Karamata's theorem for regularised Cauchy transforms

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Abstract: We prove Abelian and Tauberian theorems for regularised Cauchy transforms of positive Borel measures on the real line whose distribution functions grow at most polynomially at infinity. In particular, we relate the asymptotics of the distribution functions to the asymptotics of the regularised Cauchy transform.

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1 Introduction

Let μ be a positive measure on $\mathbb{R}$ such that $\int_{\mathbb{R}}(1+|t|)^{-1}\,\mathrm{d}\mu(t)<\infty$. The Cauchy transform of μ is the function

$$C[\mu](z) := \int_{\mathbb{R}} \frac{1}{t-z}\,\mathrm{d}\mu(t) \tag{1.1}$$

defined and analytic in the open upper half-plane $\mathbb{C}^+$. It plays an important role in many areas, such as spectral theory, moment problems, complex analysis and random matrix theory. A prominent particular case occurs when μ is supported on $[0,\infty)$. Then we speak of the Stieltjes transform of μ and write

$$S[\mu](z) := \int_{[0,\infty)} \frac{1}{t-z}\,\mathrm{d}\mu(t). \tag{1.2}$$

This function is defined and analytic in the slit plane $\mathbb{C}\setminus[0,\infty)$.

A measure μ can be reconstructed from its Cauchy (or Stieltjes) transform by means of the Stieltjes inversion formula,

$$\mu\big((\alpha,\beta)\big) = \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_{\alpha+\delta}^{\beta-\delta} \operatorname{Im} C[\mu](x+i\varepsilon)\,\mathrm{d}x \tag{1.3}$$

for $-\infty<\alpha<\beta<\infty$. This formula can be seen as relating the local behaviour of μ at a point or on a bounded interval in $\mathbb{R}$ with the local behaviour of $C[\mu]$ around this point or interval: in order to evaluate the right-hand side of (1.3) only the values of $C[\mu](z)$ for z in some rectangle $(\alpha,\beta)\times(0,\varepsilon)\subseteq\mathbb{C}^+$ have to be known.

One question that has attracted a lot of attention is the relation between the asymptotics of μ at ∞ and the asymptotics of its transform at ∞. For the case of the Stieltjes transform results were obtained already in the early 20$^{\text{th}}$ century: G. Valiron [35], E.C. Titchmarsh [34], and G.H. Hardy and J.E. Littlewood [12] proved that, for each $\gamma\in(-1,0)$,

$$S[\mu](-x)\sim cx^{\gamma} \quad\Leftrightarrow\quad \mu([0,t))\sim c' t^{\gamma+1}, \tag{1.4}$$

where c,c' are related by a certain formula. Here the symbol $\sim$ means that the quotient of the left-hand and right-hand sides tends to 1, and is understood for $x,t\to+\infty$. The asymptotics of $S[\mu]$ along the ray $e^{i\pi}(0,\infty)$ could be substituted by the asymptotics along any ray contained in the domain of analyticity $\mathbb{C}\setminus[0,\infty)$ (allowing the constant C to depend on the angle of the ray), or even non-tangentially. This early result about the Stieltjes transform was generalised in several directions, and there is a vast literature on that topic. As examples we mention [16] where J. Karamata generalised (1.4) to growth of regular

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variation instead of power asymptotics, or [27] where asymptotic expansions with infinitely many terms instead of a monomial on the right-hand side of (1.4) was considered. In the bilateral case, meaning measures that are not semi-bounded, asymptotics have to be taken along the positive imaginary axis, or a ray in $\mathbb{C}^+$ or non-tangentially. In this case there is much less known. One of the main difficulties is that contributions from the positive and negative half-axes can cancel each other.

In many applications, e.g. spectral theory of Sturm–Liouville and Schrödinger operators, measures are used that grow faster at infinity: instead of $\int_{\mathbb{R}}(1+|t|)^{-1}\,\mathrm{d}\mu(t)<\infty$ they have only power bounded tails, meaning $\int_{\mathbb{R}}(1+t^2)^{-\kappa-1}\,\mathrm{d}\mu(t)<\infty$ for some $\kappa\in\mathbb{N}_0$. For such measures the Cauchy transform (1.1) has to be redefined by including appropriate regularising summands in the integrand. The most common case is that μ is Poisson integrable, i.e. $\kappa=0$, and a commonly used regularisation in this case is

$$\widetilde{C}[\mu](z) := \int_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{t}{1+t^2}\Big)\,\mathrm{d}\mu(t), \qquad z\in\mathbb{C}^+. \tag{1.5}$$

Some Abelian and Tauberian theorems dealing with polynomial asymptotics in the bilateral case are given in [28, 29], a Tauberian theorem for the Cauchy transform (1.1) can be found in [30], and an Abelian theorem of somewhat different type (and formulated for integration on the unit circle instead of the real line) is [33]. A Tauberian theorem for the regularised Cauchy transform (1.5) is stated in [2]; however, the proof given contains a mistake. Fortunately the result itself turns out to be true; see Theorem 3.2 and the discussion preceding it.

In the current paper we prove Abelian and Tauberian theorems for higher-order regularised Cauchy transforms and growth of regular variation in Karamata's sense (see Appendix A for this notion). We relate the asymptotics of $\mu([0,t))$ and $\mu((-t,0))$ when $t\to+\infty$ to the asymptotics of the higher-order regularised Cauchy transform when $z\to+i\infty$ radially or non-tangentially. The main result of the paper is Theorem 5.1 where we give a full characterisation (including explicit formulae for constants) in the generic case. There are some boundary cases, namely when the index of regular variation is an integer, where only one direction is possible: either the Abelian direction where we deduce properties of the regularised Cauchy transform from properties of the measure, or the Tauberian direction, which is the other way round. The phenomenon that more complicated behaviour occurs at integer powers was already observed in [28, 29]. We investigate these exceptional cases more closely in Theorem 5.5.

For the proof of our results we follow common lines and consider imaginary and real parts of the integral separately. The imaginary part can be written as a Stieltjes transform, and thus inherits being well behaved; see Theorem 4.7. Contrasting this, the real part is the difference of two Stieltjes transforms, and this is the point where cancellation may happen.

Let us give a brief overview of the contents of the paper. In Section 2 we define higher-order regularised Cauchy transforms and study basic properties. In particular, we explore the relation with generalised Nevanlinna functions in the sense of M.G. Krein and H. Langer [19], characterise the range of the transform, and prove an analogue of the classical Grommer–Hamburger theorem that relates convergence of a sequence of measures to convergence of their Cauchy transforms. We use the latter theorem to prove a basic Tauberian theorem in Section 3. The proofs of Abelian theorems, which are contained in Section 4, use different methods: the main ingredients are Karamata's theorems. In Section 5 we combine the results from Sections 3 and 4 to prove our main theorems. We also provide counterexamples for the boundary cases. Finally, in Appendix A we recall and extend some results on regularly varying functions and Stieltjes transforms of measures supported on $[0,\infty)$.

Notation

Throughout the paper we use the following conventions and notations.

▷ We set $\mathbb{N}:=\{1,2,\ldots\}$, $\mathbb{N}_0:=\mathbb{N}\cup\{0\}$, and let $\mathbb{C}$ be the field of complex numbers.

▷ We always use the branches of the logarithm and complex powers which are analytic on $\mathbb{C} \setminus (-\infty, 0]$ and take the value 0 or 1, respectively, at the point 1.

▷ We set $\mathbb{C}^+ := \{z \in \mathbb{C}\colon\ \operatorname{Im} z > 0\}$ and $\mathbb{C}^- := \{z \in \mathbb{C}\colon\ \operatorname{Im} z < 0\}$.

▷ For a domain Ω denote by $\operatorname{Hol}(\Omega)$ and $\operatorname{Mer}(\Omega)$ the set of holomorphic and of meromorphic functions on Ω respectively.

▷ We use the notation $f \sim g$ to express that $\frac{f}{g} \to 1$, and the notation $f \ll g$ if $\frac{f}{g} \to 0$. Further, we write $f \lesssim g$ if there exists a constant $c > 0$ such that $f \leq cg$, and we write $f \asymp g$ if $f \lesssim g$ and $g \lesssim f$. The domain of validity will be stated or will be clear from the context.

▷ When we speak of a "measure", we always mean positive Borel measure (unless explicitly specified differently).

▷ Throughout the rest of the paper we use the Stieltjes transform $\mathcal{S}[\mu]$ as defined in (A.7), where we use a different sign convention from the one used in (1.2).

2 Regularised Cauchy integrals

2.1 Definition of higher-order regularised Cauchy integrals

To start with, let us recall the characterisations of the ranges of the transforms C and $\widetilde{C}$ introduced in (1.1) and (1.5). These are classical results going back to F. Riesz, G. Herglotz and R. Nevanlinna; for a comprehensive account see, e.g. [14] or [10].

2.1 Proposition. *Let $q \in \operatorname{Hol}(\mathbb{C}^+)$.*

(i) *The function q can be represented in the form*

$$q(z) = a + C[\mu](z) = a + \int_{\mathbb{R}} \frac{1}{t-z}\, \mathrm{d}\mu(t), \qquad z \in \mathbb{C}^+, \tag{2.1}$$

with some $a \in \mathbb{R}$ and a positive measure μ on $\mathbb{R}$ with $\int_{\mathbb{R}} (1+|t|)^{-1} \mathrm{d}\mu(t) < \infty$ if and only if

$$\forall z \in \mathbb{C}^+ :\ \operatorname{Im} q(z) \geq 0 \quad \text{and} \quad \int_1^\infty \frac{\operatorname{Im} q(iy)}{y}\, \mathrm{d}y < \infty.$$

(ii) *The function q can be represented in the form*

$$q(z) = a + bz + \widetilde{C}[\mu](z) = a + bz + \int_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{t}{1+t^2}\Big)\, \mathrm{d}\mu(t), \qquad z \in \mathbb{C}^+, \tag{2.2}$$

with some $a \in \mathbb{R}$, $b \geq 0$ and a positive measure μ on $\mathbb{R}$ with $\int_{\mathbb{R}} (1+t^2)^{-1} \mathrm{d}\mu(t) < \infty$ if and only if

$$\forall z \in \mathbb{C}^+ :\ \operatorname{Im} q(z) \geq 0.$$

2.2 Remark.

(i) Assume that q is represented in the form (2.2). Then the constants a, b are given by

$$a = \operatorname{Re} q(i), \qquad b = \lim_{y \to +\infty} \frac{1}{iy} q(iy), \tag{2.3}$$

and the measure μ is given by the Stieltjes inversion formula

$$\forall \alpha, \beta \in \mathbb{R},\ \alpha < \beta :\ \lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int_{\alpha+\delta}^{\beta-\delta} \operatorname{Im} q(x + i\varepsilon)\, \mathrm{d}x = \mu\big((\alpha, \beta)\big);$$

see, e.g. [14]. In particular, the map $(a, b, \mu) \mapsto q$ with q satisfying (2.2) is injective.

(ii) Assume that $a = b = 0$ in (2.2), i.e. $q(z) = \widetilde{C}[\mu](z)$. Then

$$|q(iy)| \ll y, \qquad y \to +\infty, \tag{2.4}$$

by (2.3), and

$$\lim_{y\to+\infty} y \operatorname{Im} q(iy) = \sup_{y>0} y \operatorname{Im}(iy) = \mu(\mathbb{R}); \tag{2.5}$$

see again [14].

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It is apparent that moving from function (2.1) to function (2.2) is only the first step on a ladder: instead of Poisson-integrable measures one may use measures whose tails have at most power growth, and instead of the term $a + bz$ one may use any polynomial with real coefficients. In the integral higher-order regularisation will become necessary.

We work with a scale of higher-order regularised Cauchy transforms which is commonly used in the framework of indefinite inner product spaces; see, e.g. [19].

2.3 Definition. Let $\kappa \in \mathbb{N}_0$.

(i) We denote by $\mathbb{E}_{\leq\kappa}$ the set of all pairs (μ, p) where

- ▷ μ is a measure on $\mathbb{R}$ that satisfies

$$\int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{(1+t^2)^{\kappa+1}} < \infty; \tag{2.6}$$

- ▷ p is a polynomial with real coefficients whose degree does not exceed $2\kappa + 1$;
- ▷ the coefficient of $z^{2\kappa+1}$ in p satisfies

$$\frac{1}{(2\kappa+1)!} p^{(2\kappa+1)}(0) \geq \int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{(1+t^2)^{\kappa+1}}. \tag{2.7}$$

(ii) The κ-regularised Cauchy transform is the map $\mathscr{C}_\kappa \colon \mathbb{E}_{\leq\kappa} \to \operatorname{Hol}(\mathbb{C}^+)$ defined by

$$\mathscr{C}_\kappa[\mu, p](z) := p(z) + (1+z^2)^{\kappa+1} \int_{\mathbb{R}} \frac{1}{t-z} \cdot \frac{\mathrm{d}\mu(t)}{(1+t^2)^{\kappa+1}}, \qquad z \in \mathbb{C}^+. \tag{2.8}$$

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In order to represent polynomials with real coefficients as regularised Cauchy integrals, we have to include pairs such as $(0, z^{2\kappa})$ in $\mathbb{E}_{\leq\kappa}$. For this reason we cannot speak of "the leading coefficient of p" in (2.7).

These maps can indeed be seen as higher-order regularised Cauchy integrals: for $k \in \mathbb{N}_0$ we have

$$\frac{1}{t-z} - (t+z) \sum_{j=0}^{k} \frac{(1+z^2)^j}{(1+t^2)^{j+1}} = \frac{1}{t-z} - \frac{t+z}{1+t^2} \cdot \frac{1 - \left(\frac{1+z^2}{1+t^2}\right)^{k+1}}{1 - \frac{1+z^2}{1+t^2}}$$

$$= \frac{1}{t-z} - (t+z) \cdot \frac{1 - \left(\frac{1+z^2}{1+t^2}\right)^{k+1}}{t^2 - z^2} = \frac{1}{t-z} \cdot \frac{(1+z^2)^{k+1}}{(1+t^2)^{k+1}}; \tag{2.9}$$

hence we obtain, with $k = \kappa$,

$$\mathscr{C}_\kappa[\mu, p](z) = p(z) + \int_{\mathbb{R}} \left[\frac{1}{t-z} - (t+z) \sum_{j=0}^{\kappa} \frac{(1+z^2)^j}{(1+t^2)^{j+1}} \right] \mathrm{d}\mu(t). \tag{2.10}$$

Note that the regularising terms in the integral on the right-hand side of (2.10) are the first $\kappa+1$ terms of an expansion of $\frac{1}{t-z} = (t+z)\cdot\frac{1}{t^2-z^2}$ in terms of powers of $\frac{1}{1+t^2}$. For $\kappa = 0$ relation (2.10) reads as

$$\mathscr{C}_0[\mu,p](z) = p(z) + \int_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{t+z}{1+t^2}\Big)\,\mathrm{d}\mu(t), \tag{2.11}$$

which yields the following connection with the previously discussed regularised Cauchy-type integral (1.5):

- ▷ $a + bz + \widetilde{C}[\mu](z) = \mathscr{C}_0[\mu,p](z)$ with $p(z) := a + \Big(b + \int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{1+t^2}\Big)z$;
- ▷ $(a,b,\mu) \in \mathbb{R}\times[0,\infty)\times\Big\{\mu\colon \text{ positive measure with } \int_{\mathbb{R}}\frac{\mathrm{d}\mu(t)}{1+t^2} < \infty\Big\}$

  $\iff \quad (\mu,p) \in \mathbb{E}_{\leq 0}$ with p related to a, b and μ as above.

2.4 Remark. Using again (2.9) with $\kappa = 0$ we obtain the following representation for $\mathscr{C}_\kappa$ for arbitrary $\kappa \in \mathbb{N}_0$:

$$\begin{aligned}\mathscr{C}_\kappa[\mu,p](z) &= p(z) + (1+z^2)^\kappa \int_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{t+z}{1+t^2}\Big)\frac{\mathrm{d}\mu(t)}{(1+t^2)^\kappa}\\ &= \Big(p(z) - z(1+z^2)^\kappa \int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{(1+t^2)^{\kappa+1}}\Big)\\ &\quad + (1+z^2)^\kappa \int_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{t}{1+t^2}\Big)\frac{\mathrm{d}\mu(t)}{(1+t^2)^\kappa}.\end{aligned} \tag{2.12}$$

If the stronger integrability condition $\int_{\mathbb{R}}(1+|t|)^{-(2\kappa+1)}\mathrm{d}\mu(t) < \infty$ is satisfied, then we can split the second integral on the right-hand side of (2.12) and rewrite it as

$$\begin{aligned}\mathscr{C}_\kappa[\mu,p](z) = \bigg(p(z) - (1+z^2)^\kappa\Big[z\int_{\mathbb{R}}\frac{\mathrm{d}\mu(t)}{(1+t^2)^{\kappa+1}} + \int_{\mathbb{R}}\frac{t}{1+t^2}\cdot\frac{\mathrm{d}\mu(t)}{(1+t^2)^\kappa}\Big]\bigg)\\ + (1+z^2)^\kappa\int_{\mathbb{R}}\frac{1}{t-z}\cdot\frac{\mathrm{d}\mu(t)}{(1+t^2)^\kappa}.\end{aligned} \tag{2.13}$$

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Before we collect some properties of $\mathbb{E}_{\leq\kappa}$ and $\mathscr{C}_\kappa$, we recall the Stieltjes–Livšic inversion formula; see, e.g. [21, Corollary II.1.2] or [11, Theorem 1.2.4]). Let σ be a finite measure on $\mathbb{R}$, let $\alpha,\beta\in\mathbb{R}$ with $\alpha<\beta$, and let f be an analytic function on a neighbourhood of $[\alpha,\beta]$. For $\delta,\varepsilon>0$ let $\Gamma_\varepsilon^\delta$ be the path consisting of the two directed line segments

$$\alpha+\delta-i\varepsilon \rightsquigarrow \beta-\delta-i\varepsilon \quad\text{and}\quad \beta-\delta+i\varepsilon \rightsquigarrow \alpha+\delta+i\varepsilon. \tag{2.14}$$

Then

$$\lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\frac{-1}{2\pi i}\int_{\Gamma_\varepsilon^\delta} f(z)\int_{\mathbb{R}}\frac{1}{t-z}\,\mathrm{d}\sigma(t)\,\mathrm{d}z = \int_{(\alpha,\beta)} f(t)\,\mathrm{d}\sigma(t). \tag{2.15}$$

2.5 Lemma. *Let $\kappa\in\mathbb{N}_0$.*

(i) *The set $\mathbb{E}_{\leq\kappa}$ is a positive cone and $\mathscr{C}_\kappa$ is a cone map, i.e. compatible with finite sums and non-negative scalar multiples.*

(ii) *The map $\mathscr{C}_\kappa$ is injective. For $q\in\operatorname{ran}\mathscr{C}_\kappa$ the element $(\mu,p) = \mathscr{C}_\kappa^{-1}q$ is obtained as follows: the polynomial p can be recovered from solving the $2\kappa+2$ equations obtained by splitting real and imaginary parts of*

$$q^{(j)}(i) = p^{(j)}(i), \qquad j\in\{0,\dots,\kappa\}; \tag{2.16}$$

the measure μ can be obtained via the Stieltjes inversion formula: for $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$ we have

$$\mu\bigl((\alpha,\beta)\bigr) = \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_{\alpha+\delta}^{\beta-\delta}\operatorname{Im} q(t+i\varepsilon)\,\mathrm{d}\mu(t). \tag{2.17}$$

(iii) *Let $\kappa' > \kappa$. Then the inclusion* $\operatorname{ran}\mathscr{C}_\kappa \subseteq \operatorname{ran}\mathscr{C}_{\kappa'}$ *holds, and, for $(\mu, p) \in \mathbb{E}_{\leq\kappa}$, we have $\mathscr{C}_\kappa[\mu, p] = \mathscr{C}_{\kappa'}[\mu, \widetilde{p}]$ with*

$$\widetilde{p}(z) = p(z) + \sum_{j=\kappa+1}^{\kappa'}(1+z^2)^j\left[z\int_{\mathbb{R}}\frac{\mathrm{d}\mu(t)}{(1+t^2)^{j+1}} + \int_{\mathbb{R}}\frac{t\,\mathrm{d}\mu(t)}{(1+t^2)^{j+1}}\right].$$

Proof.

(i) The statements are clear from the definitions of $\mathbb{E}_{\leq\kappa}$ and $\mathscr{C}_\kappa$.

(ii) Let $(\mu, p) \in \mathbb{E}_{\leq\kappa}$ and set $q = \mathscr{C}_\kappa[\mu, p]$. It follows from the definition of $\mathscr{C}_\kappa$ that (2.16) holds, which implies that p is uniquely determined by q. To show (2.17), let us first extend q to $\mathbb{C}\setminus\mathbb{R}$ by symmetry: $q(z) := \overline{q(\overline{z})}$ for $z \in \mathbb{C}^-$. Moreover, let $a, b \in \mathbb{R}$ with $a < b$ and let $\Gamma_\varepsilon^\delta$ be the path in (2.14). Then (2.15) implies that

$$\begin{aligned}
\lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_{a+\delta}^{b-\delta}\operatorname{Im} q(t+i\varepsilon)\,\mathrm{d}\mu(t) &= \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\frac{-1}{2\pi i}\int_{\Gamma_\varepsilon^\delta} q(z)\,\mathrm{d}z \\
&= \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\frac{-1}{2\pi i}\int_{\Gamma_\varepsilon^\delta}(1+z^2)^{\kappa+1}\left[\int_{\mathbb{R}}\frac{1}{t-z}\cdot\frac{\mathrm{d}\mu(t)}{(1+t^2)^{\kappa+1}}\right]\mathrm{d}z \\
&= \int_{(a,b)}(1+t^2)^{\kappa+1}\cdot\frac{\mathrm{d}\mu(t)}{(1+t^2)^{\kappa+1}} = \mu\bigl((a,b)\bigr).
\end{aligned}$$

The unique determination of μ and p shows that $\mathscr{C}_\kappa$ is injective.

(iii) Let $(\mu, p) \in \mathbb{E}_{\leq\kappa}$ and let $\kappa' > \kappa$. It follows from (2.9) that

$$\frac{1}{t-z}\cdot\frac{(1+z^2)^{\kappa+1}}{(1+t^2)^{\kappa+1}} = \frac{1}{t-z}\cdot\frac{(1+z^2)^{\kappa'+1}}{(1+t^2)^{\kappa'+1}} + (t+z)\sum_{j=\kappa+1}^{\kappa'}\frac{(1+z^2)^j}{(1+t^2)^{j+1}}, \tag{2.18}$$

which yields the statement in (iii). ❑

Note that the Stieltjes inversion formula (2.17) for the recovery of μ is independent of κ.

2.2 Determining the range of $\mathscr{C}_\kappa$

An intrinsic characterisation of the range of $\mathscr{C}_\kappa$ along the lines of Proposition 2.1 (ii) can be given. This is based on [19, 20, 22] and related to [25, Theorem 3.9] (a predecessor of the latter is [15, Lemma 3.6]).

Let us first recall the definition of generalised Nevanlinna functions in the sense of [19]. We need, in particular, functions from the subclasses $\mathcal{N}_\kappa^{(\infty)}$, which are characterised by a special behaviour at infinity and which were studied in, e.g. [6, 7, 8, 13, 24, 25, 26].

2.6 Definition. For $q \in \operatorname{Mer}(\mathbb{C}^+)$ we denote by Ω_q its domain of analyticity; for the constant $q \equiv \infty$ we set $\Omega_q = \emptyset$.

(i) For $q \in \operatorname{Mer}(\mathbb{C}^+) \cup \{\infty\}$ we denote by $\kappa_q \in \mathbb{N}_0 \cup \{\infty\}$ the number of negative squares of the Hermitian kernel

$$K_q(w, z) := \frac{q(z) - \overline{q(w)}}{z - \overline{w}}, \qquad z, w \in \Omega_q, \tag{2.19}$$

i.e. the supremum of all numbers of negative squares of the quadratic forms

$$\sum_{i,j=1}^{m} K_q(w_j, w_i)\xi_i\overline{\xi_j}$$

with $m \in \mathbb{N}$ and $w_1, \dots, w_m \in \Omega_q$.

(ii) Let $\kappa \in \mathbb{N}_0$. We denote by $\mathcal{N}_\kappa$ the set of all functions $q \in \mathrm{Mer}(\mathbb{C}^+)$ with $\kappa_q = \kappa$. Moreover, we set $\mathcal{N}_{\leq\kappa} := \bigcup_{\kappa'=0}^{\kappa} \mathcal{N}_{\kappa'}$ and $\mathcal{N}_{<\infty} := \bigcup_{\kappa'=0}^{\infty} \mathcal{N}_{\kappa'}$.

(iii) Let $\kappa \in \mathbb{N}_0$. We denote by $\mathcal{N}_\kappa^{(\infty)}$ the set of all functions $q \in \mathcal{N}_\kappa$ for which

$$\lim_{y\to+\infty} \left|\frac{q(iy)}{y^{2\kappa-1}}\right| = \infty \qquad \text{or} \qquad \lim_{y\to+\infty} \frac{q(iy)}{(iy)^{2\kappa-1}} \in (-\infty, 0).$$

Moreover, we set $\mathcal{N}_{\leq\kappa}^{(\infty)} := \bigcup_{\kappa'=0}^{\kappa} \mathcal{N}_{\kappa'}^{(\infty)}$ and $\mathcal{N}_{<\infty}^{(\infty)} := \bigcup_{\kappa'=0}^{\infty} \mathcal{N}_{\kappa'}^{(\infty)}$.

◊

2.7 Remark.

(i) Note that the classes $\mathcal{N}_0$ and $\mathcal{N}_0^{(\infty)}$ coincide with the set of all Nevanlinna functions, i.e. those functions q that are analytic on $\mathbb{C}^+$ and satisfy $\mathrm{Im}\, q(z) \geq 0$ for $z \in \mathbb{C}^+$. Further, functions in $\mathcal{N}_\kappa$ have at most κ poles and at most κ zeros in $\mathbb{C}^+$.

(ii) The classes $\mathcal{N}_\kappa^{(\infty)}$, which have also been denoted by $\mathcal{N}_\kappa^\infty$ in the literature, can also be characterised differently, namely, for $q \in \mathrm{Mer}(\mathbb{C}^+)$ the following conditions are equivalent (see [13], or also [6, 15]):

(a) $q \in \mathcal{N}_{<\infty}^{(\infty)}$;

(b) ∞ is the only (generalised) pole not of positive type (in the sense of [19, §3]), i.e. ∞ is the only (generalised) eigenvalue with a non-positive eigenvector of a representing relation in a Pontryagin space (see also [22] for an analytic characterisation of generalised poles not of positive type);

(c) there exist $m \in \mathbb{N}_0$, a real polynomial p and a measure σ on $\mathbb{R}$ such that $\int_{\mathbb{R}}(1 + t^2)^{-1}\mathrm{d}\sigma(t) < \infty$ and

$$q(z) = (1+z^2)^m \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) \mathrm{d}\sigma(t) + p(z); \tag{2.20}$$

(d) there exist $n \in \mathbb{N}_0$, $\beta_1, \dots, \beta_n \in \mathbb{C}^+ \cup \mathbb{R}$, $\rho_j \in \mathbb{N}$ and $q_0 \in \mathcal{N}_0$ such that

$$q(z) = \prod_{j=1}^{n} (z-\beta_j)^{\rho_j}(z-\overline{\beta_j})^{\rho_j} q_0(z).$$

If q is as in (2.20) and $\deg p = l$ with leading coefficient c_l, then $q \in \mathcal{N}_\kappa^{(\infty)}$ with

$$\kappa \leq \max\{m, \kappa_p\} \qquad \text{where} \quad \kappa_p = \begin{cases} \frac{l}{2}, & l \text{ even}, \\ \frac{l-\mathrm{sgn}(c_l)}{2}, & l \text{ odd}; \end{cases} \tag{2.21}$$

equality holds in (2.21) if $m = 0$ or σ is an infinite measure; see, e.g. [6, (1.16)]. Note that for a real polynomial p one has $\kappa_p \leq \kappa'$ if and only if $\deg p \leq 2\kappa' + 1$ and $p^{(2\kappa'+1)}(0) \geq 0$.

(iii) The representation (2.10) is a special case of the integral representation of $\mathcal{N}_{<\infty}$-functions given in [19, Satz 3.1].

(iv) In [25] representations of functions in $\mathcal{N}_{<\infty}^{(\infty)}$ were constructed with distributions (more precisely, distributional densities) on the one-point compactification $\mathbb{R} \cup \{\infty\}$ of $\mathbb{R}$ which act like measures on $\mathbb{R}$.

(v) Functions in $\mathcal{N}_{<\infty}^{(\infty)}$ are analytic in $\mathbb{C}^+$.

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2.8 Theorem. *For every* $\kappa \in \mathbb{N}_0$ *the equality* $\operatorname{ran} \mathscr{C}_\kappa = \mathcal{N}_{\leq\kappa}^{(\infty)}$ *holds.*

Proof. Let $\kappa \in \mathbb{N}_0$. It follows from (2.12) that $q \in \operatorname{ran} \mathscr{C}_\kappa$ if and only if it can be written as

$$q(z) = \widetilde{p}(z) + (1+z^2)^\kappa \int_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{t}{1+t^2}\Big)\, \mathrm{d}\sigma(t) \tag{2.22}$$

with a real polynomial $\widetilde{p}$ of degree at most $2\kappa+1$ with $\widetilde{p}^{(2\kappa+1)}(0) \geq 0$ and a measure σ such that $\int_{\mathbb{R}} (1+t^2)^{-1} \mathrm{d}\sigma(t) < \infty$.

First assume that $q \in \operatorname{ran} \mathscr{C}_\kappa$. Then (2.22) holds with σ and $\widetilde{p}$ as above. By Remark 2.7 (ii) we obtain that $q \in \mathcal{N}_{\leq\kappa}^{(\infty)}$.

Conversely, assume that $q \in \mathcal{N}_{\leq\kappa}^{(\infty)}$, say $q \in \mathcal{N}_{\kappa'}^{(\infty)}$ with $\kappa' \in \{0, \dots, \kappa\}$. Then there exists a representation of q as in (2.20) such that $\kappa' = \max\{m, \kappa_p\}$, where κ_p is as in (2.21); in particular $m \leq \kappa$ and $\kappa_p \leq \kappa$. The function

$$\widehat{q}(z) := (1+z^2)^m \int_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{t}{1+t^2}\Big)\, \mathrm{d}\sigma(t)$$

belongs to $\operatorname{ran} \mathscr{C}_m$ by the first paragraph of this proof. Since $m \leq \kappa$, we obtain from Lemma 2.5 (iii) that $\widehat{q} \in \operatorname{ran} \mathscr{C}_\kappa$. The relations $\deg p \leq 2\kappa+1$ and $p^{(2\kappa+1)}(0) \geq 0$ show that $(0, p) \in \mathbb{E}_{\leq\kappa}$, and hence $p = \mathscr{C}_\kappa[0, p] \in \operatorname{ran} \mathscr{C}_\kappa$. Now Lemma 2.5 (i) implies that $q = \widehat{q} + p \in \operatorname{ran} \mathscr{C}_\kappa$. ❑

2.3 $\mathscr{C}_\kappa$ as a homeomorphism: the Grommer–Hamburger theorem

Next we discuss a continuity property of $\mathscr{C}_\kappa$; see Theorem 2.12 below. This result is a variant of a classical theorem of J. Grommer and H. Hamburger; see the discussion in Remark 2.13.

Before being able to formulate a result, we have to make clear which topologies we use. On the set $\operatorname{Hol}(\mathbb{C}^+)$ we always use the topology of locally uniform convergence. Topologising $\mathbb{E}_{\leq\kappa}$ is slightly more subtle. We proceed as follows. Fix $\kappa \in \mathbb{N}_0$. The set of all positive measures μ that satisfies (2.6) is a subset of the dual space of the weighted C_0-space

$$C_0(\mathbb{R}, \omega_\kappa) := \Big\{ f \in C(\mathbb{R}) \colon \lim_{|x|\to\infty} |f(x)|\omega_\kappa(x) = 0 \Big\}, \qquad \|f\| := \sup_{x\in\mathbb{R}} |f(x)|\omega_\kappa(x),$$

where ω_κ is the weight function $\omega_\kappa(x) := (1+x^2)^{\kappa+1}$; note that

$$\|\mu\|_{C_0(\mathbb{R},\omega_\kappa)'} = \int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{(1+t^2)^{\kappa+1}} \tag{2.23}$$

for a positive measure μ that satisfies (2.6). We endow the set of all positive measures μ that satisfy (2.6) with the subspace topology of the w^*-topology in $C_0(\mathbb{R}, \omega_\kappa)'$. The set of all polynomials of degree at most $2\kappa+1$ is isomorphic to $\mathbb{R}^{2\kappa+2}$ mapping a polynomial to its coefficients, and we endow polynomials with the Euclidean norm transported via this isomorphism. The set $\mathbb{E}_{\leq\kappa}$ is now topologised as a subspace of the product.

This topology has some very nice properties, which are summarised in the following lemma.

2.9 Lemma. *Let $\kappa \in \mathbb{N}_0$.*

(i) *$\mathbb{E}_{\leq\kappa}$ is a closed subset of $C_0(\mathbb{R},\omega_\kappa)' \times \mathbb{R}^{2\kappa+2}$.*

(ii) *A subset $\mathcal{E} \subseteq \mathbb{E}_{\leq\kappa}$ is relatively compact if and only if*

$$\sup\big\{\|p\|:\ (\mu,p)\in\mathcal{E}\big\} < \infty. \tag{2.24}$$

(iii) *The sets*

$$\mathcal{E}_N := \big\{(\mu,p)\in\mathbb{E}_{\leq\kappa}:\ \|p\|\leq N\big\}$$

are compact. We have $\mathcal{E}_N \subseteq \operatorname{Int}\mathcal{E}_{N+1}$, and $\bigcup_{N\in\mathbb{N}}\mathcal{E}_N = \mathbb{E}_{\leq\kappa}$, where the interior $\operatorname{Int}\mathcal{E}_{N+1}$ of $\mathcal{E}_{N+1}$ is understood within $\mathbb{E}_{\leq\kappa}$.

Proof.

(i) Assume that $((\mu_i,p_i))_{i\in I}$ is a net in $\mathbb{E}_{\leq\kappa}$ that converges to some element $(\mu,p) \in C_0(\mathbb{R},\omega_\kappa)' \times \mathbb{R}^{2\kappa+2}$. Then μ is again a positive measure, and, by (2.23),

$$\begin{aligned}\frac{1}{(2\kappa+1)!}p^{(2\kappa+1)}(0) &= \lim_{i\in I}\frac{1}{(2\kappa+1)!}p_i^{(2\kappa+1)}(0)\\ &\geq \limsup_{i\in I}\int_{\mathbb{R}}\frac{\mathrm{d}\mu_i(t)}{(1+t^2)^{\kappa+1}} \geq \int_{\mathbb{R}}\frac{\mathrm{d}\mu(t)}{(1+t^2)^{\kappa+1}}.\end{aligned}$$

Thus $(\mu,p)\in\mathbb{E}_{\leq\kappa}$, and we see that $\mathbb{E}_{\leq\kappa}$ is indeed closed.

(ii) Let π_2 be the projection onto the second component of $C_0(\mathbb{R},\omega_\kappa)' \times \mathbb{R}^{2\kappa+2}$. Then π_2 is continuous. This already shows the implication "⇒" in (ii). Conversely, assume that (2.24) holds. It follows from (2.23) that

$$\sup\Big\{\|\mu\|_{C_0(\mathbb{R},\omega_\kappa)'}:\ (\mu,p)\in\mathcal{E}\Big\} \leq \sup\big\{\|p\|:\ (\mu,p)\in\mathcal{E}\big\} =: c < \infty.$$

We see that

$$\mathcal{E} \subseteq \{\mu\in C_0(\mathbb{R},\omega_\kappa)':\ \|\mu\|_{C_0(\mathbb{R},\omega_\kappa)'}\leq c\} \times \{p\in\mathbb{R}^{2\kappa+2}:\ \|p\|\leq c\},$$

and hence $\mathcal{E}$ is relatively compact in $C_0(\mathbb{R},\omega_\kappa)' \times \mathbb{R}^{2\kappa+2}$ by the Banach–Alaoglu Theorem. Since $\mathbb{E}_{\leq\kappa}$ is closed in this product space, $\mathcal{E}$ is also relatively compact in $\mathbb{E}_{\leq\kappa}$.

Item (iii) follows from what we have shown so far and from the fact that the continuity of π_2 implies that $\mathcal{E}_N$ is closed and that

$$\operatorname{Int}\mathcal{E}_{N+1} = \big\{(\mu,p)\in\mathbb{E}_{\leq\kappa}:\ \|p\|<N+1\big\}.$$

❑

2.10 Proposition. *The range of $\mathscr{C}_\kappa$ is closed in $\operatorname{Hol}(\mathbb{C}^+)$, and $\mathscr{C}_\kappa$ is a homeomorphism onto its range.*

Proof. Continuity of $\mathscr{C}_\kappa$ is clear from our choice of topology. Let $(q_i)_{i\in I}$ be a net in $\operatorname{ran}\mathscr{C}_\kappa$, and assume that $\lim_{i\in I} q_i = q$ in $\operatorname{Hol}(\mathbb{C}^+)$. Remembering (2.16) we find $i_0\in I$ and $N\in\mathbb{N}$ such that $\mathscr{C}_\kappa^{-1}(q_i)\in\mathcal{E}_N$ for all $i\geq i_0$. Since $\mathcal{E}_N$ is compact (and $\mathscr{C}_\kappa$ is continuous and injective), it follows that the limit $\lim_{i\in I}\mathscr{C}_\kappa^{-1}(q_i)$ exists in $\mathcal{E}_N\subseteq\mathbb{E}_{\leq\kappa}$. ❑

The following result is used in the proofs of Theorems 2.12 and 3.1.

2.11 Proposition. *Let $\kappa\in\mathbb{N}_0$ and $q_n\in\mathcal{N}_{\leq\kappa}$, $n\in\mathbb{N}_0$. Assume that*

(i) *for each compact $K\subseteq\mathbb{C}^+$ with non-empty interior O there exists $m_K\in\mathbb{N}$ such that q_n is analytic on O for all $n\geq m_K$;*

(ii) *there exists $M\subseteq\mathbb{C}^+$ with accumulation point in $\mathbb{C}^+$ such that $\lim\limits_{n\to\infty} q_n(z)$ exists for all $z\in M$.*

Then there exists $\mathring{q} \in \mathcal{N}_{\le\kappa} \cap \mathrm{Hol}(\mathbb{C}^+)$ such that $\lim_{n\to\infty} q_n = \mathring{q}$ locally uniformly in $\mathbb{C}^+$. Here we understand locally uniform convergence in the space of meromorphic functions considered as analytic functions into the Riemann sphere.

Proof. Assumptions (i) and (ii) imply, in particular, that there exist $\kappa+1$ points $z_0, \dots, z_\kappa \in \mathbb{C}^+$ such that $|q_n(z_i)| \le c$ for all $n \in \mathbb{N}$ and $i \in \{0,\dots,\kappa\}$ and some $c > 0$. By [23, Theorem 3.2] there exist a subsequence $(q_{n_k})_{k\in\mathbb{N}}$, a set $P \subseteq \mathbb{C}^+$ with $|P| \le \kappa$, and $\mathring{q} \in \mathcal{N}_{\le\kappa}$ such that

$$\lim_{k\to\infty} q_{n_k} = \mathring{q} \qquad \text{locally uniformly on } \mathbb{C}^+ \setminus P.$$

Note that $\mathring{q}$ is meromorphic on $\mathbb{C}^+$ because $\mathring{q} \in \mathcal{N}_{\le\kappa}$. Let $w \in P$. There exists a closed disc $K \subseteq \mathbb{C}^+$ around w with interior O such that $\mathring{q}$ is zero-free on $O \setminus \{w\}$. By assumption (i), q_{n_k} is analytic on O for all k with $n_k \ge m_K$. The convergence of the logarithmic residue implies that $\mathring{q}$ is analytic at w, and hence $\lim_{k\to\infty} q_{n_k} = \mathring{q}$ locally uniformly on O. Since w was arbitrary in P, this shows that $\mathring{q}$ is analytic on $\mathbb{C}^+$ and that $\lim_{k\to\infty} q_{n_k} = \mathring{q}$ locally uniformly on $\mathbb{C}^+$.

The above considerations can be done for every subsequence of (q_n) instead of (q_n) itself. Now assumption (ii) implies that $\lim_{n\to\infty} q_n = \mathring{q}$ locally uniformly on $\mathbb{C}^+$. ❑

We can now prove an analogue of the classical Grommer–Hamburger theorem for regularised Cauchy transforms.

2.12 Theorem. *Let $\kappa \in \mathbb{N}_0$, let $(\mu_n, p_n) \in \mathbb{E}_{\le\kappa}$ for $n \in \mathbb{N}$, set $q_n := \mathscr{C}_\kappa[\mu_n, p_n]$, and let $\mathring{q} \in \mathrm{Hol}(\mathbb{C}^+)$. Then the following three statements are equivalent:*

- (i) *$\exists M \subseteq \mathbb{C}^+$ such that M has an accumulation point in $\mathbb{C}^+$ and that $\lim_{n\to\infty} q_n(z) = \mathring{q}(z)$ for all $z \in M$;*
- (ii) *$\lim_{n\to\infty} q_n = \mathring{q}$ locally uniformly on $\mathbb{C}^+$;*
- (iii) *$\exists(\mathring{\mu}, \mathring{p}) \in \mathbb{E}_{\le\kappa}$ such that $\mathring{q} = \mathscr{C}_\kappa[\mathring{\mu}, \mathring{p}]$, $\lim_{n\to\infty} p_n = \mathring{p}$ and*

$$\forall a,b \in \mathbb{R}:\ a < b,\ \mathring{\mu}(\{a\}) = \mathring{\mu}(\{b\}) = 0 \quad \Longrightarrow \quad \lim_{n\to\infty} \mu_n((a,b)) = \mathring{\mu}((a,b)). \tag{2.25}$$

Proof. The equivalence of (i) and (ii) follows directly from Theorem 2.8 and Proposition 2.11 since q_n is analytic on $\mathbb{C}^+$ for every $n \in \mathbb{N}$.

Let us now prove the equivalence of (ii) and (iii). It follows from Proposition 2.10 that (ii) is equivalent to

$$\exists(\mathring{\mu}, \mathring{p}) \in \mathbb{E}_{\le\kappa}:\ \mathring{q} = \mathscr{C}_\kappa[\mathring{\mu}, \mathring{p}],\ \lim_{n\to\infty} p_n = \mathring{p} \tag{2.26}$$

together with $\lim_{n\to\infty} \mu_n = \mathring{\mu}$ w.r.t. w^* in $C_0(\mathbb{R}, \omega_\kappa)'$. Now assume that (2.26) holds. Then the convergence of (p_n) implies that

$$\|\mu_n\|_{C_0(\mathbb{R},\omega_\kappa)'} = \int_{\mathbb{R}} \frac{\mathrm{d}\mu_n(t)}{(1+t^2)^{\kappa+1}} \le \frac{1}{(2\kappa+1)!} p_n^{(2\kappa+1)}(0) \le c$$

for some $c > 0$. Since $C_{00}(\mathbb{R})$, which denotes the set of compactly supported continuous functions on $\mathbb{R}$, is dense in $C_0(\mathbb{R}, \omega_\kappa)$, the relation $\mu_n \to \mathring{\mu}$ w.r.t. w^* in $C_0(\mathbb{R}, \omega_\kappa)'$ is equivalent to $\mu_n \to \mathring{\mu}$ w.r.t. w^* in $C_{00}(\mathbb{R})'$. The latter relation is equivalent to (2.25) by the portmanteau-type theorem [1, Theorem 1]. ❑

2.13 Remark.

- (i) Let us make the connection with the original formulation of the Grommer–Hamburger theorem; see, e.g. [36, §48]. The latter is about Cauchy transforms, (1.1), of finite measures, and states the following: let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of finite measures on $\mathbb{R}$ whose total variations are uniformly bounded and let $\mathring{q} \in \mathrm{Hol}(\mathbb{C}^+)$; then the following statements are equivalent:

(a) $\lim_{n\to\infty} C[\mu_n](z) = \mathring{q}(z)$ for all $z \in \mathbb{C}^+$;

(b) there exists a finite measure $\mathring{\mu}$ such that $\mathring{q} = C[\mathring{\mu}]$ and (2.25) holds.

(ii) There has been some confusion about the formulation of the Grommer–Hamburger theorem. The condition in (2.25) says that $(\mu_n)_{n\in\mathbb{N}}$ converges vaguely, i.e. w.r.t. w^* in $C_{00}(\mathbb{R})'$. However, at some places in the literature it is claimed that (a) in item (i) implies that $(\mu_n)_{n\in\mathbb{N}}$ converges weakly, i.e. w.r.t. w^* in $C_{\mathrm{b}}(\mathbb{R})'$, where $C_{\mathrm{b}}(\mathbb{R})$ is the space of bounded continuous functions. The example $\mu_n = \delta_n$, where δ_n denotes the Dirac measure at n, shows that this is not true: $\lim_{n\to\infty} C[\delta_n] = 0$ locally uniformly, but $\lim_{n\to\infty} \delta_n = 0$ only vaguely and not weakly; in particular, mass is lost. Note that, by the portmanteau theorem (see, e.g. [18, Theorem 13.16]), a sequence of uniformly bounded measures (μ_n) on $\mathbb{R}$ converges weakly to a measure $\mathring{\mu}$ if and only if it converges vaguely to $\mathring{\mu}$ and $\lim_{n\to\infty} \mu_n(\mathbb{R}) = \mathring{\mu}(\mathbb{R})$. See also the discussion in [9].

(iii) In the original Grommer–Hamburger theorem one needs the a priori assumption that the total variations are uniformly bounded. On the other hand, in Theorem 2.12 the integrals $\int_{\mathbb{R}} (1+t^2)^{-(\kappa+1)} \mathrm{d}\mu_n(t)$ are automatically bounded by (2.7) and the convergence of the polynomials (p_n). Consider also the following example: let $\mu_n = n^2\delta_n$ and $p_n(z) = \frac{n^2}{1+n^2} z$. By (2.11) we have

$$\begin{aligned} \mathscr{C}_0[\mu_n, p_n](z) &= p_n(z) + \int_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{t+z}{1+t^2} \Big) \, \mathrm{d}\mu_n(t) \\ &= \widetilde{C}[\mu_n](z) = n^2 \Big(\frac{1}{n-z} - \frac{n}{1+n^2} \Big) \to z \end{aligned}$$

locally uniformly as $n \to \infty$. Note that the limit function belongs to $\operatorname{ran} \mathscr{C}_0$ but is not the Cauchy transform of a finite measure.

(iv) It follows from [1, Theorem 1] that (2.25) is equivalent to the following condition:

$$\text{for every bounded Borel set } A \text{ with } \mathring{\mu}(\partial A) = 0: \quad \lim_{n\to\infty} \mu_n(A) = \mathring{\mu}(A).$$

(v) Under the additional assumption that $\mathring{q} \in \mathcal{N}_\kappa^{(\infty)}$, i.e. $\mathring{q}$ has the same number of negative squares as q_n, the implication (ii) ⇒ (2.25) in Theorem 2.12 can also be deduced from [23, Corollary 3.1]; see also [26, Lemma 3.7].

◊

3 A Tauberian theorem

In this section we prove a Tauberian theorem for the transform $\mathscr{C}_\kappa$ where the asymptotic behaviour of the measure μ towards infinity can be derived from the asymptotic behaviour of the function $q = \mathscr{C}_\kappa[\mu, p]$ at infinity. As mentioned in the Introduction, there is a wide range of Tauberian theorems for Stieltjes transforms, where the measure is only supported on the positive half-line. Surprisingly, it seems there is much less known for Cauchy integrals, where the measure is allowed to be supported on the whole real line. One result, which has been frequently cited, is claimed in [2, Theorem 7.5]. The proof given in that paper contains a mistake[1]. Fortunately, the result itself turns out to be true. In this section we provide a simple and conclusive argument which allows us, at the same time, to drop one assumption made in [2] and to generalise it to higher-order regularised Cauchy transforms.

Theorem 3.2 contains the above mentioned Tauberian theorem for the transform $q = \mathscr{C}_\kappa[\mu, p]$. In most cases the asymptotic behaviour of the measure μ can be determined

[1] In particular, in that paper it is claimed that the relation $\int_{\mathbb{R}} (t-z)^{-3} \mathrm{d}\tau(t) = 0$ for all $z \in \mathbb{C}^+$ for a non-decreasing function τ on $\mathbb{R}$ implies that τ is constant at all points of continuity, which is not true as the example $\tau(t) = t$ shows.

independently for the positive and the negative real axis; see (3.6) and (3.7). The assumption about the asymptotic behaviour of q at infinity can be formulated in different ways. Theorem 3.1 shows the equivalence of these assumptions, where conditions (i) and (ii) are relatively minimal assumptions. In particular, (ii) says that, along one ray towards infinity, q behaves like a constant times a regularly varying function; for the latter notion see Appendix A. Note that we prove Theorem 3.1 for the larger class $\mathcal{N}_{\le\kappa}$ instead of the class $\mathcal{N}_{\le\kappa}^{(\infty)} = \operatorname{ran}\mathscr{C}_\kappa$.

The following functions play an important role in the current section, namely as limiting functions of rescalings of a given q:

$$Q_{\alpha,\omega}(z) := i\omega\Big(\frac{z}{i}\Big)^{\alpha}, \qquad z \in \mathbb{C}^+, \tag{3.1}$$

where $\alpha \in \mathbb{R}$ and $\omega \in \mathbb{C}\setminus\{0\}$. Note that, by the normalisation of the power function at the end of the Introduction, we have $Q_{\alpha,\omega}(i) = i\omega$.

3.1 Theorem. *Let $\kappa \in \mathbb{N}_0$ and $q \in \mathcal{N}_{\le\kappa}$, and let $f : [r_0,\infty) \to (0,\infty)$ with $r_0 > 0$ be measurable. Then the following statements are equivalent:*

(i) *there exists $M \subseteq \mathbb{C}^+$ with an accumulation point in $\mathbb{C}^+$ such that*

$$\forall z \in M: \ \lim_{r\to\infty} \frac{q(rz)}{f(r)} \quad \text{exists and is non-zero;} \tag{3.2}$$

(ii) *f is regularly varying and there exists $z_0 \in \mathbb{C}^+$ such that*

$$\lim_{r\to\infty} \frac{q(rz_0)}{f(r)} \quad \text{exists and is non-zero;} \tag{3.3}$$

(iii) *f is regularly varying with index $\alpha \in [-2\kappa-1, 2\kappa+1]$ and there exists $\omega \in \mathbb{C}\setminus\{0\}$ such that*

$$\lim_{r\to\infty} \frac{q(rz)}{f(r)} = Q_{\alpha,\omega}(z) \qquad \text{locally uniformly for } z \in \mathbb{C}^+;$$

moreover, there exists $\kappa' \in \{0,\dots,\kappa\}$ such that $Q_{\alpha,\omega} \in \mathcal{N}_{\kappa'}$ and hence

$$\big|\arg\big((-1)^{\kappa'}\omega\big)\big| \le \frac{\pi}{2}\Big(1 - \big||\alpha| - 2\kappa'\big|\Big). \tag{3.4}$$

If (i)–(iii) *are satisfied and ξ denotes the limit in* (3.3)*, then $\omega = \frac{\xi}{i}\big(\frac{z_0}{i}\big)^{-\alpha}$.*

Note that in the case when $\kappa = 0$ we must have $\kappa' = 0$ in (iii) and hence (3.4) reduces to $|\arg\omega| \le \frac{\pi}{2}(1-|\alpha|)$.

3.2 Theorem. *Let $\kappa \in \mathbb{N}_0$ and $(\mu,p) \in \mathbb{E}_{\le\kappa}$, set $q := \mathscr{C}_\kappa[\mu,p]$, and let $f : [x_0,\infty) \to (0,\infty)$ with $x_0 > 0$ be measurable. Assume that the equivalent conditions* (i)–(iii) *in Theorem 3.1 hold. Then $\alpha \in [-1, 2\kappa+1]$ and $Q_{\alpha,\omega} \in \operatorname{ran}\mathscr{C}_\kappa$. Let $\mu_{\alpha,\omega}$ be the measure component of $\mathscr{C}_\kappa^{-1}Q_{\alpha,\omega}$. For all $a,b \in \mathbb{R}$ with $a<b$ and $\mu_{\alpha,\omega}(\{a\}) = \mu_{\alpha,\omega}(\{b\}) = 0$, we have*

$$\lim_{r\to\infty} \frac{1}{rf(r)}\mu\big((ra,rb)\big) = \mu_{\alpha,\omega}\big((a,b)\big). \tag{3.5}$$

In particular, if $\alpha > -1$, then

$$\lim_{r\to\infty} \frac{1}{rf(r)}\mu\big((0,r)\big) = \frac{1}{\pi}\cdot\frac{|\omega|}{\alpha+1}\cos\Big(\frac{\alpha\pi}{2} - \arg\omega\Big), \tag{3.6}$$

$$\lim_{r\to\infty} \frac{1}{rf(r)}\mu\big((-r,0)\big) = \frac{1}{\pi}\cdot\frac{|\omega|}{\alpha+1}\cos\Big(\frac{\alpha\pi}{2} + \arg\omega\Big); \tag{3.7}$$

if $\alpha = -1$, then $\omega > 0$ and

$$\lim_{r\to\infty} \frac{1}{rf(r)}\mu\big((-r,r)\big) = \omega. \tag{3.8}$$

3.3 Remark.

(i) Any asymmetry of the limits on the right-hand sides of (3.6) and (3.7) can be seen from ω. To this end, assume that $\alpha > -1$ and write $\alpha = 2m + \alpha_0$ with $m \in \mathbb{N}_0$ and $|\alpha_0| \le 1$; if $\alpha \in 2\mathbb{N}_0 + 1$, choose m such that $(-1)^m\omega > 0$. The limits on the right-hand sides of (3.6) and (3.7) can be rewritten as follows:

$$\begin{aligned} c_\pm &:= \frac{1}{\pi} \cdot \frac{|\omega|}{\alpha+1} \cos\Big(\frac{\alpha\pi}{2} \mp \arg\omega\Big) \\ &= \frac{1}{\pi} \cdot \frac{|\omega|}{\alpha+1} \begin{cases} \cos\Big(\frac{\alpha_0\pi}{2} \mp \arg\omega\Big), & m \text{ even}, \\ \cos\Big(\frac{\alpha_0\pi}{2} + \pi \mp \arg\omega\Big), & m \text{ odd}, \end{cases} \\ &= \frac{1}{\pi} \cdot \frac{|\omega|}{\alpha+1} \cos\Big(\frac{\alpha_0\pi}{2} \mp \arg\big((-1)^m\omega\big)\Big). \end{aligned}$$

It follows from Lemma 3.4 below and its proof that $m = \kappa'$ and hence, by (3.4), that $\big|\frac{\alpha_0\pi}{2} \mp \arg\big((-1)^m\omega\big)\big| \le \frac{\pi}{2}$. From this the following equivalences follow easily, where we set $\psi := \arg((-1)^m\omega)$,

$$\begin{aligned} c_+ > c_- &\quad\Longleftrightarrow\quad \alpha_0, \psi \ne 0 \ \wedge\ \operatorname{sgn}\alpha_0 = \operatorname{sgn}\psi, \\ c_+ = c_- &\quad\Longleftrightarrow\quad \alpha_0 = 0 \ \vee\ \psi = 0, \\ c_+ < c_- &\quad\Longleftrightarrow\quad \alpha_0, \psi \ne 0 \ \wedge\ \operatorname{sgn}\alpha_0 = -\operatorname{sgn}\psi, \\ c_\pm = 0 &\quad\Longleftrightarrow\quad \big|\tfrac{\alpha_0\pi}{2} \mp \psi\big| = \tfrac{\pi}{2}, \\ c_+ = c_- = 0 &\quad\Longleftrightarrow\quad |\alpha_0| = 1 \ \vee\ \big(\alpha_0 = 0 \ \wedge\ |\psi| = \tfrac{\pi}{2}\big). \end{aligned}$$

If $c_+ \ne 0$, then the function $r \mapsto \mu((0,r))$ is regularly varying with index $\alpha + 1$; if $c_- \ne 0$, then $r \mapsto \mu((-r,0))$ is regularly varying with index $\alpha + 1$.

(ii) Example 4.9 below shows that there are situations where (i)–(iii) in Theorem 3.1 are satisfied but none of $t \mapsto \mu((-t,t))$, $t \mapsto \mu((0,t))$, $t \mapsto \mu((-t,0))$ is regularly varying. See also Example 5.3.

◊

Before we prove Theorems 3.1 and 3.2, we need some lemmas.

3.4 Lemma. *Let $\alpha \in \mathbb{R}$ and $\omega \in \mathbb{C} \setminus \{0\}$ and let $Q_{\alpha,\omega}$ be as in (3.1).*

(i) *We have $Q_{\alpha,\omega} \in \mathcal{N}_\kappa$ if and only if*

$$\big|\arg\big((-1)^\kappa\omega\big)\big| \le \frac{\pi}{2}\Big(1 - \big||\alpha| - 2\kappa\big|\Big). \tag{3.9}$$

This is possible only when

$$\kappa = \begin{cases} \big\lfloor \frac{|\alpha|+1}{2} \big\rfloor & \text{if } \alpha \notin 2\mathbb{Z}+1, \\ \frac{|\alpha|+1}{2} & \text{if } \alpha \in 2\mathbb{Z}+1 \text{ and } (-1)^{\frac{|\alpha|+1}{2}}\omega > 0, \\ \frac{|\alpha|-1}{2} & \text{if } \alpha \in 2\mathbb{Z}+1 \text{ and } (-1)^{\frac{|\alpha|+1}{2}}\omega < 0. \end{cases} \tag{3.10}$$

In particular, $\frac{|\alpha|-1}{2} \le \kappa \le \frac{|\alpha|+1}{2}$.

(ii) *Assume that $Q_{\alpha,\omega} \in \mathcal{N}_\kappa$, i.e. that (3.9) is satisfied. Then $Q_{\alpha,\omega} \in \mathcal{N}_\kappa^{(\infty)}$ if and only if $\alpha \ge -1$ and, in addition, $\omega > 0$ in the case when $\alpha = -1$.*

(iii) *Assume that* $Q_{\alpha,\omega} \in \mathcal{N}_\kappa^{(\infty)}$ *and let* $\mu_{\alpha,\omega}$ *be the measure component of* $\mathscr{C}_\kappa^{-1} Q_{\alpha,\omega}$*. If* $\alpha = -1$*, then* $\mu_{\alpha,\omega} = \omega\delta_0$*, where* δ_0 *is the Dirac measure at* 0*. If* $\alpha > -1$*, then* $\mu_{\alpha,\omega}$ *is absolutely continuous w.r.t. the Lebesgue measure and has density*

$$\frac{\mathrm{d}\mu_{\alpha,\omega}}{\mathrm{d}t}(t) = \frac{|\omega|}{\pi}|t|^\alpha \cos\Big(\frac{\alpha\pi}{2} - (\operatorname{sgn} t)\arg\omega\Big), \qquad a.e.\ t \in \mathbb{R}.$$

Proof.

(i) Write $\alpha = 2m + \alpha_0$ with $m \in \mathbb{Z}$ and $|\alpha_0| \le 1$ (note that if α is an odd integer, then m and α_0 are not unique). Then

$$Q_{\alpha,\omega}(z) = i\omega\Big(\frac{z}{i}\Big)^{2m}\Big(\frac{z}{i}\Big)^{\alpha_0} = z^{2m} q_0(z) \tag{3.11}$$

with

$$q_0(z) = i(-1)^m\omega\Big(\frac{z}{i}\Big)^{\alpha_0}.$$

Since the only generalised poles and zeros not of positive type (in the sense of [19]; see also [22]) can be 0 and ∞, it follows from [5, Corollary] or [4, Proposition 3.2 and Theorem 3.3] that $Q_{\alpha,\omega} \in \mathcal{N}_\kappa$ if and only if $|m| = \kappa$ and $q_0 \in \mathcal{N}_0$. Determining the sector onto which $\mathbb{C}^+$ is mapped under q_0 one can easily show that $q_0 \in \mathcal{N}_0$ if and only if

$$\big|\arg\big((-1)^m\omega\big)\big| \le \frac{\pi}{2}\big(1 - |\alpha_0|\big).$$

Since $|\alpha_0| = \big||\alpha| - |2m|\big| = \big||\alpha| - 2\kappa\big|$, the equivalence of $Q_{\alpha,\omega} \in \mathcal{N}_\kappa$ and (3.9) follows. The formula for κ can be derived easily from (3.9).

(ii) It follows from the factorisation (3.11) and [5, Corollary] that $q \in \mathcal{N}_\kappa^{(\infty)}$ if and only if $m \ge 0$.

(iii) If $\alpha = -1$ and $\omega > 0$, then $Q_{\alpha,\omega}(z) = -\frac{\omega}{z}$ and hence $Q_{\alpha,\omega} \in \mathcal{N}_0 = \mathcal{N}_0^{(\infty)}$ and $\mu_{\alpha,\omega} = \omega\delta_0$. Assume that $\alpha > -1$. The function $Q_{\alpha,\omega}$ can be extended to a continuous function on $(\mathbb{C}^+ \cup \mathbb{R}) \setminus \{0\}$, and, for $t > 0$, we have

$$\begin{aligned}\operatorname{Im} Q_{\alpha,\omega}(\pm t) &= \operatorname{Im}\big[i\omega(\mp it)^\alpha\big] = \operatorname{Im}\Big[|\omega|t^\alpha e^{i(\frac{\pi}{2} + \arg\omega \mp \alpha\frac{\pi}{2})}\Big] \\ &= |\omega|t^\alpha \sin\Big(\frac{\pi}{2} + \arg\omega \mp \frac{\alpha\pi}{2}\Big) = |\omega|t^\alpha\cos\Big(\arg\omega \mp \frac{\alpha\pi}{2}\Big).\end{aligned}$$

Now the assertion follows from the Stieltjes inversion formula (2.17). Note that there is no point mass at 0 if $\alpha > -1$. ❑

3.5 Remark. It follows from (3.9) and (3.10) that $Q_{\alpha,\omega} \in \mathcal{N}_{<\infty}$ if and only if

$$\alpha \notin 2\mathbb{Z} + 1 \quad \text{and} \quad \big|\arg\big((-1)^{\lfloor\frac{|\alpha|+1}{2}\rfloor}\omega\big)\big| \le \frac{\pi}{2}\operatorname{dist}(\alpha, 2\mathbb{Z}+1) \tag{3.12}$$

or

$$\alpha \in 2\mathbb{Z} + 1 \quad \text{and} \quad \omega \in \mathbb{R}. \tag{3.13}$$

◊

The significance of regular variation is that having a regularly varying asymptotics for $q(rz)$ for one single point z already suffices to get locally uniform asymptotics depending on z as a power. The reason for this is the multiplicative nature of the argument in $q(rz)$. In the following lemma we use the standard notation $z_0 M = \{z_0 z\colon z \in M\}$ with $M \subseteq \mathbb{R}$.

3.6 Lemma. *Let* $z_0 \in \mathbb{C}^+$*,* $r_0 > 0$ *and let* $q : z_0(r_0,\infty) \to \mathbb{C}\setminus\{0\}$ *be a continuous function. Further, let* $f : [x_0,\infty) \to (0,\infty)$ *with* $x_0 > 0$ *be a measurable function and let* $B \subseteq (r_0,\infty)$ *be a set with positive Lebesgue measure. Assume that*

$$\forall z \in z_0 B: \lim_{r\to\infty}\frac{q(rz)}{f(r)} \quad \textit{exists and is non-zero.}$$

Then there exist $\alpha \in \mathbb{R}$ and $\omega \in \mathbb{C} \setminus \{0\}$ such that f is regularly varying with index α and

$$\forall z \in z_0(0,\infty): \lim_{r\to\infty} \frac{q(rz)}{f(r)} = Q_{\alpha,\omega}(z), \tag{3.14}$$

where $Q_{\alpha,\omega}$ is as in (3.1).

Proof. Set $\mathring{q}(z) := \lim_{r\to\infty} \frac{q(rz)}{f(r)}$, $z \in z_0B$. Choose $s_0 \in B$. For every $\lambda \in \frac{1}{s_0}B$ we have $\lambda s_0 z_0 \in z_0 B$ and $s_0 z_0 \in z_0 B$ and hence

$$\lim_{r\to\infty} \frac{q(r \cdot \lambda s_0 z_0)}{f(r)} = \mathring{q}(\lambda s_0 z_0), \qquad \lim_{r\to\infty} \frac{q(\lambda r \cdot s_0 z_0)}{f(\lambda r)} = \mathring{q}(s_0 z_0).$$

Taking quotients of these equations we obtain

$$\lim_{r\to\infty} \frac{f(\lambda r)}{f(r)} = \frac{\mathring{q}(\lambda s_0 z_0)}{\mathring{q}(s_0 z_0)}.$$

Since the set $\frac{1}{s_0}B$ has positive measure, the Characterisation Theorem [3, Theorem 1.4.1] yields that f is regularly varying with index, say, $\alpha \in \mathbb{R}$. Hence, for every $\lambda > 0$ we have

$$\lim_{r\to\infty} \frac{q(r \cdot \lambda s_0 z_0)}{f(r)} = \lim_{r\to\infty} \frac{q(\lambda r \cdot s_0 z_0)}{f(\lambda r)} \cdot \lim_{r\to\infty} \frac{f(\lambda r)}{f(r)} = \mathring{q}(s_0 z_0)\lambda^\alpha.$$

Replacing $\lambda s_0 z_0$ by z we obtain that, for every $z \in z_0(0,\infty)$,

$$\lim_{r\to\infty} \frac{q(rz)}{f(r)} = \mathring{q}(s_0 z_0)\Big(\frac{z}{s_0 z_0}\Big)^\alpha,$$

which implies (3.14). ❑

3.7 Lemma. *Let $q \in \operatorname{ran} \mathscr{C}_\kappa$, $r > 0$, and consider the function $q_{\langle r\rangle}(z) := q(rz)$. Then the following statements hold.*

(i) $q_{\langle r\rangle} \in \operatorname{ran} \mathscr{C}_\kappa$.

(ii) *The measure components μ and $\mu_{\langle r\rangle}$ of $\mathscr{C}_\kappa^{-1}(q)$ and $\mathscr{C}_\kappa^{-1}(q_{\langle r\rangle})$, respectively, are related by*

$$\mu_{\langle r\rangle} = \tfrac{1}{r}\Sigma^r_*\mu,$$

where $\Sigma^r_*\mu$ is the push-forward of μ under the map $\Sigma^r\colon t \mapsto \frac{1}{r}t$, i.e. $\mu_{\langle r\rangle}(M) = \frac{1}{r}\mu(rM)$ for a measurable set $M \subseteq \mathbb{R}$.

Proof. The statement in (i) is obvious from Theorem 2.8. Write $q = \mathscr{C}_\kappa[\mu, p]$ and $q_{\langle r\rangle} = \mathscr{C}_\kappa[\mu_{\langle r\rangle}, p_{\langle r\rangle}]$. Making a change of variable ($t = rs$) we obtain

$$\begin{aligned} q_{\langle r\rangle}(z) = q(rz) &= p(rz) + \big(1+(rz)^2\big)^{\kappa+1} \int_{\mathbb{R}} \frac{1}{t - rz} \cdot \frac{\mathrm{d}\mu(t)}{(1+t^2)^{\kappa+1}} \\ &= p(rz) + \big(1+(rz)^2\big)^{\kappa+1} \cdot \int_{\mathbb{R}} \frac{1}{s-z} \cdot \frac{\frac{1}{r}\,\mathrm{d}(\Sigma^r_*\mu)(s)}{(1+(rs)^2)^{\kappa+1}}. \end{aligned}$$

Extend $q_{\langle r\rangle}$ to $\mathbb{C}^+ \cup \mathbb{C}^-$ by symmetry: $q_{\langle r\rangle}(z) := \overline{q_{\langle r\rangle}(\overline{z})}$, $z \in \mathbb{C}^-$, and let $\Gamma^\delta_\varepsilon$ be the path in (2.14). The Stieltjes inversion formula (2.17) and the Stieltjes–Livšic inversion formula (2.15) yield that, for all $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$, we have

$$\begin{aligned} \mu_{\langle r\rangle}\big((\alpha,\beta)\big) &= \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0} \frac{1}{\pi} \int_{\alpha+\delta}^{\beta-\delta} \operatorname{Im} q_{\langle r\rangle}(t+i\varepsilon)\,\mathrm{d}t = \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0} \frac{-1}{2\pi i} \int_{\Gamma^\delta_\varepsilon} q_{\langle r\rangle}(z)\,\mathrm{d}z \\ &= \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0} \frac{-1}{2\pi i} \int_{\Gamma^\delta_\varepsilon} \big(1+(rz)^2\big)^{\kappa+1} \bigg[\int_{\mathbb{R}} \frac{1}{s-z} \cdot \frac{\frac{1}{r}\,\mathrm{d}(\Sigma^r_*\mu)(s)}{(1+(rs)^2)^{\kappa+1}}\bigg]\,\mathrm{d}z \\ &= \int_{(\alpha,\beta)} \big(1+(rs)^2\big)^{\kappa+1} \cdot \frac{\frac{1}{r}\,\mathrm{d}(\Sigma^r_*\mu)(s)}{(1+(rs)^2)^{\kappa+1}} = \tfrac{1}{r}(\Sigma^r_*\mu)\big((a,b)\big). \end{aligned}$$

❑

Proof of Theorem 3.1. The implication (iii) ⇒ (ii) is trivial.

Next let us show the implication (ii) ⇒ (i). For each $s > 0$ the limit

$$\lim_{r\to\infty} \frac{q(r\cdot sz_0)}{f(r)} = \lim_{r\to\infty} \left(\frac{q(rs\cdot z_0)}{f(rs)} \cdot \frac{f(rs)}{f(r)} \right) = \lim_{t\to\infty} \frac{q(tz_0)}{f(t)} \cdot \lim_{r\to\infty} \frac{f(rs)}{f(r)}$$

exists and is non-zero. Hence (i) is satisfied with $M = \{sz_0\colon\ s \in (0,\infty)\}$.

Finally, we prove the implication (i) ⇒ (iii). Let $(r_n)_{n\in\mathbb{N}}$ be an arbitrary sequence of positive numbers with $r_n \to \infty$, and set

$$q_n(z) := \frac{q(r_n z)}{f(r_n)}, \qquad z \in \mathbb{C}^+. \tag{3.15}$$

It is easy to see that $q_n \in \mathcal{N}_{\le\kappa}$. Since q, as a function from $\mathcal{N}_{\le\kappa}$, has only finitely many poles in $\mathbb{C}^+$, assumption (i) in Proposition 2.11 is satisfied. By (3.2) also assumption (ii) in Proposition 2.11 is fulfilled. Hence, the latter proposition implies that there exists $\mathring{q} \in \mathcal{N}_{\le\kappa} \cap \mathrm{Hol}(\mathbb{C}^+)$ such that $\lim_{n\to\infty} q_n = \mathring{q}$ locally uniformly in $\mathbb{C}^+$. Since the sequence $(r_n)_{n\in\mathbb{N}}$ was arbitrary, it follows again from (3.2) that

$$\lim_{r\to\infty} \frac{q(rz)}{f(r)} = \mathring{q}(z)$$

locally uniformly for $z \in \mathbb{C}^+$. Now Lemma 3.6 implies that there exist $\alpha \in \mathbb{R}$ and $\omega \in \mathbb{C}\setminus\{0\}$ such that f is regularly varying with index α and that $\mathring{q} = Q_{\alpha,\omega}$. Since $\mathcal{N}_{\le\kappa}$ is closed under locally uniform convergence, we have $Q_{\alpha,\omega} \in \mathcal{N}_{\kappa'}$ with some $\kappa' \le \kappa$. By Lemma 3.4 this shows that $|\alpha| \le 2\kappa' + 1 \le 2\kappa + 1$ and that (3.4) holds. ❑

Proof of Theorem 3.2. Let $(r_n)_{n\in\mathbb{N}}$ be an arbitrary sequence of positive numbers with $r_n \to \infty$, and define q_n as in (3.15). It follows from Lemma 3.7 that $q_n \in \operatorname{ran}\mathscr{C}_\kappa$ and that the measure component μ_n of $\mathscr{C}_\kappa^{-1} q_n$ satisfies

$$\mu_n\big((a,b)\big) = \frac{1}{r_n f(r_n)} \mu\big((r_n a, r_n b)\big) \tag{3.16}$$

for all $a, b \in \mathbb{R}$ with $a < b$. Theorem 2.12 implies that $Q_{\alpha,\omega} \in \operatorname{ran}\mathscr{C}_\kappa = \mathcal{N}_{\le\kappa}^{(\infty)}$, which, by Lemma 3.4 (ii) shows that $\alpha \ge -1$. Let $\mu_{\alpha,\omega}$ be the measure component of $\mathscr{C}_\kappa^{-1} Q_{\alpha,\omega}$. Further, let $a, b \in \mathbb{R}$ be such that $a < b$ and $\mu_{\alpha,\omega}(\{a\}) = \mu_{\alpha,\omega}(\{b\}) = 0$. It follows from Theorem 2.12 that

$$\lim_{n\to\infty} \mu_n\big((a,b)\big) = \mu_{\alpha,\omega}\big((a,b)\big). \tag{3.17}$$

Since the sequence $(r_n)_{n\in\mathbb{N}}$ was arbitrary, relations (3.16) and (3.17) imply (3.5). We obtain from Lemma 3.4 (iii) that

$$\mu_{\alpha,\omega}\big(\pm(0,1)\big) = \frac{1}{\pi} \cdot \frac{|\omega|}{\alpha+1} \cos\Big(\frac{\alpha\pi}{2} \mp \arg\omega\Big)$$

if $\alpha > -1$ and $\mu_{\alpha,\omega}((-1,1)) = \omega$ if $\alpha = -1$. This, combined with (3.5), yields (3.6)–(3.8). ❑

4 The Abelian direction

In this section we consider Abelian theorems, i.e. we study the asymptotic behaviour of q at infinity using some knowledge about the asymptotic behaviour of the distribution functions $t \mapsto \mu([0,t))$ and $t \mapsto \mu((-t,0))$. Some of the theorems contain also a Tauberian direction, which complement the results in Section 3. The main tool is Karamata's theorem about Stieltjes transforms of measures that are supported on the half-axis $[0,\infty)$. We follow the common lines to pass from unilateral to bilateral theorems, e.g. [28, 30], and represent imaginary and real parts as Stieltjes transforms. To this end we use the push-forward measure μ_* of μ under the map $t \mapsto t^2$, which satisfies

$$\mu_*((-\infty,0)) = 0 \qquad \text{and} \qquad \mu_*([0,t^2)) = \mu((-t,t)), \quad t>0. \tag{4.1}$$

We shall often use a substitution to change between μ and μ_*; let us note that, for a non-negative, measurable function h on $[0,\infty)$, we have $\int_{[0,\infty)} h(s)\,\mathrm{d}\mu_*(s) = \int_{\mathbb{R}} h(t^2)\,\mathrm{d}\mu(t)$.

In order to apply Karamata's theorem in an effective way, we need a finer classification of the growth properties of the positive measure μ, namely, let us set

$$\mathscr{p}(\mu) := \inf\biggl\{ n\in\mathbb{N}\colon \int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{(1+|t|)^n} < \infty \biggr\} \in \mathbb{N}\cup\{\infty\}. \tag{4.2}$$

In the Abelian theorems we often assume that the symmetrised distribution function $t \mapsto \mu((-t,t))$ is regularly varying. The following lemma can be used to give a different characterisation of $\mathscr{p}(\mu)$ and its relation to the index of the regularly varying distribution function.

4.1 Lemma. *Let μ be a measure on $\mathbb{R}$. For $\gamma>0$ we have*

$$\int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{(1+|t|)^\gamma} < \infty \quad\Leftrightarrow\quad \int_1^\infty \frac{\mu((-t,t))}{t^{\gamma+1}}\,\mathrm{d}t < \infty.$$

If $t\mapsto \mu((-t,t))$ is regularly varying with index β, then $\mathscr{p}(\mu)$ is finite and $\beta\in[\mathscr{p}(\mu)-1, \mathscr{p}(\mu)]$.

Proof. Let the measure μ_* be defined as in (4.1), let $\gamma>0$ and define the measure ν on $[1,\infty)$ such that $\nu((t,\infty)) = t^{-\frac{\gamma}{2}}$. It follows from Lemma A.3 that the following equivalences hold:

$$\begin{aligned}
\int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{(1+|t|)^\gamma} < \infty \quad &\Leftrightarrow \quad \int_{[0,\infty)} \frac{\mathrm{d}\mu_*(s)}{(1+\sqrt{s})^\gamma} < \infty \quad \Leftrightarrow \quad \int_{[1,\infty)} s^{-\frac{\gamma}{2}}\,\mathrm{d}\mu_*(s) < \infty \\
&\Leftrightarrow \quad \int_1^\infty s^{-\frac{\gamma}{2}-1}\mu_*([1,s))\,\mathrm{d}s < \infty \qquad\qquad (4.3)\\
&\Leftrightarrow \quad \int_1^\infty \frac{\mu((-t,t))}{t^{\gamma+1}}\,\mathrm{d}t < \infty,
\end{aligned}$$

where in the last step we used the substitution $s=t^2$.

Now assume that $t\mapsto \mu((-t,t))$ is regularly varying with index β. Then $s\mapsto \mu_*[1,s)$ is regularly varying with index $\frac{\beta}{2}$. It is clear that the integral in (4.3) is finite if γ is large enough, which shows that $\mathscr{p}(\mu)$ is finite. Further, the fact that the integral in (4.3) is finite for $\gamma=\mathscr{p}(\mu)$ and infinite for $\gamma=\mathscr{p}(\mu)-1$ (unless μ itself is finite) implies that $\frac{\beta}{2}-\frac{\mathscr{p}(\mu)}{2}\le 0$ and $\frac{\beta}{2}-\frac{\mathscr{p}(\mu)-1}{2}\ge 0$, which finishes the proof; cf. Proposition A.4. Note that, when μ is finite, then $\beta=0$ and $\mathscr{p}(\mu)=1$. ❑

For the Abelian theorems we treat real and imaginary parts of $q(iy)$ separately as they have different representations in terms of Stieltjes transforms. This is done in the following two subsections.

4.1 The imaginary part

The imaginary part of $q(iy)$ is relatively well behaved as it can be written in terms of one Stieltjes transform. In order to apply Karamata's theorem, we choose κ minimal in (2.6) for a given measure μ. To this end, let us define

$$\kappa(\mu) := \inf\biggl\{ n\in\mathbb{N}_0\colon \int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{(1+|t|)^{2n+2}} < \infty \biggr\} < \infty. \tag{4.4}$$

Comparing (4.4) with (4.2) we can easily deduce that

$$\kappa(\mu) = \Big\lfloor \frac{\mathfrak{p}(\mu)-1}{2} \Big\rfloor. \tag{4.5}$$

Throughout this section we suppose that the following assumption is satisfied.

4.2 Assumption. Let μ be a measure on $\mathbb{R}$ such that $\mathfrak{p}(\mu) < \infty$ and set $\kappa := \kappa(\mu)$. Further, let $p \in \mathbb{R}[z]$ with $p(z) = c_{2\kappa+1} z^{2\kappa+1} + \ldots + c_0$ such that $(\mu, p) \in \mathbb{E}_{\leq\kappa}$, and set $q := \mathscr{C}_\kappa[\mu, p]$. ♢

Note that, by Definition 2.3 (i), we have $c_{2\kappa+1} \geq \int_\mathbb{R} (1+t^2)^{-(\kappa+1)} \mathrm{d}\mu(t)$. The two cases, equality and strict inequality, lead to different asymptotic behaviour of $\operatorname{Im} q(iy)$ as $y \to \infty$, as the next proposition shows. Naturally, the case when the polynomial dominates the integral is the simpler one.

4.3 Proposition. *Let μ, κ, p and q be as in Assumption 4.2.*

(i) *If $c_{2\kappa+1} > \int_\mathbb{R} (1+t^2)^{-(\kappa+1)} \mathrm{d}\mu(t)$, then*

$$q(iy) \sim i(-1)^\kappa \bigg(c_{2\kappa+1} - \int_\mathbb{R} \frac{\mathrm{d}\mu(t)}{(1+t^2)^{\kappa+1}} \bigg) y^{2\kappa+1}, \qquad y \to \infty.$$

(ii) *If $c_{2\kappa+1} = \int_\mathbb{R} (1+t^2)^{-(\kappa+1)} \mathrm{d}\mu(t)$, then*

$$|q(iy)| \ll y^{2\kappa+1} \qquad \textit{and} \qquad \operatorname{Im} q(iy) \sim (-1)^\kappa y^{2\kappa+1} \int_\mathbb{R} \frac{1}{t^2+y^2} \cdot \frac{\mathrm{d}\mu(t)}{(1+t^2)^\kappa}, \tag{4.6}$$

as $y \to \infty$.

(iii) *If μ is an infinite measure, then*

$$(-1)^\kappa \operatorname{Im} q(iy) \gg y^{2\kappa-1}, \qquad y \to \infty. \tag{4.7}$$

Proof.

① From (2.12) we obtain

$$\begin{aligned} q(iy) &= p(iy) + iy(1-y^2)^\kappa \int_\mathbb{R} \frac{\mathrm{d}\mu(t)}{(1+t^2)^{\kappa+1}} + (1-y^2)^\kappa \int_\mathbb{R} \bigg(\frac{1}{t-iy} - \frac{t}{1+t^2} \bigg) \frac{\mathrm{d}\mu(t)}{(1+t^2)^\kappa} \\ &= i(-1)^\kappa \bigg(c_{2\kappa+1} - \int_\mathbb{R} \frac{\mathrm{d}\mu(t)}{(1+t^2)^{\kappa+1}} \bigg) y^{2\kappa+1} + c_{2\kappa}(-1)^\kappa y^{2\kappa} + \mathrm{O}\big(y^{2\kappa-1}\big) \\ &\quad + (-1)^\kappa \Big(y^{2\kappa} + \mathrm{O}\big(y^{2\kappa-2}\big) \Big) \int_\mathbb{R} \bigg(\frac{1}{t-iy} - \frac{t}{1+t^2} \bigg) \frac{\mathrm{d}\mu(t)}{(1+t^2)^\kappa}. \end{aligned} \tag{4.8}$$

Together with (2.4), this proves the assertion in (i), relation (4.7) when $c_{2\kappa+1} > \int_\mathbb{R} (1+t^2)^{-(\kappa+1)} \mathrm{d}\mu(t)$, and the first relation in (4.6) when $c_{2\kappa+1} = \int_\mathbb{R} (1+t^2)^{-(\kappa+1)} \mathrm{d}\mu(t)$.

② For the rest of the proof assume that $c_{2\kappa+1} = \int_\mathbb{R} (1+t^2)^{-(\kappa+1)} \mathrm{d}\mu(t)$. If the measure μ is finite, then $\kappa = 0$ and

$$\operatorname{Im} q(iy) = \operatorname{Im} \int_\mathbb{R} \frac{1}{t-iy} \, \mathrm{d}\mu(t) = y \int_\mathbb{R} \frac{1}{t^2+y^2} \, \mathrm{d}\mu(t).$$

③ Let us now consider the case when μ is infinite. We can use (4.8) to write the imaginary part as

$$\operatorname{Im} q(iy) = (-1)^\kappa \Big(y^{2\kappa} + \mathrm{O}\big(y^{2\kappa-2}\big) \Big) \operatorname{Im} \bigg[\int_\mathbb{R} \bigg(\frac{1}{t-iy} - \frac{t}{1+t^2} \bigg) \frac{\mathrm{d}\mu(t)}{(1+t^2)^\kappa} \bigg] + \mathrm{O}\big(y^{2\kappa-1}\big).$$

By the definition of κ we have $\int_{\mathbb{R}}(1+t^2)^{-\kappa}\mathrm{d}\mu(t)=\infty$. Hence (2.5) implies that

$$\operatorname{Im}\left[\int_{\mathbb{R}}\left(\frac{1}{t-iy}-\frac{t}{1+t^2}\right)\frac{\mathrm{d}\mu(t)}{(1+t^2)^{\kappa}}\right]\gg\frac{1}{y},$$

from which the second relation in (4.6) follows.

❑

In the following we assume that the leading asymptotics of $q(iy)$ is not given by a polynomial term but the measure μ. More precisely, we suppose that the following assumption is satisfied.

4.4 Assumption. Let μ, κ, p and q satisfy the conditions in Assumption 4.2. Further, assume that

$$c_{2\kappa+1}=\int_{\mathbb{R}}\frac{\mathrm{d}\mu(t)}{(1+t^2)^{\kappa+1}}. \tag{4.9}$$

◊

4.5 Remark. Suppose that Assumption 4.2 holds and that $\kappa=\kappa(\mu)=0$, with κ defined in (4.4). According to (2.11) we can write

$$q(z)=\mathscr{C}_0[\mu,p]=\left(c_1-\int_{\mathbb{R}}\frac{\mathrm{d}\mu(t)}{1+t^2}\right)z+c_0+\int_{\mathbb{R}}\left(\frac{1}{t-z}-\frac{t}{1+t^2}\right)\mathrm{d}\mu(t). \tag{4.10}$$

Assumption 4.4 is equivalent to the coefficient of z in (4.10) vanishing. Hence with Assumption 4.4 being satisfied we have

$$q(z)=c_0+\int_{\mathbb{R}}\left(\frac{1}{t-z}-\frac{t}{1+t^2}\right)\mathrm{d}\mu(t)=c_0+\widetilde{C}[\mu](z), \tag{4.11}$$

where $\widetilde{C}[\mu]$ is defined in (1.5). ◊

The next lemma shows that the imaginary part of $q(iy)$ can be written — at least asymptotically — in terms of a Stieltjes transform. For the definition of the Stieltjes transform see (A.7).

4.6 Lemma. *Let p, κ, p and q satisfy the conditions in Assumption 4.2 and Assumption 4.4. Further, let μ_* be the push-forward measure of μ as in (4.1) and define the measure τ_κ on $[0,\infty)$ by*

$$\mathrm{d}\tau_\kappa(s)=\frac{\mathrm{d}\mu_*(s)}{(1+s)^{\kappa}},\qquad s\in[0,\infty). \tag{4.12}$$

Then the Stieltjes transform $\mathcal{S}[\tau_\kappa]$ is well defined and

$$\operatorname{Im}q(iy)\sim(-1)^{\kappa}y^{2\kappa+1}\mathcal{S}[\tau_\kappa](y^2),\qquad y\to\infty. \tag{4.13}$$

Proof. Since

$$\int_{[0,\infty)}\frac{\mathrm{d}\tau_\kappa(s)}{1+s}=\int_{[0,\infty)}\frac{\mathrm{d}\mu_*(s)}{(1+s)^{\kappa+1}}=\int_{\mathbb{R}}\frac{\mathrm{d}\mu(t)}{(1+t^2)^{\kappa+1}}<\infty,$$

the Stieltjes transform $\mathcal{S}[\tau_\kappa]$ is well defined. It follows from Proposition 4.3 (ii) that

$$\begin{aligned}\operatorname{Im}q(iy)&\sim(-1)^{\kappa}y^{2\kappa+1}\int_{\mathbb{R}}\frac{1}{t^2+y^2}\cdot\frac{\mathrm{d}\mu(t)}{(1+t^2)^{\kappa}}\\&=(-1)^{\kappa}y^{2\kappa+1}\int_{[0,\infty)}\frac{1}{s+y^2}\cdot\frac{\mathrm{d}\mu_*(s)}{(1+s)^{\kappa}}\\&=(-1)^{\kappa}y^{2\kappa+1}\int_{[0,\infty)}\frac{1}{s+y^2}\,\mathrm{d}\tau_\kappa(s)=(-1)^{\kappa}y^{2\kappa+1}\mathcal{S}[\tau_\kappa](y^2),\end{aligned}$$

which proves (4.13). ❑

In the following theorem we prove that the imaginary part of $q(iy)$ is related to the symmetric distribution function $\mu((-t,t))$ of the measure μ. In most cases $|\operatorname{Im} q(iy)|$ is regularly varying if and only if $t \mapsto \mu((-t,t))$ is regularly varying.

4.7 Theorem. *Let μ, κ, p and q satisfy the conditions in Assumptions 4.2 and 4.4. Further, let $\beta \geq 0$ and consider the following two statements:*

(a) *the symmetrised distribution function $t \mapsto \mu((-t,t))$ is regularly varying with index β;*

(b) *the function $y \mapsto (-1)^\kappa \operatorname{Im} q(iy)$ is regularly varying with index $\beta - 1$.*

Then we have the following relations.

(i) *The implication* (a) $\Rightarrow$ (b) *holds.*

(ii) *Unless $\kappa > 0$ and $\beta = 2\kappa$, also* (b) $\Rightarrow$ (a) *holds.*

(iii) *Assume that* (a) *and* (b) *are satisfied. Then $\beta \in [2\kappa, 2\kappa+2]$ and*

$$\operatorname{Im} q(iy) \sim \begin{cases} \dfrac{\frac{\pi\beta}{2}}{\sin\frac{\pi\beta}{2}} \cdot \dfrac{\mu((-y,y))}{y}, & \beta \in [0,\infty) \setminus \{2,4,\ldots\}, \\ (-1)^\kappa \beta y^{\beta-1} \displaystyle\int_1^y \frac{\mu((-t,t))}{t^{\beta+1}}\,\mathrm{d}t, & \beta = 2\kappa \ \wedge\ \kappa > 0, \\ (-1)^\kappa \beta y^{\beta-1} \displaystyle\int_y^\infty \frac{\mu((-t,t))}{t^{\beta+1}}\,\mathrm{d}t, & \beta = 2\kappa + 2, \end{cases} \tag{4.14}$$

as $y \to \infty$, where the first fraction in the first case on the right-hand side is understood as 1 *when $\beta = 0$. In particular, if $\beta \in 2\mathbb{N}$, then*

$$(-1)^\kappa \operatorname{Im} q(iy) \gg \frac{\mu((-y,y))}{y}. \tag{4.15}$$

4.8 Remark. In the situation of Theorem 4.7 assume that (a) is satisfied and that $\beta \in 2\mathbb{N}$. It follows from the definition of κ in (4.4) and from Lemma 4.1 that

$$\beta = 2\kappa \quad \Leftrightarrow \quad \int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{(1+|t|)^\beta} = \infty \quad \Leftrightarrow \quad \int_1^\infty \frac{\mu((-t,t))}{t^{\beta+1}}\,\mathrm{d}t = \infty,$$

$$\beta = 2\kappa + 2 \quad \Leftrightarrow \quad \int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{(1+|t|)^\beta} < \infty \quad \Leftrightarrow \quad \int_1^\infty \frac{\mu((-t,t))}{t^{\beta+1}}\,\mathrm{d}t < \infty.$$

$\Diamond$

Proof of Theorem 4.7. Let μ_* be the push-forward measure of μ as in (4.1) and define τ_κ as in (4.12). Then (4.13) holds. We prove the theorem in several steps.

① Let us first consider the case when $\kappa = 0$. Then $\tau_0 = \mu_*$ and, by Theorem A.7, we have the following equivalences:

$$\begin{aligned} \text{(a)} \quad &\Leftrightarrow \quad s \mapsto \mu_*([0,s)) \text{ is regularly varying with index } \tfrac{\beta}{2} \\ &\Leftrightarrow \quad \mathcal{S}[\mu_*] \text{ is regularly varying with index } \tfrac{\beta}{2} - 1 \\ &\Leftrightarrow \quad \text{(b)}. \end{aligned}$$

Assume now that (a) and (b) hold. Another application of Theorem A.7 implies that $\frac{\beta}{2} \in [0,1]$. When $\beta \in [0,2)$, Remark A.8 yields

$$\operatorname{Im} q(iy) = y\,\mathcal{S}[\mu_*](y^2) \sim y \frac{\frac{\pi\beta}{2}}{\sin\frac{\pi\beta}{2}} \cdot \frac{\mu_*([0,y^2))}{y^2} \sim \frac{\frac{\pi\beta}{2}}{\sin\frac{\pi\beta}{2}} \cdot \frac{\mu((-y,y))}{y}.$$

When $\beta = 2$, we obtain from Theorem A.7 and the substitution $s = t^2$ that

$$\operatorname{Im} q(iy) = y\,\mathcal{S}[\mu_*](y^2) \sim y \int_{y^2}^{\infty} \frac{\mu_*([0,s))}{s^2}\,\mathrm{d}s = 2y \int_y^{\infty} \frac{\mu((-t,t))}{t^3}\,\mathrm{d}t,$$

which proves (4.14) when $\kappa = 0$.

② Now let us consider the case when $\kappa > 0$. Set $h(s) := \frac{1}{(1+s)^{\kappa}} \sim s^{-\kappa}$, $s \to \infty$. By the definition of κ we have

$$\int_{[0,\infty)} h(s)\,\mathrm{d}\mu_*(s) = \int_{[0,\infty)} \frac{\mathrm{d}\mu_*(s)}{(1+s)^{\kappa}} = \int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{(1+t^2)^{\kappa}} = \infty,$$

$$\int_{[0,\infty)} \frac{h(s)}{1+s}\,\mathrm{d}\mu_*(s) = \int_{[0,\infty)} \frac{\mathrm{d}\mu_*(s)}{(1+s)^{\kappa+1}} = \int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{(1+t^2)^{\kappa+1}} < \infty,$$

which shows that (A.20) with $\nu = \mu_*$ is satisfied.

③ Assume that $\kappa > 0$ and (a) holds. Then $s \mapsto \mu_*([0,s))$ is regularly varying with index $\frac{\beta}{2}$. By Proposition A.11 (i) with $\alpha = \frac{\beta}{2}$ and $\gamma = -\kappa$ we have $\alpha + \gamma = \frac{\beta}{2} - \kappa \in [0,1]$ and hence $\beta > 0$. Therefore we can apply Proposition A.11 (i) again to obtain that $y \mapsto y^{2\kappa+1}\mathcal{S}[\tau_\kappa](y^2)$ is regularly varying with index $2\kappa + 1 + 2\big(\frac{\beta}{2} - \kappa - 1\big) = \beta - 1$, i.e. (b) holds.

If $\beta \notin 2\mathbb{N}$, then $\alpha + \gamma = \frac{\beta}{2} - \kappa \in (0,1)$ and (A.22) implies that

$$\begin{aligned}
\operatorname{Im} q(iy) &\sim (-1)^{\kappa} y^{2\kappa+1} \mathcal{S}[\tau_\kappa](y^2) \\
&\sim (-1)^{\kappa} y^{2\kappa+1} \frac{\pi \frac{\beta}{2}}{\sin\big(\pi\big(\frac{\beta}{2} - \kappa\big)\big)} \cdot \big(y^2\big)^{-\kappa+1} \mu_*([0,y)) \\
&= \frac{\frac{\pi\beta}{2}}{\sin\frac{\pi\beta}{2}} \cdot \frac{\mu((-y,y))}{y}.
\end{aligned}$$

Now assume that $\beta \in 2\mathbb{N}$. The relation $\frac{\beta}{2} - \kappa \in [0,1]$ implies that either $\kappa = \frac{\beta}{2}$ or $\kappa = \frac{\beta}{2} - 1$. Let us consider the former case; the other case is similar. It follows again from (A.22) that, with the substitution $s = t^2$,

$$\begin{aligned}
\operatorname{Im} q(iy) &\sim (-1)^{\kappa} y^{2\kappa+1} \mathcal{S}[\tau_\kappa](y^2) \\
&\sim (-1)^{\kappa} y^{2\kappa+1} \cdot \frac{\beta}{2} \cdot \frac{1}{y^2} \int_1^{y^2} s^{-\kappa-1} \mu_*([0,s))\,\mathrm{d}s \\
&= (-1)^{\kappa} \beta y^{2\kappa-1} \int_1^{y} \frac{\mu_*([0,t^2))}{t^{2\kappa+1}}\,\mathrm{d}t = (-1)^{\kappa} \beta y^{\beta-1} \int_1^{y} \frac{\mu((-t,t))}{t^{\beta+1}}\,\mathrm{d}t,
\end{aligned}$$

which proves (4.14) also in the case $\kappa > 0$.

④ Now assume that $\kappa > 0$, $\beta \neq 2\kappa$ and (b) holds. It follows from (4.13) that $\mathcal{S}[\tau_\kappa]$ is regularly varying. Since $\alpha + \gamma = \frac{\beta}{2} - \kappa > 0$, we can use Proposition A.11 (ii) to deduce that $s \mapsto \mu_*([0,s))$ is regularly varying, which implies that (a) holds.

⑤ Finally, assume that $\beta \in 2\mathbb{N}$. Relation (4.15) follows from (4.14) and Theorem A.2.

❏

The following example shows that the implication (b) ⇒ (a) is, in general, not valid when $\beta = 2\kappa > 0$.

4.9 Example. Let $\kappa \in \mathbb{N}$ and set $h(s) = \frac{1}{(1+s)^\kappa}$, $s \in [0,\infty)$. Choose the measures σ and ν as in Example A.12, and let μ be the symmetric measure on $\mathbb{R}$ such that $\mu_* = \nu$, i.e. μ is the discrete measure with point masses

$$\mu(\{e^{n/2}\}) = \mu(\{-e^{n/2}\}) = \nu(\{e^n\}) = (1+e^n)^\kappa, \qquad n \in \mathbb{N}.$$

According to Example A.12 the distribution function $t \mapsto \mu((-t,t)) = \nu([0,t^2))$ is not regularly varying, which means that (a) does not hold. On the other hand, $t \mapsto \tau_\kappa([0,t)) = \sigma([0,t))$ is slowly varying, again by Example A.12. It follows from Theorem A.7 that $\mathcal{S}[\tau_\kappa]$ is regularly varying with index -1, and hence (b) holds with $\beta = 2\kappa$; see (4.13). Note that we have

$$\operatorname{Im} q(iy) \sim 2(-1)^\kappa y^{2\kappa-1} \log y, \qquad y \to \infty,$$

by (A.27).

Since μ is symmetric, we have $\operatorname{Re} q(iy) = 0$ for $y > 0$. Hence (3.3) is satisfied with $z_0 = i$ and $f(r) = r^{2\kappa-1} \log r$. Theorem 3.2 implies that also (3.6) and (3.7) are satisfied. However, since $\omega = 2(-1)^\kappa \in \mathbb{R}$ and $\alpha = 2\kappa - 1$ is odd, the right-hand sides of (3.6) and (3.7) vanish. This shows that, in some cases, one cannot use Theorem 3.2 to deduce from the validity of (i)–(iii) in Theorem 3.1 that $t \mapsto \mu((-t,t))$ is regularly varying. ◊

4.10 Example. Let $a, b \geq 0$ with $a \neq b$, and consider the function

$$q(z) = a \log z - b \log(-z),$$

which belongs to the Nevanlinna class $\mathcal{N}_0$. Since $q(ri) = (a-b)\log r + i(a+b)\frac{\pi}{2}$, conditions (i)–(iii) in Theorem 3.1 are satisfied with $f(r) = \log r$, $\alpha = 0$ and $\omega = i(b-a)$. Let μ be the measure in the representation $q = \mathscr{C}_0[\mu, p]$. Theorem 3.2 only yields that

$$\lim_{r\to\infty} \frac{\mu((0,r))}{r \log r} = 0, \qquad \lim_{r\to\infty} \frac{\mu((-r,0))}{r \log r} = 0.$$

On the other hand, since $\operatorname{Im} q(ri) = (a+b)\frac{\pi}{2}$, we can apply Theorem 4.7 to obtain that $r \mapsto \mu((-r,r))$ is regularly varying and that

$$\mu((-r,r)) \sim \frac{2}{\pi} r \operatorname{Im} q(ir) = (a+b)r.$$

Note that, actually, $\mu((0,r)) = br$ and $\mu((-r,0)) = ar$. ◊

The next proposition shows that, in the case $\kappa = 0$, the validity of the first asymptotic relation in (4.14) implies already (a) and (b) in Theorem 4.7.

4.11 Proposition. *Let μ be a measure on $\mathbb{R}$ such that $\int_\mathbb{R} (1+t^2)^{-1} d\mu(t) < \infty$, let $c_0 \in \mathbb{R}$ and set $q := \widetilde{C}[\mu] + c_0$. Assume that the limit*

$$\lim_{y\to\infty} \left(\operatorname{Im} q(iy) \Big/ \frac{\mu((-y,y))}{y} \right)$$

exists and is positive. Then (a) *and* (b) *in Theorem 4.7 are satisfied.*

Proof. Let μ_* be as in (4.1). By (4.13), the following limit

$$\lim_{t\to\infty} \frac{\mu_*([0,t))}{t \mathcal{S}[\mu_*](t)} = \lim_{y\to\infty} \frac{\mu_*([0,y^2))}{y^2 \mathcal{S}[\mu_*](y^2)} = \lim_{y\to\infty} \frac{\mu((-y,y))}{y \operatorname{Im} q(iy)}$$

exists and is positive. Hence [32, Theorem B] implies that $t \mapsto \mu_*([0,t))$ is regularly varying, and hence (a) and (b) hold. ❑

4.2 The real part

The real part of $q(iy)$ is more subtle since it can be written only in terms of a difference of two Stieltjes transforms and cancellations can arise. We introduce the following notation for the main part of $\operatorname{Re} q(iy)$. Let μ be a measure on $\mathbb{R}$ such that there exists $\ell \in \mathbb{N}_0$ with

$$\int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{(1+|t|)^{2\ell+1}} < \infty, \tag{4.16}$$

and define

$$\mathsf{RC}_\ell[\mu](y) := (1-y^2)^\ell \int_{\mathbb{R}} \frac{t}{t^2+y^2} \cdot \frac{\mathrm{d}\mu(t)}{(1+t^2)^\ell}, \qquad y>0. \tag{4.17}$$

4.12 Lemma. *Let μ, κ, p and q be as in Assumption 4.2, and let $\wp(\mu)$ and RC_ℓ be as in (4.2) and (4.17) respectively.*

(i) *If $\wp(\mu)$ is odd, then there exists a real, even polynomial $\widetilde{p}$ of degree at most $2\kappa-2$ such that*

$$\operatorname{Re} q(iy) = (-1)^\kappa \left(c_{2\kappa} - \int_{\mathbb{R}} \frac{t}{(1+t^2)^{\kappa+1}} \,\mathrm{d}\mu(t) \right) y^{2\kappa} + \mathsf{RC}_\kappa[\mu](y) + \widetilde{p}(y). \tag{4.18}$$

(ii) *If $\wp(\mu)$ is even, then there exists a real, even polynomial $\widetilde{p}$ of degree at most 2κ such that*

$$\operatorname{Re} q(iy) = \mathsf{RC}_{\kappa+1}[\mu](y) + \widetilde{p}(y) \tag{4.19}$$

with $\widetilde{p}(y) = (-1)^\kappa c_{2\kappa} y^{2\kappa} + \mathrm{O}(y^{2\kappa-2})$.

Proof.
(i) When $\wp(\mu)$ is odd, then $\int_{\mathbb{R}} (1+|t|)^{-(2\kappa+1)} \mathrm{d}\mu(t) < \infty$. Hence from (2.13) we obtain

$$\operatorname{Re} q(iy) = \operatorname{Re}(p(iy)) - (1-y^2)^\kappa \int_{\mathbb{R}} \frac{t}{(1+t^2)^{\kappa+1}} \,\mathrm{d}\mu(t) + (1-y^2)^\kappa \int_{\mathbb{R}} \frac{t}{t^2+y^2} \cdot \frac{\mathrm{d}\mu(t)}{(1+t^2)^\kappa},$$

which yields (4.18).
(ii) When $\wp(\mu)$ is even, then we use (2.8) to write

$$\operatorname{Re} q(iy) = \operatorname{Re}(p(iy)) + (1-y^2)^{\kappa+1} \int_{\mathbb{R}} \frac{t}{t^2+y^2} \cdot \frac{\mathrm{d}\mu(t)}{(1+t^2)^{\kappa+1}},$$

which gives (4.19). ❑

4.13 Assumption. Let μ, κ, p and q be as in Assumption 4.2. Assume that, if $\wp(\mu)$ is odd, then

$$c_{2\kappa} = \int_{\mathbb{R}} \frac{t}{(1+t^2)^{\kappa+1}} \,\mathrm{d}\mu(t).$$

◊

4.14 Remark. Assume that $\wp(\mu) = 1$ and that Assumptions 4.2 and 4.4 are satisfied. We then have $\kappa = \kappa(\mu) = 0$ and, by Remark 4.5, we can write q as in (4.11), i.e. $q(z) = c_0 + \widetilde{C}[\mu](z)$. If, in addition, Assumption 4.13 is satisfied, then $c_0 = \int_{\mathbb{R}} \frac{t}{1+t^2} \,\mathrm{d}\mu(t)$ and hence q is the Cauchy transform of μ, i.e.

$$q(z) = \int_{\mathbb{R}} \frac{1}{t-z} \,\mathrm{d}\mu(t) = C[\mu](z),$$

where $C[\mu]$ is defined in (1.1). ◊

To investigate the behaviour of the real part, we often choose ℓ minimal so that (4.17) makes sense, i.e. let us set

$$\ell(\mu) := \inf\left\{ n \in \mathbb{N}_0 \colon \int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{(1+|t|)^{2n+1}} < \infty \right\}. \tag{4.20}$$

It follows easily that $\ell(\mu) = \left\lfloor \frac{\mathscr{p}(\mu)}{2} \right\rfloor$, where $\mathscr{p}(\mu)$ is as in (4.2).

4.15 Remark. Let μ be a measure on $\mathbb{R}$ and define $\mathscr{p}(\mu)$, $\kappa(\mu)$ and $\ell(\mu)$ as in (4.2), (4.4) and (4.20) respectively. If $\mathscr{p}(\mu)$ is odd, then $\kappa(\mu) = \ell(\mu) = \frac{\mathscr{p}(\mu)-1}{2}$. If $\mathscr{p}(\mu)$ is even, then $\kappa(\mu)+1 = \ell(\mu) = \frac{\mathscr{p}(\mu)}{2}$. In both cases we have $\kappa(\mu)+\ell(\mu) = \mathscr{p}(\mu)-1$. ♢

4.16 Lemma. *Let μ, κ, p and q be as in Assumption 4.2 and $\ell = \ell(\mu)$. Then there exists a real, even polynomial $\widetilde{p}$ of degree at most $2\ell-2$ such that*

$$\operatorname{Re} q(iy) = \mathsf{RC}_\ell[\mu](y) + \widetilde{p}(y).$$

Proof. If $\mathscr{p}(\mu)$ is odd, then $\ell(\mu) = \kappa(\mu)$ and the statement follows from Lemma 4.12 (i). If $\mathscr{p}(\mu)$ is even, then $\ell(\mu) = \kappa(\mu)+1$ and we can apply Lemma 4.12 (ii). ❑

4.17 Definition. Let μ be a measure on $\mathbb{R}$, let μ_* be the push-forward measure as in (4.1), and let $\ell \in \mathbb{N}_0$ be such that (4.16) is satisfied, i.e. $\ell \geq \ell(\mu)$. Define the measure σ_ℓ on $[0,\infty)$ by

$$\mathrm{d}\sigma_\ell(s) = \frac{\sqrt{s}}{(1+s)^\ell}\,\mathrm{d}\mu_*(s), \qquad s \in [0,\infty), \tag{4.21}$$

and set

$$\mathcal{F}_\ell[\mu](y) := (1-y^2)^\ell \mathcal{S}[\sigma_\ell](y^2), \qquad y > 0,$$

where $\mathcal{S}$ is the Stieltjes transform defined in (A.7). ♢

4.18 Lemma. *Let μ be a measure on $\mathbb{R}$ and define the measures μ^+ and μ^- on $\mathbb{R}$ by*

$$\mathrm{d}\mu^+(t) := \mathbb{1}_{[0,\infty)}(t)\,\mathrm{d}\mu(t), \qquad \mathrm{d}\mu^-(t) := \mathbb{1}_{(-\infty,0)}(t)\,\mathrm{d}\mu(t), \qquad t \in \mathbb{R}. \tag{4.22}$$

Let $\ell \in \mathbb{N}_0$ such that (4.16) holds and let $\mathcal{F}_\ell$ be as in Definition 4.17. Then

$$\mathsf{RC}_\ell[\mu](y) = \mathcal{F}_\ell[\mu^+](y) - \mathcal{F}_\ell[\mu^-](y), \tag{4.23}$$

$$\left|\mathsf{RC}_\ell[\mu](y)\right| \leq \left|\mathcal{F}_\ell[\mu](y)\right| \tag{4.24}$$

for $y > 0$.

Proof. Let μ_* and $\mu_*^\pm$ be the push-forward measures of μ and $\mu^\pm$ respectively as in (4.1), and let σ_ℓ, $\sigma_\ell^\pm$ be as in (4.21).

The definition of ℓ in (4.20) implies that

$$\int_{[0,\infty)} \frac{\mathrm{d}\sigma_\ell^\pm(s)}{1+s} = \int_{[0,\infty)} \frac{\sqrt{s}}{(1+s)^{\ell+1}}\,\mathrm{d}\mu_*^\pm(s) = \int_{\mathbb{R}} \frac{|t|}{(1+t^2)^{\ell+1}}\,\mathrm{d}\mu^\pm(t) < \infty,$$

and hence $\mathcal{S}[\sigma_\ell^+]$ and $\mathcal{S}[\sigma_\ell^-]$ are well defined. Moreover,

$$\begin{aligned}
\mathsf{RC}_\ell[\mu](y) &= (1-y^2)^\ell \left[\int_{\mathbb{R}} \frac{1}{t^2+y^2} \cdot \frac{|t|}{(1+t^2)^\ell}\,\mathrm{d}\mu^+(t) - \int_{\mathbb{R}} \frac{1}{t^2+y^2} \cdot \frac{|t|}{(1+t^2)^\ell}\,\mathrm{d}\mu^-(t) \right] \\
&= (1-y^2)^\ell \left[\int_{[0,\infty)} \frac{1}{s+y^2} \cdot \frac{\sqrt{s}}{(1+s)^\ell}\,\mathrm{d}\mu_*^+(s) - \int_{[0,\infty)} \frac{1}{s+y^2} \cdot \frac{\sqrt{s}}{(1+s)^\ell}\,\mathrm{d}\mu_*^-(s) \right] \\
&= (1-y^2)^\ell \Big[\mathcal{S}[\sigma_\ell^+](y^2) - \mathcal{S}[\sigma_\ell^-](y^2) \Big],
\end{aligned}$$

which yields (4.23). In a similar way as above one shows that $\mathcal{S}[\sigma_\ell]$ is well defined. Further, we have

$$\begin{aligned}\big|\mathsf{RC}_\ell[\mu](y)\big| &\le \big|(1-y^2)^\ell\big|\int_{\mathbb{R}}\frac{1}{t^2+y^2}\cdot\frac{|t|}{(1+t^2)^\ell}\mathrm{d}\mu(t)\\ &= \big|(1-y^2)^\ell\big|\int_{[0,\infty)}\frac{1}{s+y^2}\cdot\frac{\sqrt{s}}{(1+s)^\ell}\,\mathrm{d}\mu_*(s) = \big|\mathcal{S}[\sigma_\ell](y^2)\big|,\end{aligned}$$

which proves (4.24). ❑

In the following key proposition the asymptotic behaviour of $\mathcal{F}_\ell[\mu]$ is determined. It is used in Proposition 4.22 and in the Abelian implications in Theorems 5.1 and 5.5.

4.19 Proposition. *Let μ be a measure on $\mathbb{R}$, set $\ell := \ell(\mu)$ and define $\mathcal{F}_\ell[\mu]$ as in Definition 4.17. Further, assume that $t \mapsto \mu((-t,t))$ is regularly varying with index β. Then $\beta \in [2\ell-1, 2\ell+1]$. Moreover, if $\beta = 0$, then*

$$\mathcal{F}_\ell[\mu](y) \ll \frac{\mu((-y,y))}{y}, \qquad y \to \infty. \tag{4.25}$$

If $\beta > 0$, then $(-1)^\ell\mathcal{F}_\ell[\mu]$ is regularly varying with index $\beta - 1$ and satisfies

$$\mathcal{F}_\ell[\mu](y) \sim \begin{cases} \dfrac{\frac{\pi\beta}{2}}{\cos\frac{\pi\beta}{2}}\cdot\dfrac{\mu((-y,y))}{y}, & \beta \notin 2\mathbb{N}-1,\\[2ex] (-1)^\ell\beta y^{\beta-1}\displaystyle\int_1^y \frac{\mu((-t,t))}{t^{\beta+1}}\,\mathrm{d}t, & \beta = 2\ell-1,\\[2ex] (-1)^\ell\beta y^{\beta-1}\displaystyle\int_y^\infty \frac{\mu((-t,t))}{t^{\beta+1}}\,\mathrm{d}t, & \beta = 2\ell+1,\end{cases} \tag{4.26}$$

and

$$y^{2\ell-2} \ll \big|\mathcal{F}_\ell[\mu](y)\big| \ll y^{2\ell} \tag{4.27}$$

as $y \to \infty$. In particular, if $\beta \in 2\mathbb{N}-1$, then

$$\big|\mathcal{F}_\ell[\mu](y)\big| \gg \frac{\mu((-y,y))}{y}. \tag{4.28}$$

4.20 Remark.

(i) In the situation of Proposition 4.19 assume that $\beta \in 2\mathbb{N}-1$ and that $t \mapsto \mu((-t,t))$ is regularly varying with index β. It follows from the definition of $\ell(\mu)$ in (4.20) and from Lemma 4.1 that, with $\ell = \ell(\mu)$,

$$\beta = 2\ell-1 \quad\Leftrightarrow\quad \int_{\mathbb{R}}\frac{\mathrm{d}\mu(t)}{(1+|t|)^\beta} = \infty \quad\Leftrightarrow\quad \int_1^\infty \frac{\mu((-t,t))}{t^{\beta+1}}\,\mathrm{d}t = \infty,$$

$$\beta = 2\ell+1 \quad\Leftrightarrow\quad \int_{\mathbb{R}}\frac{\mathrm{d}\mu(t)}{(1+|t|)^\beta} < \infty \quad\Leftrightarrow\quad \int_1^\infty \frac{\mu((-t,t))}{t^{\beta+1}}\,\mathrm{d}t < \infty.$$

(ii) Instead of the assumption $\ell = \ell(\mu)$ in the proposition, let us consider the case when $\ell > \ell(\mu)$. Then

$$\sigma_\ell([0,\infty)) = \int_{[0,\infty)}\frac{\sqrt{s}}{(1+s)^\ell}\,\mathrm{d}s = \int_{\mathbb{R}}\frac{|t|}{(1+t^2)^\ell}\,\mathrm{d}\mu(t) < \infty,$$

and, by Remark A.6, we have

$$\mathcal{F}_\ell[\mu](y) \sim (-1)^\ell y^2\mathcal{S}[\sigma_\ell](y^2) \sim (-1)^\ell\sigma_\ell([0,\infty))y^{2\ell-2}$$

as $y \to \infty$.

$\diamondsuit$

Proof of Proposition 4.19. Let μ_* be the push-forward measure of μ as (4.1), and let σ_ℓ be as in (4.21). Then $s \mapsto \mu_*([0,s)) = \mu((-\sqrt{s}, \sqrt{s}))$ is regularly varying with index $\frac{\beta}{2}$. Set $h(s) := \frac{\sqrt{s}}{(1+s)^\ell} \sim s^{\frac{1}{2}-\ell}$ as $s \to \infty$. Then

$$\int_{[0,\infty)} h(s)\,\mathrm{d}\mu_*(s) = \int_{\mathbb{R}} \frac{|t|}{(1+t^2)^\ell}\,\mathrm{d}\mu(t). \tag{4.29}$$

Let us first consider the case when the integrals in (4.29) are finite. Then $\ell = 0$ by the definition of ℓ, and σ_0 and μ are finite measures, which implies that $\beta = 0$. From Remark A.6 we obtain that

$$\mathcal{F}_0[\mu](y) = \mathcal{S}[\sigma_0](y^2) \sim \frac{\sigma_0([0,\infty))}{y^2} \ll \frac{\mu(\mathbb{R})}{y} \sim \frac{\mu((-y,y))}{y},$$

which proves (4.25) in this case.

For the rest of the proof assume that the integrals in (4.29) are infinite. The definition of ℓ also implies that

$$\int_{[0,\infty)} \frac{h(s)}{1+s}\,\mathrm{d}\mu_*(s) = \int_{\mathbb{R}} \frac{|t|}{(1+t^2)^{\ell+1}}\,\mathrm{d}\mu(t) < \infty,$$

which shows that (A.20) with $\nu = \mu_*$ is satisfied. Hence we can apply Proposition A.11 with $\alpha = \frac{\beta}{2}$ and $\gamma = \frac{1}{2} - \ell$, which yields $\alpha + \gamma = \frac{\beta}{2} + \frac{1}{2} - \ell \in [0,1]$, i.e. $\beta \in [2\ell-1, 2\ell+1]$. Proposition A.11 also implies that, if $\beta > 0$, then $(-1)^\ell \mathcal{F}_\ell[\mu](y) \sim y^{2\ell}\mathcal{S}[\mu_\ell](y^2)$ is regularly varying with index $2\ell + 2\big(\frac{\beta}{2} + \frac{1}{2} - \ell - 1\big) = \beta - 1$.

Let us first consider the case when $\beta \notin 2\mathbb{N}-1$ and $\beta > 0$. Then $\alpha+\gamma = \frac{\beta}{2}+\frac{1}{2}-\ell \in (0,1)$, and from Proposition A.11 we obtain

$$\begin{aligned}\mathcal{F}_\ell[\mu](y) &\sim (-1)^\ell y^{2\ell}\mathcal{S}[\sigma_\ell](y^2) \sim (-1)^\ell y^{2\ell} \frac{\pi\frac{\beta}{2}}{\sin\big(\pi\big(\frac{\beta}{2}+\frac{1}{2}-\ell\big)\big)} \big(y^2\big)^{-\frac{1}{2}-\ell}\mu_*([0,y^2)) \\ &= \frac{\frac{\pi\beta}{2}}{\sin\big(\frac{\pi\beta}{2}+\frac{\pi}{2}\big)} \cdot \frac{\mu((-y,y))}{y} = \frac{\frac{\pi\beta}{2}}{\cos\frac{\pi\beta}{2}} \cdot \frac{\mu((-y,y))}{y}\end{aligned}$$

as $y \to \infty$.

Next assume that $\beta = 2\ell - 1$. Then $\alpha + \gamma = \frac{\beta}{2} + \frac{1}{2} - \ell = 0$, and Proposition A.11 and the substitution $s = t^2$ yield

$$\begin{aligned}\mathcal{F}_\ell[\mu](y) &\sim (-1)^\ell y^{2\ell}\mathcal{S}[\sigma_\ell](y^2) \sim (-1)^\ell y^{2\ell}\frac{\beta}{2}\cdot\frac{1}{y^2}\int_1^{y^2} s^{-\frac{1}{2}-\ell}\mu_*([0,s))\,\mathrm{d}s \\ &= (-1)^\ell\frac{\beta}{2}y^{2\ell-2}\int_1^y t^{-1-2\ell}\mu_*([0,t^2))\,2t\,\mathrm{d}t = (-1)^\ell\beta y^{2\ell-2}\int_1^y \frac{\mu((-t,t))}{t^{2\ell}}\,\mathrm{d}t,\end{aligned}$$

which shows (4.26) in this case.

The proofs of (4.25) when $\beta = 0$ the integrals in (4.29) are infinite and of (4.26) in the remaining case $\beta = 2\ell + 1$ are similar.

We can use (A.21) to obtain

$$\big|\mathcal{F}_\ell[\mu](y)\big| \sim y^{2\ell}\mathcal{S}[\sigma_\ell](y^2) \qquad \text{and} \qquad \frac{1}{y^2} \ll \mathcal{S}[\sigma_\ell](y^2) \ll 1,$$

which yields (4.27).

Finally, the relation in (4.28) follows easily from (4.26) and Theorem A.2. ❑

We also need the following comparison result.

4.21 Lemma. *Let μ_1 and μ_2 be measures on $\mathbb{R}$ and set $\ell := \ell(\mu_2)$. Further, assume that $t \mapsto \mu_2((-t,t))$ is regularly varying with index $\beta > 0$ and that the limit*

$$\lim_{t\to\infty} \frac{\mu_1((-t,t))}{\mu_2((-t,t))}$$

exists in $[0,\infty)$. Then $\mathcal{F}_\ell[\mu_1]$ is well defined, i.e. $\ell \geq \ell(\mu_1)$, and

$$\lim_{y\to\infty} \frac{\mathcal{F}_\ell[\mu_1](y)}{\mathcal{F}_\ell[\mu_2](y)} = \lim_{t\to\infty} \frac{\mu_1((-t,t))}{\mu_2((-t,t))}.$$

Proof. Since $\beta > 0$, it follows as in the proof of Proposition 4.19 that (A.28) with $\nu_2 = (\mu_2)_*$ and $h(s) = \frac{\sqrt{s}}{(1+s)^\ell}$ is satisfied. Now the claim follows from Lemma A.13. ❑

When $t \mapsto \mu((-t,t))$ is regularly varying with an index that is not an odd integer, the real part of $q(iy)$ is dominated by the imaginary part, as the following proposition shows.

4.22 Proposition. *Suppose that μ, p and q satisfy Assumptions 4.2, 4.4 and 4.13. Further, assume that $t \mapsto \mu((-t,t))$ is regularly varying with index β such that $\beta \notin 2\mathbb{N}-1$. Then*

$$\limsup_{y\to\infty} \left|\frac{\operatorname{Re} q(iy)}{\operatorname{Im} q(iy)}\right| \leq \left|\tan\frac{\pi\beta}{2}\right|. \tag{4.30}$$

Proof. Set $\ell := \ell(\mu)$. It follows from Lemmas 4.16 and 4.18 that

$$\left|\frac{\operatorname{Re} q(iy)}{\operatorname{Im} q(iy)}\right| = \left|\frac{\mathsf{RC}_\ell[\mu](y) + \widetilde{p}(y)}{\operatorname{Im} q(iy)}\right| \leq \frac{|\mathcal{F}_\ell[\mu](y)|}{|\operatorname{Im} q(iy)|} + \mathrm{O}\biggl(\frac{y^{2\ell-2}}{|\operatorname{Im} q(iy)|}\biggr). \tag{4.31}$$

Theorem 4.7 implies that $y \mapsto |\operatorname{Im} q(iy)|$ is regularly varying with index $\beta - 1$. Further, we obtain from Proposition 4.19 that $\beta \in [2\ell-1, 2\ell+1]$. Since, by assumption, $\beta \notin 2\mathbb{N}-1$, we have $\beta - 1 > 2\ell - 2$ and hence $|\operatorname{Im} q(iy)| \gg y^{2\ell-2}$, i.e. the O-term on the right-hand side of (4.31) converges to 0 as $y \to \infty$.

From Proposition 4.19 and Theorem 4.7 we obtain

$$|\mathcal{F}_\ell[\mu](y)| \begin{cases} \ll \dfrac{\mu((-y,y))}{y}, & \beta = 0, \\[2ex] \sim \dfrac{\frac{\pi\beta}{2}}{|\cos\frac{\pi\beta}{2}|} \cdot \dfrac{\mu((-y,y))}{y}, & \beta \in (0,\infty) \setminus (2\mathbb{N}-1), \end{cases}$$

$$|\operatorname{Im} q(iy)| \begin{cases} \sim \dfrac{\frac{\pi\beta}{2}}{|\sin\frac{\pi\beta}{2}|} \cdot \dfrac{\mu((-y,y))}{y}, & \beta \in [0,\infty) \setminus 2\mathbb{N}, \\[2ex] \gg \dfrac{\mu((-y,y))}{y}, & \beta \in 2\mathbb{N}, \end{cases}$$

which yields

$$\lim_{y\to\infty} \frac{|\mathcal{F}_\ell[\mu](y)|}{|\operatorname{Im} q(iy)|} = \left|\tan\frac{\pi\beta}{2}\right|$$

and hence (4.30). ❑

5 Abelian–Tauberian theorems

In the following theorem, one of the main results of this paper, we combine the Tauberian and the Abelian theorems from the previous sections. In most cases we can give a full characterisation when the asymptotic behaviour of the regularised Cauchy transform is described by a regularly varying function. Thereby, we assume that the behaviour of $q = \mathscr{C}_\kappa[\mu,p]$ is not governed by the polynomial summand.

5.1 The generic situation

Recall the notation $\mathfrak{p}(\mu)$ from (4.2) and $\kappa(\mu)$ from (4.4).

5.1 Theorem. *Let μ be a measure on $\mathbb{R}$ such that $\mathfrak{p}(\mu) < \infty$ and set $\kappa := \kappa(\mu)$. Further, let $p \in \mathbb{R}[z]$ with $p(z) = c_{2\kappa+1}z^{2\kappa+1} + \ldots + c_0$ such that $(\mu, p) \in \mathbb{E}_{\leq\kappa}$, and assume that*

$$c_{2\kappa+1} = \int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{(1+t^2)^{\kappa+1}}.$$

and, if $\mathfrak{p}(\mu)$ is odd, that

$$c_{2\kappa} = \int_{\mathbb{R}} \frac{t}{(1+t^2)^{\kappa+1}}\,\mathrm{d}\mu(t).$$

Set $q := \mathscr{C}_\kappa[\mu, p]$.

Let $\beta \in [0, \infty)$ and consider the following statements:

(a) *the symmetrised distribution function $t \mapsto \mu((-t, t))$ is regularly varying with index β;*

(a)$'$ *the limit*

$$\zeta := \lim_{t\to\infty} \frac{\mu((-t, 0))}{\mu([0, t))} \tag{5.1}$$

exists in $[0, \infty]$;

(b) *there exist a regularly varying function $f\colon [x_0, \infty) \to (0, \infty)$ with $x_0 > 0$ and a constant $\omega \in \mathbb{C} \setminus \{0\}$ such that*

$$\lim_{r\to\infty} \frac{q(rz)}{f(r)} = i\omega\Big(\frac{z}{i}\Big)^{\beta-1} \tag{5.2}$$

holds locally uniformly for $z \in \mathbb{C}^+$.

Then the following relations hold.

(i) *If $\beta \notin \mathbb{N}_0$, then*

$$\text{(a)} \;\wedge\; \text{(a)}' \quad\Leftrightarrow\quad \text{(b)}.$$

(ii) *Assume that $\beta \in 2\mathbb{N}_0$. Then*

$$\text{(a)} \;\Rightarrow\; \text{(b)}.$$

If $\beta = 0$ or $\int_{\mathbb{R}}(1 + |t|)^{-\beta}\mathrm{d}\mu(t) < \infty$, then

$$\text{(b)} \;\Rightarrow\; \text{(a)}.$$

(iii) *If $\beta \in 2\mathbb{N} - 1$, then*

$$\text{(a)} \;\wedge\; \text{(a)}' \;\wedge\; \zeta \neq 1 \quad\Rightarrow\quad \text{(b)}.$$

Further, assume that either (a) *and* (b) *hold and $\beta \in 2\mathbb{N}_0$, or that* (a), (a)$'$ *and* (b) *hold and $\beta \notin \mathbb{N}_0$, or that* (a), (a)$'$ *with $\zeta \neq 1$ and* (b) *hold and $\beta \in 2\mathbb{N} - 1$; then ω and f can be chosen as*

$$\omega = (-1)^{\mathfrak{p}(\mu)-1}\Big(\cos\frac{\pi\beta}{2} + i\frac{\zeta-1}{\zeta+1}\cdot\sin\frac{\pi\beta}{2}\Big), \tag{5.3}$$

where $\frac{\zeta-1}{\zeta+1}$ is understood as 1 when $\zeta = \infty$, where $\mathfrak{p}(\mu)$ is defined in (4.2), *and*

$$f(r) = \begin{cases} \dfrac{\pi\beta}{|\sin(\pi\beta)|}\cdot\dfrac{\mu((-r, r))}{r}, & \beta \in [0, \infty)\setminus\mathbb{N}, \\[2ex] \beta r^{\beta-1}\displaystyle\int_1^r \frac{\mu((-t, t))}{t^{\beta+1}}\,\mathrm{d}t, & \beta \in \mathbb{N} \;\wedge\; \displaystyle\int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{(1+|t|)^\beta} = \infty, \\[2ex] \beta r^{\beta-1}\displaystyle\int_r^\infty \frac{\mu((-t, t))}{t^{\beta+1}}\,\mathrm{d}t, & \beta \in \mathbb{N} \;\wedge\; \displaystyle\int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{(1+|t|)^\beta} < \infty; \end{cases} \tag{5.4}$$

here $\frac{\pi\beta}{|\sin(\pi\beta)|}$ is understood as 1 when $\beta = 0$.

Before we prove the theorem, let us add a couple of comments.

5.2 Remark.

(i) Note that, in the case $\beta \in \mathbb{N}$, one has $f(r) \gg \frac{\mu((-r,r))}{r}$ by Theorem A.2.

(ii) When $\beta \in 2\mathbb{N}_0$, then $\omega \in \mathbb{R}$ and hence $|\operatorname{Im} q(iy)| \gg |\operatorname{Re} q(iy)|$. When $\beta \in 2\mathbb{N}-1$ in the theorem, then $\omega \in i\mathbb{R}$ and hence $|\operatorname{Re} q(iy)| \gg |\operatorname{Im} q(iy)|$.

(iii) It follows from the proof of Theorem 5.1 (see (5.12)) that, if $\beta \in 2\mathbb{N}-1$ and (a) and (a)′ are satisfied with $\zeta = 1$, then $|q(iy)| \ll f(y)$ with f as in (5.4). A more detailed discussion of some cases in this situation are contained in Theorem 5.5 below.

(iv) In the case when $\beta \notin \mathbb{N}_0$, condition (a) is not sufficient to guarantee (b) since one can easily construct measures μ that satisfy (a) but not (a)′, e.g. by distributing the mass in an alternating way on the positive and negative half-axes.

(v) In Example 5.3 below we show that the converse implication in Theorem 5.1 (iii) does not hold. However, see Theorem 5.5 (ii) below for a Tauberian implication in the case when $\omega \notin i\mathbb{R}$.

(vi) Example 5.4 below deals with the situation when $\beta = 1$ and $\zeta = 1$, where Theorem 5.1 is not applicable. This example shows that, in general, the asymptotic behaviour of $q(iy)$ is not determined by the leading asymptotic behaviour of $\mu([0,t))$ and $\mu((-t,0))$. However, in this example statement (b) in Theorem 5.1 still holds. We do not know whether there exist a measure μ and a function q such that $\beta \in 2\mathbb{N}-1$ and (a) and (a)′ hold with $\zeta = 1$ but (b) does not hold.

◊

Proof of Theorem 5.1. Let $\kappa = \kappa(\mu)$ and $\ell = \ell(\mu)$.

① It follows from Theorem 3.1 that (b) is equivalent to

(c) there exist a regularly varying function $f\colon [x_0,\infty) \to (0,\infty)$ with $x_0 > 0$ and a constant $\omega \in \mathbb{C} \setminus \{0\}$ such that

$$\lim_{y\to\infty} \frac{q(iy)}{f(y)} = i\omega. \tag{5.5}$$

② Let us first consider the case when $\beta \in 2\mathbb{N}_0$. It follows from Proposition 4.22 that $|\operatorname{Re} q(iy)| \ll |\operatorname{Im} q(iy)|$ and hence, (5.5) is equivalent to

$$\lim_{y\to\infty} \frac{\operatorname{Im} q(iy)}{f(y)} = \omega.$$

From Theorem 4.7 and Remark 4.8 we therefore obtain the implications in (ii). Assume now that (a) and (b) hold. It follows from Proposition 4.19 that $\beta \in [2\ell-1, 2\ell+1]$ and hence $\ell = \frac{\beta}{2}$. With f as in (5.4) we obtain from (4.14) and Remark 4.8 that

$$\lim_{y\to\infty} \frac{\operatorname{Im} q(iy)}{f(y)} = (-1)^{\kappa}. \tag{5.6}$$

On the other hand, by Remark 4.15, ω from (5.3) equals $\omega = (-1)^{p(\mu)-1} \cos\frac{\pi\beta}{2} = (-1)^{\kappa+\ell}(-1)^{\ell} = (-1)^{\kappa}$, which coincides with the limit in (5.6).

③ Next we assume that $\beta \notin \mathbb{N}_0$ and (b) holds. It follows from Theorem 3.1 that the index of the regularly varying function f is $\alpha := \beta - 1$, and hence $r \mapsto rf(r)$ is regularly varying with index β. Now Theorem 3.2 implies that (3.6) and (3.7) hold. Since $\alpha \notin \mathbb{N}_0$, at least one of the right-hand sides of (3.6), (3.7) is non-zero. Taking the quotient of (3.7) and (3.6) we obtain that the limit in (5.1) exists; further, $t \mapsto \mu((-t,t))$ is regularly varying with index β, i.e. (a) and (a)′ hold.

④ For the rest of the proof we assume that $\beta \notin 2\mathbb{N}_0$, that (a) and (a)′ hold, and if $\beta \in 2\mathbb{N}-1$, then also $\zeta \neq 1$. Let μ^+ and μ^- be as in (4.22) and f as in (5.4). From Lemmas 4.16 and 4.18 we obtain that

$$\begin{aligned}\frac{1}{i}\cdot\frac{q(iy)}{f(y)} &= \frac{\mathcal{F}_\ell[\mu](y)}{f(y)}\cdot\frac{\operatorname{Im} q(iy) - i\operatorname{Re} q(iy)}{\mathcal{F}_\ell[\mu](y)}\\ &= \frac{\mathcal{F}_\ell[\mu](y)}{f(y)}\left[\frac{\operatorname{Im} q(iy)}{\mathcal{F}_\ell[\mu](y)} - i\left(\frac{\mathcal{F}_\ell[\mu^+](y)}{\mathcal{F}_\ell[\mu](y)} - \frac{\mathcal{F}_\ell[\mu^+](y)}{\mathcal{F}_\ell[\mu](y)} + \frac{\widetilde{p}(y)}{\mathcal{F}_\ell[\mu](y)}\right)\right],\end{aligned} \tag{5.7}$$

where $\widetilde{p}$ is a real, even polynomial of degree at most $2\ell-2$. In the next couple of steps we evaluate the limits of parts of this expression.

⑤ First we show that

$$\lim_{y\to\infty}\frac{\mathcal{F}_\ell[\mu](y)}{f(y)} = (-1)^{\mathfrak{p}(\mu)-1}\sin\frac{\pi\beta}{2}. \tag{5.8}$$

We start with the case when $\beta \notin 2\mathbb{N}-1$. It follows from Lemma 4.1 that $\beta \in (\mathfrak{p}(\mu)-1, \mathfrak{p}(\mu))$ and hence $\operatorname{sgn}(\sin(\pi\beta)) = (-1)^{\mathfrak{p}(\mu)-1}$. Further, we obtain from Proposition 4.19 and (5.4) that

$$\lim_{y\to\infty}\frac{\mathcal{F}_\ell[\mu](y)}{f(y)} = \frac{|\sin(\pi\beta)|}{2\cos\frac{\pi\beta}{2}} = \frac{(-1)^{\mathfrak{p}(\mu)-1}\sin(\pi\beta)}{2\cos\frac{\pi\beta}{2}} = (-1)^{\mathfrak{p}(\mu)-1}\sin\frac{\pi\beta}{2}.$$

Now assume that $\beta \in 2\mathbb{N}-1$. Then $\lim_{y\to\infty}\frac{\mathcal{F}_\ell[\mu](y)}{f(y)} = (-1)^\ell$. On the other hand, by Theorem 4.7 we have $\beta \in [2\kappa, 2\kappa+2]$ and hence $\beta = 2\kappa+1$, which, by Remark 4.15, implies that

$$(-1)^{\mathfrak{p}(\mu)-1}\sin\frac{\pi\beta}{2} = (-1)^{\kappa+\ell}\sin\Big(\pi\kappa + \frac{\pi}{2}\Big) = (-1)^{\kappa+\ell}\cos(\pi\kappa) = (-1)^\ell,$$

which proves (5.8) also in this case.

⑥ Next we show that

$$\lim_{y\to\infty}\frac{\operatorname{Im} q(iy)}{\mathcal{F}_\ell[\mu](y)} = \cot\frac{\pi\beta}{2}. \tag{5.9}$$

When $\beta \notin 2\mathbb{N}-1$, we can use Theorem 4.7 and Proposition 4.19 to obtain (5.9). When $\beta \in 2\mathbb{N}-1$, Theorem 4.7 and (4.28) imply that $\lim_{y\to\infty}\frac{\operatorname{Im} q(iy)}{\mathcal{F}_\ell[\mu](y)} = 0$, and again (5.9) holds.

⑦ Let us consider the expressions within the round brackets on the right-hand side of (5.7). Assume first that μ^+ is not the zero measure. The relation $\mu = \mu^+ + \mu^-$ implies that

$$\frac{\mu((-t,t))}{\mu^+((-t,t))} = 1 + \frac{\mu^-((-t,t))}{\mu^+((-t,t))} = 1 + \frac{\mu((-t,0))}{\mu([0,t))} \to 1+\zeta, \qquad t\to\infty,$$

and hence

$$\lim_{t\to\infty}\frac{\mu^+((-t,t))}{\mu((-t,t))} = \frac{1}{1+\zeta}, \qquad \lim_{t\to\infty}\frac{\mu^-((-t,t))}{\mu((-t,t))} = \frac{\zeta}{1+\zeta}.$$

If μ^+ is the zero measure, then these relations also hold with $\zeta = \infty$. Together with Lemma 4.21 this yields

$$\lim_{y\to\infty}\frac{\mathcal{F}_\ell[\mu^+](y)}{\mathcal{F}_\ell[\mu](y)} = \frac{1}{1+\zeta}, \qquad \lim_{y\to\infty}\frac{\mathcal{F}_\ell[\mu^-](y)}{\mathcal{F}_\ell[\mu](y)} = \frac{\zeta}{1+\zeta}. \tag{5.10}$$

Further, (4.27) implies that

$$\lim_{y\to\infty}\frac{\widetilde{p}(y)}{\mathcal{F}_\ell[\mu](y)} = 0. \tag{5.11}$$

⑧ Combining (5.7), (5.8), (5.9), (5.10) and (5.11) we arrive at

$$\frac{1}{i}\lim_{y\to\infty}\frac{q(iy)}{f(y)}=(-1)^{\mathfrak{p}(\mu)-1}\sin\frac{\pi\beta}{2}\Big(\cot\frac{\pi\beta}{2}-i\frac{1-\zeta}{1+\zeta}\Big), \tag{5.12}$$

which is equal to the expression on the right-hand side of (5.3); this proves (5.5). Since, by assumption, either $\beta\notin 2\mathbb{N}-1$ or $\zeta\neq 1$, we have $\omega\neq 0$. It follows from (5.8) that

$$f(y)\sim\big|\sin\tfrac{\pi\beta}{2}\big|\cdot\big|\mathcal{F}_\ell[\mu](y)\big|.$$

By Proposition 4.19 the right-hand side, and hence also f, is regularly varying. This shows that (c) is satisfied, which, in turn, implies (b).

❑

The following example shows that the converse of the implication in Theorem 5.1 (iii) does not hold.

5.3 Example. Let $\ell\in\mathbb{N}$. With $h(s)=\frac{\sqrt{s}}{(1+s)^\ell}$, $s\in[0,\infty)$, choose σ and ν as in Example A.12. Moreover, let μ be the measure on $\mathbb{R}$ such that $\mu((-\infty,0))=0$ and $\mu_*=\mu_*^+=\nu$. Then

$$\mu\big(\{e^{\frac{k}{2}}\}\big)=\mu_*^+(\{e^k\})=\frac{(1+e^k)^\ell}{e^{\frac{k}{2}}},\qquad k\in\mathbb{N}.$$

From the relation

$$\int_{\mathbb{R}}\frac{\mathrm{d}\mu(t)}{(1+|t|)^n}=\sum_{k=1}^\infty\frac{1}{\big(1+e^{\frac{k}{2}}\big)^n}\cdot\frac{(1+e^k)^\ell}{e^{\frac{k}{2}}}$$

we can easily deduce that $\mathfrak{p}(\mu)=2\ell$ and hence $\ell=\ell(\mu)$. Now set $q=\mathscr{C}_\kappa[\mu,p]$ with $\kappa=\kappa(\mu)$ and a polynomial p such that (4.9) holds. Then Assumptions 4.2, 4.4 and 4.13 are satisfied since $\mathfrak{p}(\mu)$ is even. It follows from Example A.12 that $t\mapsto\mu((-t,t))=\mu^+([0,t))=\nu([0,t^2))$ is not regularly varying, i.e. (a) in Theorem 5.1 is not satisfied.

Let us now consider the behaviour of $q(iy)$ as $y\to\infty$. It is clear that $\sigma_\ell=\sigma$ with σ_ℓ as in (4.21). From Lemma 4.18 and (A.27) we obtain

$$\mathsf{RC}_\ell[\mu](y)=\mathcal{F}_\ell[\mu^+](y)\sim(-1)^\ell y^{2\ell}\mathcal{S}[\sigma](y^2)\sim 2(-1)^\ell y^{2\ell-2}\log y,$$

and hence

$$\operatorname{Re}q(iy)\sim 2(-1)^\ell y^{2\ell-2}\log y,\qquad y\to\infty, \tag{5.13}$$

by Lemma 4.16. Next we show that the imaginary part is dominated by the real part. Let τ_κ be as in (4.12). Since $\kappa(\mu)=\ell(\mu)-1$ by Remark 4.15, we have

$$\tau_\kappa(\{e^k\})=\frac{1}{(1+e^k)^\kappa}\cdot\frac{(1+e^k)^\ell}{e^{\frac{k}{2}}}=\frac{1+e^k}{e^{\frac{k}{2}}},\qquad k\in\mathbb{N}.$$

Let $\tilde\tau$ be the measure on $[0,\infty)$ with $\tilde\tau([0,s))=\frac{2\sqrt{e}}{\sqrt{e}-1}\big(s^{\frac12}-1\big)$ for $s\in(1,\infty)$ and $\tilde\tau([0,1])=0$. For $t\in(e^n,e^{n+1}]$, $n\in\mathbb{N}$, we have

$$\tau_\kappa([0,t))=\sum_{k=1}^n\tau_k(\{e^k\})\le 2\sum_{k=1}^n e^{\frac{k}{2}}=\frac{2\sqrt{e}}{\sqrt{e}-1}\big(e^{\frac{n}{2}}-1\big)\le\frac{2\sqrt{e}}{\sqrt{e}-1}\big(t^{\frac12}-1\big)=\tilde\tau([0,t)).$$

By Lemma A.9 (i) and Remark A.8 we have

$$\begin{aligned}|\operatorname{Im}q(iy)|&\sim y^{2\kappa+1}\mathcal{S}[\tau_\kappa](y^2)\le y^{2\kappa+1}\mathcal{S}[\tilde\tau](y^2)\sim y^{2\kappa+1}\frac{\pi}{2}\cdot\frac{\tilde\tau([0,y^2))}{y^2}\\&\sim\frac{\pi\sqrt{e}}{\sqrt{e}-1}y^{2\kappa}=\frac{\pi\sqrt{e}}{\sqrt{e}-1}y^{2\ell-2},\end{aligned} \tag{5.14}$$

which shows that $|\operatorname{Re} q(iy)| \gg |\operatorname{Im} q(iy)|$. This, together with (5.13), implies that (3.2) with $f(r) = 2r^{2\ell-2}\log r$ is satisfied and therefore also (b) in Theorem 5.1 with $\beta = 2\ell - 1$ and $\omega = i(-1)^{\ell+1}$. Note also that (a)′ is satisfied with $\zeta = 0$.

For $\ell = 1$ we obtain a function $q \in \mathcal{N}_0$ since $\kappa = 0$ in this case. We can choose p such that

$$q(z) = \widetilde{C}[\mu](z) = \sum_{k=1}^{\infty}\left(\frac{1}{e^{\frac{k}{2}} - z} - \frac{e^{\frac{k}{2}}}{1+e^k}\right) \cdot \frac{1+e^k}{e^{\frac{k}{2}}}.$$

According to (5.13) and (5.14) it satisfies $q(iy) \sim 2\log y$ as $y \to \infty$. ◊

5.4 Example. Let $\gamma \in (0,1)$ and define the measure $\mu = \lambda + \nu$ on $\mathbb{R}$ where λ is the Lebesgue measure and ν is the measure such that $\nu((-\infty,0)) = 0$ and

$$\nu([0,t)) = \begin{cases} t, & t \in (0,e], \\ \dfrac{t}{(\log t)^\gamma}, & t \in (e,\infty). \end{cases}$$

Further, set $q(z) := \widetilde{C}[\mu](z)$. Clearly, $\mu([0,t)) \sim t$ and $\mu((-t,0)) = t$ and hence $\mu((-t,t)) \sim 2t$, which shows that (a) and (a)′ in Theorem 5.1 are satisfied with $\beta = 1$ and $\zeta = 1$. Further, Theorem 4.7 implies that $\operatorname{Im} q(iy) \to \pi$ as $y \to \infty$. For the real part we obtain from Lemma 4.18 and Proposition 4.19 that

$$\begin{aligned} \operatorname{Re} q(iy) &= \mathsf{RC}_1[\mu](y) = \mathcal{F}_1[\lambda^+ + \nu](y) - \mathcal{F}_1[\lambda^-](y) = \mathcal{F}_1[\nu](y) \\ &\sim -\int_1^y \frac{\nu((-t,t))}{t^2}\,\mathrm{d}t \sim -\int_1^y \frac{1}{t(\log t)^\gamma}\,\mathrm{d}t = -\frac{1}{1-\gamma}(\log y)^{1-\gamma}. \end{aligned}$$

This shows that $q(iy) \sim -\frac{1}{1-\gamma}(\log y)^{1-\gamma}$ as $y \to \infty$, and hence (b) in Theorem 5.1 is satisfied. However, the asymptotics of $q(iy)$ at infinity is not determined by the leading term of the asymptotics of $\mu([0,t))$ or $\mu((-t,0))$ as the former depends on γ whereas the latter does not. ◊

5.2 The exceptional case

In the following theorem we consider certain situations when $\beta \in 2\mathbb{N} - 1$ and $\zeta = 1$, a case that is not covered by Theorem 5.1. When $\beta \in 2\mathbb{N} - 1$ and $\zeta \neq 1$, the real part of $q(iy)$ dominates the imaginary part and is strictly larger than $\frac{\mu((-y,y))}{y}$; see Theorem 5.1 and Remark 5.2 (ii). In the next theorem we consider cases when there is cancellation between contributions from the measure μ on the positive and negative axes to the real part and where the growth of $q(iy)$ is determined by the imaginary part.

5.5 Theorem. *Let μ, κ, p and q be as in Assumptions 4.2 and 4.4 and let $\beta \in 2\mathbb{N} - 1$. Then the following statements are true.*

(i) *Assume that $\mu = \mu_0 + \mu_1 + \mu_2$ with measures μ_0, μ_1 and μ_2 such that the following conditions are satisfied:*

  (a) *μ_0 is symmetric and $t \mapsto \mu_0((-t,t))$ is regularly varying with index β;*

  (b) *either $\mu_1([0,\infty)) = 0$ or $t \mapsto \mu_1([0,t))$ is regularly varying with index β;*
  either $\mu_1((-\infty,0)) = 0$ or $t \mapsto \mu_1((-t,0))$ is regularly varying with index β;

  (c) *$\mu_2((-t,t)) \lesssim t^\gamma$ as $t \to \infty$ with some $\gamma < \beta$;*

  (d) *if $\int_{\mathbb{R}}(1+|t|)^{-\beta}\mathrm{d}\mu_1(t) < \infty$, then*

$$c_{2\kappa} = \int_{\mathbb{R}} \frac{t}{(1+t^2)^{\kappa+1}}\,\mathrm{d}(\mu_1+\mu_2)(t);$$

(e) *with the notation in* (4.22) *the following two limits (for* $+$ *and* $-$*) exist in* $\mathbb{R}$*:*

$$\eta_\pm := \begin{cases} \lim\limits_{t\to\infty}\biggl(\int_1^t \frac{\mu_1^\pm((-s,s))}{s^{\beta+1}}\,\mathrm{d}s \Big/ \frac{\mu_0((-t,t))}{t^\beta}\biggr) & \text{if } \int_{\mathbb{R}} \frac{\mathrm{d}\mu_1(t)}{(1+|t|)^\beta} = \infty, \\ -\lim\limits_{t\to\infty}\biggl(\int_t^\infty \frac{\mu_1^\pm((-s,s))}{s^{\beta+1}}\,\mathrm{d}s \Big/ \frac{\mu_0((-t,t))}{t^\beta}\biggr) & \text{if } \int_{\mathbb{R}} \frac{\mathrm{d}\mu_1(t)}{(1+|t|)^\beta} < \infty. \end{cases}$$

Then $\mu((-t,t)) \sim \mu_0((-t,t))$*,* $\beta = 2\kappa+1$*, and* (5.2) *holds locally uniformly for* $z \in \mathbb{C}^+$ *with*

$$f(r) = \beta\frac{\mu((-r,r))}{r}, \qquad \omega = (-1)^\kappa\Bigl(\frac{\pi}{2} + i\bigl(\eta_+ - \eta_-\bigr)\Bigr). \tag{5.15}$$

(ii) *Assume that there exist a regularly varying function* $f : [x_0,\infty) \to (0,\infty)$ *with* $x_0 > 0$ *and a constant* $\omega \notin i\mathbb{R}$ *such that* (5.2) *holds uniformly for* $z \in \mathbb{C}^+$*. Then* $t \mapsto \mu((-t,t))$ *is regularly varying with index* β *and* (5.1) *holds with* $\zeta = 1$*. One can choose* f *as in* (5.15)*, in which case* ω *satisfies* $\operatorname{Re}\omega = (-1)^\kappa\frac{\pi}{2}$*.*

5.6 Remark. Example 5.3 shows that the condition $\omega \notin i\mathbb{R}$ in (ii) is essential. ◊

Proof of Theorem 5.5. We split the proof into a couple of steps; the first five steps deal with the proof of (i).

① First we note that either $\mu_1 = 0$ or $t \mapsto \mu_1((-t,t))$ is regularly varying with index β. In the latter case we obtain from Theorem A.2 (i) and assumptions (b) and (e) that, when $\int_{\mathbb{R}}(1+|t|)^{-\beta}\mathrm{d}\mu_1(t) = \infty$,

$$\frac{\mu_1((-t,t))}{t^\beta} \ll \int_1^t \frac{\mu_1((-s,s))}{s^{\beta+1}}\,\mathrm{d}s = \int_1^t \frac{\mu_1^+((-s,s))}{s^{\beta+1}}\,\mathrm{d}s + \int_1^t \frac{\mu_1^-((-s,s))}{s^{\beta+1}}\,\mathrm{d}s \lesssim \frac{\mu_0((-t,t))}{t^\beta}$$

as $t \to \infty$. A similar calculation and the use of Theorem A.2 (ii) show that also in the case when $\int_{[0,\infty)}(1+|t|)^{-\beta}\mathrm{d}\mu_1(t) < \infty$, we have $\mu_1((-t,t)) \ll \mu_0((-t,t))$. Together with assumption (c), this implies that $\mu((-t,t)) \sim \mu_0((-t,t))$.

It follows from Theorem 4.7 that $\beta = 2\kappa+1$ and

$$\operatorname{Im} q(iy) \sim \frac{\frac{\pi\beta}{2}}{\sin\bigl(\pi\kappa + \frac{\pi}{2}\bigr)} \cdot \frac{\mu((-y,y))}{y} = (-1)^\kappa\frac{\pi}{2}f(y). \tag{5.16}$$

② Set $\ell := \ell(\mu)$ and $\widehat{\mu} := \mu_1 + \mu_2$. By Remark 4.15 we have either $\ell = \kappa$ or $\ell = \kappa+1$. It follows from assumption (c) and Lemma 4.1 that $\int_{\mathbb{R}}(1+|t|)^{-\beta}\mathrm{d}\mu_2(t) < \infty$ and hence $\ell(\mu_2) \le \kappa$. If $\mathfrak{p}(\mu)$ is odd (i.e. $\ell = \kappa$), then

$$\int_{\mathbb{R}} \frac{\mathrm{d}\mu_1(t)}{(1+|t|)^\beta} \le \int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{(1+|t|)^\beta} < \infty$$

and hence, by the symmetry of μ_0 and assumption (d),

$$\int_{\mathbb{R}} \frac{t}{(1+t^2)^{\kappa+1}}\,\mathrm{d}\mu(t) = \int_{\mathbb{R}} \frac{t}{(1+t^2)^{\kappa+1}}\,\mathrm{d}\widehat{\mu}(t) = c_{2\kappa}.$$

This shows that Assumption 4.13 is satisfied. It follows from Lemma 4.16 and the symmetry of μ_0 that

$$\operatorname{Re} q(iy) = \mathsf{RC}_\ell[\mu](y) + \widetilde{p}(y) = \mathsf{RC}_\ell[\widehat{\mu}](y) + \widetilde{p}(y) \tag{5.17}$$

with $\widetilde{p}$ as in Lemma 4.16.

Set $\hat{\ell} := \kappa$ when $\mu_1 = 0$ and $\hat{\ell} := \ell(\widehat{\mu})$ otherwise. It follows in a similar way as above that, by assumption (b), either $\mu_1^+ = 0$ or $\ell(\mu_1^+) = \kappa$ or $\ell(\mu_1^+) = \kappa+1$, and similarly for μ_1^-. Since $\ell(\mu_2) \le \kappa$, we have $\kappa \le \hat{\ell} \le \ell \le \kappa+1$.

Let us rewrite the expression in (5.17) in the case when $\hat{\ell} = \kappa$ and $\ell = \kappa + 1$. We obtain from (2.18) with κ and κ' there replaced by $\hat{\ell} - 1$ and $\ell - 1$ respectively that

$$\begin{aligned}\frac{t}{t^2+y^2} \cdot \frac{(1-y^2)^\ell}{(1+t^2)^\ell} &= \operatorname{Re}\biggl[\frac{1}{t-iy} \cdot \frac{(1+(iy)^2)^\ell}{(1+t^2)^\ell}\biggr] \\ &= \operatorname{Re}\biggl[\frac{1}{t-iy} \cdot \frac{(1+(iy)^2)^{\hat{\ell}}}{(1+t^2)^{\hat{\ell}}}\biggr] - \operatorname{Re}\biggl[(t+iy) \cdot \frac{(1+(iy)^2)^{\hat{\ell}}}{(1+t^2)^{\hat{\ell}+1}}\biggr] \\ &= \frac{t}{t^2+y^2} \cdot \frac{(1-y^2)^{\hat{\ell}}}{(1+t^2)^{\hat{\ell}}} - (1-y^2)^{\hat{\ell}} \cdot \frac{t}{(1+t^2)^{\hat{\ell}+1}}.\end{aligned}$$

Hence

$$\operatorname{Re} q(iy) = \mathsf{RC}_{\hat{\ell}}[\widehat{\mu}](y) - (1-y^2)^\kappa \int_{\mathbb{R}} \frac{t}{(1+t^2)^{\kappa+1}}\,\mathrm{d}\widehat{\mu}(t) + \widetilde{p}(y) = \mathsf{RC}_{\hat{\ell}}[\widehat{\mu}])(y) + \widehat{p}(y),$$

where the polynomial $\widehat{p}$ satisfies

$$\widehat{p}(y) = (-1)^\kappa \biggl[c_{2\kappa} - \int_{\mathbb{R}} \frac{t}{(1+t^2)^{\kappa+1}}\,\mathrm{d}\widehat{\mu}(t)\biggr] y^{2\kappa} + \mathrm{O}\bigl(y^{2\kappa-2}\bigr) = \mathrm{O}\bigl(y^{2\kappa-2}\bigr) = \mathrm{O}\bigl(y^{2\hat{\ell}-2}\bigr)$$

by assumption (d).

In all cases we can write

$$\operatorname{Re} q(iy) = \mathsf{RC}_{\hat{\ell}}[\widehat{\mu}](y) + \widehat{p}(y) = \mathcal{F}_{\hat{\ell}}[\mu_1^+](y) - \mathcal{F}_{\hat{\ell}}[\mu_1^-](y) + \mathsf{RC}_{\hat{\ell}}[\mu_2](y) + \widehat{p}(y)$$

with an even polynomial $\widehat{p}$ of degree at most $2\hat{\ell} - 2$; note that we have used Lemma 4.18 in the last step and that $\widehat{p} = \widetilde{p}$ when $\hat{\ell} = \ell$. We therefore have

$$\frac{1}{i} \cdot \frac{q(iy)}{f(y)} = \frac{\operatorname{Im} q(iy)}{f(y)} - i\frac{\mathcal{F}_{\hat{\ell}}[\mu_1^+](y)}{f(y)} + i\frac{\mathcal{F}_{\hat{\ell}}[\mu_1^-](y)}{f(y)} - i\frac{\mathsf{RC}_{\hat{\ell}}[\mu_2](y)}{f(y)} - i\frac{\widehat{p}(y)}{f(y)}. \tag{5.18}$$

③ We show that

$$\lim_{y\to\infty} \frac{\mathcal{F}_{\hat{\ell}}[\mu_1^\pm](y)}{f(y)} = -(-1)^\kappa \eta_\pm. \tag{5.19}$$

Let us start with μ_1^+. When $\mu_1^+ = 0$, the equality in (5.19) is obvious. Assume now that $\mu_1^+ \neq 0$. Let us first consider the case when $\ell(\mu_1^+) = \hat{\ell}$. Then the assumptions of Proposition 4.19 are satisfied for μ replaced by μ_1^+. Hence that lemma and assumption (e) yield that, when $\int_{\mathbb{R}}(1+|t|)^{-\beta}\mathrm{d}\mu_1(t) = \infty$, i.e. $\ell = \kappa+1$,

$$\lim_{y\to\infty} \frac{\mathcal{F}_{\hat{\ell}}[\mu_1^+](y)}{f(y)} = \lim_{y\to\infty}\biggl[(-1)^{\hat{\ell}}\beta y^{\beta-1} \int_1^y \frac{\mu_1^+((-s,s))}{s^{\beta+1}}\,\mathrm{d}s \bigg/ \biggl(\beta\frac{\mu_0((-y,y))}{y}\biggr)\biggr] = -(-1)^\kappa \eta_+;$$

a similar calculation proves (5.19) also in the case when $\int_{\mathbb{R}}(1+|t|)^{-\beta}\mathrm{d}\mu_1(t) < \infty$. The same considerations can be applied to μ_1^- when $\mu_1^- = 0$ or $\ell(\mu_1^-) = \hat{\ell}$.

It remains to prove (5.19) for μ_1^+ when $\ell(\mu_1^+) < \ell(\mu_1^-) = \hat{\ell} = \kappa + 1$ or for μ_1^- when $\ell(\mu_1^-) < \ell(\mu_1^+)$. We consider only the first case. It follows from Remark 4.20 (ii) applied to μ_1^+, the first inequality in (4.27) and the already proved relation (5.19) for μ_1^- (since $\ell(\mu_1^-) = \hat{\ell}$) that, with some $c > 0$,

$$\bigl|\mathcal{F}_{\hat{\ell}}[\mu_1^+](y)\bigr| \sim cy^{2\hat{\ell}-2} = cy^{2\kappa} \ll \bigl|\mathcal{F}_{\hat{\ell}}[\mu_1^-](y)\bigr| \lesssim f(y).$$

On the other hand, the relations $\ell(\mu_1^-) = \kappa+1$, $\ell(\mu_1^+) < \kappa+1$ and Lemma 4.1 imply that

$$\int_1^\infty \frac{\mu_1^+((-s,s))}{s^{\beta+1}}\,\mathrm{d}s < \infty, \qquad \int_1^\infty \frac{\mu_1^-((-s,s))}{s^{\beta+1}}\,\mathrm{d}s = \infty,$$

and hence

$$\int_1^t \frac{\mu_1^+((-s,s))}{s^{\beta+1}}\,\mathrm{d}s \ll \int_1^t \frac{\mu_1^-((-s,s))}{s^{\beta+1}}\,\mathrm{d}s.$$

This and the existence of the limit for η_- yield $\eta_+ = 0$, which shows that (5.19) holds also for μ_1^+ in this case.

④ Next let us show that

$$\lim_{y\to\infty}\left(\frac{\mathsf{RC}_{\hat\ell}[\mu_2](y)}{f(y)} + \frac{\widehat{p}(y)}{f(y)}\right) = 0. \tag{5.20}$$

Let us consider the two possible cases for $\hat\ell$ separately. First assume that $\hat\ell = \kappa$. Choose $\rho \in (\max\{\gamma, \beta-1\}, \beta)$ and define the measure ν by $\nu((-\infty,0)) = 0$ and $\nu([0,t)) = t^\rho$ for $t > 0$. Since $\rho \in (\beta-1,\beta)$, we have $\ell(\nu) = \hat\ell$. It follows from Lemma 4.21 and assumption (c) that

$$\lim_{y\to\infty}\frac{\mathcal{F}_{\hat\ell}[\mu_2](y)}{\mathcal{F}_{\hat\ell}[\nu](y)} = \lim_{t\to\infty}\frac{\mu_2((-t,t))}{\nu((-t,t))} = 0.$$

From (4.24) and (4.26) we obtain

$$\big|\mathsf{RC}_{\hat\ell}[\mu_2](y)\big| \le \big|\mathcal{F}_{\hat\ell}[\mu_2](y)\big| \ll \big|\mathcal{F}_{\hat\ell}[\nu](y)\big| \sim \frac{\frac{\pi\rho}{2}}{\cos\frac{\pi\rho}{2}}\, y^{\rho-1}.$$

Since f is regularly varying with index $\beta - 1 = 2\hat\ell$ and $\widehat{p}$ is a polynomial of degree at most $2\hat\ell - 2$, relation (5.20) follows in this case; see (A.1).

Now let us consider the case when $\hat\ell = \kappa+1$. Then $\ell(\mu_2) \le \kappa < \hat\ell$, and hence Lemma 4.18 and Remark 4.20 (ii) imply that

$$\big|\mathsf{RC}_{\hat\ell}[\mu_2](y)\big| \le \big|\mathcal{F}_{\hat\ell}[\mu_2](y)\big| \sim c y^{2\hat\ell-2}$$

with some $c > 0$. Without loss of generality assume that $\ell(\mu_1^+) = \hat\ell$ (the case $\ell(\mu_1^-) = \hat\ell$ is similar). It follows from assumption (e) and Proposition 4.19 that

$$f(y) \gtrsim y^{\beta-1}\int_1^y \frac{\mu_1^+([0,s))}{s^{\beta+1}}\,\mathrm{d}s \sim \frac{1}{\beta}\big|\mathcal{F}_{\hat\ell}[\mu_1^+](y)\big| \gg y^{2\hat\ell-2},$$

which yields (5.20) also in this case.

⑤ Combining (5.18), (5.16), (5.19) and (5.20) we arrive at

$$\frac{1}{i}\lim_{y\to\infty}\frac{q(iy)}{f(y)} = (-1)^\kappa\Big(\frac{\pi}{2} + i\big(\eta_+ - \eta_-\big)\Big),$$

which, by Theorem 3.1, shows the remaining assertions in (i).

⑥ Let us now prove item (ii). Assume that f and $\omega \notin i\mathbb{R}$ are such that (5.2) holds. We obtain from Theorem 3.2 and Remark 3.3 (i) that

$$\lim_{t\to\infty}\frac{\mu((0,t))}{tf(t)} = \lim_{t\to\infty}\frac{\mu((-t,0))}{tf(t)} = \frac{\omega}{\pi(\alpha+1)}\cos\big(\arg\big((-1)^m\omega\big)\big) \tag{5.21}$$

where $\alpha = \beta - 1 = 2m$ with $m \in \mathbb{N}_0$ and $\big|\arg\big((-1)^m\omega\big)\big| \le \frac{\pi}{2}$. Since, by assumption, $\omega \notin i\mathbb{R}$, we have $\mathrm{Re}((-1)^m\omega) > 0$ and hence

$$|\omega|\cos\big(\arg\big((-1)^m\omega\big)\big) = \mathrm{Re}\big((-1)^m\omega\big) = (-1)^m\,\mathrm{Re}\,\omega.$$

In particular, the limits in (5.21) are non-zero, and therefore (5.1) holds with $\zeta = 1$ and

$$\lim_{t\to\infty}\frac{\mu((-t,t))}{tf(t)} = (-1)^m\frac{2}{\pi\beta}\,\mathrm{Re}\,\omega. \tag{5.22}$$

This implies that $t \mapsto \mu((-t,t))$ is regularly varying with index β, which, in turn, yields that $\beta = 2\kappa + 1$ and hence $m = \kappa$. Choosing f as in (5.15) we obtain from (5.22) that $\operatorname{Re}\omega = (-1)^\kappa \frac{\pi}{2}$.

❑

To illustrate the Abelian direction of Theorem 5.5, let us consider the following example.

5.7 Example. Let μ be a measure on $\mathbb{R}$ that satisfies

$$\begin{aligned} \mu([0,t)) &= t\log t + a_+ t + \mathrm{O}(t^\gamma), \\ \mu((-t,0)) &= t\log t + a_- t + \mathrm{O}(t^\gamma) \end{aligned} \tag{5.23}$$

as $t \to +\infty$ with $a_+, a_- \geq 0$ and $\gamma < 1$. We can write μ as $\mu = \mu_0 + \mu_1 + \mu_2$ where μ_0, μ_1, μ_2 are measures satisfying

$$\begin{aligned} &\mu_0([-1,1]) = 0; && \mu_0([1,t)) = \mu_0((-t,-1)) = t\log t, \qquad t > 1; \\ &\mu_1([0,t)) = a_+ t, && \mu_1((-t,0)) = a_- t, \qquad t > 0; \\ &\mu_2((-t,t)) = \mathrm{O}(t^\gamma), && t \to +\infty. \end{aligned}$$

Since $\int_{\mathbb{R}} (1+|t|)^{-1} \mathrm{d}\mu(t) = \infty$ (see Lemma 4.1), assumptions (a)–(d) in Theorem 5.5 are fulfilled with $\beta = 1$. For (e) we consider

$$\eta_\pm = \lim_{t\to\infty} \left(\int_1^t \frac{a_\pm s}{s^2}\,\mathrm{d}s \Big/ \frac{2t\log t}{t} \right) = \frac{a_\pm}{2},$$

which shows that (e) is also satisfied. Now let $q(z) = c_0 + \widetilde{C}[\mu](z)$ with $c_0 \in \mathbb{R}$. Then, by Theorem 5.5 (i), relation (5.2) holds with

$$f(r) \sim \frac{\mu_0((-r,r))}{r} = 2\log r, \qquad \omega = \frac{\pi}{2} + i\frac{a_+ - a_-}{2},$$

i.e.

$$\lim_{r\to\infty} \frac{q(rz)}{\log r} = -a_+ + a_- + i\pi$$

locally uniform for $z \in \mathbb{C}^+$. An example of a function q with a measure μ as in (5.23) is

$$q(z) = (a_- - a_+ + i\pi)\log(z+i) + \pi(a_+ + 1)i, \qquad z \in \mathbb{C}^+,$$

where the $\mathrm{O}(t^\gamma)$ term is actually $\mathrm{O}(\log t)$. ◊

Appendix A. Regularly varying functions and some theorems of Karamata

In this appendix we provide some classical results about regularly varying functions in slightly extended or rounded-off formulations. A very good source for the theory of regular variation is [3]; this is our standard reference.

Recall the definition of regular variation in Karamata's sense.

A.1 Definition. A function $f\colon [x_0,\infty) \to (0,\infty)$ with $x_0 > 0$ is called *regularly varying* with *index* $\alpha \in \mathbb{R}$ if it is measurable and

$$\forall \lambda > 0\colon\ \lim_{r\to\infty} \frac{f(\lambda r)}{f(r)} = \lambda^\alpha.$$

A regularly varying function f with index 0 is also called *slowly varying.* ◊

Examples include functions f behaving for large r like

$$f(r) = r^\alpha \cdot \big(\log r\big)^{\beta_1} \cdot \big(\log\log r\big)^{\beta_2} \cdot \ldots \cdot \big(\underbrace{\log\cdots\log}_{m^{\text{th}} \text{ iterate}} r\big)^{\beta_m},$$

where $\alpha, \beta_1, \ldots, \beta_m \in \mathbb{R}$. Other examples are $f(r) = r^\alpha e^{(\log r)^\beta}$ with $\beta \in (0,1)$, or $f(r) = r^\alpha e^{\frac{\log r}{\log\log r}}$; see [3, §1.3]. In many respects, regularly varying functions of index α behave like the power function r^α. For instance, if f is regularly varying with index α, then

$$f(r) \gg r^{\alpha-\varepsilon}, \qquad f(r) \ll r^{\alpha+\varepsilon} \qquad \text{as } r \to \infty \tag{A.1}$$

for every $\varepsilon > 0$, which follows, e.g. from the Potter bounds; see [3, Theorem 1.5.6 (iii)].

Another property that reflects the power-like behaviour is a fundamental result by J. Karamata about primitives of regularly varying functions. We state a comprehensive formulation collecting what is proved in [3, Section 1.5.6]. More precisely, item (i) in Theorem A.2 follows from [3, Theorem 1.5.11 (i) and Proposition 1.5.9a]; item (ii) follows from [3, Theorem 1.5.11 (ii) and Proposition 1.5.9b].

A.2 Theorem (Karamata)**.** *Let $x_0 > 0$ and let $f : [x_0, \infty) \to (0, \infty)$ be measurable and locally bounded. Further, assume that f is regularly varying with index $\alpha \in \mathbb{R}$.*

(i) *Suppose that $\alpha + 1 \geq 0$. Then the function $x \mapsto \int_{x_0}^x f(t)\,\mathrm{d}t$ is regularly varying with index $\alpha + 1$, and*

$$\lim_{x\to\infty} \left(x f(x) \Big/ \int_{x_0}^x f(t)\,\mathrm{d}t \right) = \alpha + 1.$$

(ii) *Suppose that $\int_{x_0}^\infty f(t)\,\mathrm{d}t < \infty$. Then $\alpha+1 \leq 0$, the function $x \mapsto \int_x^\infty f(t)\,\mathrm{d}t$ is regularly varying with index $\alpha + 1$, and*

$$\lim_{x\to\infty} \left(x f(x) \Big/ \int_x^\infty f(t)\,\mathrm{d}t \right) = -(\alpha + 1).$$

In the following we often use integration by parts in its proper measure-theoretic form as stated in the following lemma.

A.3 Lemma. *Let $-\infty < a < b \leq \infty$, let μ and ν be measures on $[a, b)$. Then*

$$\int_{[a,b)} \mu([a,t))\,\mathrm{d}\nu(t) = \int_{[a,b)} \nu((t,b))\,\mathrm{d}\mu(t). \tag{A.2}$$

If these integrals are finite, then $\lim_{x\to b} \mu([a,x))\nu([x,b)) = 0$.

Proof. If $\nu([a,b)) = \infty$, then either both sides are zero (when μ is the zero measure) or both sides are infinite (otherwise). In the case when ν is a finite measure, relation (A.2) follows from Fubini's theorem. To show the last assertion, assume that both sides of (A.2) are finite and let $x \in (a, b)$. Then (A.2) applied to $[a, x)$ instead of $[a, b)$ yields

$$\begin{aligned} \int_{[a,x)} \mu([a,t))\,\mathrm{d}\nu(t) &= \int_{[a,x)} \nu((t,x))\,\mathrm{d}\mu(t) \\ &= \int_{[a,x)} \nu((t,b))\,\mathrm{d}\mu(t) - \mu([a,x))\nu([x,b)). \end{aligned}$$

Letting $x \to b$ we obtain the claimed limit relation. ❑

From Theorem A.2 we obtain the next proposition, which relates integrals with respect to a measure to integrals involving the corresponding distribution function.

A.4 Proposition. *Let σ be a measure on $[1,\infty)$, which is not the zero measure. Assume that the distribution function $t \mapsto \sigma([1,t))$ is regularly varying with index α (since $t \mapsto \sigma([1,t))$ is non-decreasing, we have $\alpha \geq 0$).*

(i) *Let $\gamma \in \mathbb{R} \setminus \{0\}$. Then*

$$\lim_{x\to\infty}\Big(\int_{[1,x)} t^\gamma \, \mathrm{d}\sigma(t) \Big/ \int_1^x t^{\gamma-1}\sigma([1,t))\,\mathrm{d}t\Big) = \max\{\alpha,-\gamma\} = \begin{cases} \alpha, & \alpha+\gamma \geq 0, \\ |\gamma|, & \alpha+\gamma < 0. \end{cases} \tag{A.3}$$

The function $x \mapsto \int_1^x t^{\gamma-1}\sigma([1,t))\,\mathrm{d}t$ is regularly varying with index $\alpha+\gamma$ if $\alpha+\gamma \geq 0$, and non-decreasing and bounded (in particular, slowly varying) if $\alpha+\gamma<0$; if $\alpha \neq 0$ or $\gamma \leq 0$, the same holds for $x \mapsto \int_{[1,x)} t^\gamma\,\mathrm{d}\sigma(t)$.

In particular, if $\int_{[1,\infty)} t^\gamma\,\mathrm{d}\sigma(t) = \infty$, then $\alpha+\gamma \geq 0$. Moreover, if $\int_{[1,\infty)} t^\gamma\,\mathrm{d}\sigma(t) < \infty$ and σ is an infinite measure, then $\alpha+\gamma \leq 0$.

(ii) *Let $\rho > 0$ and assume that $\int_{[1,\infty)} t^{-\rho}\mathrm{d}\sigma(t) < \infty$. Then $\sigma([1,x)) \ll x^\rho$, and hence $\alpha \leq \rho$. Further, we have*

$$\lim_{x\to\infty}\Big(\int_{[x,\infty)} \frac{\mathrm{d}\sigma(t)}{t^\rho} \Big/ \int_x^\infty \frac{\sigma([1,t))}{t^{\rho+1}}\,\mathrm{d}t\Big) = \alpha.$$

The function $x \mapsto \int_x^\infty t^{-(\rho+1)}\sigma([1,t))\,\mathrm{d}t$ is regularly varying with index $\alpha-\rho$; if $\alpha \neq 0$, the same holds for $x \mapsto \int_x^\infty t^{-\rho}\mathrm{d}\sigma(t)$.

Proof.

① For the proof of (i) we integrate by parts (using the measure $\mathrm{d}\nu(t) := t^{\gamma-1}\,\mathrm{d}t$ in Lemma A.3) to obtain

$$\begin{aligned}\int_{[1,x)} \sigma([1,t))t^{\gamma-1}\,\mathrm{d}t &= \int_{[1,x)} \Big(\int_t^x s^{\gamma-1}\,\mathrm{d}s\Big)\,\mathrm{d}\sigma(t) \\ &= \frac{1}{\gamma}x^\gamma\sigma([1,x)) - \frac{1}{\gamma}\int_{[1,x)} t^\gamma\,\mathrm{d}\sigma(t),\end{aligned} \tag{A.4}$$

and therefore

$$\int_{[1,x)} t^\gamma\,\mathrm{d}\sigma(t) \Big/ \int_{[1,x)} t^{\gamma-1}\sigma([1,t))\,\mathrm{d}t = \Big(x^\gamma\sigma([1,x)) \Big/ \int_{[1,x)} t^{\gamma-1}\sigma([1,t))\,\mathrm{d}t\Big) - \gamma. \tag{A.5}$$

② First we consider the case when $\alpha+\gamma \geq 0$. We can apply Theorem A.2 (i) to obtain that the integral $\int_{[1,x)} \sigma([1,t))t^{\gamma-1}\,\mathrm{d}t$ is regularly varying with index $\alpha+\gamma$, and that the quotient on the right-hand side of (A.5) tends to $\alpha+\gamma$. The asserted limit relation follows. In particular, if $\alpha \neq 0$, also the integral $\int_1^x t^\gamma\,\mathrm{d}\sigma(t)$ is regularly varying with index $\alpha+\gamma$.

③ Now assume that $\alpha+\gamma<0$. Since $\alpha \geq 0$, we have $\gamma < 0$ and hence $\max\{\alpha,-\gamma\} = -\gamma = |\gamma|$. The integral $\int_1^\infty t^{\gamma-1}\sigma([1,t))\,\mathrm{d}t$ converges, and hence $\lim_{x\to\infty} x^\gamma\sigma([1,x)) = 0$ and by (A.4)

$$\int_{[1,\infty)} t^\gamma\,\mathrm{d}\sigma(t) = |\gamma|\int_{[1,\infty)} t^{\gamma-1}\sigma([1,t))\,\mathrm{d}t < \infty.$$

In particular, the function $x \mapsto \int_{[1,x)} t^\gamma\,\mathrm{d}\sigma(t)$ is slowly varying and the asserted limit relation holds.

④ For the last statement in (i) assume that $\int_{[0,\infty)} t^\gamma \,\mathrm{d}\sigma(t) < \infty$, that σ is an infinite measure, and suppose that $\alpha + \gamma > 0$. If $\alpha > 0$, then $\int_{[1,x)} t^\gamma \,\mathrm{d}\sigma(t)$ is regularly varying with positive index $\alpha + \gamma$ and hence unbounded, a contradiction. If $\alpha = 0$, then $\gamma > 0$, and therefore $\int_{[1,x)} t^\gamma \,\mathrm{d}\sigma(t) \geq \sigma([1,x)) \to \infty$ as $x \to \infty$, again a contradiction.

⑤ For the proof of (ii) we argue in a similar way. Integrate by parts (using the measure $\mathrm{d}\nu(t) = \frac{\mathrm{d}t}{t^{\rho+1}}$) to obtain

$$\int_1^\infty \sigma([1,t))\frac{\mathrm{d}t}{t^{\rho+1}} = \frac{1}{\rho}\int_{[1,\infty)} \frac{\mathrm{d}\sigma(t)}{t^\rho}, \qquad \lim_{x\to\infty} \frac{\sigma([1,x))}{x^\rho} = 0.$$

The second relation shows, in particular, that $\alpha \leq \rho$. We integrate by parts again to obtain

$$\begin{aligned}\int_x^\infty \sigma([1,t))\frac{\mathrm{d}t}{t^{\rho+1}} &= \int_x^\infty \sigma([1,x))\frac{\mathrm{d}t}{t^{\rho+1}} + \int_x^\infty \sigma([x,t))\frac{\mathrm{d}t}{t^{\rho+1}} \\ &= \frac{1}{\rho}\cdot\frac{\sigma([1,x))}{x^\rho} + \frac{1}{\rho}\int_{[x,\infty)} \frac{\mathrm{d}\sigma(t)}{t^\rho},\end{aligned}$$

and hence

$$\int_{[x,\infty)} \frac{\mathrm{d}\sigma(t)}{t^\rho} \bigg/ \int_x^\infty \frac{\sigma([1,t))}{t^{\rho+1}}\,\mathrm{d}t = \rho - \left(\frac{\sigma([1,x))}{x^\rho} \bigg/ \int_x^\infty \frac{\sigma([1,t))}{t^{\rho+1}}\,\mathrm{d}t\right).$$

By Theorem A.2 (ii) the integral $\int_{[1,x)} t^{-(\rho+1)}\sigma([1,t))\,\mathrm{d}t$ is regularly varying with index $\alpha-\rho$, and the quotient on the right-hand side tends to $\rho - \alpha$. The assertions made in (ii) follow.

❑

The following example shows that, when $\alpha = 0$ and $\gamma > 0$ in Proposition A.4 (i), the function $x \mapsto \int_{[1,x)} t^\gamma \,\mathrm{d}\sigma(t)$ may fail to be regularly varying. Instead of t^γ we consider an arbitrary function g with $g(t) \sim t^\gamma$ as $t \to \infty$. This more general example is used in Example A.12.

A.5 Example. Define the discrete measure σ supported on $\{e^k : k \in \mathbb{N}\}$ with the following point masses:

$$\sigma(\{e^k\}) = 1, \qquad k \in \mathbb{N}.$$

The distribution function satisfies $\sigma([1,t)) = 0$ if $t \leq e$ and $\sigma([1,t)) = n$ if $t \in (e^n, e^{n+1}]$ for $n \in \mathbb{N}$. The relation $t \in (e^n, e^{n+1}]$ is equivalent to $\log t - 1 \leq n < \log t$, and hence $\sigma([1,t)) \sim \log t$ as $t \to \infty$. This shows that the distribution function $t \mapsto \sigma([1,t))$ is slowly varying.

Now let $g : [1,\infty) \to (0,\infty)$ be a function such that $g(t) \sim t^\gamma$ as $t \to \infty$ with $\gamma > 0$, and define the function

$$f(x) := \int_{[1,x)} g(t)\,\mathrm{d}\sigma(t), \qquad x \in (1,\infty).$$

For $x \in (e^{n-1}, e^n]$ we have $f(x) = \sum_{k=1}^{n-1} g(e^k)$. It follows easily that, as $n \to \infty$,

$$\sum_{k=1}^{n-1} g(e^k) \sim \sum_{k=1}^{n-1} e^{\gamma k} = \frac{e^{\gamma n} - e^\gamma}{e^\gamma - 1} \sim \frac{e^{\gamma n}}{e^\gamma - 1}. \tag{A.6}$$

Now choose $\lambda = \sqrt{e}$. From (A.6) we obtain

$$\frac{f(\lambda e^m)}{f(e^m)} = \frac{\sum_{k=1}^m g(e^k)}{\sum_{k=1}^{m-1} g(e^k)} \to e^\gamma \qquad \text{as } m \to \infty,$$

$$\frac{f(\lambda e^{m+\frac{1}{2}})}{f(e^{m+\frac{1}{2}})} = \frac{\sum_{k=1}^m g(e^k)}{\sum_{k=1}^m g(e^k)} = 1,$$

which implies that f is not regularly varying.

Note that $f(x) \asymp x^\gamma$ as $x \to \infty$, which can be seen from (A.6). ◊

The next topic we discuss is the asymptotic behaviour of Stieltjes transforms: let μ be a measure on $[0,\infty)$ which satisfies $\int_{[0,\infty)}(1+t)^{-1}\,\mathrm{d}\mu(t)<\infty$; then the Stieltjes transform of μ is defined by

$$\mathcal{S}[\mu](x) := \int_{[0,\infty)} \frac{\mathrm{d}\mu(t)}{t+x}, \qquad x>0. \tag{A.7}$$

As is common practice in the literature, we use a sign convention that is different from the one in (1.2) (i.e. different from the convention for the Cauchy transform) in order to obtain functions that are defined on the positive half-line.

A.6 Remark. Let μ be a measure on $[0,\infty)$ which satisfies $\int_{[0,\infty)}(1+t)^{-1}\,\mathrm{d}\mu(t)<\infty$. Then, by the dominated and monotone convergence theorems,

$$\lim_{x\to\infty}\mathcal{S}[\mu](x)=0, \qquad \lim_{x\to\infty} x\mathcal{S}[\mu](x)=\mu\big([0,\infty)\big).$$

In particular, if μ is not the zero measure, then, as $x\to\infty$,

$$\begin{aligned}\frac{1}{x} &\asymp \mathcal{S}[\mu](x) \ll 1 \quad \text{if } \mu \text{ is finite,}\\ \frac{1}{x} &\ll \mathcal{S}[\mu](x) \ll 1 \quad \text{if } \mu \text{ is infinite.}\end{aligned}$$

◊

Karamata's theorem about the Stieltjes transform [16, 17] characterises regular variation of the Stieltjes transform and gives precise information about the size of $\mathcal{S}[\mu](x)$ also when μ is infinite. We use it in a formulation that includes a boundary case; this is often excluded, e.g. in [3, Theorem 1.7.4] or [31].

A.7 Theorem (Karamata)**.** *Let μ be a measure on $[0,\infty)$, which is not the zero measure and satisfies $\int_{[0,\infty)}(1+t)^{-1}\,\mathrm{d}\mu(t)<\infty$. Then the following two statements are equivalent:*

(i) *the distribution function $t\mapsto\mu([0,t))$ is regularly varying with index α;*

(ii) *$\mathcal{S}[\mu]$ is regularly varying with index $\alpha-1$.*

If (i) *and* (ii) *hold, then $\alpha\in[0,1]$ and*

$$\mathcal{S}[\mu](x) \sim C_\alpha \int_x^\infty \frac{\mu([0,t))}{t^2}\,\mathrm{d}t, \qquad x\to\infty. \tag{A.8}$$

with

$$C_\alpha := \begin{cases} \dfrac{\pi\alpha(1-\alpha)}{\sin(\pi\alpha)}, & \alpha\in(0,1);\\ 1, & \alpha\in\{0,1\}.\end{cases} \tag{A.9}$$

The integral in (A.8) *is finite for every $x>0$.*

As in the usual presentations in textbooks we follow the lines of [16] and reduce the problem to the Laplace–Stieltjes transform. Recall that the Laplace–Stieltjes transform of a positive measure ν on $[0,\infty)$ is the function $\mathscr{L}[\nu]\colon\mathbb{R}\to[0,\infty]$ defined as

$$\mathscr{L}[\nu](x) := \int_{[0,\infty)} e^{-xt}\,\mathrm{d}\nu(t), \qquad x\in\mathbb{R}.$$

In the proof of Theorem A.7 we also need the concept of a regularly varying function at 0: a function $g:(0,x_0]\to(0,\infty)$ with $x_0>0$ is called *regularly varying at* 0 *with index β* if $x\mapsto g\big(\frac{1}{x}\big)$ is regularly varying with index $-\beta$ in the sense of Definition A.1.

Proof of Theorem A.7.

① The relation with the Stieltjes transform is established by Fubini's theorem: for $x > 0$ we have

$$\begin{aligned}\mathcal{S}[\mu](x) &= \int_{[0,\infty)} \frac{1}{t+x}\,\mathrm{d}\mu(t) = \int_{[0,\infty)} \Big(\int_{[0,\infty)} e^{-(t+x)s}\,\mathrm{d}s\Big)\,\mathrm{d}\mu(t) \\ &= \int_{[0,\infty)} e^{-xs}\Big(\int_{[0,\infty)} e^{-ts}\,\mathrm{d}\mu(t)\Big)\,\mathrm{d}s = \int_{[0,\infty)} e^{-xs}\,\mathcal{L}[\mu](s)\,\mathrm{d}s. \qquad (A.10)\end{aligned}$$

Let σ be the measure on $[0,\infty)$ with density $\mathcal{L}[\mu]$, i.e.

$$\sigma([0,t)) := \int_0^t \mathcal{L}[\mu](s)\,\mathrm{d}s, \qquad t > 0.$$

The latter integral is finite since, again by Fubini's theorem,

$$\int_0^t \mathcal{L}[\mu](s)\,\mathrm{d}s = \int_0^t \int_{[0,\infty)} e^{-sr}\,\mathrm{d}\mu(r)\,\mathrm{d}s = \int_{[0,\infty)} \frac{1-e^{-tr}}{r}\,\mathrm{d}\mu(r);$$

the finiteness of the last integral follows from the assumption $\int_{[0,\infty)}(1+t)^{-1}\mathrm{d}\mu(t) < \infty$. Now (A.10) can be written as $\mathcal{S}[\mu] = \mathcal{L}[\sigma]$.

② Assume first that $x \mapsto \mu([0,x))$ is regularly varying with index α. Since $x \mapsto \mu([0,x))$ is non-decreasing, $\alpha \geq 0$; by Proposition A.4 (ii) we have $\alpha \leq 1$. It follows from [3, Theorem 1.7.1] in the form of [3, (1.7.3)] that

$$\mathcal{L}[\mu]\Big(\frac{1}{x}\Big) \sim \Gamma(1+\alpha)\mu([0,x)), \qquad x \to \infty. \qquad (A.11)$$

In particular, the function $x \mapsto \mathcal{L}[\mu](\frac{1}{x})$ is regularly varying with index α. Note that in [3] the right-continuous distribution function $x \mapsto \mu([0,x])$ is used. However, the latter is regularly varying if and only if $x \mapsto \mu([0,x))$ is regularly varying and $\mu([0,x]) \sim \mu([0,x))$, $x \to \infty$ if these functions are regularly varying.

Before we apply the Laplace–Stieltjes transform a second time, we need the asymptotic behaviour of $x \mapsto \sigma\big([0,\frac{1}{x})\big)$ as $x \to \infty$. From (A.11) we obtain

$$\sigma\big([0,\tfrac{1}{x})\big) = \int_0^{\frac{1}{x}} \mathcal{L}[\mu](s)\,\mathrm{d}s = \int_x^\infty \mathcal{L}[\mu]\Big(\frac{1}{t}\Big)\frac{1}{t^2}\,\mathrm{d}t \sim \Gamma(1+\alpha)\int_x^\infty \frac{\mu([0,t))}{t^2}\,\mathrm{d}t, \qquad x \to \infty.$$

It follows from Proposition A.4 (ii) that the function $x \mapsto \sigma\big([0,\frac{1}{x})\big)$ is regularly varying with index $\alpha - 1$, and hence, $t \mapsto \sigma([0,t))$ is regularly varying at 0 with index $1-\alpha$. Now [3, Theorem 1.7.1$'$] implies that

$$\begin{aligned}\mathcal{S}[\mu](x) &= \mathcal{L}[\sigma](x) \sim \Gamma(2-\alpha)\sigma\big([0,\tfrac{1}{x})\big) \\ &\sim \Gamma(1+\alpha)\Gamma(2-\alpha)\int_{[x,\infty)} \frac{\mu([0,t))}{t^2}\,\mathrm{d}t, \qquad x \to \infty,\end{aligned}$$

and that $\mathcal{S}[\mu]$ is regularly varying (at infinity) with index $\alpha-1$. It follows from the reflection formula for the Gamma function that $\Gamma(1+\alpha)\Gamma(2-\alpha) = \alpha(1-\alpha)\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi(1-\alpha)\alpha}{\sin(\pi\alpha)}$ when $\alpha \in (0,1)$.

③ Conversely, assume that (ii) holds. Again by [3, Theorem 1.7.1$'$], the function $t \mapsto \sigma([0,t))$ is regularly varying at 0 with index $1-\alpha$. Since $\mathcal{L}[\mu]$ is non-increasing, we can apply [3, Theorem 1.7.2b] to deduce that $\mathcal{L}[\mu]$ is regularly varying at 0 with index $-\alpha$. Finally, we obtain from [3, Theorem 1.7.1] that (i) holds.

❑

A.8 Remark. In the case when $\alpha < 1$ we can use Theorem A.2 (ii) to rewrite the right-hand side of (A.8) to obtain the standard formulation as in [3, Theorem 1.7.4]; namely, under the assumption of Theorem A.7 we have

$$\mathcal{S}[\mu](x) \sim \frac{\pi\alpha}{\sin(\pi\alpha)} \cdot \frac{\mu([0,x))}{x}, \qquad x \to \infty,$$

where the first fraction on the right-hand side is understood as 1 if $\alpha = 0$. ◊

In the proofs of our main results we often need the following elementary facts where we compare the Stieltjes transforms of two measures.

A.9 Lemma. *Let μ_1 and μ_2 be measures on $[0,\infty)$ such that μ_2 is not the zero measure and that $\int_{[0,\infty)}(1+t)^{-1}\mathrm{d}\mu_i(t) < \infty$ for $i \in \{1,2\}$.*

(i) *If $\mu_1([0,t)) \le \mu_2([0,t))$ for all $t \in (0,\infty)$, then $\mathcal{S}[\mu_1](x) \le \mathcal{S}[\mu_2](x)$ for all $x \in (0,\infty)$.*

(ii) *We have*

$$\limsup_{x\to\infty} \frac{\mathcal{S}[\mu_1](x)}{\mathcal{S}[\mu_2](x)} \le \limsup_{r\to\infty} \frac{\mu_1([0,r))}{\mu_2([0,r))}. \tag{A.12}$$

Proof. We use Lemma A.3 with the measure $\mathrm{d}\nu(t) := \frac{\mathrm{d}t}{(t+x)^2}$ for $x > 0$ to rewrite the Stieltjes transforms:

$$\mathcal{S}[\mu_i](x) = \int_{[0,\infty)} \frac{1}{t+x}\,\mathrm{d}\mu_i(t) = \int_0^\infty \frac{\mu_i([0,t))}{(t+x)^2}\,\mathrm{d}t, \qquad x > 0,\ i \in \{1,2\}, \tag{A.13}$$

which immediately yields the assertion in (i).

Let us now prove the statement in (ii). If $\mu_1 = 0$ or the right-hand side of (A.12) is infinite, then there is nothing to prove. Hence we assume that μ_1 is not the zero measure and the right-hand side of (A.12) is finite. Let $r_0 > 0$ such that $\mu_i([0,r_0)) > 0$ for $i \in \{1,2\}$. From (A.13) we obtain, for $x > 0$ (and the asymptotic relations as $x \to \infty$),

$$\begin{aligned}
\mathcal{S}[\mu_1](x) &= \int_0^\infty \frac{\mu_1([0,t))}{(t+x)^2}\,\mathrm{d}t = \underbrace{\int_0^{r_0} \frac{\mu_1([0,t))}{(t+x)^2}\,\mathrm{d}t}_{\le \frac{1}{x^2}\cdot r_0\mu_1([0,r_0))} + \underbrace{\int_{r_0}^\infty \frac{\mu_1([0,t))}{(t+x)^2}\,\mathrm{d}t}_{\ge \mu_1([0,r_0))\cdot\frac{1}{r_0+x}} \\
&\sim \int_{r_0}^\infty \frac{\mu_1([0,t))}{(t+x)^2}\,\mathrm{d}t \le \sup_{t\ge r_0} \frac{\mu_1([0,t))}{\mu_2([0,t))} \cdot \int_{r_0}^\infty \frac{\mu_2([0,t))}{(t+x)^2}\,\mathrm{d}t \\
&\sim \sup_{t\ge r_0} \frac{\mu_1([0,t))}{\mu_2([0,t))} \cdot \int_0^\infty \frac{\mu_2([0,t))}{(t+x)^2}\,\mathrm{d}t = \sup_{t\ge r_0} \frac{\mu_1([0,t))}{\mu_2([0,t))} \cdot \mathcal{S}[\mu_2](x).
\end{aligned}$$

With this we can deduce that $\limsup_{x\to\infty} \frac{\mathcal{S}[\mu_1](x)}{\mathcal{S}[\mu_2](x)} \le \sup_{t\ge r_0} \frac{\mu_1([0,t))}{\mu_2([0,t))}$. Since r_0 was arbitrary, the assertion follows. ❑

We also need a comparison result when we integrate powers with respect to two different measures.

A.10 Lemma. *Let ν_1, ν_2 be measures on $[0,\infty)$ and let $\gamma \in \mathbb{R} \setminus \{0\}$.*

(i) *Assume that $\gamma < 0$ and that $\nu_1([0,x)) \lesssim \nu_2([0,x))$ as $x \to \infty$. Then*

$$\int_{[1,x)} t^\gamma\,\mathrm{d}\nu_1(t) \lesssim \int_{[1,x)} t^\gamma\,\mathrm{d}\nu_2(t), \qquad x \to \infty. \tag{A.14}$$

If, in addition, $\int_{[1,\infty)} t^\gamma\,\mathrm{d}\nu_2(t) = \infty$, then

$$\limsup_{x\to\infty} \frac{\int_{[1,x)} t^\gamma\,\mathrm{d}\nu_1(t)}{\int_{[1,x)} t^\gamma\,\mathrm{d}\nu_2(t)} \le \limsup_{x\to\infty} \frac{\nu_1([0,x))}{\nu_2([0,x))}. \tag{A.15}$$

(ii) *Assume that* $\gamma > 0$ *and that* $x \mapsto \nu_2([0,x))$ *is regularly varying with index* $\alpha > 0$*. Then*

$$\limsup_{x\to\infty} \frac{\int_{[1,x)} t^\gamma \,\mathrm{d}\nu_1(t)}{\int_{[1,x)} t^\gamma \,\mathrm{d}\nu_2(t)} \le \frac{\alpha+\gamma}{\alpha} \limsup_{x\to\infty} \frac{\nu_1([0,x))}{\nu_2([0,x))}. \tag{A.16}$$

(iii) *Assume that* $\gamma > 0$ *and that* $x \mapsto \nu_i([0,x))$ *are regularly varying with index* $\alpha_i > 0$ *for* $i \in \{1,2\}$*. Then*

$$\frac{\int_{[1,x)} t^\gamma \,\mathrm{d}\nu_1(t)}{\int_{[1,x)} t^\gamma \,\mathrm{d}\nu_2(t)} \sim \frac{\alpha_1(\alpha_2+\gamma)}{\alpha_2(\alpha_1+\gamma)} \cdot \frac{\nu_1([0,x))}{\nu_2([0,x))}, \qquad x\to\infty.$$

Proof. If the limit superior on the right-hand side of (A.16) is $+\infty$, then there is nothing to prove for (ii). Hence, for the proof (ii) we assume that the right-hand side of (A.16) is finite. In (i) the right-hand side of (A.15) is finite by assumption. Let $M > 0$ and $x_0 > 1$ be such that

$$\forall x \ge x_0: \ \frac{\nu_1([0,x))}{\nu_2([0,x))} \le M. \tag{A.17}$$

(i) Assume that $\gamma < 0$. For $i \in \{1,2\}$, integration by parts yields

$$\begin{aligned}\int_{[1,x)} t^\gamma \,\mathrm{d}\nu_i(t) &= x^\gamma \nu_i\big([0,x)\big) - \nu_i\big([0,1)\big) - \gamma \int_1^x t^{\gamma-1}\nu_i([0,t))\,\mathrm{d}t \\ &= x^\gamma \nu_i\big([0,x)\big) - \nu_i\big([0,1)\big) + |\gamma| \int_1^{x_0} t^{\gamma-1}\nu_i([0,t))\,\mathrm{d}t + |\gamma|\int_{x_0}^x t^{\gamma-1}\nu_i([0,t))\,\mathrm{d}t \\ &= x^\gamma \nu_i\big([0,x)\big) + |\gamma|\int_{x_0}^x t^{\gamma-1}\nu_i([0,t))\,\mathrm{d}t + c_i\end{aligned}$$

with some $c_i \in \mathbb{R}$. Together with (A.17) we obtain, for $x \ge x_0$,

$$\begin{aligned}\int_{[1,x)} t^\gamma \,\mathrm{d}\nu_1(t) &= x^\gamma \nu_1\big([0,x)\big) + |\gamma|\int_{x_0}^x t^{\gamma-1}\nu_1([0,t))\,\mathrm{d}t + c_1 \\ &\le M x^\gamma \nu_2\big([0,x)\big) + M|\gamma|\int_{x_0}^x t^{\gamma-1}\nu_2([0,t))\,\mathrm{d}t + c_1 \\ &= M\int_{[1,x)} t^\gamma \,\mathrm{d}\nu_2(t) - Mc_2 + c_1.\end{aligned}$$

This proves (A.14). Now assume that $\int_{[1,\infty)} t^\gamma \,\mathrm{d}\nu_2(t) = \infty$. Then

$$\frac{\int_{[1,x)} t^\gamma \,\mathrm{d}\nu_1(t)}{\int_{[1,x)} t^\gamma \,\mathrm{d}\nu_2(t)} \le M + \frac{c_1 - Mc_2}{\int_{[1,x)} t^\gamma \,\mathrm{d}\nu_2(t)} \to M, \qquad x\to\infty,$$

from which we can deduce that

$$\limsup_{x\to\infty} \frac{\int_{[1,x)} t^\gamma \,\mathrm{d}\nu_1(t)}{\int_{[1,x)} t^\gamma \,\mathrm{d}\nu_2(t)} \le \sup_{t\ge x_0} \frac{\nu_1([0,t))}{\nu_2([0,t))}.$$

Since x_0 was arbitrary, the inequality in (A.15) follows.

(ii) Now we assume that $\gamma > 0$ and that $x \mapsto \nu_2([0,x))$ is regularly varying with index α. The latter, together with Proposition A.4 (i) and Theorem A.2 (i), implies

$$\int_{[1,x)} t^\gamma \,\mathrm{d}\nu_2(t) \sim \frac{\alpha}{\alpha+\gamma} x^\gamma \nu_2([1,x)), \qquad x\to\infty. \tag{A.18}$$

The monotonicity of $t \mapsto t^\gamma$ and (A.17) yield, for $x \geq x_0$,

$$\int_{[1,x)} t^\gamma \,\mathrm{d}\nu_1(t) \leq x^\gamma \nu_1\big([1,x)\big) \leq x^\gamma \nu_1\big([0,x)\big) \leq M x^\gamma \nu_2\big([0,x)\big)$$

$$= M x^\gamma \nu_2\big([1,x)\big)\left(1 + \frac{\nu_2([0,1))}{\nu_2([1,x))}\right). \tag{A.19}$$

Since, by assumption, $\alpha > 0$, we have $\nu_2([1,x)) \to \infty$. Together with (A.18) and (A.19), this shows that

$$\limsup_{x\to\infty} \frac{\int_{[1,x)} t^\gamma \,\mathrm{d}\nu_1(t)}{\int_{[1,x)} t^\gamma \,\mathrm{d}\nu_2(t)} \leq M \frac{\alpha+\gamma}{\alpha}.$$

Now the assertion follows as in the proof of (i).

(iii) In the same way as in the proof of (ii), we obtain from Proposition A.4 (i) and Theorem A.2 (i) that

$$\int_{[1,x)} t^\gamma \,\mathrm{d}\nu_i(t) \sim \frac{\alpha_i}{\alpha_i+\gamma} x^\gamma \nu_i([1,x)), \qquad x \to \infty,$$

for $i \in \{1,2\}$. From this the result follows since $\nu_i([1,x)) \sim \nu_i([0,x))$. ❑

The next proposition contains asymptotic results about Stieltjes transforms of certain measures; it plays a key role in the proofs of some of the main results of the paper.

A.11 Proposition. *Let ν be a measure on $[0,\infty)$. Further, let $h : [0,\infty) \to [0,\infty)$ be a continuous function such that $h(t) > 0$ for $t > 0$ and $h(t) \sim t^\gamma$, $t \to \infty$, with $\gamma \in \mathbb{R} \setminus \{0\}$. Assume that*

$$\int_{[0,\infty)} h(t) \,\mathrm{d}\nu(t) = \infty \qquad \textit{and} \qquad \int_{[0,\infty)} \frac{h(t)}{1+t} \,\mathrm{d}\nu(t) < \infty \tag{A.20}$$

and define the measure σ on $[0,\infty)$ by

$$\mathrm{d}\sigma(t) = h(t)\,\mathrm{d}\nu(t), \qquad t \in [0,\infty).$$

Then the Stieltjes transform $\mathcal{S}[\sigma]$ is well defined and satisfies

$$\frac{1}{x} \ll \mathcal{S}[\sigma](x) \ll 1, \qquad x \to \infty. \tag{A.21}$$

Now let $\alpha \geq 0$ and consider the following two statements:

(a) *$t \mapsto \nu([0,t))$ is regularly varying with index α;*

(b) *$\mathcal{S}[\sigma]$ is regularly varying with index $\alpha + \gamma - 1$.*

Then the following relations are true.

(i) *Assume that* (a) *holds. Then $\alpha + \gamma \in [0,1]$.*

If $\alpha > 0$, then (b) *holds and, as $x \to \infty$,*

$$\mathcal{S}[\sigma](x) \sim \begin{cases} \dfrac{\pi\alpha}{\sin(\pi(\alpha+\gamma))} \cdot x^{\gamma-1}\nu([0,x)), & \alpha+\gamma \in (0,1), \\[2ex] \dfrac{\alpha}{x}\displaystyle\int_1^x t^{\gamma-1}\nu([0,t))\,\mathrm{d}t, & \alpha+\gamma = 0, \\[2ex] \alpha\displaystyle\int_x^\infty t^{\gamma-2}\nu([0,t))\,\mathrm{d}t, & \alpha+\gamma = 1. \end{cases} \tag{A.22}$$

If $\alpha = 0$ and $\gamma \in (0,1)$, then

$$\mathcal{S}[\sigma](x) \ll x^{\gamma-1}\nu([0,x)), \qquad x \to \infty. \tag{A.23}$$

(ii) *If* (b) *holds and* $\alpha + \gamma > 0$*, then also* (a) *holds.*

Proof. The relations in (A.20) imply that the Stieltjes transform $\mathcal{S}[\sigma]$ is well defined and that the measure σ is infinite. Hence (A.21) follows from Remark A.6.

(i) Assume that (a) holds. It follows from (A.20) that $\int_{[1,\infty)} t^\gamma \,\mathrm{d}\nu(t) = \infty$ and $\int_{[1,\infty)} t^{\gamma-1} \,\mathrm{d}\nu(t) < \infty$, which, by Proposition A.4 (i), yields that $\alpha+\gamma \geq 0$ and $\alpha+\gamma-1 \leq 0$.

Let us first consider the case when $\alpha > 0$. The first relation in (A.20) and Proposition A.4 (i) imply that, as $x \to \infty$,

$$\sigma([0,x)) \sim \int_{[1,x)} t^\gamma \,\mathrm{d}\nu(t) \sim \alpha \int_1^x t^{\gamma-1}\nu([0,t))\,\mathrm{d}t \tag{A.24}$$

and that $x \mapsto \sigma([0,x))$ is regularly varying with index $\alpha+\gamma$. Hence we can apply Theorem A.7 to deduce that (b) holds and that

$$\mathcal{S}[\sigma](x) \sim C_{\alpha+\gamma} \int_x^\infty \frac{\sigma([0,t))}{t^2}\,\mathrm{d}t, \tag{A.25}$$

where $C_{\alpha+\gamma}$ is defined in (A.9). If $\alpha+\gamma < 1$, then Theorem A.2 (ii) and (A.24) yield

$$\mathcal{S}[\sigma](x) \sim \frac{C_{\alpha+\gamma}}{1-\alpha-\gamma} \cdot \frac{\sigma([0,x))}{x} \sim \alpha \frac{C_{\alpha+\gamma}}{1-\alpha-\gamma} \cdot \frac{1}{x} \int_1^x t^{\gamma-1}\nu([0,t))\,\mathrm{d}t.$$

In the case when $\alpha+\gamma = 0$, the first fraction in front of the integral is equal to 1. If $\alpha+\gamma \in (0,1)$, we can apply Theorem A.2 (i) to arrive at

$$\mathcal{S}[\sigma](x) \sim \frac{\alpha C_{\alpha+\gamma}}{1-\alpha-\gamma} \cdot \frac{1}{x} \cdot \frac{1}{\alpha+\gamma} \cdot x^\gamma \nu([0,x)) = \frac{\pi\alpha}{\sin(\pi(\alpha+\gamma))} \cdot x^{\gamma-1}\nu([0,x)),$$

which proves (A.22) in the first two cases.

Now assume that $\alpha+\gamma = 1$. We obtain from (A.24) and Theorem A.2 (i) that

$$\sigma([0,x)) \sim \alpha x^\gamma \nu([0,x)),$$

which, together with (A.25), implies that

$$\mathcal{S}[\sigma](x) \sim \alpha C_1 \int_x^\infty t^{\gamma-2}\nu([0,t))\,\mathrm{d}t,$$

which proves (A.22) in the third case.

Let us now assume that $\alpha = 0$ and $\gamma \in (0,1)$. It follows again from Proposition A.4 (i) that

$$\sigma([0,x)) \sim \int_{[1,x)} t^\gamma \,\mathrm{d}\nu(t) \ll \int_1^x t^{\gamma-1}\nu([0,t))\,\mathrm{d}t. \tag{A.26}$$

Define the measure μ on $[0,\infty)$ by $\mathrm{d}\mu(t) = t^{\gamma-1}\nu([0,t))\,\mathrm{d}t$. Then

$$\int_{[1,\infty)} \frac{1}{t}\,\mathrm{d}\mu(t) = \int_{[1,\infty)} t^{\gamma-2}\nu([0,t))\,\mathrm{d}t < \infty$$

since the integrand in the second integral is regularly varying with index $\gamma - 2 < -1$. From Theorem A.2 (i) we obtain that $x \mapsto \mu([0,x))$ is regularly varying with index γ, and (A.26) implies that

$$\mu([0,x)) = \int_0^x t^{\gamma-1}\nu([0,t))\,\mathrm{d}t \gg \sigma([0,x)).$$

Hence we can apply Lemma A.9 and Remark A.8 to deduce that

$$\begin{aligned}\mathcal{S}[\sigma](x) \ll \mathcal{S}[\mu](x) &\sim \frac{\pi\gamma}{\sin(\pi\gamma)} \cdot \frac{\mu([0,x))}{x} \\ &= \frac{\pi\gamma}{\sin(\pi\gamma)} \cdot \frac{1}{x}\int_0^x t^{\gamma-1}\nu([0,t))\,\mathrm{d}t \sim \frac{\pi}{\sin(\pi\gamma)} \cdot \frac{1}{x} x^\gamma \nu([0,x)),\end{aligned}$$

which proves (A.23).

(ii) Assume that (b) holds and that $\alpha + \gamma > 0$. It follows from Theorem A.7 that $x \mapsto \sigma([0,x))$ is regularly varying with index $\alpha+\gamma$. Since $\mathrm{d}\nu(t) = \frac{1}{h(t)}\,\mathrm{d}\sigma(t)$, $t \in (0,\infty)$, we obtain from Proposition A.4 (i) that (a) holds. ❑

The following example shows that the implication (b) ⇒ (a) is not valid in general if $\alpha+\gamma = 0$. The example is also used in Examples 4.9 and 5.3.

A.12 Example. Let $h : [0,\infty) \to [0,\infty)$ be a continuous function such that $h(t) > 0$ for $t > 0$ and $h(t) \sim t^\gamma$ with $\gamma < 0$. Further, let σ be the discrete measure on $[0,\infty)$ as in Example A.5, i.e. with point masses $\sigma(\{e^k\}) = 1$, $k \in \mathbb{N}$. Let ν be the measure on $[0,\infty)$ such that $\mathrm{d}\sigma(t) = h(t)\,\mathrm{d}\nu(t)$, $t \in (0,\infty)$ and $\nu(\{0\}) = 0$; it is also discrete and has point masses

$$\nu(\{e^k\}) = \frac{1}{h(e^k)}, \qquad k \in \mathbb{N}.$$

Since

$$\int_{[0,\infty)} h(t)\,\mathrm{d}\nu(t) = \int_{[0,\infty)} \mathrm{d}\sigma(t) = \infty,$$

$$\int_{[0,\infty)} \frac{h(t)}{1+t}\,\mathrm{d}\nu(t) = \int_{[0,\infty)} \frac{1}{1+t}\,\mathrm{d}\sigma(t) = \sum_{k=1}^{\infty} \frac{1}{1+e^k} < \infty,$$

condition (A.20) is satisfied. Example A.5 shows that the distribution function $t \mapsto \sigma([0,t))$ is slowly varying. Hence, by Theorem A.7, $\mathcal{S}[\sigma]$ is regularly varying with index -1, i.e. (b) in Proposition A.11 is satisfied with $\alpha+\gamma-1 = -1$. Note that $\sigma([0,t)) \sim \log t$ and therefore

$$\mathcal{S}[\sigma](x) \sim \frac{\log x}{x}, \qquad x \to \infty, \tag{A.27}$$

by Remark A.8. On the other hand, Example A.5 with g such that $g(t) = \frac{1}{h(t)}$, $t \in [1,\infty)$, also implies that

$$x \mapsto \nu([0,x)) = \int_{[0,x)} g(t)\,\mathrm{d}\sigma(t) = f(x),$$

with f from Example A.5, is not regularly varying, i.e. (a) in Proposition A.11 is not satisfied. Hence the implication (b) ⇒ (a) does not hold in general when $\alpha+\gamma = 0$. ♢

Finally, we need a comparison result for Stieltjes transforms of measures as in Proposition A.11.

A.13 Lemma. *Let ν_1, ν_2 be measures on $[0,\infty)$, let $h : [0,\infty) \to [0,\infty)$ be a continuous function such that $h(t) > 0$ for $t > 0$ and $h(t) \sim t^\gamma$, $t \to \infty$, with some $\gamma \in \mathbb{R}\setminus\{0\}$. Assume that*

$$\int_{[0,\infty)} h(t)\,\mathrm{d}\nu_2(t) = \infty \qquad \text{and} \qquad \int_{[0,\infty)} \frac{h(t)}{1+t}\,\mathrm{d}\nu_2(t) < \infty \tag{A.28}$$

and define the measures σ_i on $[0,\infty)$ by $\mathrm{d}\sigma_i(t) = h(t)\,\mathrm{d}\nu_i(t)$, $t \in [0,\infty)$, $i \in \{1,2\}$. Further, assume that $t \mapsto \nu_2([0,t))$ is regularly varying with strictly positive index and that the limit

$$\lim_{t\to\infty} \frac{\nu_1([0,t))}{\nu_2([0,t))} \tag{A.29}$$

exists in $[0,\infty)$. Then $\mathcal{S}[\sigma_1]$ is well defined and

$$\lim_{x\to\infty} \frac{\mathcal{S}[\sigma_1](x)}{\mathcal{S}[\sigma_2](x)} = \lim_{t\to\infty} \frac{\nu_1([0,t))}{\nu_2([0,t))}. \tag{A.30}$$

Proof. Since $\int_{[1,\infty)} t^{\gamma-1}\,\mathrm{d}\nu_2(t) < \infty$ by (A.28), we can use the existence of the limit in (A.29) and Lemma A.10 (i) (when $\gamma < 1$) or Lemma A.10 (ii) (when $\gamma > 1$) to deduce that $\int_{[1,\infty)} t^{\gamma-1}\,\mathrm{d}\nu_1(t) < \infty$ (for $\gamma = 1$ this follows directly). Hence $\mathcal{S}[\sigma_1]$ is well defined.

Denote the limit in (A.29) by c. Let us first consider the case when $c > 0$. Then $t \mapsto \nu_1([0,t))$ is regularly varying with the same index as $t \mapsto \nu_2([0,t))$, and $\int_{[1,\infty)} t^{\gamma}\,\mathrm{d}\nu_1(t) = \infty$ (again by Lemma A.10). We now obtain from Lemma A.10 (i) (when $\gamma < 0$) or Lemma A.10 (iii) (when $\gamma > 0$) that

$$\lim_{x\to\infty} \frac{\sigma_1([0,x))}{\sigma_2([0,x))} = \lim_{x\to\infty} \frac{\int_{[1,x)} t^{\gamma}\,\mathrm{d}\nu_1(t)}{\int_{[1,x)} t^{\gamma}\,\mathrm{d}\nu_2(t)} = c.$$

By Lemma A.9, this implies that (A.30) holds.

Now assume that $c = 0$. From Lemma A.10 (i) and (ii) we can deduce that

$$\sigma_1([0,x)) \asymp \int_{[1,x)} t^{\gamma}\,\mathrm{d}\nu_1(t) \ll \int_{[1,x)} t^{\gamma}\,\mathrm{d}\nu_2(t) \sim \sigma_2([0,x)),$$

which, together with Lemma A.9, yields (A.30). ❑

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