\Lambda_{b_j^\#}^{d_j^\#}(p,g)\Big)^*\,.$$

Again (9.1.3)-continuity yields that $\Lambda_{b_j}^{d_j}(p^\#,\overline{g}) = (\Lambda_{b_j^\#}^{d_j^\#}(p,g))^*$ for all $p \in \mathbb{H}(\mathbb{R}_\infty)$, $g \in \mathrm{BM}(\mathbb{R}_\infty)$.

For (iii), assume that $(p,g) \in \ker\pi_d$, i.e. $p + dg = 0$. Then $g = -d^{-1}p$, and hence g is the restriction to $\mathbb{R}_\infty$ of a meromorphic function defined in some neighbourhood of $\mathbb{R}_\infty$. However, since $g \in \mathrm{BM}(\mathbb{R}_\infty)$, actually $-d^{-1}p \in \mathbb{H}(\mathbb{R}_\infty)$. Thus

$$\Lambda_{b_j}^{d_j}(p,g) = \Phi_{\mathrm{RD}}^{A,\sigma_0}\big(\pi_d(p,g)\big) = 0\,.$$

Finally, assume that additionally d_j' and b_j' are given. For $p \in \mathbb{H}(\mathbb{R}_\infty)$ and $g \in \mathbb{C}(z)\cap\mathbb{H}(\mathbb{R}_\infty)$ we can compute

$$\Lambda_{b_j'}^{d_j'}\big(p, \frac{d}{d'}g\big) = \Phi_{\mathrm{RD}}^{A,\sigma_0}\big(p + d'\frac{d}{d'}g\big) = \Phi_{\mathrm{RD}}^{A,\sigma_0}(p+dg) = \Lambda_{b_j}^{d_j}(p,g)\,.$$

Both functions

$$g \mapsto \Lambda_{b_j'}^{d_j'}\big(p, \frac{d}{d'}g\big) \quad \text{and} \quad g \mapsto \Lambda_{b_j}^{d_j}(p,g)$$

are (9.1.3)-continuous. It follows that they coincide on all of $\mathbb{H}(\mathbb{R}_\infty)\times\mathrm{BM}(\mathbb{R}_\infty)$. ❑

<u>*Step 4; Definition of Φ_A*</u>

Let d be a greatest common divisor of the set of all $\mathbb{R}_\infty$-definitizing elements of A, and choose $\mathbb{R}_\infty$-definitizing elements $d_1,\ldots,d_n\mathbb{H}(\mathbb{R}_\infty)$ and elements $b_1,\ldots,b_n \in \mathbb{H}(\mathbb{R}_\infty)$ such that $d = \sum_{i=1}^n b_i d_i$. Since $\mathfrak{d}_d = \mathfrak{d}_{\mathrm{crt}}^A$, we have $\mathfrak{A}_A = \mathfrak{A}(\mathfrak{d}_d)$.

Due to Proposition 9.4.9, (iii), there exists a continuous algebra homomorphism Φ_A with

$$\begin{tikzcd} \mathbb{H}(\mathbb{R}_\infty)\times\mathrm{BM}(\mathbb{R}_\infty) \arrow[r, "\Lambda_{b_j}^{d_j}"] \arrow[d, "\pi_d"'] & \mathcal{B}(\mathcal{K}) \\ \mathfrak{A}_A \arrow[ur, dashed, "\Phi_A"'] & \end{tikzcd}$$

LEH29 **9.4.10 Lemma.**

(*i*) *The map Φ_A does not depend on the choice of d.*

(*ii*) *We have* $\Phi_A(\overline{f}) = \Phi_A^*$, $f \in \mathfrak{A}_A$.

(*iii*) *Whenever* $F \in \mathbb{H}(\mathbb{R}_\infty)$, *then* $\Phi_A(F) = \Phi_{\mathrm{RD}}^{A,\sigma_0}(F)$.

Proof. Write $d = \sum_{i=1}^n b_i d_i$. Assume that d' is another greatest common divisor of the set of all $\mathbb{R}_\infty$-definitizing elements of A, write $d' = \sum_{i=1}^m b_i' d_i'$, and let Φ_A' be the correspondingly defined map. We have

$$\begin{tikzcd}
 & \mathfrak{A}_A \arrow[dd, dashed, bend right=80, "\Phi_A"'] \arrow[dd, dashed, bend left=80, "\Phi_A'"] & \\
\mathbb{H}(\mathbb{R}_\infty)\times\mathrm{BM}(\mathbb{R}_\infty) \arrow[ur, "\pi_d"] \arrow[rr, "\mathrm{id}\times\left(\frac{d}{d'}\cdot\right)"] \arrow[dr, "\Lambda_{b_j}^{d_j}"'] & & \mathbb{H}(\mathbb{R}_\infty)\times\mathrm{BM}(\mathbb{R}_\infty) \arrow[ul, "\pi_{d'}"'] \arrow[dl, "\Lambda_{b_j'}^{d_j'}"] \\
 & \mathcal{B}(\mathcal{K}) &
\end{tikzcd}$$

and hence Φ_A and Φ_A' coincide. This proves (*i*).

To show (*ii*), note that with d also the element $d^\#$ is a greatest common divisor of the set of all $\mathbb{R}_\infty$-definitizing elements of A. We have $d^\# = \sum_{i=1}^n b_i^\# d_i^\#$ and, due to the already proved item (*i*), the diagram

$$\begin{tikzcd}
\mathfrak{A}_A \arrow[rr, "\overline{\cdot}"] \arrow[dd, dashed, bend right=80, "\Phi_A"'] & & \mathfrak{A}_A \arrow[dd, dashed, bend left=80, "\Phi_A"] \\
\mathbb{H}(\mathbb{R}_\infty)\times\mathrm{BM}(\mathbb{R}_\infty) \arrow[u, "\pi_d"] \arrow[rr, ".^\#"] \arrow[d, "\Lambda_{b_j}^{d_j}"'] & & \mathbb{H}(\mathbb{R}_\infty)\times\mathrm{BM}(\mathbb{R}_\infty) \arrow[u, "\pi_{d^\#}"'] \arrow[d, "\Lambda_{b_j^\#}^{d_j^\#}"] \\
\mathcal{B}(\mathcal{K}) \arrow[rr, ".^*"'] & & \mathcal{B}(\mathcal{K})
\end{tikzcd}$$

Finally, we turn to (*iii*). Let $F \in \mathbb{H}(\mathbb{R}_\infty)$ be given. The definitions of Φ_A and $\Lambda_{B_j}^{d_j}$ give

$$\Phi_A(F) = \Lambda_{b_j}^{d_j}(F, 0) = \Phi_{\mathrm{RD}}^{A,\sigma_0}(F)\,.$$

❑

<u>*Step 5; The properties (i) and (ii)*</u>

Let $f \in \mathfrak{A}_A$ be given. We have $\Phi_{\mathrm{RD}}^{A,\sigma_0}(1) = \Phi_A(1)$, and

$$\Phi_A(f)\Phi_A(1) = \Phi_A(1)\Phi_A(f) = \Phi_A(f)\,.$$

This already shows that the decomposition $\mathcal{K} = \mathcal{K}_1[\dot{+}]\mathcal{K}_2$ reduces $\Phi_A(f)$ and that $\Phi_A(f)|_{\mathcal{K}_2} = 0$.

Next, let $T \in \mathcal{B}(\mathcal{K}_1)$ be given, and assume that T commutes with all operators $(A - w)^{-1}|_{\mathcal{K}_1}$. Set $A_1 := A \cap \mathcal{K}_1^2$, then $\sigma(A_1) = \sigma(A) \cap \mathbb{R}_\infty$ and $(A_1 - w)^{-1} = (A - w)^{-1}|_{\mathcal{K}_1}$, $w \in \rho(A) \cap \mathbb{C}$. The set $O := \mathbb{C}_\infty \setminus (\sigma(A) \setminus \mathbb{R}_\infty)$ is an open subset of $\mathbb{C}_\infty$ and contains $\sigma(A_1)$. Moreover, $O \cap \rho(A_1) = O \cap \rho(A)$. Theorem 8.3.1 implies that T commutes with all operators $\Phi_{\mathrm{RD}}^{A_1}(F)$, $F \in \mathbb{H}(\sigma(A_1))$. However, if $f \in \mathbb{H}(\mathbb{R}_\infty)$, then $\Phi_{\mathrm{RD}}^{A,\sigma_0}(f) = \Phi_{\mathrm{RD}}^{A_1}(f)$, and we conclude that

$$T\Phi_{\mathrm{RD}}^{A,\sigma_0}(f) = \Phi_{\mathrm{RD}}^{A,\sigma_0}(f)T, \quad f \in \mathbb{H}(\mathbb{R}_\infty)\,.$$

Let $d \in \mathbb{H}(\mathbb{R}_\infty)$ be $\mathbb{R}_\infty$-definitizing for A. By the above relation T commutes with all operators $\Psi^d_{\mathrm{rat}}(f)$, $f \in \mathbb{C}(z) \cap C(\mathbb{R}_\infty)$, and by Proposition 9.4.3 hence with all operators $\Psi^d(f)$, $f \in \mathrm{BM}(\mathbb{R}_\infty)$. The definition of $\Lambda^{d_j}_{b_j}$ shows immediately that T commutes with all $\Lambda^{d_j}_{b_j}(p,g)$, $(p,g) \in \mathbb{H}(\mathbb{R}_\infty) \times \mathrm{BM}(\mathbb{R}_\infty)$. This gives, by the definition of Φ_A, that T commutes with all $\Phi_A(f)$, $f \in \mathfrak{A}_A$.

Conversely, assume that $T\Phi_A(f) = \Phi_A(f)T$, $f \in \mathfrak{A}_A$. Let $w \in \rho(A) \setminus \mathbb{R}_\infty$, then the function $f(z) := \frac{1}{z-w}$ belongs to $\mathbb{H}(\mathbb{R}_\infty)$, and we have

$$\Phi_A(f) = \Phi^{A,\sigma_0}_{\mathrm{RD}}(f) = (A-w)^{-1}\Phi_A(1)\,.$$

Hence, T commutes with $(A-w)^{-1}|_{\mathcal{K}_1}$. Since the resolvent $(A-w)^{-1}$ depends continuously on w, and $\rho(A)\setminus\mathbb{R}_\infty$ is dense in $\rho(A)\cap\mathbb{C}$, the operator T commutes also with all operators $(A-w)^{-1}$, $w \in \rho(A) \cap \mathbb{R}$.

We turn to the proof of (ii). Let d be $\mathbb{R}_\infty$-definitizing for A, and let $f \in \mathfrak{A}_A$ be such that $\frac{f}{d} \in \mathrm{BM}(\mathbb{R}_\infty)$ and takes nonnegative values. According to the definition of Φ_A choose $d'_j, b'_j \in \mathbb{H}(\mathbb{R}_\infty)$, $d' := \sum_{j=1}^n b'_j d'_j$, such that $\Phi_A \circ \pi_{d'} = \Lambda^{d'_j}_{b'_j}$. Then $d'|d$ in $\mathbb{H}(\mathbb{R}_\infty)$, and hence using Proposition 9.4.9, (iv),

$$\Phi_A(f) = \Lambda^{d'_j}_{b'_j}\big(0, \frac{d}{d'}\frac{f}{d}\big) = \Lambda^d_1\big(0, \frac{f}{d}\big) = \Psi^d\big(\frac{f}{d}\big)\,.$$

However, Ψ^d has the property to map nonnegative functions to nonnegative operators.

<u>*Step 6; The Spectral Mapping Theorem*</u>

Let f be given according to Theorem 9.4.2, (iii). According to the definition of Φ_A choose $d_j, b_j \in \mathbb{H}(\mathbb{R}_\infty)$ such that $(d := \sum_{i=1}^n b_j d_j)$

$$\Phi_A \circ \pi_d = \Lambda^{d_j}_{b_j}\,.$$

Moreover, write $f = p + dg$ with $p \in \mathbb{H}(\mathbb{R}_\infty)$ and $g \in \mathrm{BM}(\mathbb{R}_\infty)$. Due to the continuity assumptions put on f, the function g is continuous at each point $w \in \operatorname{supp}\mathfrak{d}_d$. If $w \in \mathbb{R}_\infty \setminus \operatorname{supp}\mathfrak{d}_d$, then $g(z) = \frac{f(z)-p(z)}{d(z)}$ in a neighbourhood of w with $d(z)$ being nonzero, and hence is continuous at w. Altogether, $g \in C(\mathbb{R}_\infty)$.

Choose a sequence $g_n \in \mathbb{C}(z) \cap C(\mathbb{R}_\infty)$ which converges to g uniformly on $\mathbb{R}_\infty$. We can write

$$\Phi_A(f) = \Phi_A(p + dg_n) + \Phi_A(d(g-g_n))\,.$$

Continuity of $\Lambda^{d_j}_{b_j}$ gives

$$\lim_{n\to\infty} \Phi_A(d(g-g_n)) = \lim_{n\to\infty} \Lambda^{d_j}_{b_j}(g-g_n) = 0\,,$$

and hence we have

$$\Phi_A(f) = \lim_{n\to\infty}\big[\Phi_A(f) - \Phi_A(d(g-g_n))\big] = \lim_{n\to\infty}\Phi_A(p+dg_n)\,.$$

However, since $g_n \in \mathbb{H}(\mathbb{R}_\infty)$, we have $\Phi_A(p+dg_n) = \Phi^{A,\sigma_0}_{\mathrm{RD}}(p+dg_n)$.

By the Spectral Mapping Theorem for the Riesz-Dunford calculus we thus have $\sigma(\Phi_A(p+dg_n)) = (p+dg_n)(\sigma(A))$. Since $p+dg_n$ converges to f uniformly, $(p+dg_n)(\sigma(A))$ converges to $f(\sigma(A))$ in the Hausdorff metric. Since $\Phi_A(p+dg_n)$ converges to $\Phi_A(f)$ in the norm of $\mathcal{B}(\mathcal{K})$, and the perturbation $\Phi_A(p+dg_n) - \Phi_A(f)$ commutes with $\Phi_A(f)$, [K, IV.Theorem 3.6] implies that $\sigma(\Phi_A(p+dg_n))$ converges to $\sigma(\Phi_A(f))$ with respect to the Hausdorff metric. Putting together these pieces, we obtain $\sigma(\Phi_A(f)) = f(\sigma(A))$.

Step 7; The support of Φ_A

The crucial construction is to assign to each function $f \in \mathfrak{A}_A$ an analytic function $\Xi_f : \rho(A) \cup (\mathbb{C}_\infty \setminus \operatorname{supp} f) \to \mathcal{B}(\mathcal{K})$. To this end, fix a point $z \in \rho(A) \cap \mathbb{C}$, and consider the functions

$$\Xi_1(w) := \begin{cases} -(w-z)(A-w)^{-1}\Phi_A(f)\,, & w \in \rho(A) \cap \mathbb{C} \\ \Phi_A(f) \qquad\qquad\qquad\qquad , & w \in \rho(A) \cap \{\infty\} \end{cases}$$

$$\Xi_2(w) := \Phi_A\big(\xi_f(w,.)\big),\ w \in \mathbb{C}_\infty \setminus \operatorname{supp} f$$

Both functions are analytic on their domains. For Ξ_2 this follows from Proposition 9.3.3, for Ξ_1 we only have to remember that $\lim_{w\to\infty} w(A-w)^{-1} = -I$ in case $\infty \in \rho(A)$.

Next, we show that Ξ_1 and Ξ_2 coincide on the intersection of their domains. First we consider a point $w \in \mathbb{C} \setminus \mathbb{R}$ which belongs to $\rho(A) \cap (\mathbb{C}_\infty \setminus \operatorname{supp} f)$. Then the function $\frac{1}{x-w}$ belongs to $\mathbb{H}(\mathbb{R}_\infty)$ as a function of x, and

$$\Phi_A\Big(\frac{1}{x-w}\Big) = \Phi_{\mathrm{RD}}^{A,\sigma_0}\Big(\frac{1}{x-w}\Big) = (A-w)^{-1}\Phi_{\mathrm{RD}}^{A,\sigma_0}(1) = (A-w)^{-1}\Phi_A(1)\,.$$

Since $\xi_f(w,x) = \frac{w-z}{w-x} f(x)$, we find

$$\Xi_2(w) = \Phi_A\big(\xi_f(w,.)\big) = -(w-z)\cdot(A-w)^{-1}\Phi_A(1)\cdot\Phi_A(f) = \Xi_1(w)\,.$$

The set $[\mathbb{C} \setminus \mathbb{R}] \cap [\rho(A) \cap (\mathbb{C}_\infty \setminus \operatorname{supp} f)]$ is dense in $\rho(A) \cap (\mathbb{C}_\infty \setminus \operatorname{supp} f)$, and hence by continuity $\Xi_1(w) = \Xi_2(w)$ for all $w \in \rho(A) \cap (\mathbb{C}_\infty \setminus \operatorname{supp} f)$.

Due to what we just showed a function $\Xi_f : \rho(A) \cup (\mathbb{C}_\infty \setminus \operatorname{supp} f) \to \mathcal{B}(\mathcal{K})$ is well-defined by

$$\Xi_f(w) := \begin{cases} \Xi_1(w)\,, & w \in \rho(A) \\ \Xi_2(w)\,, & w \in \mathbb{C}_\infty \setminus \operatorname{supp} f \end{cases}$$

and is analytic.

After this preparation, we come to the actual proof of Theorem 9.4.2, (iv). First, we show that the set $\sigma(A) \cap \mathbb{R}_\infty$ indeed has the property (9.4.1). Let $f \in \mathfrak{A}_A$ with $\operatorname{supp} f \cap (\sigma(A) \cap \mathbb{R}_\infty) = \emptyset$ be given. Since, by definition, $\operatorname{supp} f \subseteq \mathbb{R}_\infty$, this means that $\operatorname{supp} f \subseteq \rho(A)$. Hence $\rho(A) \cup (\mathbb{C}_\infty \setminus \operatorname{supp} f) = \mathbb{C}_\infty$, and thus Ξ_f is an analytic map defined on all of $\mathbb{C}_\infty$. By Liouville's Theorem it is thus constant. The actual value of this constant can be computed by taking limits:

$$\Xi_f(w) = \Xi_f(\infty) = \lim_{u\to\infty} \Xi_f(u) = \begin{cases} \lim_{u\to\infty} \Xi_1(u)\,, & \infty \in \rho(A) \\ \lim_{u\to\infty} \Xi_2(u)\,, & \infty \notin \operatorname{supp} f \end{cases}$$

If $\infty \in \rho(A)$, then $\lim_{u\to\infty}[-(w-z)(A-w)^{-1}] = I$ and hence $\lim_{u\to\infty} \Xi_1(u) = \Phi_A(f)$. If $\infty \notin \operatorname{supp} f$, then $\lim_{u\to\infty} \xi_f(u,.) = f(.)$ and again $\lim_{u\to\infty} \Xi_2(u) = \Phi_A(f)$. Thus

$$\Xi_f(w) = \Phi_A(f), \quad w \in \mathbb{C}_\infty .$$

However, we have $\Xi_f(z) = \Xi_1(z) = 0$, and it follows that $\Phi_A(f) = 0$.

To show that $\sigma(A)\cap\mathbb{R}_\infty$ is the smallest set with (9.4.1), let some closed subset C of $\mathbb{R}_\infty$ with this property be given. We have to show that $C \supseteq \sigma(A)\cap\mathbb{R}_\infty$, in other words, $\mathbb{R}_\infty \setminus C \subseteq \rho(A)$. To this end, we first separate the real and nonreal parts of the spectrum of A. Set

$$\mathcal{X}_1 := \operatorname{ran} \Phi_{\mathrm{RD}}^{A,\sigma_0}(1),\ \mathcal{X}_2 := \ker \Phi_{\mathrm{RD}}^{A,\sigma_0}(1), \quad A_1 := A\cap \mathcal{X}_1^2,\ A_2 := A\cap\mathcal{X}_2^2 .$$

Then $A = A_1 \dot{+} A_2$ and $\sigma(A_1) = \sigma(A)\cap\mathbb{R}_\infty$, $\sigma(A_2) = \sigma(A)\setminus\mathbb{R}_\infty$. Moreover,

$$(A_1-w)^{-1} = (A-w)^{-1}\Phi_{\mathrm{RD}}^{A,\sigma_0}(1)\big|_{\mathcal{X}_1} = (A-w)^{-1}\Phi_A(1)\big|_{\mathcal{X}_1}, \quad w\in\rho(A) .$$

Next, for each open subset O of $\mathbb{C}_\infty$ with $C\subseteq O$, choose a partition of unity $\chi_O, \chi_2 \in C^\infty(\mathbb{R}_\infty)$ subordinate to the open cover $\{O\cap\mathbb{R}_\infty, \mathbb{R}_\infty\setminus C\}$ of $\mathbb{R}_\infty$. Since C satisfies (9.4.1), we then have

$$\Phi_A(f) = \Phi_A(\chi_O f) + \Phi_A(\chi_2 f) = \Phi_A(\chi_O f), \quad f\in\mathfrak{A}_A . \tag{9.4.7}$$ H60

In particular, $\Phi_A(1) = \Phi_A(\chi_O)$. Set

$$D := \mathbb{C}\cap\left[\rho(A)\cup(\mathbb{C}_\infty\setminus\operatorname{supp}\chi_O)\right],$$

and consider the analytic function $\tilde{R}(w) := \frac{\Xi_{\chi_O}(w)}{z-w}|_{\mathcal{X}_1}$, $w\in D$. If $w\in\rho(A)\cap\mathbb{C}$, then $\tilde{R}(w) = (A-w)^{-1}\Phi_A(1)|_{\mathcal{X}_1} = (A_1-w)^{-1}$. For $u,w\in\rho(A)\cap\mathbb{C}$, thus the resolvent identity

$$\tilde{R}(u)-\tilde{R}(w) = (u-w)\tilde{R}(u)\tilde{R}(w)$$

holds. The set D contains $\mathbb{C}\setminus\mathbb{R}$, and hence is connected. Keeping $w\in\rho(A)\cap\mathbb{C}$ fixed, and applying the Identity Theorem, we obtain that the resolvent identity holds in fact for all $u\in D$ and $w\in\rho(A)\cap\mathbb{C}$. Keeping $u\in D$ fixed and again applying the Identity Theorem, yields that $\tilde{R}$ satisfies the resolvent identity for all $u,w\in D$. By Corollary 7.3.3, this implies that $\rho(A_1)\supseteq D$. Since $\operatorname{supp}\chi_O\subseteq O$, thus

$$\mathbb{C}\setminus O \subseteq \rho(A_1)\cap\mathbb{R}_\infty = \rho(A)\cap\mathbb{R}_\infty\subseteq\rho(A) .$$

Consider the point ∞ and assume that $\infty\notin O$. We are going to show that A_1 is a bounded operator, i.e. that $\infty\in\rho(A)$. Fix $z\in\rho(A)\setminus\mathbb{R}_\infty$. For $w\in\mathbb{C}\setminus\mathbb{R}$, $w\neq z$, define a function $g_w:\mathbb{R}_\infty\to\mathbb{C}$ as

$$g_w(x) := \begin{cases} \frac{w-x}{w-z}\chi_O(x)\,, & x\in\mathbb{R} \\ 0 \quad\quad\quad , & x=\infty \end{cases}$$

Then $g_w\in C^\infty(\mathbb{R}_\infty)$, and $g_w(x)\cdot\xi_{\chi_0}(w,x) = \chi_O(x)$, $x\in\mathbb{R}_\infty$. Remembering (9.4.7), thus

$$\Phi_A(g_w)\cdot\Phi_A\big(\xi_{\chi_O}(w,.)\big) = \Phi_A(\chi_O) = \Phi_A(1) . \tag{9.4.8}$$ H61

Let $R > \|\phi_A(zg_0)\|$, and choose $w\in\rho(A)\setminus\mathbb{R}_\infty$ with $|w|>R$. Note here that this choice of w is possible, since $\sigma(A)\cap\mathbb{R}_\infty$ and $\sigma(A)\setminus\mathbb{R}_\infty$ are disjoint and relatively open subsets of $\sigma(A)$ and hence ∞ is not an accumulation point of $\sigma(A)\setminus\mathbb{R}_\infty$. We have

$$\Phi_A(g_w)=-\frac{1}{w-z}\big(\Phi_A(zg_0)-w\Phi_A(1)\big)=-\frac{1}{w-z}\big(\Phi_A(zg_0)-w\big)\Phi_A(1)\,,$$

and hence $\Phi_A(g_w)|_{\mathcal{K}_1}$ is boundedly invertible. Moreover, by (9.4.8), we have

$$\big(\Phi_A(zg_0)|_{\mathcal{K}_1}-w\big)^{-1}=-\frac{1}{w-z}\Big(\Phi_A(g_w)|_{\mathcal{K}_1}\Big)^{-1}=-\frac{1}{w-z}\Phi_A\big(\xi_{\chi_O}(w,.)\big)\big|_{\mathcal{K}_1}=$$

$$=\frac{\Xi_2(w)}{z-w}\Big|_{\mathcal{K}_1}=\frac{\Xi_1(w)}{z-w}\Big|_{\mathcal{K}_1}=(A-w)^{-1}\Phi_A(\chi_O)|_{\mathcal{K}_1}=(A_1-w)^{-1}\,.$$

We conclude that $A_1=\Phi_A(zg_0)|_{\mathcal{K}_1}$, and in particular that thus $A_1\in\mathcal{B}(\mathcal{K}_1)$.

We have so far established that $\mathbb{C}_\infty\setminus O\subseteq\rho(A)$, in other words that $O\supseteq\sigma(A)$. However, since C is closed, we have

$$C=\bigcap\Big\{O:\ O\subseteq\mathbb{C}_\infty\text{ open}, C\subseteq O\Big\}\,,$$

and hence $C\supseteq\sigma(A)$.

The proof of Theorem 9.4.2 is finished. ✌

REH33 *9.4.11 Remark.* The Langer-Jonas functional calculus can be extended immediately so to include the nonreal spectrum. Simply, by mapping an element $(f,r)\in\mathfrak{A}_A\times\mathbb{H}(\sigma(A)\setminus\mathbb{R}_\infty)$ to $\Phi_A(f)+\Phi_{\text{RD}}^{A,\sigma(A)\setminus\mathbb{R}_\infty}(r)$. This again gives a continuous *-algebra homomorphism with properties corresponding to the respective properties of Φ_A and $\Phi_{\text{RD}}^{A,\sigma(A)\setminus\mathbb{R}_\infty}$. //

Let us use the Langer-Jonas functional calculus to obtain more knowledge on the set of $\mathbb{R}_\infty$-definitizing functions.

PRH36 **9.4.12 Proposition.** *Let $\mathcal{K}$ be a Krein space, and A a selfadjoint relation in $\mathcal{K}$ such that $\sigma(A)\cap\mathbb{R}_\infty$ and $\sigma(A)\setminus\mathbb{R}_\infty$ are relatively open subsets of $\sigma(A)$. Then A is definitizable along $\mathbb{R}_\infty$ if and only if there exists a rational function $d\in\mathbb{C}(z)\cap C(\mathbb{R}_\infty)$ with $d=d^\#$ which is $\mathbb{R}_\infty$-definitizing for A.*

Proof.
Step 1: Our first aim is to show that there exists a $\mathbb{R}_\infty$-definitizing function d_1 with $d_1^\#=d_1$. Let d_0 be any $\mathbb{R}_\infty$-definitizing function. If $d_0=d_0^\#$, we set $d_1:=d_0$ and are done. Otherwise, consider $d_1:=i(d_0-d_0^\#)$. Then $d_1\in\mathbb{H}(\mathbb{R}_\infty)\setminus\{0\}$ and $d_1=d_1^\#$. We have

$$0\leq\big[\Phi_{\text{RD}}^{A,\sigma_0}(d_0)x,x\big]=\big[x,\Phi_{\text{RD}}^{A,\sigma_0}(d_0^\#)x\big]=\big[\Phi_{\text{RD}}^{A,\sigma_0}(d_0^\#)x,x\big],\quad x\in\mathcal{K}\,,$$

and hence

$$\big[\Phi_{\text{RD}}^{A,\sigma_0}\big(i(d_0-d_0^\#)\big)x,x\big]=0,\quad x\in\mathcal{K}\,.$$

We see that d_1 is $\mathbb{R}_\infty$-definitizing for A.

Step 2: Choose a $\mathbb{R}_\infty$-definitizing function d with $d^\#=d$,and $M\in\text{GL}(2,\mathbb{R})$ with $\phi_M(\infty)\notin\operatorname{supp}\mathfrak{d}_d$. Set $\tilde d:=d\circ\phi_M$, then $\tilde d\in\mathbb{H}(\mathbb{R}_\infty)$ and $\operatorname{supp}\mathfrak{d}_{\tilde d}\subseteq\mathbb{R}$. Set

$$\tilde g(z):=\prod_{w\in\operatorname{supp}\mathfrak{d}_{\tilde d}}(z-w)^{\mathfrak{d}_{\tilde d}(w)}\,,$$

and $g := \tilde{g} \circ \phi_M^{-1}$. Then $g \in \mathbb{C}(z) \cap \mathbb{H}(\mathbb{R}_\infty)$, satisfies $\mathfrak{d}_g = \mathfrak{d}_d$, and $g^\# = g$. The function $\frac{g}{d}$ is therefore analytic in $\mathbb{H}(\mathbb{R}_\infty)$, has no zeros, and takes real values. Thus it is either positive on all of $\mathbb{R}_\infty$ or negative on all of $\mathbb{R}_\infty$. Set

$$d_1 := \begin{cases} g \;\;, & \frac{g}{d} \text{ positive} \\ -g\,, & \frac{g}{d} \text{ negative} \end{cases}$$

then by Theorem 9.4.2, (ii), it follows that

$$\Phi_{\mathrm{RD}}^{A,\sigma_0}(d_1) = \Phi_A(d_1) \geq 0\,.$$

❏

It is apparent from the definition of the algebra $\mathfrak{A}_A$ that the points ofsupp $\mathfrak{d}_{\mathrm{crt}}^A$ play a particular role.

DEH12 **9.4.13 Definition.** Let $\mathcal{K}$ be a Krein space, and A a selfadjoint relation in $\mathcal{K}$ which is definitizable along $\mathbb{R}_\infty$. A point $x \in \mathbb{R}_\infty$ is called a *critical point*, if $x \in \operatorname{supp} \mathfrak{d}_{\mathrm{crt}}^A$, and we will use the notation $\operatorname{crt}(A) := \operatorname{supp} \mathfrak{d}_{\mathrm{crt}}^A$. //

PRH35 **9.4.14 Proposition.** *Let $\mathcal{K}$ be a Krein space, and A a selfadjoint relation in $\mathcal{K}$ which is definitizable along $\mathbb{R}_\infty$.*

(i) *If $x_0 \in \rho(A) \cap \mathbb{R}_\infty$, then there exists a $\mathbb{R}_\infty$-definitizing element d with $\mathfrak{d}_d(x_0) \in \{0,1\}$.*

(ii) *We have* $\operatorname{crt}(A) \subseteq \sigma(A)$.

Proof. Choose a $\mathbb{R}_\infty$-definitizing element $d_0 \in \mathbb{H}(\mathbb{R}_\infty)$. Let us first consider the case that $x_0 \in \mathbb{R}$. Set $\alpha := [\frac{\mathfrak{d}_{d_0}(x_0)}{2}]$, and

$$d(z) := \frac{d_0(z)}{(z-x_0)^{2\alpha}}\,.$$

Then $d \in \mathbb{H}(\mathbb{R}_\infty)$, and $\mathfrak{d}_d(x_0) \in \{0,1\}$. Choose a partition of unity $\chi_1, \chi_2 \in C^\infty(\mathbb{R}_\infty)$ subordinate to the open cover $\{\mathbb{R}_\infty \setminus \{x_0\}, \mathbb{R}_\infty \setminus \sigma(A)\}$. Then $\chi_1 d \in C^\infty(\mathbb{R}_\infty)$, and the function

$$\frac{\chi_1(x)d(x)}{d_0(x)} = \frac{\chi_1(x)}{(x-x_0)^{2\alpha}}$$

belongs to $C^\infty(\mathbb{R}_\infty)$ and takes nonnegative values. Thus, by Theorem 9.4.2, (ii), we have $\Phi_A(\chi_1 d) \geq 0$. However, by Theorem 9.4.2, (iv),

$$\Phi_A(\chi_1 d) = \Phi_A(d) = \Phi_{\mathrm{RD}}^{A,\sigma_0}(d)\,.$$

This shows that d is $\mathbb{R}_\infty$-definitizing. The case that $x_0 = \infty$ is treated in the same way using $d(x) := x^{2\alpha} d_0(x)$. This finishes the proof of (i).

We come to the proof of (ii). Assume that $x_0 \in \rho(A) \cap \mathbb{R}_\infty$, then we already know that there exists a $\mathbb{R}_\infty$-definitizing d with $\mathfrak{d}_d(x_0) \in \{0,1\}$. If $\mathfrak{d}_d(x_0) = 0$, it already follows that $x_0 \notin \operatorname{crt}(A)$. Assume that $\mathfrak{d}_d(x_0) = 1$, and consider first the case that $x_0 \in \mathbb{R}$. Let χ_1, χ_2 be as in the above part of this proof, and choose $x_1 > x_0$ such that $[x_0, x_1] \cap \operatorname{supp} \chi_1 = \emptyset$. Set

$$\tilde{d}(z) := \frac{z-x_1}{z-x_0} d(z)\,,$$

then $\tilde{d} \in \mathbb{H}(\mathbb{R}_\infty)$ and $\mathfrak{d}_{\tilde{d}}(x_0) = 0$. The function

$$\frac{\chi_1(x)\tilde{d}(x)}{d(x)} = \chi_1(x)\frac{x - x_1}{x - x_0}$$

belongs to $C^\infty(\mathbb{R}_\infty)$ and takes nonnegative values. Hence,

$$\Phi_{\mathrm{RD}}^{A,\sigma_0}(\tilde{d}) = \Phi_A(\chi_1 \tilde{d}) \geq 0\,,$$

i.e. $\tilde{d}$ is $\mathbb{R}_\infty$-definitizing. The case that $x_0 = \infty$ is treated in the same way choosing $x_1 < \min\operatorname{supp}\chi_1$ and using $\tilde{d}(x) := (x - x_1)d(x)$. This finishes the proof of (ii). ❑

Let us conclude this chapter with a remark on fractional linear transformations.

PRH67 **9.4.15 Proposition.** *Let $\mathcal{K}$ be a Krein space and A a $\mathbb{R}_\infty$-definitizing selfadjoint relation in $\mathcal{K}$. Moreover, let $N \in \mathrm{GL}(2,\mathbb{R})$. Then $\tilde{A} := \phi_N(A)$ is selfadjoint and $\mathbb{R}_\infty$-definitizing. We have $\mathfrak{d}_{\mathrm{crt}}^{\tilde{A}} = \mathfrak{d}_{\mathrm{crt}}^{A} \circ \phi_N^{-1}$. The composition map $\circ\phi_N$ is an homeomorphic *-algebra isomorphism of $\mathfrak{A}_{\tilde{A}}$ onto $\mathfrak{A}_A$, and we have*

$$\Phi_{\tilde{A}}(\tilde{f}) = \Phi_A(\tilde{f} \circ \phi_N), \quad \tilde{f} \in \mathfrak{A}_{\tilde{A}}\,. \tag{9.4.9}$$ H68

Proof. First of all, by Lemma 7.4.6, certainly $\tilde{A}$ is selfadjoint. Moreover, by the Spectral Mapping Theorem, $\sigma(\tilde{A}) = \phi_N(\sigma(A))$. Since ϕ_N is a homeomorphism of $\mathbb{C}_\infty$ onto itself, thus $\sigma(\tilde{A})$ is the disjoint union of its relatively open subsets $\phi_N(\sigma(A) \cap \mathbb{R}_\infty)$ and $\phi_N(\sigma(A) \setminus \mathbb{R}_\infty)$. However, since ϕ_N maps $\mathbb{R}_\infty$ onto itself, we have

$$\phi_N(\sigma(A) \setminus \mathbb{R}_\infty) = \phi_N(\sigma(A)) \setminus \mathbb{R}_\infty = \sigma(\tilde{A}) \setminus \mathbb{R}_\infty\,,$$
$$\phi_N(\sigma(A) \cap \mathbb{R}_\infty) = \phi_N(\sigma(A)) \cap \mathbb{R}_\infty = \sigma(\tilde{A}) \cap \mathbb{R}_\infty\,.$$

This shows that the requirement (i) of Definition 9.4.1 is satisfied by $\tilde{A}$.

Set $\tilde{\sigma_0} := \sigma(\tilde{A}) \cap \mathbb{R}_\infty$, the we know from (8.3.5) that

$$\Phi_{\mathrm{RD}}^{\tilde{A},\tilde{\sigma_0}}(\tilde{f}) = \Phi_{\mathrm{RD}}^{A,\sigma_0}(f \circ \phi_N), \quad f \in \mathbb{H}(\mathbb{R}_\infty)\,. \tag{9.4.10}$$ H69

Remember here that composition with ϕ_N is a homeomorphic *-algebra isomorphism of $\mathbb{H}(\mathbb{R}_\infty)$ onto itself. The above relation shows that $d \in \mathbb{H}(\mathbb{R}_\infty)$ is $\mathbb{R}_\infty$-definitizing for A if and only if $\tilde{d} := d \circ \phi_N^{-1}$ is $\mathbb{R}_\infty$-definitizing for $\tilde{A}$. It follows that $\tilde{A}$ is $\mathbb{R}_\infty$-definitizable and

$$\mathfrak{d}_{\mathrm{crt}}^{\tilde{A}} = \mathfrak{d}_{\mathrm{crt}}^{A} \circ \phi_N^{-1}\,.$$

By Lemma 9.2.10, thus, composition with ϕ_N^{-1} is a homeomorphic *-algebra isomorphism of $\mathfrak{A}_A$ onto $\mathfrak{A}_{\tilde{A}}$.

For the proof of (9.4.9) we need some preparation. Choose $d \in \mathbb{H}(\mathbb{R}_\infty)$ with $\mathfrak{d}_d = \mathfrak{d}_{\mathrm{crt}}^{A}$ and let $d_j \in \mathbb{H}(\mathbb{R}_\infty)$ be $\mathbb{R}_\infty$-definitizing for A and $b_j \in \mathbb{H}(\mathbb{R}_\infty)$ such that $d = \sum_{j=1}^n b_j d_j$. Set $\tilde{d}_j := d_j \circ \phi_N^{-1}$, $\tilde{b}_j := b_j \circ \phi_N^{-1}$, $\tilde{d} := d \circ \phi_N^{-1}$, then $\tilde{d}_j$ are $\mathbb{R}_\infty$-definitizing for $\tilde{A}$, $\tilde{d} = \sum_{j=1}^n \tilde{b}_j \tilde{d}_j$, and $\mathfrak{d}_{\tilde{d}} = \mathfrak{d}_{\mathrm{crt}}^{\tilde{A}}$.

Let $x, y \in \mathcal{K}$ be fixed, and choose measures μ_1, μ_2 such that

$$h_n \to h \ \mu_1\text{-boundedly pointwise} \ \Rightarrow \ [\Lambda_{b_j}^{d_j}(p, h_n)x, y] \to [\Lambda_{b_j}^{d_j}(p, h)x, y]\,,$$

$$h_n \to h \ \mu_2\text{-boundedly pointwise} \ \Rightarrow \ [\Lambda_{\tilde{b}_j}^{\tilde{d}_j}(\tilde{p}, h_n)x, y] \to [\Lambda_{\tilde{b}_j}^{\tilde{d}_j}(\tilde{p}, h)x, y]\,,$$

and set $\mu := \mu_1 + \mu_2 + [\mu_1 \circ \phi_N] + [\mu_2 \circ \phi_N]$.

Now we come to the actual proof of (9.4.9). Let $\tilde{f} \in \mathfrak{A}_{\tilde{A}}$ be given, and set $f := \tilde{f} \circ \phi_N$. Let $(\tilde{p}, \tilde{g}) \in \mathbb{H}(\mathbb{R}_\infty) \times \mathrm{BM}(\mathbb{R}_\infty)$ be such that $\tilde{f} = \pi_{\tilde{d}}(\tilde{p}, \tilde{g})$, and set $p := \tilde{p} \circ \phi_N$ and $g := \tilde{g} \circ \phi_N$. Then $f = \pi_d(p, g)$. Choose $\tilde{g}_n \in \mathbb{H}(\mathbb{R}_\infty)$ such that $\tilde{g}_n \to \tilde{g}$ μ-boundedly pointwise, and set $g_n := \tilde{g}_n \circ \phi_N$. Then, due to the definition of μ, also $g_n \to g$ μ-boundedly pointwise. It follows that

$$[\Phi_A(f)x, y] = [\Lambda_{b_j}^{d_j}(p, g)x, y] = \lim_{n\to\infty} [\Lambda_{b_j}^{d_j}(p, g_n)x, y] = \lim_{n\to\infty} [\Lambda_{\tilde{b}_j}^{\tilde{d}_j}(\tilde{p}, \tilde{g}_n)x, y] =$$

$$= [\Lambda_{\tilde{b}_j}^{\tilde{d}_j}(\tilde{p}, \tilde{g})x, y] = [\Phi_{\tilde{A}}(\tilde{f})x, y]\,.$$

Thereby the third equality sign holds because of (9.4.4) and (9.4.10). Since x, y were arbitrary, we conclude that $\Phi_A(f) = \Phi_{\tilde{A}}(\tilde{f})$. ❑

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Index

LABELS:

A:

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
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11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
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