Homogeneous spaces of entire functions

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Abstract: Homogeneous spaces are de Branges' Hilbert spaces of entire functions with the property that certain weighted rescaling transforms induce isometries of the space into itself. A classical example of a homogeneous space is the Paley-Wiener space of entire functions with exponential type at most a being square integrable on the real axis. Other examples occur in the theory of the Bessel equation. Being homogeneous is a strong property, and one can describe all homogeneous spaces, their structure Hamiltonians, and the measures associated with chains of such spaces, explicitly in terms of powers, logarithms, and confluent hypergeometric functions.

The theory of homogeneous spaces was in large parts settled by L.de Branges in the early 1960's. However, in his work some connections and explicit formulae are not given, some results are stated without a proof, and last but not least a mistake occurs which seemingly remained unnoticed up to the day.

In this paper we give a detailed account on homogeneous spaces. We provide explicit proofs for all formulae and relations between the mentioned objects, and correct the mentioned mistake.

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1 Introduction

The theory of *Hilbert spaces of entire functions* was founded by L.de Branges in the late 1950's and further developed in a series of papers. A comprehensive presentation appeared as the monograph [Bra68]. These spaces are reproducing kernel Hilbert spaces whose elements are entire functions and which possess certain additional properties.

1.1 Definition. A *de Branges space* is a Hilbert space $\mathcal{H}$ which satisfies the following axioms.

- ▷ The elements of $\mathcal{H}$ are entire functions and for each $w \in \mathbb{C} \setminus \mathbb{R}$ the point evaluation functional $F \mapsto F(w)$, $F \in \mathcal{H}$, is linear and continuous in the norm $\|.\|_{\mathcal{H}}$ of $\mathcal{H}$.
- ▷ For each $F \in \mathcal{H}$, also the function $F^{\#}(z) := \overline{F(\overline{z})}$ belongs to $\mathcal{H}$ and $\|F^{\#}\|_{\mathcal{H}} = \|F\|_{\mathcal{H}}$.
- ▷ If $w \in \mathbb{C} \setminus \mathbb{R}$ and $F \in \mathcal{H}$ with $F(w) = 0$, then

$$\frac{F(z)}{z-w} \in \mathcal{H} \quad \text{and} \quad \Big\| \frac{z-\overline{w}}{z-w} F(z) \Big\|_{\mathcal{H}} = \|F\|_{\mathcal{H}}.$$

Throughout this paper the notion of a de Branges space shall additionally include the following requirement:

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▷ For each $t \in \mathbb{R}$ there exists $F \in \mathcal{H}$ with $F(t) \neq 0$.

The first three properties imply that point evaluations are also continuous at each real point w, and together with the fourth property it follows that the third axiom holds also for all real points w.

Hilbert spaces of entire functions are an – in essence – equivalent view on entire operators in the sense of M.G.Krein, e.g. [GG97], which makes them relevant in an operator theoretic context. Further, they are equivalent to certain shift-coinvariant subspaces of the Hardy space, which makes them relevant in a function theoretic context. Typical areas where de Branges spaces occur are the spectral theory of Sturm-Liouville or Krein-Feller operators, e.g. [Rem02; KWW07], models for symmetric operators [Mar11; AMR13], interpolation and sampling e.g. [OS02; BB11], or Beurling-Malliavin type theorems e.g. [HM03a; HM03b]. Our standard reference is [Bra68]; other references are e.g. [BW15; Rom14; Rem18].

One can think of de Branges' theory as a generalisation of Fourier analysis. The maybe best known examples of de Branges spaces are the Paley-Wiener spaces: given $a > 0$ the set

$$\mathcal{PW}_a := \Big\{F \;\Big|\; F \text{ entire with exponential type} \leq a, \text{ and } \int_{\mathbb{R}} |F(x)|^2 \,\mathrm{d}x < \infty\Big\}$$

is a de Branges space when endowed with the L^2-scalar product induced by the Lebesgue measure on $\mathbb{R}$. These spaces, and de Branges' structure theory applied with them, are closely related to the exponential function and the classical sine- and cosine transforms.

Paley-Wiener spaces have several very specific structural properties. One of them is that each space $\mathcal{PW}_a$ is well-behaved with respect to a weighted rescaling transform: for all $c \in (0, 1]$ the map

$$F(z) \mapsto c^{\frac{1}{2}} F(cz) \tag{1.1}$$

maps $\mathcal{PW}_a$ isometrically into itself. The same phenomenon occurs in the theory of Bessel functions and, more generally, confluent hypergeometric functions. The only difference being that for the spaces related with such functions the power $\frac{1}{2}$ in (1.1) has to be replaced with another power. This fact served as a motivation for L.de Branges to formulate the following axiomatic definition, cf. [Bra68, §50].

1.2 Definition. Let $\nu > -1$ and let $\mathcal{H}$ be a de Branges space. Then $\mathcal{H}$ is called *homogeneous of order* ν, if for all $c \in (0, 1]$ the map

$$F(z) \mapsto c^{\nu+1} F(cz) \tag{1.2}$$

maps $\mathcal{H}$ isometrically into itself.

Being homogeneous is a very strong property, and not many de Branges spaces possess it.

The source of homogeneity of a space $\mathcal{H}$ can be pinpointed in different ways.

(1) The power series coefficients of a certain entire function determining $\mathcal{H}$ satisfy a recurrence of a particular form (a linear recurrence for the vector made up from the real- and imaginary parts of the coefficients).

(2) The family of reproducing kernels of de Branges subspaces of $\mathcal{H}$ (cf. Definition 1.7) satisfies a functional equation involving a weight and rescaling.

(3) The canonical system given by the chain of de Branges subspaces of $\mathcal{H}$ (Theorem 1.9) has a Hamiltonian of a particular form (involving powers and, possibly, logarithms).

(4) The norm of $\mathcal{H}$ is the L^2-norm given by a measure which is absolutely continuous w.r.t. the Lebesgue measure and has power density (on the left- and right half-axis separately), and the functions in the space are of bounded type in the upper half-plane.

Most of these topics were investigated in [Bra62] and [Bra68, Theorem 50], although some of them are not made explicit. For example it is stated on [Bra62, p.205] that "homogeneous spaces of entire functions are related to Bessel functions and more general confluent hypergeometric functions", but the actual formulae are not given. Also the passage from measures with power density to homogeneous de Branges spaces is not treated.

Homogeneous spaces appeared in some places in the literature. We mention [HV96; CL14; Vaa23] who use the theory in the context of an extremal problem from number theory, and [Gon17; GL18] who study invariance under differentiation and give interpolation formulae. It should be noted that in those references only homogeneous spaces occur which enjoy an additional symmetry property and are related to Bessel functions.

In the recent manuscript [ELW24] homogeneous spaces in their full generality corresponding to confluent hypergeometric functions play a decisive role. And it was during writing of that paper, that we found out that [Bra62; Bra68] contains a mistake. In fact, in the description of all homogeneous spaces of order $-\frac{1}{2}$ (which contains the Paley-Wiener case), a whole 1-parameter family of spaces was forgotten; details are explained in Remark 6.4. It seems that previously this mistake was not noticed; fortunately it also does not affect the earlier literature mentioned above due to the symmetry present in the spaces of those papers.

The purpose of our present paper is twofold. One, we provide a detailed account on homogeneity in de Branges spaces and all the viewpoints listed above, round off the picture by taking some slightly more general or more systematic viewpoints, and discuss side results which were not touched upon in de Branges' original work. Two, we correct the mentioned mistake. We should say it very clearly that our aim is to provide a comprehensive structured account on homogeneous spaces and to make the results accessible also to non-specialists. Hence, fully elaborated proofs of all assertions are included, also when some of them simply follow what was done by de Branges.

Let us briefly describe the structuring of the paper. In the second part of this introduction we set up our notation concerning de Branges spaces and recall several facts which are needed in order to make the presentation self-contained. Then we introduce the main players of the paper and provide some of their elementary properties (Section 2). There follow two sections (Sections 3 and 4) where we investigate the solutions of the canonical systems occurring in the present context: first, approaching them via the power series coefficient recurrence (Theorem 3.1) and, second, giving explicit formulae in terms of special functions (Theorem 4.1). In Section 5 we systematically investigate the group action of power-weighted rescalings in the context of de Branges spaces. Putting together all those results, this culminates in Sections 6 and 7 in a complete description of homogeneous spaces. We determine the structure of their chain of subspaces (Theorem 6.2), and the measures associated with chains of homogeneous spaces (Theorem 7.2).

1.1 De Branges spaces, chains, and measures

We recall the basics of the theory of Hilbert spaces of entire functions. This compilation is extracted almost exclusively from [Bra68].

№1

De Branges spaces via Hermite-Biehler functions

In Definition 1.1 we used an axiomatic way to introduce de Branges spaces. On a more concrete level, these objects may also be introduced via certain entire functions. The reason being that the reproducing kernel of a de Branges space has a very particular form.

1.3 Definition. A *Hermite-Biehler function* is an entire function E which satisfies

$$\forall z \in \mathbb{C}_+ : \quad |E(\overline{z})| < |E(z)|,$$

and has no real zeroes[1].

We denote the set of all Hermite-Biehler functions as $\mathcal{HB}$.

For any entire function E we use the generic notation

$$A := \frac{1}{2}(E + E^{\#}), \; B := \frac{i}{2}(E - E^{\#}), \qquad E = A - iB, \tag{1.3}$$

and denote

$$K_E(z, w) := \frac{B(z)A(\overline{w}) - B(\overline{w})A(z)}{z - \overline{w}}.$$

Then an entire function E without real zeroes belongs to $\mathcal{HB}$ if and only if K_E is a positive kernel and is not identically equal to zero.

The connection between de Branges spaces and the Hermite-Biehler class can be summarised as follows.

1.4 Theorem.

(i) *Let $E \in \mathcal{HB}$. Then the reproducing kernel Hilbert space $\mathcal{H}(E)$ generated by the positive kernel K_E is a de Branges space.*

(ii) *Let $\mathcal{H}$ be a de Branges space. Then there exists $E \in \mathcal{HB}$ such that $\mathcal{H} = \mathcal{H}(E)$.*

[1] We require absence of real zeroes in order to fit our convention from Definition 1.1 that for all real points a de Branges space should contain elements which do not vanish at that point.

(iii) *Let* $E, \tilde{E} \in \mathcal{HB}$. *Then* $\mathcal{H}(E) = \mathcal{H}(\tilde{E})$ *if and only if there exists* $M \in \mathrm{SL}(2, \mathbb{R})$ *such that*

$$(A, B) = (\tilde{A}, \tilde{B})M.$$

This theorem allows to switch between abstract and concrete levels. While the abstract – axiomatic – viewpoint is suitable to make the connection with operator theory, the concrete viewpoint allows to invoke classical function theory.

The description of a de Branges space $\mathcal{H}$ via the reproducing kernel K_E is implicit, since it involves a completion process to pass from the linear span of kernels to the whole space. An explicit description reads as follows.

1.5 Theorem. *Let* $E \in \mathcal{HB}$. *Then an entire function* F *belongs to the space* $\mathcal{H}(E)$, *if and only if*

$$\int_{\mathbb{R}} \Big|\frac{F(t)}{E(t)}\Big|^2 \,\mathrm{d}t < \infty \;\wedge\; \forall z \in \mathbb{C}\colon\;\; |F(z)|^2 \leq K_E(z,z) \cdot \int_{\mathbb{R}} \Big|\frac{F(t)}{E(t)}\Big|^2 \,\mathrm{d}t.$$

If $F \in \mathcal{H}(E)$, *then* $\|F\|^2 = \int_{\mathbb{R}} \big|\frac{F(t)}{E(t)}\big|^2 \,\mathrm{d}t$.

Note here that $K_E(z,z)^{\frac{1}{2}}$ is the norm of the point evaluation functional at z.

The freedom of choice of E expressed by Theorem 1.4 (*iii*) can be used to impose certain normalisations. For example, it is always possible to choose E with $E(0) = 1$.

When working with Hermite-Biehler functions, the following notion is often used.

1.6 Remark. For a Hermite-Biehler function E there exists a continuously differentiable function φ_E with $\varphi_E' > 0$, such that $E(t)e^{i\varphi_E(t)} \in \mathbb{R}$ for all $t \in \mathbb{R}$. Such a function is unique up to an additive constant in $\pi\mathbb{Z}$, and each such function is called *phase function* of E.

№2
The chain of de Branges subspaces

A central role in the theory of de Branges spaces is played by those subspaces of a given space which are themselves de Branges spaces.

1.7 Definition. Let $\mathcal{H}$ be a de Branges space. A linear subspace $\mathcal{L}$ of $\mathcal{H}$ is called a *de Branges subspace*, if it is closed, invariant under the involution $F \mapsto F^{\#}$, and invariant under division of zeroes.

We denote the set of all de Branges subspaces of $\mathcal{H}$ as $\operatorname{Sub}\mathcal{H}$.

Note that $\mathcal{L}$ is a de Branges subspace of $\mathcal{H}$ if and only if it is with the inner product inherited from $\mathcal{H}$ itself a de Branges space. It is a significant result that $\operatorname{Sub}\mathcal{H}$ has a very particular order structure.

1.8 Theorem. *Let* $\mathcal{H}$ *be a de Branges space. Then* $\operatorname{Sub}\mathcal{H}$ *is totally ordered with respect to inclusion. We have*

$$\forall \mathcal{L} \in \operatorname{Sub}\mathcal{H} \setminus \{\mathcal{H}\}\colon\quad \dim \Big[\bigcap \{\mathcal{L}' \mid \mathcal{L}' \in \operatorname{Sub}\mathcal{H}, \mathcal{L}' \supsetneq \mathcal{L}\} \Big/ \mathcal{L}\Big] \leq 1,$$

$$\forall \mathcal{L} \in \operatorname{Sub}\mathcal{H}, \dim\mathcal{L} > 1\colon\quad \dim \Big[\mathcal{L} \Big/ \operatorname{cls}\bigcup \{\mathcal{L}' \mid \mathcal{L}' \in \operatorname{Sub}\mathcal{H}, \mathcal{L}' \subsetneq \mathcal{L}\}\Big] \leq 1.$$

The fact that $\operatorname{Sub}\mathcal{H}$ is a chain is known as *de Branges' ordering theorem*, and its proof heavily relies on function theoretic tools.

№3
The structure Hamiltonian

Let $\mathcal{H}$ be a de Branges space. When passing to the concrete description via Hermite-Biehler functions the chain $\operatorname{Sub}\mathcal{H}$ can be described by means of a differential equation.

To explain this, we need to make a small excursion to the theory of canoncial systems. A two-dimensional *canonical system* is a differential equation of the form (for practical reasons we write the equation for row vectors)

$$\frac{\partial}{\partial t}(y_1(t), y_2(t))J = z(y_1(t), y_2(t))H(t), \qquad t \in (l_-, l_+) \text{ a.e.}, \tag{1.4}$$

where $-\infty \leq l_- < l_+ \leq \infty$, J is the symplectic matrix $J := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $z \in \mathbb{C}$ is the eigenvalue parameter, and where $H \in L^1_{\text{loc}}((l_-, l_+), \mathbb{R}^{2\times 2})$. The function H is called the *Hamiltonian* of the system[2].

The equation (1.4) can be rewritten in integral form. In fact, a function $(y_1, y_2)\colon (l_-, l_+) \to \mathbb{C}^{1\times 2}$ is locally absolutely continuous and satisfies (1.4), if and only if it is measurable, locally bounded, and satisfies

$$\forall l_- < a < b < l_+\colon \ \ (y_1(b), y_2(b))J - (y_1(a), y_2(a))J = z\int_a^b (y_1(t), y_2(t))H(t)\,\mathrm{d}t.$$

An interval $(a, b) \subseteq (l_-, l_+)$ is called *indivisible of type* ϕ, if

$$\ker H(t) = \operatorname{span}\left\{\begin{pmatrix} -\sin\phi \\ \cos\phi \end{pmatrix}\right\}, \qquad t \in (a, b) \text{ a.e.}$$

Given a Hamiltonian H on an interval (l_-, l_+) and a point $t \in (l_-, l_+)$, we denote by $W_H(t, s, z)$, $s \in (l_-, l_+)$, the unique 2×2-matrix solution of the initial value problem

$$\begin{cases} \frac{\partial}{\partial s}W_H(t, s, z)J = zW_H(t, s, z)H(s), & s \in (l_-, l_+) \text{ a.e.}, \\ W_H(t, t, z) = I. \end{cases}$$

We refer to $W_H(t, s, z)$ as the family of *transfer matrices* associated with H. Note that

$$\begin{aligned} &\forall t, s, r \in (l_-, l_+)\colon \ \ W_H(t, s, z)W_H(s, r, z) = W_H(t, r, z), \\ &\forall t, s \in (l_-, l_+)\colon \ \ W_H(t, s, 0) = I. \end{aligned}$$

1.9 Theorem. *Let $E \in \mathcal{HB}$ with $E(0) = 1$. Then there exists a unique Hamiltonian H_E defined on the interval $(-\infty, 0)$ with*

$$H_E(t) \geq 0 \ \text{a.e.}, \quad \operatorname{tr} H_E(t) = 1 \ \text{a.e.}, \quad \int_{-\infty}^0 \begin{pmatrix} 1 \\ 0 \end{pmatrix}^* H_E(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathrm{d}t < \infty,$$

[2] We deliberately do not assume that H is positive semidefinite or that H is integrable up to one or both endpoints of the interval.

such that the solution of the initial value problem

$$\begin{cases} \frac{\partial}{\partial t}(A(t,z),B(t,z))J = z(A(t,z),B(t,z))H_E(t), & t\in(-\infty,0) \text{ a.e.},\\ (A(0,z),B(0,z)) = (A(z),B(z)), \end{cases}$$

satisfies (following the generic notation (1.3) we write $E(t,.) := A(t,.)-iB(t,.)$*)*

$$\forall t\in(-\infty,0]\colon \quad E(t,.)\in\mathcal{HB}\cup\{1\},$$

$$\operatorname{Sub}\mathcal{H}(E) = \big\{\mathcal{H}(E(t,.))\mid t\in(-\infty,0], E(t,.)\neq 1, \\ t \text{ is not inner point of an indivisible interval}\big\}.$$

The Hamiltonian H_E, granted uniquely by this theorem, is called the *structure Hamiltonian* associated with E, and we write the transfer matrices of H_E as $W_E(t,s,z)$.

The freedom in the choice of E when representing a de Branges space $\mathcal{H}$ as $\mathcal{H}(E)$ expressed by Theorem 1.4 (iii) translates easily to structure Hamiltonians. Given the normalisation that $E(0)=1$, each two Hermite-Biehler functions $E,\tilde{E}$ with $\mathcal{H}(E)=\mathcal{H}(\tilde{E})$ are related as

$$(A,B) = (\tilde{A},\tilde{B})\begin{pmatrix} 1 & 0\\ \gamma & 1\end{pmatrix}$$

with some real constant γ. The corresponding structure Hamiltonians are then related as

$$H_E = \begin{pmatrix} 1 & 0\\ \gamma & 1\end{pmatrix} H_{\tilde{E}} \begin{pmatrix} 1 & \gamma\\ 0 & 1\end{pmatrix}.$$

№4
Unbounded chains and measures

Given a de Branges space $\mathcal{H}$, there always exist positive Borel measures on the real line, such that $\mathcal{H}$ is contained isometrically in $L^2(\mu)$[3]. For example, choose $E\in\mathcal{HB}$ with $\mathcal{H}=\mathcal{H}(E)$, then $\mathcal{H}\subseteq L^2(\frac{\mathrm{d}t}{|E(t)|^2})$ isometrically. Other examples are obtained using orthonormal bases of $\mathcal{H}$, and such can also be constructed explicitly from E.

In the description of all measures μ such that $\mathcal{H}$ is contained isometrically in $L^2(\mu)$, chains of de Branges spaces occur which, unlike $\operatorname{Sub}\mathcal{H}$, do not have a maximal element. In the present context, the theory of such chains is not needed in its full generality; the reason being that in the context of homogeneous spaces the dimensions occuring in Theorem 1.8 are always equal to 0. In the following discussion we restrict to what is needed at present.

1.10 Definition. We call a set $\mathcal{C}$ of de Branges spaces an *unbounded chain*, if

- ▷ $\mathcal{C}$ is totally ordered with respect to isometric inclusion;
- ▷ $\forall\mathcal{H}\in\mathcal{C}\colon \quad \operatorname{Sub}\mathcal{H}\subseteq\mathcal{C}$;

[3]To make it explicit: we say that $\mathcal{H}\subseteq L^2(\mu)$ isometrically, if

$$\forall F\in\mathcal{H}\colon \quad \|F\|^2 = \int_{\mathbb{R}} \Big|\frac{F(t)}{E(t)}\Big|^2\,\mathrm{d}t.$$

▷ $\lim\limits_{\mathcal{H}\in\mathcal{C}} K_{\mathcal{H}}(0,0)=\infty$, where $K_{\mathcal{H}}$ denotes the reproducing kernel of $\mathcal{H}$, and $\mathcal{C}$ is understood as a directed set w.r.t. inclusion.

1.11 Theorem. *Let $\mathcal{C}$ be an unbounded chain of de Branges spaces. Then the following statements hold.*

(i) *There exists a unique positive Borel measure $\mu_{\mathcal{C}}$ on $\mathbb{R}$, such that*

$$\forall \mathcal{H}\in\mathcal{C}\colon\ \ \mathcal{H}\subseteq L^2(\mu) \text{ isometrically,}$$
$$\bigcup\big\{\mathcal{H}\mid \mathcal{H}\in\mathcal{C}\big\} \text{ is dense in } L^2(\mu).$$

(ii) *If $(\mu_{\mathcal{H}})_{\mathcal{H}\in\mathcal{C}}$ is a net of positive Borel measures such that $\mathcal{H}\subseteq L^2(\mu_{\mathcal{H}})$ isometrically for all $\mathcal{H}\in\mathcal{C}$, then $\lim_{\mathcal{H}\in\mathcal{C}}\mu_{\mathcal{H}}=\mu_{\mathcal{C}}$ in the sense of vague convergence of measures*[4]*.*

Passing to the concrete level of descriptions via Hermite-Biehler functions, also unbounded chains $\mathcal{C}$ and the measure $\mu_{\mathcal{C}}$ can be understood with the help of canoncial systems. To formulate this fact, we need to recall the notions of Nevanlinna functions and the Weyl coefficient of a limit point system.

A function q is called a *Nevanlinna function*, if it is analytic on $\mathbb{C}\setminus\mathbb{R}$, satisfies $\operatorname{Im} q(z)\geq 0$ for all $z\in\mathbb{C}_+$, and $q(\overline{z})=\overline{q(z)}$ for all $z\in\mathbb{C}\setminus\mathbb{R}$. Assume now we have a Hamiltonian on some interval (l_-,l_+) which is locally integrable at l_- but not integrable on the whole interval. Then for every family $(q_t)_{t\in[l_-,l_+)}$ of Nevanlinna functions the limit[5] [6]

$$q_H(z):=\lim_{t\uparrow l_+} W_H(l_-,t,z)\star q_t$$

exists locally uniformly on $\mathbb{C}\setminus\mathbb{R}$, is independent of the choice of q_t, and is a Nevanlinna function. This limit is called the *Weyl coefficient* of H.

1.12 Theorem. *Let $\mathcal{C}$ be an unbounded chain of de Branges spaces, and let $E\in\mathcal{HB}$ with $E(0)=1$ and $\mathcal{H}(E)\in\mathcal{H}$. Then the following statements hold.*

(i) *There exists a unique Hamiltonian H on $(0,\infty)$ with*

$$H(t)\geq 0 \text{ a.e.,}\qquad \operatorname{tr} H(t)=1 \text{ a.e.,}$$

such that the solution of the initial value problem

$$\begin{cases} \frac{\partial}{\partial t}(A(t,z),B(t,z))J=z(A(t,z),B(t,z))H(t), & t\in(0,\infty) \text{ a.e.,}\\ (A(0,z),B(0,z))=(A(z),B(z)), \end{cases}$$

satisfies

$$\forall t\in[0,\infty)\colon\ \ E(t,.)\in\mathcal{HB},$$
$$\big\{\mathcal{L}\in\mathcal{C}\mid \mathcal{H}(E)\subseteq\mathcal{L}\big\}=\big\{\mathcal{H}(E(t,.))\mid t\in[0,\infty),$$
$$t \text{ is not inner point of an indivisible interval}\big\}.$$

[4]For this notion of convergence see, e.g., [Kle20, §13.2].

[5]Due to the assumption that H is locally integrable at l_-, the transfer matrix exists also starting with inital node l_-.

[6]The symbol "$\star$" denotes the usual action of $\mathrm{GL}(2,\mathbb{C})$ on the Riemann sphere $\mathbb{C}_\infty$:

$$\begin{pmatrix} m_{11} & m_{12}\\ m_{21} & m_{22}\end{pmatrix}\star\tau:=\frac{m_{11}\tau+m_{12}}{m_{21}\tau+m_{22}}.$$

(ii) *Let q_H be the Weyl coefficient of H, and set*

$$q_{E,\mathcal{C}} := \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \star \Big(\frac{-1}{q_H}\Big). \tag{1.5}$$

Then there exists $\beta \geq 0$ such that

$$\operatorname{Im} q_{E,\mathcal{C}}(x+iy) = \beta y + \frac{1}{\pi} \int\limits_{\mathbb{R}} \frac{y}{(t-x)^2+y^2} \cdot |E(t)|^2 \, \mathrm{d}\mu_{\mathcal{C}}(t), \quad x \in \mathbb{R}, y > 0.$$

If $\bigcup \{\mathcal{L} \in \operatorname{Sub} \mathcal{H} \mid \mathcal{L} \subsetneq \mathcal{H}\}$ is dense in $\mathcal{H}$, then $\beta = 0$.

The representation of $\mu_{\mathcal{C}}$ given in Theorem 1.12 (ii) yields a continuity property.

1.13 Lemma. *Let $\mathcal{C}_n$, $n \in \mathbb{N}_0 \cup \{\infty\}$, be unbounded chains. Let $E_n \in \mathcal{HB}$, $n \in \mathbb{N}_0 \cup \{\infty\}$ be such that $E_n(0) = 1$ and $\mathcal{H}(E_n) \in \mathcal{C}_n$ for all $n \in \mathbb{N}_0 \cup \{\infty\}$, and denote by H_n the corresponding Hamiltonians granted by Theorem 1.12 (i).*

Assume that $\lim_{n\to\infty} E_n = E_\infty$ locally uniformly on $\mathbb{C}$, and $\lim_{n\to\infty} H_n = H_\infty$ locally weak-L^1 on $[0,\infty)$. Then $\lim_{n\to\infty} \mu_{\mathcal{C}_n} = \mu_{\mathcal{C}_\infty}$ vaguely.

Since results of this kind are not discussed in [Bra68], we provide the argument.

Proof. Consider the respective functions $q_{E_n,\mathcal{C}_n}$, $n \in \mathbb{N}_0 \cup \{\infty\}$ introduced in (1.5). By our assumptions on convergence of E_n and H_n we have $\lim_{n\to\infty} q_{E_n,\mathcal{C}_n} = q_{E_\infty,\mathcal{C}_\infty}$ locally uniformly on $\mathbb{C} \setminus \mathbb{R}$ (recall here that, by a theorem about canonical systems, convergence of Hamiltonians implies convergence of Weyl coefficients, e.g. [Rem18]). The Grommer-Hamburger theorem implies that $\lim_{n\to\infty} |E_n(t)|^2 \, \mathrm{d}\mu_{\mathcal{C}_n} = |E_\infty(t)|^2 \, \mathrm{d}\mu_{\mathcal{C}_\infty}$ vaguely. Since the functions E_n are continuous and have no real zeroes, it follows that $\lim_{n\to\infty} \mu_{\mathcal{C}_n} = \mu_{\mathcal{C}_\infty}$ vaguely. ☐

2 Introduction to the main players

2.1 Weighted rescaling

For each $p \in \mathbb{R}$ we define a continuous group action $\odot_p$ of the positive real numbers on a function space $X^{\mathbb{C}}$, where at first X is just any normed space; later on it will mainly be $\mathbb{C}$ or $\mathbb{C}^2$. Here (and always) $X^{\mathbb{C}}$ is endowed with the topology of locally uniform convergence.

2.1 Definition. Let $p \in \mathbb{R}$ and X a normed space. Then $\odot_p : \mathbb{R}^+ \times X^{\mathbb{C}} \to X^{\mathbb{C}}$ is defined as

$$[a \odot_p F](z) := a^p F(az) \quad \text{for } z \in \mathbb{C}. \tag{2.1}$$

It is obvious that $\odot_p$ indeed is a continuous group action.

2.2 Remark. When speaking of weighted rescalings one could also think of using other weights $k(a)$ than powers in (1.2) and (2.1). For the following two reasons this does in the present context not lead to greater generality.

(i) If we want to get a group action, the weight function k must be a solution of the multiplicative Cauchy functional equation $k(ab) = k(a)k(b)$.

(ii) If for all $a \in (0,1]$ the map $F(z) \mapsto k(a)F(az)$ maps some de Branges space isometrically into itself, then k is a solution of the multiplicative Cauchy functional equation. This is proven in [Bra62].

Making some weak assumption on k, for instance that k is measurable, it is thus no loss of generality to restrict attention to weights $k(a) = a^p$ where $p \in \mathbb{R}$.

The group action $\odot_p$ fits well with the construction of de Branges spaces from Hermite-Biehler functions.

2.3 Lemma. *Let $p \in \mathbb{R}$ and $E \in \mathcal{HB}$, and let further $a \in \mathbb{R}^+$. Then $a \odot_p E \in \mathcal{HB}$, and the reproducing kernels of $\mathcal{H}(E)$ and $\mathcal{H}(a \odot_p E)$ are related as*

$$K_{a\odot_p E}(z,w) = a^{2p+1} K_E(az, aw) \quad \textit{for } z, w \in \mathbb{C}. \tag{2.2}$$

The map $F \mapsto a \odot_{p+\frac{1}{2}} F$ is an isometric isomorphism of $\mathcal{H}(E)$ onto $\mathcal{H}(a \odot_p E)$.

Proof. The fact that $a \odot_p E$ is a Hermite-Biehler function is obvious. The decomposition of $a \odot_p E$ in real- and imaginary components is

$$a \odot_p E = (a \odot_p A) - i(a \odot_p B),$$

and from this we immediately obtain the kernel relation (2.2).

Considering $w \in \mathbb{C}$ as a fixed parameter, (2.2) says that

$$\big[a \odot_{p+\frac{1}{2}} K_E(.,w)\big](z) = a^{p+\frac{1}{2}} K_E(az,w) = a^{-p-\frac{1}{2}} K_{a\odot_p E}\big(z, \frac{w}{a}\big) \quad \text{for } z \in \mathbb{C}.$$

Thus $a \odot_{p+\frac{1}{2}} K_E(.,w) \in \mathcal{H}(a \odot_p E)$, and for each two points $w, w' \in \mathbb{C}$,

$$\begin{aligned}
&\Big(a\odot_{p+\frac{1}{2}} K_E(.,w), a \odot_{p+\frac{1}{2}} K_E(.,w')\Big)_{\mathcal{H}(a\odot_p E)} \\
&\qquad = a^{-2p-1}\Big(K_{a\odot_p E}\big(., \frac{w}{a}\big), K_{a\odot_p E}\big(., \frac{w'}{a}\big)\Big)_{\mathcal{H}(a\odot_p E)} \\
&\qquad = a^{-2p-1} K_{a\odot_p E}\big(\frac{w'}{a}, \frac{w}{a}\big) = K_E(w', w) = \big(K_E(.,w), K_E(.,w')\big)_{\mathcal{H}(E)}.
\end{aligned}$$

We see that $F \mapsto a \odot_{p+\frac{1}{2}} F$ maps the linear span of reproducing kernels of $\mathcal{H}(E)$ isometrically onto the linear span of reproducing kernels of $\mathcal{H}(a \odot_p E)$. Hence, it extends to an isometric isomorphism of $\mathcal{H}(E)$ onto $\mathcal{H}(a \odot_p E)$. Since point evaluations are continuous in both spaces, this extension acts again as $F \mapsto a \odot_{p+\frac{1}{2}} F$. □

2.2 The canonical system

We define a class of Hamiltonians having a very particular form.

2.4 Definition. Let $p \in \mathbb{R}$ and $(P, \psi) \in \mathbb{R}^{2\times 2} \times \mathbb{R}$. Then we define functions

$\mathscr{D}_\psi, H_{P,\psi} \colon (0,\infty) \to \mathbb{R}^{2\times 2}$ as

$$\mathscr{D}_\psi(a) := \begin{cases} \begin{pmatrix} 1 & 0 \\ \frac{\psi}{2p} & 1 \end{pmatrix} \begin{pmatrix} a^p & 0 \\ 0 & a^{-p} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{\psi}{2p} & 1 \end{pmatrix} & \text{if } p \neq 0, \\ \begin{pmatrix} 1 & 0 \\ \psi \log a & 1 \end{pmatrix} & \text{if } p = 0, \end{cases}$$

$$H_{P,\psi}(a) := \mathscr{D}_\psi(a) P \mathscr{D}_\psi(a)^T.$$

Here we slightly overload notation by not explicitly denoting dependence on p.

Note that $\mathscr{D}_\psi$ and $H_{P,\psi}$ are continuous (in fact, infinitely differentiable) functions of $a \in (0,\infty)$, and that $\mathscr{D}_\psi(a) \in \mathrm{SL}(2,\mathbb{R})$ for all $a \in (0,\infty)$.

Let us have a closer look at the function $H_{P,\psi}$. First, we consider $\ker H_{P,\psi}(a)$. It is clear that $\operatorname{rank} H_{P,\psi}(a) = \operatorname{rank} P$ for all $a > 0$. In particular, if $H_{P,\psi}(a) = 0$ or $H_{P,\psi}(a)$ is invertible for one $a > 0$, then the respective property holds for all $a > 0$. The behaviour of $\ker H_{P,\psi}(a)$ when $\operatorname{rank} P = 1$ is slightly more complex.

2.5 Lemma. *Let $p \in \mathbb{R}$ and $(P,\psi) \in \mathbb{R}^{2\times 2} \times \mathbb{R}$, and assume that* $\ker P = \operatorname{span}\{\xi\}$ *with some nonzero vector ξ.*

(i) *Assume that $p \neq 0$. If ξ is a scalar multiple of either $\binom{1}{0}$ or $\binom{-\psi}{2p}$, then* $\ker H_{P,\psi}(a) = \operatorname{span}\{\xi\}$ *for all $a > 0$. Otherwise,* $\ker H_{P,\psi}(a) \neq \ker H_{P,\psi}(b)$ *for any two a, b with $0 < a < b < \infty$.*

(ii) *Assume that $p = 0$ and $\psi \neq 0$. If ξ is a scalar multiple of $\binom{1}{0}$, then* $\ker H_{P,\psi}(a) = \operatorname{span}\{\xi\}$ *for all $a > 0$. Otherwise,* $\ker H_{P,\psi}(a) \neq \ker H_{P,\psi}(b)$ *for any two a, b with $0 < a < b < \infty$.*

(iii) *Assume that $p = \psi = 0$. Then* $\ker H_{P,\psi}(a) = \operatorname{span}\{\xi\}$ *for all $a > 0$.*

Proof. Since $\mathscr{D}_\psi(a)$ is invertible, we have

$$\ker H_{P,\psi}(a) = \operatorname{span}\left\{\mathscr{D}_\psi(a)^{-T}\xi\right\}.$$

Consider the case that $p \neq 0$. Then

$$\mathscr{D}_\psi(a)^{-T} = \begin{pmatrix} 1 & -\frac{\psi}{2p} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a^{-p} & 0 \\ 0 & a^p \end{pmatrix} \begin{pmatrix} 1 & \frac{\psi}{2p} \\ 0 & 1 \end{pmatrix}.$$

For each two a, b with $0 < a < b < \infty$ and $\eta \in \mathbb{R}^2$ the set

$$\left\{ \begin{pmatrix} a^{-p} & 0 \\ 0 & a^p \end{pmatrix} \eta, \begin{pmatrix} b^{-p} & 0 \\ 0 & b^p \end{pmatrix} \eta \right\}$$

is linearly dependent, if and only if

$$\eta \in \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} \cup \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

It follows that $\{\mathscr{D}_\psi(a)^{-T}\xi, \mathscr{D}_\psi(b)^{-T}\xi\}$ is linearly dependent, if and only if

$$\xi \in \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} \cup \operatorname{span}\left\{ \begin{pmatrix} -\psi \\ 2p \end{pmatrix} \right\}.$$

Clearly, we have

$$\mathscr{D}_\psi(a)^{-T}\begin{pmatrix}1\\0\end{pmatrix}=a^{-p}\begin{pmatrix}1\\0\end{pmatrix},\quad \mathscr{D}_\psi(a)^{-T}\begin{pmatrix}-\psi\\2p\end{pmatrix}=a^{p}\begin{pmatrix}-\psi\\2p\end{pmatrix},$$

and the proof of (i) is complete.

Consider next the case that $p=0$ and $\psi\neq 0$. Then

$$\mathscr{D}_\psi(a)^{-T}=\begin{pmatrix}1 & -\psi\log a\\0 & 1\end{pmatrix},$$

and the assertion of (ii) follows immediately. Also (iii) is clear, since for $p=\psi=0$ we have $\mathscr{D}_\psi(a)=I$ for all $a>0$. ☐

Second, we investigate integrability of $H_{P,\psi}$ at the endpoints of the interval $(0,\infty)$. The proof is simply by explicit consideration.

2.6 Lemma. *Let $(P,\psi)\in\mathbb{R}^{2\times 2}\times\mathbb{R}$. Concerning integrability at 0 we have*

$$\begin{aligned}
&\forall p\leq -\frac{1}{2}:\ \ H_{P,\psi}\in L^1\big((0,1),\mathbb{R}^{2\times 2}\big)\ \Leftrightarrow\ \begin{pmatrix}1\\0\end{pmatrix}^*P\begin{pmatrix}1\\0\end{pmatrix}=0,\\
&p\in(-\frac{1}{2},\frac{1}{2})\ \Longrightarrow\ H_{P,\psi}\in L^1\big((0,1),\mathbb{R}^{2\times 2}\big),\\
&\forall p\geq \frac{1}{2}:\ \ H_{P,\psi}\in L^1\big((0,1),\mathbb{R}^{2\times 2}\big)\ \Leftrightarrow\ \begin{pmatrix}-\psi\\2p\end{pmatrix}^*P\begin{pmatrix}-\psi\\2p\end{pmatrix}=0,
\end{aligned}$$

and concerning integrability at ∞

$$\begin{aligned}
&\forall p\geq -\frac{1}{2},p\neq 0:\ \ H_{P,\psi}\in L^1\big((1,\infty),\mathbb{R}^{2\times 2}\big)\ \Leftrightarrow\ \begin{pmatrix}1\\0\end{pmatrix}\in\ker P\cap\ker P^T,\\
&p=0\ \Longrightarrow\ \Big[H_{P,\psi}\in L^1\big((1,\infty),\mathbb{R}^{2\times 2}\big)\ \Leftrightarrow\ P=0\Big],\\
&\forall p\leq \frac{1}{2},p\neq 0:\ \ H_{P,\psi}\in L^1\big((1,\infty),\mathbb{R}^{2\times 2}\big)\ \Leftrightarrow\ \begin{pmatrix}-\psi\\2p\end{pmatrix}\in\ker P\cap\ker P^T.
\end{aligned}$$

Proof. Let us first settle the case that $p=0$. We write $P=\begin{pmatrix}p_{11} & p_{12}\\ p_{21} & p_{22}\end{pmatrix}$ and write out the definition of $H_{P,\psi}(a)$. This yields

$$H_{P,\psi}(a)=\begin{pmatrix}p_{11} & p_{11}\psi\log a+p_{12}\\ p_{11}\psi\log a+p_{21} & p_{11}(\psi\log a)^2+(p_{12}+p_{21})\psi\log a+p_{22}\end{pmatrix}.$$

From this it is clear that $H_{P,\psi}\in L^1(0,1)$ but $H_{P,\psi}\notin L^1(1,\infty)$ unless $P=0$.

From now on assume that $p\neq 0$. Then we have

$$H_{P,\psi}(a)=$$

$$\begin{pmatrix}1 & 0\\ \frac{\psi}{2p} & 1\end{pmatrix}\cdot\underbrace{\begin{pmatrix}a^p & 0\\ 0 & a^{-p}\end{pmatrix}\cdot\overbrace{\begin{pmatrix}1 & 0\\ -\frac{\psi}{2p} & 1\end{pmatrix}P\begin{pmatrix}1 & -\frac{\psi}{2p}\\ 0 & 1\end{pmatrix}}^{=:Q}\cdot\begin{pmatrix}a^p & 0\\ 0 & a^{-p}\end{pmatrix}}_{=:L(a)}\cdot\begin{pmatrix}1 & \frac{\psi}{2p}\\ 0 & 1\end{pmatrix}.$$

Clearly, $H_{P,\psi}$ is integrable at 0 or at ∞ if and only if $L(a)$ has the respective property. We have

$$\binom{1}{0}^* L(a)\binom{1}{0} = a^{2p} \cdot \binom{1}{0}^* Q\binom{1}{0}, \quad \binom{0}{1}^* L(a)\binom{0}{1} = a^{-2p} \cdot \binom{0}{1}^* Q\binom{0}{1},$$
$$\binom{0}{1}^* L(a)\binom{1}{0} = \binom{0}{1}^* Q\binom{1}{0}, \quad \binom{1}{0}^* L(a)\binom{0}{1} = \binom{1}{0}^* Q\binom{0}{1}.$$

and

$$\binom{1}{0}^* Q\binom{1}{0} = \binom{1}{0}^* P\binom{1}{0}, \quad \binom{0}{1}^* Q\binom{0}{1} = \frac{1}{4p^2} \cdot \binom{-\psi}{2p}^* P\binom{-\psi}{2p},$$
$$\binom{0}{1}^* Q\binom{1}{0} = \frac{1}{2p} \cdot \binom{-\psi}{2p}^* P\binom{1}{0}, \quad \binom{1}{0}^* Q\binom{0}{1} = \frac{1}{2p} \cdot \binom{1}{0}^* P\binom{-\psi}{2p}.$$

Let us now go through the different cases.

$\triangleright$ Concerning integrability at 0: The off-diagonal entries of $L(a)$ are always integrable. Further,

$$\begin{aligned}
&p > -\tfrac{1}{2} \;\Rightarrow\; \binom{1}{0}^* L(a)\binom{1}{0} \in L^1(0,1),\\
&p \le -\tfrac{1}{2} \;\Rightarrow\; \Big[\binom{1}{0}^* L(a)\binom{1}{0} \in L^1(0,1) \Leftrightarrow \binom{1}{0}^* Q\binom{1}{0} = 0 \Leftrightarrow \binom{1}{0}^* P\binom{1}{0} = 0\Big],\\
&p < \tfrac{1}{2} \;\Rightarrow\; \binom{0}{1}^* L(a)\binom{0}{1} \in L^1(0,1),\\
&p \ge \tfrac{1}{2} \;\Rightarrow\; \Big[\binom{0}{1}^* L(a)\binom{0}{1} \in L^1(0,1) \Leftrightarrow \binom{0}{1}^* Q\binom{0}{1} = 0 \Leftrightarrow \binom{-\psi}{2p}^* P\binom{-\psi}{2p} = 0\Big].
\end{aligned}$$

$\triangleright$ Concerning integrability at ∞: The off-diagonal entries of $L(a)$ are not integrable unless they vanish. Moreover, we have

$$\begin{aligned}
&p < -\tfrac{1}{2} \;\Rightarrow\; \binom{1}{0}^* L(a)\binom{1}{0} \in L^1(1,\infty),\\
&p \ge -\tfrac{1}{2} \;\Rightarrow\; \Big[\binom{1}{0}^* L(a)\binom{1}{0} \in L^1(1,\infty) \Leftrightarrow \binom{1}{0}^* Q\binom{1}{0} = 0 \Leftrightarrow \binom{1}{0}^* P\binom{1}{0} = 0\Big],\\
&p > \tfrac{1}{2} \;\Rightarrow\; \binom{0}{1}^* L(a)\binom{0}{1} \in L^1(1,\infty),\\
&p \le \tfrac{1}{2} \;\Rightarrow\; \Big[\binom{0}{1}^* L(a)\binom{0}{1} \in L^1(1,\infty) \Leftrightarrow \binom{0}{1}^* Q\binom{0}{1} = 0 \Leftrightarrow \binom{-\psi}{2p}^* P\binom{-\psi}{2p} = 0\Big].
\end{aligned}$$

If $p \ge -\frac{1}{2}$ we thus have

$$\begin{aligned}
L(a) \in L^1(1,\infty) \;&\Leftrightarrow\; \binom{1}{0}^* Q\binom{0}{1} = \binom{0}{1}^* Q\binom{1}{0} = \binom{1}{0}^* Q\binom{1}{0} = 0\\
&\Leftrightarrow\; \binom{1}{0}^* P\binom{-\psi}{2p} = \binom{-\psi}{2p}^* P\binom{1}{0} = \binom{1}{0}^* P\binom{1}{0} = 0\\
&\Leftrightarrow\; \binom{1}{0} \in \ker P \cap \ker P^T,
\end{aligned}$$

and if $p \le \frac{1}{2}$

$$\begin{aligned}
L(a) \in L^1(1,\infty) \;&\Leftrightarrow\; \binom{1}{0}^* Q\binom{0}{1} = \binom{0}{1}^* Q\binom{1}{0} = \binom{0}{1}^* Q\binom{0}{1} = 0\\
&\Leftrightarrow\; \binom{1}{0}^* P\binom{-\psi}{2p} = \binom{-\psi}{2p}^* P\binom{1}{0} = \binom{-\psi}{2p}^* P\binom{-\psi}{2p} = 0\\
&\Leftrightarrow\; \binom{-\psi}{2p} \in \ker P \cap \ker P^T.
\end{aligned}$$

$\square$

2.3 The recurrence relation

Given parameters $p \in \mathbb{R} \setminus (-\frac{1}{2}(\mathbb{N}_0+1))$ and $(P,\psi) \in \mathbb{R}^{2\times 2} \times \mathbb{R}$, we consider the linear recurrence for a sequence $((\alpha_n,\beta_n))_{n\in\mathbb{N}_0}$ of pairs of real numbers given by

$$\begin{cases} (\alpha_{n+1},\beta_{n+1}) = (\alpha_n,\beta_n) \cdot \frac{-1}{(n+1)(2p+n+1)} PJ \begin{pmatrix} 2p+n+1 & 0 \\ \psi & n+1 \end{pmatrix} & \text{for } n \in \mathbb{N}_0, \\ (\alpha_0,\beta_0) = (1,0), \end{cases} \tag{2.3}$$

where again $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

The solution $((\alpha_n,\beta_n))_{n\in\mathbb{N}_0}$ can be estimated. We have[7]

$$C := \sup_{n\in\mathbb{N}_0} \left\| \begin{pmatrix} 1 & 0 \\ \frac{\psi}{2p+n+1} & \frac{n+1}{2p+n+1} \end{pmatrix} \right\| < \infty,$$

and hence obtain inductively that

$$\forall n \in \mathbb{N}_0 \colon \ \big\|(\alpha_n,\beta_n)\big\| \le \frac{C^n\|P\|^n}{n!}. \tag{2.4}$$

This shows that the generating functions

$$A(z) := \sum_{n=0}^{\infty} \alpha_n z^n, \quad B(z) := \sum_{n=0}^{\infty} \beta_n z^n,$$

of the sequences $(\alpha_n)_{n\in\mathbb{N}_0}$ and $(\beta_n)_{n\in\mathbb{N}_0}$ are entire and that

$$|A(z)|, |B(z)| \le e^{C\|P\|\cdot|z|} \quad \text{for } z \in \mathbb{C},$$

i.e., A and B are of finite exponential type not exceeding $C\|P\|$.

In the following we denote by $\mathrm{Hol}(\mathbb{C})$ the set of all entire functions.

2.7 Definition. Let $p \in \mathbb{R} \setminus (-\frac{1}{2}(\mathbb{N}_0+1))$. We define maps

$$\begin{aligned} &\Xi_p \colon \ \mathbb{R}^{2\times 2} \times \mathbb{R} \to \mathrm{Hol}(\mathbb{C}) \times \mathrm{Hol}(\mathbb{C}) \\ &\widehat{\Xi}_p \colon \ \mathbb{R}^{2\times 2} \times \mathbb{R} \to \mathrm{Hol}(\mathbb{C}) \end{aligned}$$

by assigning to a parameter (P,ψ) the pair (A,B) of generating functions of the solution of (2.3), and setting $\widehat{\Xi}_p(P,\psi) := A - iB$.

We start with a simple but practical observation. Here (and always) we topologise domain and codomain of Ξ_p and $\widehat{\Xi}_p$ naturally with the euclidean topology and the topology of locally uniform convergence, respectively.

2.8 Lemma. *The map*

$$\begin{cases} \mathbb{R} \setminus (-\frac{1}{2}(\mathbb{N}_0+1)) \times \mathbb{R}^{2\times 2} \times \mathbb{R} & \to & \mathrm{Hol}(\mathbb{C}) \times \mathrm{Hol}(\mathbb{C}) \\ (p,P,\psi) & \mapsto & \Xi_p(P,\psi) \end{cases}$$

is continuous.

[7] We always use the euclidean norm on $\mathbb{C}^d$, and the corresponding operator norm for matrices.

Proof. Assume we have $((p_i, P_i, \psi_i))_{i\in\mathbb{N}_0}$ and (p, P, ψ) with $\lim_{i\to\infty}(p_i, P_i, \psi_i) = (p, P, \psi)$, and let $(\alpha_{i,n}, \beta_{i,n})$ and (α_n, β_n) be the corresponding solutions of (2.3). We have

$$C_1 := \sup_{i\in\mathbb{N}_0} \|P_i\| < \infty, \quad C_2 := \sup_{i\in\mathbb{N}_0} \sup_{n\in\mathbb{N}_0} \left\| \begin{pmatrix} 1 & 0 \\ \frac{\psi_i}{2p_i+n+1} & \frac{n+1}{2p_i+n+1} \end{pmatrix} \right\| < \infty,$$

and get the uniform bound

$$\forall i, n \in \mathbb{N}_0\colon \quad \|(\alpha_{i,n}, \beta_{i,n})\|, \|(\alpha_n, \beta_n)\| \le \frac{(C_1 C_2)^n}{n!}.$$

Clearly, $\lim_{i\to\infty}(\alpha_{i,n}, \beta_{i,n}) = (\alpha_n, \beta_n)$ for all $n \in \mathbb{N}_0$, and it follows that $\lim_{i\to\infty} \Xi_{p_i}(P_i, \psi_i) = \Xi_p(P, \psi)$ locally uniformly. □

Let us collect some facts about the recurrence (2.3) and its generating functions. The case that the left upper corner of the parameter P vanishes is exceptional.

2.9 Lemma. *Let $p \in \mathbb{R} \setminus (-\frac{1}{2}(\mathbb{N}_0 + 1))$ and $(P, \psi) \in \mathbb{R}^{2\times 2} \times \mathbb{R}$, and let $((\alpha_n, \beta_n))_{n\in\mathbb{N}_0}$ be the unique solution of (2.3). Then the following statements are equivalent.*

(i) $\binom{1}{0}^* P \binom{1}{0} = 0$.

(ii) $\beta_1 = 0$.

(iii) $B = 0$.

Proof. Set $\kappa_{11} := \binom{1}{0}^* P \binom{1}{0}$ and $\kappa_{21} := \binom{0}{1}^* P \binom{1}{0}$. Equating the second component of (2.3) for $n = 0$ gives $\beta_1(1 + 2p) = \kappa_{11}$. This shows "$(i) \Leftrightarrow (ii)$". Assume that $\kappa_{11} = 0$. Multiplying (2.3) from the right with $\binom{0}{1}$ yields

$$\forall n \in \mathbb{N}_0\colon \quad \beta_{n+1} = \frac{\kappa_{21}}{2p + n + 1} \beta_n,$$

and we conclude that $\beta_n = 0$ for all $n \in \mathbb{N}_0$. □

It is an important property of a parameter P in (2.3) whether or not P is symmetric. One reason is that under the assumption of symmetry, the parameter (P, ψ) can be recovered from the solution $((\alpha_n, \beta_n))_{n\in\mathbb{N}_0}$ of (2.3) by simple formulae.

2.10 Definition. We denote

$$\mathbb{P} := \Big\{ (P, \psi) \in \mathbb{R}^{2\times 2} \times \mathbb{R} \mid P = P^T \text{ and } \binom{1}{0}^* P \binom{1}{0} \neq 0 \Big\}.$$

2.11 Lemma. *Let $p \in \mathbb{R} \setminus (-\frac{1}{2}(\mathbb{N}_0 + 1))$ and $(P, \psi) \in \mathbb{P}$. and write $P = \begin{pmatrix} \kappa_1 & \kappa_3 \\ \kappa_3 & \kappa_2 \end{pmatrix}$. Then (P, ψ) can be recovered from the first two terms $(\alpha_1, \beta_1), (\alpha_2, \beta_2)$ of the solution $((\alpha_n, \beta_n))_{n\in\mathbb{N}_0}$ of (2.3) by the formulae*

$$\begin{aligned}
\kappa_1 &= \beta_1(1 + 2p), \\
\kappa_2 &= -\frac{\alpha_1 \beta_2}{\beta_1^2} 2(1 + 2p) + \frac{\alpha_1^2}{\beta_1}(1 + 2p) + \frac{\beta_2^2}{\beta_1^3}(2 + 2p) - 2\frac{\alpha_2}{\beta_1}, \\
\kappa_3 &= \frac{\beta_2}{\beta_1}(2 + 2p) - \alpha_1(1 + 2p), \\
\psi &= \frac{\beta_2}{\beta_1^2}(2 + 2p) - \frac{\alpha_1}{\beta_1} 2p.
\end{aligned}$$

Note here that $\beta_1 \neq 0$.

Proof. The relation (2.3) written for $n = 0$ and $n = 1$ gives the four equations

$$\alpha_1 - \beta_1\psi = -\kappa_3 \qquad \beta_1(1+2p) = \kappa_1$$
$$2\alpha_2 - \beta_2\psi = -\alpha_1\kappa_3 - \beta_1\kappa_2 \qquad \beta_2(2+2p) = \alpha_1\kappa_1 + \beta_1\kappa_3$$

The second equation is the asserted formula for κ_1, and the fourth equation gives κ_3. Then we use the first equation to compute ψ, and the third for κ_2. □

This lemma has the following obvious consequence.

2.12 Corollary. *Let $p \in \mathbb{R} \setminus (-\frac{1}{2}(\mathbb{N}_0 + 1))$. Then $\Xi_p|_{\mathbb{P}}$ is a homeomorphism of $\mathbb{P}$ onto its image $\Xi_p(\mathbb{P})$. The same holds for $\widehat{\Xi}_p$.*

Next, we present a simple transformation which is often practical.

2.13 Lemma. *Let $p \in \mathbb{R} \setminus (-\frac{1}{2}(\mathbb{N}_0 + 1))$ and $(P, \psi) \in \mathbb{R}^{2\times 2} \times \mathbb{R}$. Then, for all $\gamma \in \mathbb{R}$,*

$$\Xi_p(P,\psi) = \Xi_p\Big(\begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} P \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}, \psi + 2p\gamma\Big)\begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}.$$

Proof. Let $((\alpha_n, \beta_n))_{n\in\mathbb{N}_0}$ be the solution of (2.3) for (P, ψ). Then we compute

$$-(n+1)\,(2p+n+1)\cdot(\alpha_{n+1},\beta_{n+1})\begin{pmatrix} 1 & 0 \\ -\gamma & 1 \end{pmatrix} = (\alpha_n,\beta_n)PJ\begin{pmatrix} 2p+n+1 & 0 \\ \psi & n+1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\gamma & 1 \end{pmatrix}$$
$$= (\alpha_n,\beta_n)\cdot\begin{pmatrix} 1 & 0 \\ -\gamma & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}\cdot P\cdot\begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}J\begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}\cdot\begin{pmatrix} 2p+n+1 & 0 \\ \psi & n+1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\gamma & 1 \end{pmatrix}$$
$$= (\alpha_n,\beta_n)\begin{pmatrix} 1 & 0 \\ -\gamma & 1 \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}P\begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}\cdot J\cdot\begin{pmatrix} 2p+n+1 & 0 \\ \psi+2p\gamma & n+1 \end{pmatrix}.$$

□

Using this transformation we can characterise a symmetry property of the generating functions. This result is of relevance in the context of de Branges spaces which are symmetric about the origin in the sense of [Bra68, Chapter 47].

2.14 Lemma. *Let $p \in \mathbb{R}\setminus(-\frac{1}{2}(\mathbb{N}_0+1))$ and $(P,\psi) \in \mathbb{P}$, and write $P = \begin{pmatrix} \kappa_1 & \kappa_3 \\ \kappa_3 & \kappa_2 \end{pmatrix}$. Moreover, set*

$$\sigma := 2p\kappa_3 - \psi\kappa_1.$$

Then the following statements are equivalent.

(i) *$\sigma = 0$;*

(ii) *B is odd and there exists $\gamma \in \mathbb{R}$ such that $A(z) - A(-z) = \gamma B(z)$;*

(iii) *$B''(0) = 0$.*

Proof. The proof of "$(i) \Rightarrow (ii)$" uses Lemma 2.13. Applying this lemma with $\gamma := -\frac{\kappa_3}{\kappa_1}$ yields

$$\Xi_p(P,\psi) = \Xi_p\Big(\begin{pmatrix} \kappa_1 & 0 \\ 0 & \frac{\det P}{\kappa_1}\end{pmatrix}, -\frac{\sigma}{\kappa_1}\Big)\begin{pmatrix} 1 & 0 \\ -\frac{\kappa_3}{\kappa_1} & 1\end{pmatrix}.$$

Assume that $\sigma = 0$, then

$$\frac{-1}{(n+1)(2p+n+1)}\begin{pmatrix} \kappa_1 & 0 \\ 0 & \frac{\det P}{\kappa_1}\end{pmatrix} J \begin{pmatrix} 2p+n+1 & 0 \\ 0 & n+1\end{pmatrix} = \begin{pmatrix} 0 & \frac{\kappa_1}{2p+n+1} \\ -\frac{\det P}{\kappa_1(n+1)} & 0\end{pmatrix}.$$

We obtain inductively that the solution $((\tilde{\alpha}_n, \tilde{\beta}_n))_{n\in\mathbb{N}_0}$ of the recurrence (2.3) with the matrix $\begin{pmatrix} 0 & \frac{\kappa_1}{2p+n+1} \\ -\frac{\det P}{\kappa_1(n+1)} & 0\end{pmatrix}$ and the parameter 0 satisfies

$$\forall n \in \mathbb{N}_0\colon \;\; \tilde{\alpha}_{2n+1} = 0 \wedge \tilde{\beta}_{2n} = 0.$$

This means that the corresponding generating function $\tilde{A}$ is even and $\tilde{B}$ is odd. It remains to notice that

$$A = \tilde{A} - \frac{\kappa_3}{\kappa_1}\tilde{B}, \quad B = \tilde{B}.$$

The implication "$(ii) \Rightarrow (iii)$" is trivial. Finally, the equivalence of (iii) and (i) follows from Lemma 2.11. Namely, plugging the formulae of this lemma into the definition of σ gives

$$\begin{aligned}\sigma &= 2p\Big(\frac{\beta_2}{\beta_1}(2+2p) - \alpha_1(1+2p)\Big) - \Big(\frac{\beta_2}{\beta_1^2}(2+2p) - \frac{\alpha_1}{\beta_1}2p\Big)\cdot\beta_1(1+2p) \\ &= -\frac{\beta_2}{\beta_1}(2+2p).\end{aligned}$$

□

3 Solution of the canonical system via power series coefficients

In this section we give the connection between the group action $\odot_p$, canonical systems with Hamiltonians of the form $H_{P,\psi}$, and recurrences of the form (2.3). This is the approach used in [Bra68].

3.1 Theorem. *Let $p \in \mathbb{R}\setminus(-\frac{1}{2}(\mathbb{N}_0+1))$ and $(P,\psi) \in \mathbb{R}^{2\times 2}\times\mathbb{R}$, and let $(\alpha_n)_{n\in\mathbb{N}_0}$ and $(\beta_n)_{n\in\mathbb{N}_0}$ be sequences of real numbers. Then the following statements are equivalent.*

(i) *The sequence $((\alpha_n,\beta_n))_{n\in\mathbb{N}_0}$ satisfies (2.3).*

(ii) *The power series $A(z) := \sum_{n=0}^\infty \alpha_n z^n$ and $B(z) := \sum_{n=0}^\infty \beta_n z^n$ represent entire functions with $A(0) = 1$ and $B(0) = 0$. Using the notation*

$$\big(A(a,z), B(a,z)\big) := \big([a\odot_p A](z), [a\odot_p B](z)\big)\mathfrak{D}_\psi(a)^{-1} \tag{3.1}$$

for $a>0$, it holds that

$$\forall 0<a<b<\infty: \ \big(A(b,z),B(b,z)\big)J-\big(A(a,z),B(a,z)\big)J \\ = z\int_a^b \big(A(c,z),B(c,z)\big)H_{P,\psi}(c)\,\mathrm{d}c. \qquad (3.2)$$

(iii) *The power series $A(z):=\sum_{n=0}^\infty \alpha_n z^n$ and $B(z):=\sum_{n=0}^\infty \beta_n z^n$ have positive radius of convergence, we have $A(0)=1$ and $B(0)=0$, and there exist a,b with $0<a<b<\infty$ such that the equality in (3.2) holds.*

Proof. We are going to show that "$(i)\Rightarrow(ii)$" and "$(iii)\Rightarrow(i)$". The implication "$(ii)\Rightarrow(iii)$" is trivial.

We already saw in Section 2 that (i) implies that A and B are entire functions. To proceed with the proof it is convenient to rewrite (3.2): using the definition of $H_{P,\psi}(a)$ and the fact that $\mathfrak{D}_\psi(a)^TJ=J\mathfrak{D}_\psi(a)^{-1}$, (3.2) is equivalent to

$$\big([b\odot_p A](z),[b\odot_p B](z)\big)\mathfrak{D}_\psi(b)^{-1}-\big([a\odot_p A](z),[a\odot_p B](z)\big)\mathfrak{D}_\psi(a)^{-1} \\ = -z\int_a^b \big([c\odot_p A](z),[c\odot_p B](z)\big)PJ\mathfrak{D}_\psi(c)^{-1}\,\mathrm{d}c. \qquad (3.3)$$

Plugging the power series into this relation and comparing power series coefficients yields that (3.3) is equivalent to

$$\forall n\in\mathbb{N}_0: \ (\alpha_{n+1},\beta_{n+1})\big[b^{p+n+1}\mathfrak{D}_\psi(b)^{-1}-a^{p+n+1}\mathfrak{D}_\psi(a)^{-1}\big] \\ = -(\alpha_n,\beta_n)PJ\cdot\int_a^b c^{p+n}\mathfrak{D}_\psi(c)^{-1}\,\mathrm{d}c. \qquad (3.4)$$

The square bracket on the left side of this relation computes as

$$\begin{cases} \begin{pmatrix} 1 & 0\\ \frac{\psi}{2p} & 1\end{pmatrix}\begin{pmatrix} b^{n+1}-a^{n+1} & 0\\ 0 & b^{2p+n+1}-a^{2p+n+1}\end{pmatrix}\begin{pmatrix} 1 & 0\\ -\frac{\psi}{2p} & 1\end{pmatrix} & \text{if } p\neq 0,\\[2ex] \begin{pmatrix} b^{n+1}-a^{n+1} & 0\\ -\psi\big(b^{n+1}\log b-a^{n+1}\log a\big) & b^{n+1}-a^{n+1}\end{pmatrix} & \text{if } p=0,\end{cases}$$

and the integral on the right side as

$$\begin{cases} \begin{pmatrix} 1 & 0\\ \frac{\psi}{2p} & 1\end{pmatrix}\begin{pmatrix} \frac{b^{n+1}-a^{n+1}}{n+1} & 0\\ 0 & \frac{b^{2p+n+1}-a^{2p+n+1}}{2p+n+1}\end{pmatrix}\begin{pmatrix} 1 & 0\\ -\frac{\psi}{2p} & 1\end{pmatrix} & \text{if } p\neq 0,\\[2ex] \begin{pmatrix} \frac{b^{n+1}-a^{n+1}}{n+1} & 0\\ -\psi\Big(\frac{b^{n+1}}{n+1}\big(\log b-\frac{1}{n+1}\big)-\frac{a^{n+1}}{n+1}\big(\log a-\frac{1}{n+1}\big)\Big) & \frac{b^{n+1}-a^{n+1}}{n+1}\end{pmatrix} & \text{if } p=0.\end{cases}$$

For $p\neq 0$ we can rewrite the formula for the integral as

$$\begin{pmatrix} 1 & 0\\ \frac{\psi}{2p} & 1\end{pmatrix}\begin{pmatrix} \frac{1}{n+1} & 0\\ 0 & \frac{1}{2p+n+1}\end{pmatrix}\begin{pmatrix} 1 & 0\\ -\frac{\psi}{2p} & 1\end{pmatrix} \\ \cdot\begin{pmatrix} 1 & 0\\ \frac{\psi}{2p} & 1\end{pmatrix}\begin{pmatrix} b^{n+1}-a^{n+1} & 0\\ 0 & b^{2p+n+1}-a^{2p+n+1}\end{pmatrix}\begin{pmatrix} 1 & 0\\ -\frac{\psi}{2p} & 1\end{pmatrix}, \qquad (3.5)$$

and for $p = 0$ as

$$\frac{1}{n+1}\begin{pmatrix} 1 & 0 \\ \frac{\psi}{n+1} & 1 \end{pmatrix} \cdot \begin{pmatrix} b^{n+1} - a^{n+1} & 0 \\ -\psi\big(b^{n+1}\log b - a^{n+1}\log a\big) & b^{n+1} - a^{n+1} \end{pmatrix}. \tag{3.6}$$

The matrix from the square bracket in (3.4) now appears on both sides. It is invertible since $a < b$, and after cancelling out there remains a matrix on the right side which does not depend on a and b. It equals

$$\frac{1}{(n+1)(2p+n+1)}\begin{pmatrix} 2p+n+1 & 0 \\ \psi & n+1 \end{pmatrix}.$$

□

From the above theorem we obtain two simple but important properties of A and B.

3.2 Corollary. *Let $p \in \mathbb{R} \setminus (-\frac{1}{2}(\mathbb{N}_0 + 1))$ and $(P, \psi) \in \mathbb{R}^{2\times 2} \times \mathbb{R}$, and denote (as usual) $(A, B) := \Xi_p(P, \psi)$. Then A and B have no common zeroes.*

Proof. The functions $A(a, z)$ and $B(a, z)$ from (3.1) compute explicitly as

$$A(a,z) = \begin{cases} A(az) + z \cdot \frac{\psi}{2p}\big(a - a^{2p+1}\big)\frac{B(az)}{az} & \text{if } p \neq 0, \\ A(az) - z \cdot \psi a \log a \frac{B(az)}{az} & \text{if } p = 0, \end{cases} \tag{3.7}$$

$$B(a,z) = z \cdot a^{2p+1}\frac{B(az)}{az}. \tag{3.8}$$

We see that

$$\begin{cases} \lim_{a\downarrow 0}\big[A(a,z) + \frac{\psi}{2p}B(a,z)\big] = A(0) = 1 & \text{if } p \neq 0, \\ \lim_{a\downarrow 0} A(a,z) = A(0) = 1 & \text{if } p = 0. \end{cases} \tag{3.9}$$

Let $W(a, z)$ be the unique solution of the initial value problem

$$\begin{cases} \frac{\partial}{\partial a}W(a,z)J = zW(a,z)H_{P,\psi}(a) & \text{for } a \in (0,\infty), \\ W(1,z) = I. \end{cases}$$

Note here that the initial value is prescribed at the point $a = 1$, and that $A(a, z) = A(z)$ and $B(a, z) = B(z)$. Then, by uniqueness of solutions,

$$\forall a \in (0,\infty):\ \big(A(a,z), B(a,z)\big) = \big(A(z), B(z)\big) \cdot W(a,z).$$

Assume now that $z \in \mathbb{C}$ with $A(z) = B(z) = 0$. Then $A(a, z) = B(a, z) = 0$ for all $a > 0$. This contradicts (3.9). □

3.3 Corollary. *Let $p > -\frac{1}{2}$ and $(P, \psi) \in \mathbb{R}^{2\times 2} \times \mathbb{R}$, denote again $(A, B) := \Xi_p(P, \psi)$, and let $(A(a, z), B(a, z))$ for $a > 0$ be as in (3.1). Then*

$$\lim_{a\downarrow 0} A(a, .) = 1, \quad \lim_{a\downarrow 0} B(a, .) = 0.$$

In particular, the canonical system with Hamiltonian $H_{P,\psi}$ has a solution whose limit at 0 exists and is equal to $(1, 0)$.

Proof. This is obvious from (3.7) and (3.8). $\square$

3.4 Remark. In this context let us point out that any canonical system can have at most one solution $(y_1(a), y_2(a))$ with $\lim_{a\downarrow l_-}(y_1(a), y_2(a)) = (1, 0)$. This is a standard consequence of constancy of the Wronskian; for completeness we recall the argument.

Let $-\infty \leq l_- < l_+ \leq \infty$ and let H be a Hamiltonian on (l_-, l_+). Further, let $(\eta_1, \eta_2)^T \in \mathbb{C}^2 \setminus \{0\}$. Then there exists at most one solution $(y_1(a), y_2(a))$ of the canonical system with Hamiltonian H such that $\lim_{a\downarrow l_-}(y_1(a), y_2(a)) = (\eta_1, \eta_2)$.

To see this let (y_1, y_2) and $(\tilde{y}_1, \tilde{y}_2)$ be two solutions of (1.4), and assume that

$$\lim_{a\downarrow l_-}(y_1(a), y_2(a)) = \lim_{a\downarrow l_-}(\tilde{y}_1(a), \tilde{y}_2(a)) = (\eta_1, \eta_2).$$

Using the differential equation we obtain that the derivative of the Wronskian

$$\det\begin{pmatrix} \tilde{y}_1(a) & \tilde{y}_2(a) \\ y_1(a) & y_2(a) \end{pmatrix} = (y_1(a), y_2(a))J(\tilde{y}_1(a), \tilde{y}_2(a))^T$$

is equal to zero. Hence, this determinant is independent of $a \in (l_-, l_+)$. Evaluating the limit at l_- shows that it is equal to 0. Now choose $c \in (a, b)$, then $(y_1(c), y_2(c))$ and $(\tilde{y}_1(c), \tilde{y}_2(c))$ are linearly dependent. Uniqueness of solutions gives that the functions $(y_1(a), y_2(a))$ and $(\tilde{y}_1(a), \tilde{y}_2(a))$ are linearly dependent. Again evaluating the limit at l_- yields that they are equal.

Another corollary of Theorem 3.1 is that positivity is inherited.

3.5 Corollary. *Let $p > -\frac{1}{2}$ and $(P, \psi) \in \mathbb{R}^{2\times 2} \times \mathbb{R}$. If $P \geq 0$, then $\widehat{\Xi}_p(P, \psi) \in \mathcal{HB}$.*

Proof. We actually are going to show that for all $a > 0$ the function $a \odot_p A - ia \odot_p B$ is a Hermite-Biehler function. Clearly, this is equivalent to the statement that all functions $E_a(z) := A(a, z) - iB(a, z)$ where $A(a, z)$ and $B(a, z)$ are as in (3.1) are Hermite-Biehler functions. We know from Corollary 3.2 that E_a has no real zeroes, and we must proof positivity of the reproducing kernel K_{E_a}.

For $b > 0$ let $W_b(a, z)$ be the unique solution of the initial value problem

$$\begin{cases} \frac{\partial}{\partial a} W_b(a, z)J = zW_b(a, z)H_{P,\psi}(a) & \text{for } a \in (0, \infty), \\ W_b(b, z) = I. \end{cases}$$

Then, by uniqueness of solutions,

$$\forall a \in (0, \infty): \ \big(A(a, z), B(a, z)\big) = \big(A(b, z), B(b, z)\big) \cdot W_b(a, z).$$

We have the kernel relation

$$K_{E_a}(z, w) - K_{E_b}(z, w) = \\ \big(A(b, z), B(b, z)\big)\frac{W_b(a, z)JW_b(a, w)^* - J}{z - \overline{w}}\big(A(b, w), B(b, w)\big)^*.$$

Our assumption that $P \geq 0$ implies that $H_{P,\psi}(a) \geq 0$ for all $a \in (0, \infty)$, and hence the kernel on the right side is positive semidefinite for all $b \leq a$. By Corollary 3.3 we have $\lim_{b\downarrow 0} K_{E_b}(z, w) = 0$, and it follows that $K_{E_a}(z, w)$ is positive semidefinite. $\square$

4 Solution of the canonical system via special functions

It is stated in [Bra62, p.205] that "homogeneous spaces of entire functions are related to Bessel functions and more general confluent hypergeometric functions", however, "it becomes tedious and awkward in handling entire functions", and thus actual formulae are not proven. In this section we provide the formulae for $\Xi_p(P,\psi)$ including all necessary computations, and things turn out not as awkward as one might expect.

Let us recall the definition of confluent hypergeometric (limit) functions.

$$M(\alpha,\beta,z) := \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\beta)_n}\cdot\frac{z^n}{n!}, \quad {}_0F_1(\beta,z) := \sum_{n=0}^{\infty} \frac{1}{(\beta)_n}\cdot\frac{z^n}{n!},$$

where $\alpha, z \in \mathbb{C}$ and $\beta \in \mathbb{C}\setminus(-\mathbb{N}_0)$. The symbol $(\_)_n$ denotes the rising factorial, i.e.,

$$(\alpha)_0 = 1, \quad (\alpha)_{n+1} = (\alpha)_n(\alpha+n) \quad \text{for } n \in \mathbb{N}_0.$$

The function ${}_0F_1$ is indeed a limit of M, namely, it holds that

$${}_0F_1(\beta,z) = \lim_{|\alpha|\to\infty} M(\alpha,\beta,\tfrac{z}{\alpha}) \tag{4.1}$$

locally uniformly in z and β.

By Theorem 3.1 the solution of the canonical system with Hamiltonian $H_{P,\psi}$ is known once $(A,B) = \Xi_p(P,\psi)$ has been computed. Our aim in this section is to prove the following explicit formulae.

4.1 Theorem. *Let $p \in \mathbb{R}\setminus(-\frac{1}{2}(\mathbb{N}_0+1))$, let $P = \begin{pmatrix} \kappa_1 & \kappa_3 \\ \kappa_3 & \kappa_2 \end{pmatrix} \in \mathbb{R}^{2\times 2}$ be a symmetric matrix, and let $\psi \in \mathbb{R}$. Let $\kappa \in \mathbb{C}$ be a square root of $\det P$, and set*

$$\sigma := 2p\kappa_3 - \psi\kappa_1, \qquad \alpha := \tfrac{\sigma}{2i\kappa} + p \text{ if } \kappa \neq 0.$$

As usual, we write $\Xi_p(P,\psi) = (A,B)$.

(i) *If $\det P \neq 0$, then we have*

$$\begin{aligned} A(z) &= e^{i\kappa z}\Big[\frac{1}{2}M(\alpha,2p+1,-2i\kappa z) + \frac{1}{2}M(\alpha+1,2p+1,-2i\kappa z) \\ &\quad - \frac{\kappa_3}{2p+1} z M(\alpha+1,2p+2,-2i\kappa z)\Big], \end{aligned} \tag{4.2}$$

$$B(z) = e^{i\kappa z}\frac{\kappa_1}{2p+1} z M(\alpha+1,2p+2,-2i\kappa z). \tag{4.3}$$

(ii) *If $\det P = 0$, then we have*

$$A(z) = {}_0F_1(2p+1,-\sigma z) - \frac{\kappa_3}{2p+1} z\, {}_0F_1(2p+2,-\sigma z), \tag{4.4}$$

$$B(z) = \frac{\kappa_1}{2p+1} z\, {}_0F_1(2p+2,-\sigma z). \tag{4.5}$$

Let us point out that the functions written in Theorem 4.1 on the right sides of (4.2)–(4.5) do not depend on the choice of the square root κ. This is easy to check using the Kummer transformation [Olv+10, 13.2.39].

4.2 Remark. The function ${}_0F_1$ can be expressed in terms of Bessel functions of the first kind

$$J_\nu(z) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\nu+1)} \Big(\frac{z}{2}\Big)^{2n+\nu}.$$

The formula establishing this reads as

$$J_\nu(z) = \frac{(\frac{z}{2})^\nu}{\Gamma(\nu+1)}\, {}_0F_1(\nu+1, -\tfrac{z^2}{4}).$$

Based on this fact, the formulae in the boundary case Theorem 4.1 (ii) could also be written in terms of Bessel functions.

For a certain particular case, namely when $\beta = 2\alpha$, the Kummer function M is related to modified Bessel functions

$$I_\nu(z) := \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(n+\nu+1)} \Big(\frac{z}{2}\Big)^{2n+\nu}.$$

The formula is

$$I_\nu(z) = \frac{(\frac{z}{2})^\nu}{\Gamma(\nu+1)} e^{-z} M(\nu+\tfrac{1}{2}, 2\nu+1, 2z).$$

If $\sigma = 0$ this allows to rewrite the function from (4.3) to an expression involving only modified Bessel functions. If $\sigma = 0$ and $p \neq 0$, the same holds for the function from (4.2). This follows from a representation obtained in an intermediate step of the proof of Theorem 4.1, namely (4.14).

Before we go into the proof of Theorem 4.1, let us give one noteworthy corollary.

4.3 Corollary. *Let $p \in \mathbb{R} \setminus (-\frac{1}{2}(\mathbb{N}_0+1))$, let $P = \begin{pmatrix} \kappa_1 & \kappa_3 \\ \kappa_3 & \kappa_2 \end{pmatrix} \in \mathbb{R}^{2\times 2}$ be a symmetric matrix, and let $\psi \in \mathbb{R}$. Assume that $\det P \geq 0$, and write as usual $\Xi_p(P,\psi) = (A,B)$. Then A and B are of bounded type in the upper and the lower half-plane.*

Proof. If $\det P = 0$, we know that A and B are of order $\frac{1}{2}$, since the Bessel functions are of exponential type. Assume that $\det P \neq 0$. We know that A and B are of finite exponential type, cf. (2.4). By Krein's Theorem (e.g. [RR94, Theorem 6.17]), it is thus enough to check convergence of the logarithmic integrals

$$\int_{-\infty}^{\infty} \frac{\log^+ |A(x)|}{1+x^2}\, \mathrm{d}x \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{\log^+ |B(x)|}{1+x^2}\, \mathrm{d}x.$$

We use the known asymptotics for confluent hypergeometric functions, see e.g. [AS64, 13.5.1]:

$$\frac{M(\alpha,\beta,x)}{\Gamma(\beta)} = \frac{e^{\pm i\pi\alpha} x^{-\alpha}}{\Gamma(\beta-\alpha)} \cdot \big(1 + \mathrm{o}(\tfrac{1}{|x|})\big) + \frac{e^x x^{\alpha-\beta}}{\Gamma(\alpha)} \cdot \big(1 + \mathrm{o}(\tfrac{1}{|x|})\big).$$

Since $\det P \geq 0$ we have $\kappa \in \mathbb{R}$. The above asymptotic expansions thus show that $|A(x)|$ and $|B(x)|$ are bounded by some power for $x \in \mathbb{R}$. $\square$

The core computation

We follow the lines of [LPW22, Section 3] where a particular case was treated. The core of the argument is that in sufficiently many cases the canonical system with power Hamiltonian $H_{P,\psi}$ can be reduced to Kummer's equation

$$xy''(x) + (\beta - x)y'(x) - \alpha y(x) = 0,$$

with a certain choice of parameters $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{C} \setminus (-\mathbb{N}_0)$.

4.4 Proposition. *Let $p \in \mathbb{R} \setminus (-\frac{1}{2}(\mathbb{N}_0 + 1))$, let $P = \begin{pmatrix} \kappa_1 & \kappa_3 \\ \kappa_3 & \kappa_2 \end{pmatrix} \in \mathbb{R}^{2\times 2}$ be a symmetric matrix, and let $\psi \in \mathbb{R}$. Assume that*

$$p \neq 0, \quad \det P \neq 0, \quad \kappa_2 \neq 0, \quad \psi = 0.$$

Let $\kappa \in \mathbb{C}$ be a square root of $\det P$, and set $\alpha := \frac{p\kappa_3}{i\kappa} + p$. Then the functions

$$A(a, z) = e^{i\kappa az} \cdot M(\alpha, 2p, -2i\kappa az), \tag{4.6}$$

$$B(a, z) = e^{i\kappa az} \cdot \frac{a^{2p+1}}{2p+1} \kappa_1 z M(\alpha + 1, 2p + 2, -2i\kappa az), \tag{4.7}$$

satisfy

$$\frac{\partial}{\partial a}\big(A(a, z), B(a, z)\big)J = z\big(A(a, z), B(a, z)\big)H_{P,0}(a) \quad \text{for } a > 0.$$

The proof relies on the following simple fact, see e.g. [LPW22, Lemma 3.5].

4.5 Lemma. *Let $H = \begin{pmatrix} h_1 & h_3 \\ h_3 & h_2 \end{pmatrix} \in C^1((0,\infty), \mathbb{R}^{2\times 2})$ be symmetric with h_2 zerofree, let $y_1, y_2 \in C^2((0,\infty), \mathbb{C})$, and let $z \in \mathbb{C} \setminus \{0\}$. Then*

$$\forall x \in (0,\infty): \quad \big(y_1'(x), y_2'(x)\big)J = z\big(y_1(x), y_2(x)\big)H(x)$$

if and only if the following two equations hold for all $x \in (0,\infty)$:

$$\frac{1}{h_2(x)}y_1''(x) + \Big(\frac{1}{h_2(x)}\Big)' y_1'(x) + \Big[z\Big(\frac{h_3(x)}{h_2(x)}\Big)' + z^2\Big(h_1(x) - \frac{h_3(x)^2}{h_2(x)}\Big)\Big]y_1(x) = 0, \tag{4.8}$$

$$y_2(x) = -\frac{1}{z}\frac{1}{h_2(x)}y_1'(x) - \frac{h_3(x)}{h_2(x)}y_1(x). \tag{4.9}$$

Proof of Proposition 4.4. The Hamiltonian $H_{P,0}$ is explicitly given as

$$H_{P,0}(x) = \begin{pmatrix} \kappa_1 x^{2p} & \kappa_3 \\ \kappa_3 & \kappa_2 x^{-2p} \end{pmatrix}.$$

Let $w \in \mathbb{C} \setminus \{0\}$. The equation (4.8) for $H_{P,0}$ and $z := w \cdot \frac{-1}{2i\kappa}$ reads as

$$xy_1''(x) + 2p \cdot y_1'(x) - \Big(w\frac{\kappa_3 p}{i\kappa} + \frac{w^2}{4}x\Big)y_1(x) = 0. \tag{4.10}$$

Now let $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C} \setminus (-\mathbb{N}_0)$, $w \in \mathbb{C} \setminus \{0\}$, and consider the function

$$f(x) := e^{-\frac{wx}{2}} M(\alpha, \beta, wx).$$

Then $e^{\frac{x}{2}} f(\frac{x}{w}) = M(\alpha, \beta, x)$, and Kummer's equation gives

$$\frac{e^{\frac{x}{2}}}{w} \cdot \Big[\tfrac{x}{w} f''(\tfrac{x}{w}) + \beta f'(\tfrac{x}{w}) + w\big(- \tfrac{x}{4} + \tfrac{\beta}{2} - \alpha\big) f(\tfrac{x}{w})\Big] = 0$$

for all $x \in \mathbb{C}$. Equivalently, substituting x by $x \cdot w$,

$$x f''(x) + \beta f'(x) - \Big(w\big(\alpha - \tfrac{\beta}{2}\big) + \frac{w^2}{4} x\Big) f(x) = 0 \quad \text{for } x \in \mathbb{C}. \tag{4.11}$$

We observe that the equations (4.10) and (4.11) coincide when choosing the parameters α, β as

$$\beta := 2p, \quad \alpha := p(1 + \tfrac{\kappa_3}{i\kappa}).$$

We see that the function $A(a, z)$ defined in (4.6) satisfies (4.8).

To show the asserted formula (4.7) for the function $B(a, z)$, it remains to plug $A(a, z)$ into the right side of (4.9) and compute the outcome.

Recall the differentiation formula

$$\frac{\partial}{\partial x} M(\alpha, \beta, x) = \frac{\alpha}{\beta} M(\alpha + 1, \beta + 1, x),$$

and the following linear dependency between contigous hypergeometric functions, cf. [AS64, §13.4]:

$$M(\alpha, \beta, x) - M(\alpha + 1, \beta + 1, x) = \frac{\alpha - \beta}{(\beta + 1)\beta} x M(\alpha + 1, \beta + 2, x).$$

Having in mind (4.9), we use these formulae to compute

$$\begin{aligned}
&-\frac{1}{z} \frac{a^{2p}}{\kappa_2} \frac{\partial}{\partial a} A(a, z) - \kappa_3 \frac{a^{2p}}{\kappa_2} A(a, z) \\
&= -\frac{1}{z} \frac{a^{2p}}{\kappa_2} e^{i\kappa az} i\kappa z \Big[M(\alpha, 2p, -2i\kappa az) - 2 \frac{\alpha}{2p} M(\alpha + 1, 2p + 1, -2i\kappa az)\Big] \\
&\quad - \kappa_3 \frac{a^{2p}}{\kappa_2} e^{i\kappa az} M(\alpha, 2p, -2i\kappa az) \\
&= -\frac{a^{2p}}{\kappa_2} e^{i\kappa az} (i\kappa + \kappa_3) \Big[M(\alpha, 2p, -2i\kappa az) - M(\alpha + 1, 2p + 1, -2i\kappa az)\Big] \\
&= -\frac{a^{2p}}{\kappa_2} e^{i\kappa az} (i\kappa + \kappa_3) \cdot \frac{\alpha - 2p}{(2p + 1)2p} (-2i\kappa az) \cdot M(\alpha + 1, 2p + 2, -2i\kappa az) \\
&= e^{i\kappa az} \cdot \frac{a^{2p+1}}{2p + 1} \kappa_1 z M(\alpha + 1, 2p + 2, -2i\kappa az).
\end{aligned}$$

□

4.6 Corollary. *Let $p \in \mathbb{R} \setminus (-\frac{1}{2}(\mathbb{N}_0 + 1))$, let $P = \begin{pmatrix} \kappa_1 & \kappa_3 \\ \kappa_3 & \kappa_2 \end{pmatrix} \in \mathbb{R}^{2\times 2}$ be a symmetric matrix, and let $\psi \in \mathbb{R}$. Assume that*

$$p \neq 0, \quad \det P \neq 0, \quad \kappa_2 \neq 0, \quad \psi = 0.$$

Let $\kappa \in \mathbb{C}$ be a square root of $\det P$*, and set* $\alpha := \frac{p\kappa_3}{i\kappa} + p$*. Writing* $\Xi_p(P,\psi) = (A,B)$*, we have*

$$A(z) = e^{i\kappa z} \cdot M(\alpha, 2p, -2i\kappa z), \tag{4.12}$$

$$B(z) = e^{i\kappa z} \cdot \frac{\kappa_1}{2p+1} z M(\alpha+1, 2p+2, -2i\kappa z). \tag{4.13}$$

Proof. Let $\tilde{A}(z)$ and $\tilde{B}(z)$ be the right-hand sides of (4.12) and (4.13), respectively. Then $\tilde{A}(0) = 1$, $\tilde{B}(0) = 0$, and

$$\begin{aligned}&\big([a \odot_p \tilde{A}](z), [a \odot_p \tilde{B}](z)\big)\mathfrak{D}_\psi(a)^{-1} = \big(a^{-p} \cdot a^p \tilde{A}(az), a^p \cdot a^p \tilde{B}(az)\big)\\ &= \Big(e^{i\kappa az} M(\alpha, 2p, -2i\kappa az), e^{i\kappa az} a^{2p+1} \frac{\kappa_1}{2p+1} z M(\alpha{+}1, 2p{+}2, -2i\kappa az)\Big).\end{aligned}$$

These are the functions from (4.6) and (4.7), and hence satisfy the canonical system. Now we apply Theorem 3.1, "$(iii) \Rightarrow (i)$". □

Pushing further the formulae from Corollary 4.6

First we extend Corollary 4.6 to more general values of ψ. This is done with help of the transformation from Lemma 2.13.

4.7 Corollary. *Let* $p \in \mathbb{R} \setminus (-\frac{1}{2}(\mathbb{N}_0 + 1))$*, let* $P = \begin{pmatrix} \kappa_1 & \kappa_3 \\ \kappa_3 & \kappa_2 \end{pmatrix} \in \mathbb{R}^{2\times 2}$ *be a symmetric matrix, and let* $\psi \in \mathbb{R}$*. Assume that*

$$p \neq 0, \quad \det P \neq 0, \quad \frac{\kappa_1}{4p^2}\psi^2 - \frac{\kappa_3}{p}\psi + \kappa_2 \neq 0.$$

Let $\kappa \in \mathbb{C}$ *be a square root of* $\det P$*, set* $\sigma := 2p\kappa_3 - \psi\kappa_1$ *and* $\alpha := \frac{\sigma}{2i\kappa} + p$*. Writing* $\Xi_p(P,\psi) = (A,B)$*, we have*

$$\begin{aligned}A(z) &= e^{i\kappa z} \cdot \Big[M(\alpha, 2p, -2i\kappa z) - \frac{\psi\kappa_1}{2p(2p+1)} z M(\alpha+1, 2p+2, -2i\kappa z)\Big],\\ B(z) &= e^{i\kappa z} \cdot \frac{\kappa_1}{2p+1} z M(\alpha+1, 2p+2, -2i\kappa z).\end{aligned} \tag{4.14}$$

Proof. We use Lemma 2.13 with $\gamma := -\frac{\psi}{2p}$. Then

$$\tilde{P} := \begin{pmatrix} 1 & 0 \\ -\frac{\psi}{2p} & 1 \end{pmatrix} P \begin{pmatrix} 1 & -\frac{\psi}{2p} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \kappa_1 & \frac{\sigma}{2p} \\ \frac{\sigma}{2p} & \frac{\kappa_1}{4p^2}\psi^2 - \frac{\kappa_3}{p}\psi + \kappa_2 \end{pmatrix},$$

$$\tilde{\psi} := \psi + 2p(-\tfrac{\psi}{2p}) = 0, \quad \det \tilde{P} = \det P.$$

Then

$$\Xi_p(P,\psi) = \Xi_p(\tilde{P},\tilde{\psi}) \begin{pmatrix} 1 & 0 \\ -\frac{\psi}{2p} & 1 \end{pmatrix}$$

and the assertion follows by plugging in the formulae from Corollary 4.6. □

Now we use a continuity argument. Recall Lemma 2.8, which said that the function $(p, P, \psi) \mapsto \Xi_p(P,\psi)$ is continuous on its whole domain.

Proof of Theorem 4.1, (4.3). The function on the right-hand side of (4.3) is continuous in the region

$$\mathbb{R} \setminus (-\tfrac{1}{2}(\mathbb{N}_0 + 1)) \times \Big\{ P \in \mathbb{R}^{2\times 2} \mid P \text{ symmetric}, \det P \neq 0 \Big\} \times \mathbb{R}.$$

By Corollary 4.7 the equality (4.3) holds on the dense subset described by the restrictions that $p \neq 0$ and $\frac{\kappa_1}{4p^2}\psi^2 - \frac{\kappa_3}{p}\psi + \kappa_2 \neq 0$. Note here that in the considered region always $\det P \neq 0$. ☐

In order to prove (4.2) we rewrite the right-hand side of (4.14) in a way suitable to see continuity in p also at $p = 0$. This is done by using some relations among contigous confluent hypergeometric functions.

4.8 Lemma. *Let $p \in \mathbb{R} \setminus (-\frac{1}{2}(\mathbb{N}_0 + 1))$, let $P = \begin{pmatrix} \kappa_1 & \kappa_3 \\ \kappa_3 & \kappa_2 \end{pmatrix} \in \mathbb{R}^{2\times 2}$ be a symmetric matrix, and let $\psi \in \mathbb{R}$. Assume that*

$$p \neq 0, \quad \det P \neq 0.$$

Let $\kappa \in \mathbb{C}$ be a square root of $\det P$, set $\sigma := 2p\kappa_3 - \psi\kappa_1$ and $\alpha := \frac{\sigma}{2i\kappa} + p$. Then

$$\begin{aligned} M(\alpha, 2p, -2i\kappa z) &- \frac{\psi\kappa_1}{2p(2p+1)} z M(\alpha+1, 2p+2, -2i\kappa z) \\ &= \frac{1}{2} M(\alpha, 2p+1, -2i\kappa z) + \frac{1}{2} M(\alpha+1, 2p+1, -2i\kappa z) \\ &\quad - \frac{\kappa_3}{2p+1} z M(\alpha+1, 2p+2, -2i\kappa z)\Big]. \end{aligned} \tag{4.15}$$

Proof. To shorten notation set $\beta := 2p$ and $w := -2i\kappa z$. We can rewrite

$$\begin{aligned} &\Big[M(\alpha,\beta,w) - \frac{\psi\kappa_1}{\beta(\beta+1)} z M(\alpha+1,\beta+2,w)\Big] \\ &\quad - \Big[\frac{1}{2}M(\alpha,\beta+1,w) + \frac{1}{2}M(\alpha+1,\beta+1,w) - \frac{\kappa_3}{\beta+1} z M(\alpha+1,\beta+2,w)\Big] \\ &= \frac{1}{\beta}\Big[\beta M(\alpha,\beta,w) - \alpha M(\alpha+1,\beta+1,w) + (\alpha-\beta) M(\alpha,\beta+1,w)\Big] \\ &\quad + \frac{p-\alpha}{\beta}\Big[M(\alpha,\beta+1,w) - M(\alpha+1,\beta+1,w) + \frac{w}{\beta+1} M(\alpha+1,\beta+2,w)\Big]. \end{aligned}$$

The first square bracket of the right-hand side vanishes by [Olv+10, 13.3.3], and the second by [Olv+10, 13.3.4]. ☐

Proof of Theorem 4.1, (4.2). The function on the right-hand side of (4.2) is continuous in the region

$$\mathbb{R} \setminus (-\tfrac{1}{2}(\mathbb{N}_0 + 1)) \times \Big\{ P \in \mathbb{R}^{2\times 2} \mid P \text{ symmetric}, \det P \neq 0 \Big\} \times \mathbb{R}.$$

By (4.15) of the previous lemma and Corollary 4.7, the equality (4.2) holds on the dense subset described by the restrictions that $p \neq 0$ and $\frac{\kappa_1}{4p^2}\psi^2 - \frac{\kappa_3}{p}\psi + \kappa_2 \neq 0$. Note again that in the considered region always $\det P \neq 0$. ☐

It remains to establish the boundary case (ii) in Theorem 4.1.

Proof of Theorem 4.1, (ii). Assume first that $\sigma \neq 0$. The formulae (4.4), (4.5) are obtained by passing to the limit "$\kappa \to 0$" in (4.2), (4.3) using the formula (4.1). Note here that, clearly, the set of all invertible symmetric matrices is dense in the set of all symmetric matrices.

We have

$$\lim_{\kappa\to 0}(-2i\kappa)\alpha = \lim_{\kappa\to 0}(-2i\kappa)(\alpha+1) = -\sigma.$$

Since $\sigma \neq 0$, we have $|\alpha| \to \infty$ when $\kappa \to 0$. Hence

$$\lim_{\kappa\to 0} M(\alpha, 2p+1, -2i\kappa z) = \lim_{\kappa\to 0} M\big(\alpha, 2p+1, \tfrac{(-2i\kappa)\alpha}{\alpha}\big) = {}_0F_1(2p+1, -\sigma z),$$

and analogously

$$\lim_{\kappa\to 0} M(\alpha+1, 2p+1, -2i\kappa z) = {}_0F_1(2p+1, -\sigma z),$$

$$\lim_{\kappa\to 0} M(\alpha+1, 2p+2, -2i\kappa z) = {}_0F_1(2p+2, -\sigma z).$$

This gives (4.4) and (4.5).

The left and right sides of (4.4) and (4.5) depend continuously on σ, and hence the equality holds also for $\sigma = 0$. $\square$

5 Isometric inclusions of rescaled spaces

It is a structural property of a space $\mathcal{H}(E)$ whether or not there are spaces $\mathcal{H}(a \odot_p E)$ which belong to its chain of de Branges subspaces.

5.1 Definition. Let $p \in \mathbb{R}$ and $E \in \mathcal{HB}$. Then we denote

$$\mathcal{O}_p(E) := \big\{a \in \mathbb{R}^+ \mid \mathcal{H}(a \odot_p E) \subseteq \mathcal{H}(E) \text{ isometrically}\big\}.$$

Trivially, $1 \in \mathcal{O}_p(E)$. By Lemma 2.3 we have $a \in \mathcal{O}_p(E)$ if and only if the map $F \mapsto a \odot_{p+\frac{1}{2}} F$ maps $\mathcal{H}(E)$ isometrically into itself. In particular, the set $\mathcal{O}_p(E)$ depends only on the space $\mathcal{H}(E)$ and not on the particular Hermite-Biehler function generating it.

Since we have an underlying continuous group action, it can be expected that the set $\mathcal{O}_p(E)$ has some structure. To investigate it, we start with a basic fact.

5.2 Lemma. *Let $p \in \mathbb{R}$ and $E \in \mathcal{HB}$, and let further $a, b \in \mathbb{R}^+$. Then the following statements hold.*

(i) $\mathcal{H}(a \odot_p E) \subseteq \mathcal{H}(b \odot_p E)$ *isometrically*

$$\iff \quad \forall c \in \mathbb{R}^+: \ \mathcal{H}(ca \odot_p E) \subseteq \mathcal{H}(cb \odot_p E) \text{ isometrically}$$

$$\iff \quad \frac{a}{b} \in \mathcal{O}_p(E).$$

(ii) *The set $\mathcal{O}_p(E)$ is a subsemigroup of $\mathbb{R}^+$.*

(iii) $\forall a \in \mathbb{R}^+: \ \mathcal{O}_p(a \odot_p E) = \mathcal{O}_p(E)$.

Proof. For the proof of (i) let $a, b, c > 0$. By Lemma 2.3 the map $F \mapsto c \odot_{p+\frac{1}{2}} F$ is an isometric isomorphism of $\mathcal{H}(a \odot_p E)$ onto $\mathcal{H}(c \odot_p (a \odot_p E))$. Now note that $c \odot_p (a \odot_p E) = (ca) \odot_p E$. The same holds for b in place of a, and we see that the first condition implies the second. The converse implication is trivial; just use "$c = 1$". To prove that the first and third conditions are equivalent, apply the already proven with $c := \frac{1}{b}$ for "$\Rightarrow$" and with $c := b$ for "$\Leftarrow$".

To show that $\mathcal{O}_p(E)$ is closed under multiplication, let $a, b \in \mathcal{O}_p(E)$. Then $\mathcal{H}(b \odot_p E) \subseteq \mathcal{H}(E)$ isometrically, and hence also $\mathcal{H}(ab \odot_p E) \subseteq \mathcal{H}(a \odot_p E)$ isometrically. Since $\mathcal{H}(a \odot_p E) \subseteq \mathcal{H}(E)$ isometrically, it follows in turn that $\mathcal{H}(ab \odot_p E) \subseteq \mathcal{H}(E)$ isometrically.

Finally, to show (iii), note that by the already proven equivalence in (i) we have

$$\begin{aligned} c \in \mathcal{O}_p(a \odot_p E) \quad &\Longleftrightarrow \quad \mathcal{H}\big(\underbrace{c \odot_p (a \odot_p E)}_{= ca \odot_p E}\big) \subseteq \mathcal{H}(a \odot_p E) \text{ isometrically} \\ &\Longleftrightarrow \quad c \in \mathcal{O}_p(E), \end{aligned}$$

and the same for b in place of a. ☐

Our next statement is that generically isometric inclusions can occur only for $p > -\frac{1}{2}$ and $a \leq 1$. The case of one-dimensional spaces is exceptional and corresponds to the boundary case $p = -\frac{1}{2}$.

5.3 Proposition. *Let $E \in \mathcal{HB}$. Then the following statements hold.*

(i) *If $\dim \mathcal{H}(E) > 1$ and $p \in \mathbb{R}$ with $\mathcal{O}_p(E) \neq \{1\}$, then $p > -\frac{1}{2}$ and $\mathcal{O}_p(E) \subseteq (0, 1]$.*

(ii) *If $\dim \mathcal{H}(E) = 1$ and $p \in \mathbb{R}$ with $\mathcal{O}_p(E) \neq \{1\}$, then $p = -\frac{1}{2}$ and $\mathcal{H}(E) = \operatorname{span}\{1\}$.*

(iii) *If $\mathcal{H}(E) = \operatorname{span}\{1\}$, then $\mathcal{O}_{-\frac{1}{2}}(E) = \mathbb{R}^+$.*

Proof. For the proof of (i) assume that $\dim \mathcal{H}(E) > 1$. In the first step we prove that

$$\forall p \in \mathbb{R}: \quad \mathcal{O}_p(E) \subseteq (0, 1] \tag{5.1}$$

Choose a phase function φ_E associated with E (cf. Remark 1.6). By [Bra68, Theorem 22, Problem 46] our assumption that $\mathcal{H}(E)$ is at least two-dimensional implies that

$$\lim_{x \to \infty} \big(\varphi_E(x) - \varphi_E(-x)\big) > \pi.$$

Hence, we find $x_0 > 0$ with $\varphi_E(x_0) - \varphi_E(-x_0) = \pi$.

Let $p \in \mathbb{R}$ and $a \in \mathcal{O}_p(E)$. Clearly, the function $x \mapsto \varphi_E(ax)$ is a phase function associated with $a \odot_p E$. Since $\mathcal{H}(a \odot_p E) \subseteq \mathcal{H}(E)$ isometrically, [Bra68, Problem 93] yields

$$\varphi_E(ax_0) - \varphi_E(-ax_0) = \varphi_{a \odot_p E}(x_0) - \varphi_{a \odot_p E}(-x_0) \leq \varphi_E(x_0) - \varphi_E(-x_0).$$

Since φ_E is strictly increasing, it follows that $a \leq 1$. The proof of (5.1) is complete.

As a consequence, we obtain that

$$\forall p \in \mathbb{R}: \quad \mathcal{O}_p(E) \neq \{1\} \;\Rightarrow\; p \geq -\frac{1}{2} \tag{5.2}$$

To see this, recall the kernel relation (2.2). It implies in particular that

$$K_{a \odot_p E}(0,0) = a^{2p+1} K_E(0,0).$$

If $\mathcal{H}(a \odot_p E) \subseteq \mathcal{H}(E)$ isometrically, the reproducing kernels satisfy $K_{a\odot_p E}(0,0) \leq K_E(0,0)$, since these quantities are the (square of the) norm of point evaluation at 0. It follows that $a^{2p+1} \leq 1$. Knowing that a cannot exceed 1, (5.2) follows.

It remains to exclude the case that $p = -\frac{1}{2}$. Assume that $a \in \mathcal{O}_{-\frac{1}{2}}(E)$. By Lemma 5.2 we have the chain of isometric inclusions

$$\mathcal{H}(a^2 \odot_{-\frac{1}{2}} E) \subseteq \mathcal{H}(a \odot_{-\frac{1}{2}} E) \subseteq \mathcal{H}(E). \tag{5.3}$$

Since $K_{a^2\odot_{-\frac{1}{2}} E}(0,0) = K_E(0,0)$, i.e., the interval between these two de Branges subspaces is indivisible of type $\frac{\pi}{2}$, we have

$$\dim \Big(\mathcal{H}(E) \Big/ \mathcal{H}(a^2 \odot_{-\frac{1}{2}} E) \Big) \leq 1.$$

In the above chain (5.3) thus at least one inclusion must hold with equality. Using the appropriate isometry ($F \mapsto \frac{1}{a} \odot_0 F$ or $F \mapsto a \odot_0 F$), this implies that equality holds throughout. Hence, certainly $\mathcal{H}(a \odot_{-\frac{1}{2}} E) = \mathcal{H}(E)$, and we see that $F \mapsto a \odot_0 F$ is an isometric bijection of $\mathcal{H}(E)$ onto itself. Its inverse, which is $F \mapsto \frac{1}{a} \odot_0 F$, thus has the same property. Now (5.1) implies that $a = 1$. The proof of (i) is complete.

We come to the proof of (ii). Assume that $\mathcal{H}(E) = \operatorname{span}\{G\}$ with some entire function G, and assume further that $p \in \mathbb{R}$ and $a \in \mathcal{O}_p(E) \setminus \{1\}$. The function $a \odot_{p+\frac{1}{2}} G$ belongs to the space $\mathcal{H}(E)$ and has the same norm as G. Thus, $a \odot_{p+\frac{1}{2}} G = G$. Writing the power series expansion of G as $G(z) = \sum_{n=0}^{\infty} \gamma_n z^n$, this gives

$$\sum_{n=0}^{\infty} a^{p+\frac{1}{2}+n} \gamma_n z^n = \sum_{n=0}^{\infty} \gamma_n z^n,$$

and comparing coefficients yields

$$\forall n \in \mathbb{N}_0: \quad \big(a^{p+\frac{1}{2}+n} - 1\big)\gamma_n = 0$$

Since $a \neq 1$ and $\gamma_0 = G(0) \neq 0$, recall here the fourth property in Definition 1.1, this implies that

$$p = -\frac{1}{2} \quad \text{and} \quad \forall n \geq 1: \;\; \gamma_n = 0.$$

Thus, G is constant and $\mathcal{H}(E) = \operatorname{span}\{1\}$.

For the proof of (iii) it suffices to note that each map $F \mapsto a \odot_0 F$ acts as the identity on $\operatorname{span}\{1\}$. □

Now the structure of the set $\mathcal{O}_p(E)$ can be clarified.

5.4 Proposition. *Let $p > -\frac{1}{2}$ and $E \in \mathcal{HB}$. Then one of the following statements holds.*

- $\triangleright$ $\mathcal{O}_p(E) = \{1\}$;
- $\triangleright$ $\exists a_0 \in (0,1)\colon\ \ \mathcal{O}_p(E) = \{(a_0)^n \mid n \in \mathbb{N}_0\}$;
- $\triangleright$ $\mathcal{O}_p(E) = (0,1]$.

Proof. We have already seen that $\mathcal{O}_p(E)$ is a subsemigroup of $(0,1]$. The first step in the proof is to show that $\mathcal{O}_p(E)$ is also invariant under suitable quotients:

$$\forall a, b \in \mathcal{O}_p(E), a \leq b\colon\ \ \frac{a}{b} \in \mathcal{O}_p(E). \tag{5.4}$$

Let $a, b \in \mathcal{O}_p(E)$. If $a = b$, there is nothing to prove, hence assume that $a < b$. Both spaces $\mathcal{H}(a \odot_p E)$ and $\mathcal{H}(b \odot_p E)$ are contained isometrically in $\mathcal{H}(E)$. By the Ordering Theorem (e.g. [Bra68, Theorem 35]), either $\mathcal{H}(a \odot_p E) \subseteq \mathcal{H}(b \odot_p E)$ or $\mathcal{H}(b \odot_p E) \subseteq \mathcal{H}(a \odot_p E)$. Since

$$K_{a \odot_p E}(0,0) = a^{2p+1} K_E(0,0) < b^{2p+1} K_E(0,0) = K_{b \odot_p E}(0,0).$$

the first case must take place. Now Lemma 5.2 applies, and we obtain that $\frac{a}{b} \in \mathcal{O}_p(E)$.

Knowing (5.4), it follows that the set $\mathcal{G} := \mathcal{O}_p(E) \cup \mathcal{O}_p(E)^{-1}$ is a subgroup of $\mathbb{R}^+$ and satisfies $\mathcal{G} \cap (0,1] = \mathcal{O}_p(E)$. A subgroup of $\mathbb{R}^+$ is either

- $\triangleright$ trivial, i.e., $\mathcal{G} = \{1\}$, or
- $\triangleright$ nontrivial and cyclic, i.e., $\mathcal{G} = \{(a_0)^n \mid n \in \mathbb{Z}\}$ for some $a_0 \in (0,1)$, or
- $\triangleright$ dense in $\mathbb{R}^+$.

To complete the proof of the present assertion, it is thus enough to show that $\mathcal{O}_p(E)$ is closed in $(0,1]$. Let $a_n \in \mathcal{O}_p(E)$, $n \in \mathbb{N}_0$, and $a \in (0,1]$ with $\lim_{n\to\infty} a_n = a$. If $a = 1$, there is nothing to prove, hence we may assume that $a < 1$ and w.l.o.g. that $a_n < 1$ for all $n \in \mathbb{N}_0$. Choose $N \in \mathbb{N}_0$, such that $(a_0)^N < \inf_{n\in\mathbb{N}_0} a_n$.

Let H_E be the structure Hamiltonian of E, and $(E(t,.))_{t\leq 0}$ the corresponding chain of Hermite-Biehler functions. Let $t_n \in (-\infty, 0]$ be such that $\mathcal{H}(a_n \odot_p E) = \mathcal{H}(E(t_n,.))$, and $s \in (-\infty,0]$ such that $\mathcal{H}(a_0^N \odot_p E) = \mathcal{H}(E(s,.))$. Since $K_{a_0^N \odot_p E}(0,0) < K_{a_n \odot_p E}(0,0)$ for all $n \in \mathbb{N}_0$, we also have $s < t_n$ for all $n \in \mathbb{N}_0$. Choose a convergent subsequence $(t_{n_j})_{j\in\mathbb{N}_0}$ with $t_\infty := \lim_{j\to\infty} t_{n_j} \in [s, 0]$. The set of points $t \in (-\infty,0]$ for which $\mathcal{H}(E(t,.))$ is contained isometrically in $\mathcal{H}(E)$ is closed since it is the complement of the union of all indivisible intervals. Hence, $\mathcal{H}(E(t_\infty,.))$ is contained in $\mathcal{H}(E)$ isometrically. We have

$$\begin{aligned} K_{E(t_\infty,.)}(z,w) &= \lim_{j\to\infty} K_{E(t_{n_j},.)}(z,w) = \lim_{j\to\infty} K_{a_{n_j}\odot_p E}(z,w) \\ &= \lim_{j\to\infty} a_{n_j}^{2p+1} K_E(a_{n_j} z, a_{n_j} w) = a^{2p+1} K_E(az, aw) = K_{a\odot_p E}(z,w), \end{aligned}$$

and see that $\mathcal{H}(E(t_\infty,.)) = \mathcal{H}(a \odot_p E)$. Thus, $a \in \mathcal{O}_p(E)$. $\square$

Let us show by examples that each of the three alternatives can occur.

5.5 Example. Let $\mathcal{H}(E)$ be a de Branges space with $\dim \mathcal{H}(E) > 1$ and $1 \in \mathcal{H}(E)$. We assert that $\mathcal{O}_p(E) = \{1\}$ for all p. To see this, assume towards a contradiction that there exists $a \in \mathcal{O}_p(E) \setminus \{1\}$. We have $a \odot_{p+\frac{1}{2}} 1 = a^{p+\frac{1}{2}}$, and hence

$$\|1\| = \|a \odot_{p+\frac{1}{2}} 1\| = a^{p+\frac{1}{2}} \cdot \|1\|.$$

This contradicts the fact that $p > -\frac{1}{2}$ by Proposition 5.3.

5.6 Example. Let $\mathring{H}$ be a Hamiltonian on $(0, 1)$ with $\operatorname{tr} \mathring{H} = 1$ be such that the interval $(0, 1)$ is not indivisible, and define a Hamiltonian H on $(0, \infty)$ by

$$H(t) := \mathring{H}\Big(\frac{t}{2^n} - 1\Big) \quad \text{for } n \in \mathbb{Z}, t \in (2^n, 2^{n+1}).$$

Then H is multiplicatively periodic with period 2, i.e., $H(2t) = H(t)$ for all $t > 0$. Let $(A(t, z), B(t, z))$ be the unique solution of the initial value problem

$$\begin{cases} \frac{\partial}{\partial t}(A(t, z), B(t, z))J = z(A(t, z), B(t, z))H(t) & \text{for } t > 0, \\ (A(0, z), B(0, z)) = (1, 0). \end{cases}$$

Since H is periodic, uniqueness of the solution implies that

$$\forall t \geq 0, z \in \mathbb{C}\colon \quad \big(A(2t, z), B(2t, z)\big) = \big(A(t, 2z), B(t, 2z)\big).$$

In other words, the functions $E(t, z) := A(t, z) - iB(t, z)$ satisfy

$$\forall t \geq 0\colon \quad E(2t, .) = 2 \odot_0 E(t, .).$$

Choose $t_0 \in (0, 1)$ such that t_0 is not inner point of an $\mathring{H}$-indivisible interval. Then each point $2^n(t_0 + 1)$ with $n \in \mathbb{Z}$ is not inner point of an H-indivisible interval, and hence

$$\forall n, m \in \mathbb{Z}, n \leq m\colon \quad \mathcal{H}\big(E(2^n(t_0+1), .)\big) \subseteq \mathcal{H}\big(E(2^m(t_0+1), .)\big) \text{ isometrically.}$$

Since $E(2^n(t_0 + 1), .) = 2^n \odot_0 E(t_0 + 1, .)$, we see that

$$\{2^n \mid n \in -\mathbb{N}_0\} \subseteq \mathcal{O}_0(E(t_0 + 1, .)).$$

Now we want to impose some condition on $\mathring{H}$ which guarantees that

$$\mathcal{O}_0(E(t_0 + 1, .)) \neq (0, 1].$$

For example, assuming that $\mathring{H}$ consists of a sequence of indivisible intervals does the job. Assume that we have $N \geq 2$ and

$$0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = 1,$$

such that each interval (t_{i-1}, t_i) is $\mathring{H}$-indivisible and that the types of each two successive intervals are different. Then all but countably many points of $(0, \infty)$ are inner point of some H-indivisible interval, and hence the chain of de Branges subspaces of $\mathcal{H}(E(t_0 + 1, .))$ is countable. Because of (2.2) the map $a \mapsto \mathcal{H}(a \odot_0 E(t_0 + 1, .))$ maps $\mathcal{O}_0(E(t_0 + 1, .))$ injectively into this chain. Thus $\mathcal{O}_0(E(t_0 + 1, .))$ cannot be equal to $(0, 1]$.

5.7 Example. The classical Paley-Wiener space $\mathcal{PW}_1$ is generated by the Hermite-Biehler function $E(z) := e^{-iz}$. Let $F \in \mathcal{H}(E)$. For all $p \in \mathbb{R}$ and $a \in (0,1)$ the function $a \odot_{p+\frac{1}{2}} F$ belongs to $\mathcal{H}(E)$. However, its norm coincides with the norm of F, if and only if $p = 0$:

$$\|a \odot_{p+\frac{1}{2}} F\|^2_{\mathcal{H}(E)} = \int_{\mathbb{R}} |a^{p+\frac{1}{2}} F(ax)|^2 \,\mathrm{d}x = \int_{\mathbb{R}} a^{2p} |F(y)|^2 \,\mathrm{d}y.$$

Thus

$$\mathcal{O}_p(E) = \begin{cases} (0,1] & \text{if } p = 0, \\ \{1\} & \text{otherwise.} \end{cases}$$

Homogeneous de Branges spaces

Let us now turn to homogeneous spaces in the sense of Definition 1.2. Note that a space $\mathcal{H}(E)$ is homogeneous of order ν (with some $\nu > -1$), if and only if $\mathcal{O}_{\nu+\frac{1}{2}}(E) = (0,1]$. We had decided to stick to the terminology introduced by L.de Branges in [Bra68, Chapter 50] in Definition 1.2, and this is why we have a shift by $\frac{1}{2}$ in the parameters, i.e., ν from that definition and p from above are related as $p = \nu + \frac{1}{2}$.

For $E \in \mathcal{HB}$ such that $\mathcal{H}(E)$ is homogeneous of order ν, the chain of de Branges spaces

$$\big\{\mathcal{H}(a \odot_{\nu+\frac{1}{2}} E) \mid a \in (0,1]\big\}$$

is contained in the chain of all de Branges subspaces of $\mathcal{H}(E)$. It follows from continuity that it exhausts this chain (cf. Corollary 5.9 below). To prove this, we recall the following general and probably folklore fact.

5.8 Lemma. *Let Ω be a nonempty set, $\mathcal{I} \subseteq \mathbb{R}$ an interval, and $(\mathcal{H}_t)_{t\in\mathcal{I}}$ a family of reproducing kernel Hilbert spaces of complex valued functions on Ω. Denote the reproducing kernel of $\mathcal{H}_t$ as K_t. Assume that*

$$\forall s,t \in \mathcal{I}\colon\ \ t \le s \ \Rightarrow\ \mathcal{H}_t \subseteq \mathcal{H}_s \text{ isometrically},$$
$$\forall w \in \Omega\colon\ \ t \mapsto K_t(w,w) \text{ is continuous.}$$

Then

$$\forall s \in \mathcal{I}\setminus\{\sup\mathcal{I}\}\colon \quad \bigcap_{t\in\mathcal{I},t>s} \mathcal{H}_t = \mathcal{H}_s, \tag{5.5}$$

$$\forall s \in \mathcal{I}\setminus\{\inf\mathcal{I}\}\colon \quad \operatorname{Clos} \bigcup_{t\in\mathcal{I},t<s} \mathcal{H}_t = \mathcal{H}_s. \tag{5.6}$$

Proof. For $s,t \in \mathcal{I}$ with $s \le t$ we denote by P^t_s the orthogonal projection of $\mathcal{H}_t$ onto $\mathcal{H}_s$.

To prove (5.5), let $s \in \mathcal{I}\setminus\{\sup\mathcal{I}\}$ be given. Choose $t_0 \in \mathcal{I}$ with $t_0 > s$, denote $\mathcal{H}_{s+} := \bigcap_{t\in\mathcal{I},t>s} \mathcal{H}_t$ and let K_{s+} be the reproducing kernel of $\mathcal{H}_{s+}$. The limit $\lim_{t\downarrow s} P^{t_0}_t$ exists in the strong operator topology and is the orthogonal projection of $\mathcal{H}_{t_0}$ onto $\mathcal{H}_{s+}$. Thus

$$\forall w \in \Omega\colon\ \ K_{s+}(.,w) = \lim_{t\downarrow s} K_t(.,w)$$

in norm. We conclude that, for each $w \in \Omega$,

$$\|K_{s+}(.,w)\| = \lim_{t \downarrow s} \|K_t(.,w)\| = \lim_{t \downarrow s} K_t(w,w)^{\frac{1}{2}} = K_s(w,w)^{\frac{1}{2}} = \|K_s(.,w)\|.$$

Since $K_s(.,w)$ is the orthogonal projection of $K_{s+}(.,w)$ onto $\mathcal{H}_s$, it follows that $K_{s+}(.,w) = K_s(.,w)$. We see that $\mathcal{H}_{s+} = \mathcal{H}_s$.

The proof of (5.6) is carried out in the same way. $\square$

5.9 Corollary. *Let $E \in \mathcal{HB}$, $\nu > -1$, and assume that $\mathcal{H}(E)$ is homogeneous of order ν. Then the chain of de Branges subspaces of $\mathcal{H}(E)$ is equal to*

$$\big\{\mathcal{H}(a \odot_{\nu+\frac{1}{2}} E) \mid a \in (0,1]\big\}.$$

Proof. To shorten notation, set $p := \nu + \frac{1}{2}$. Let $\mathcal{H}(\tilde{E})$ be a de Branges subspace of $\mathcal{H}(E)$. For each $a \in (0,1]$ we must have either $\mathcal{H}(\tilde{E}) \subseteq \mathcal{H}(a \odot_p E)$ or $\mathcal{H}(a \odot_p E) \subseteq \mathcal{H}(\tilde{E})$. If $K_{a \odot_p E}(0,0) > K_{\tilde{E}}(0,0)$ or $a = 1$, then the first case must take place. On the other hand, the second case must take place whenever $K_{a \odot_p E}(0,0) < K_{\tilde{E}}(0,0)$, and this inequality holds for all sufficiently small a since $\lim_{a \downarrow 0} K_{a \odot_p E}(z,w) = 0$ by (2.2). Set

$$b := \Big(\frac{K_{\tilde{E}}(0,0)}{K_E(0,0)}\Big)^{\frac{1}{2p+1}} \in (0,1].$$

The function $a \mapsto a \odot_p E$ is continuous, and Lemma 5.8 implies

$$\begin{aligned} \mathcal{H}(b \odot_p E) = \operatorname{Clos} \bigcup_{a \in (0,b)} \mathcal{H}(a \odot_p E) &\subseteq \mathcal{H}(\tilde{E}) \\ &\subseteq \mathcal{H}(E) \cap \bigcap_{a \in (b,1) \cup \{1\}} \mathcal{H}(a \odot_p E) = \mathcal{H}(b \odot_p E). \end{aligned}$$

$\square$

5.10 Corollary. *Let $\mathcal{H}$ be a de Branges space and $\nu > -1$. If $\mathcal{H}$ is homogeneous of order ν, then every de Branges subspace of $\mathcal{H}$ is homogeneous of order ν.*

Proof. Choose $E \in \mathcal{HB}$ such that $\mathcal{H} = \mathcal{H}(E)$. We know that each de Branges subspace of $\mathcal{H}(E)$ is of the form $\mathcal{H}(a \odot_{\nu+\frac{1}{2}} E)$ with some $a \leq 0$. Now remember Lemma 5.2 (*iii*). $\square$

6 Structure of homogeneous spaces

In this section we give the connection between homogeneous de Branges spaces, canonical systems with Hamiltonians of the form $H_{P,\psi}$, and recurrence relations of the form (2.3).

6.1 Definition. Let $p > -\frac{1}{2}$. Then we set

$$\mathbb{P}_p := \Big\{(P,\psi) \in \mathbb{R}^{2\times 2} \times \mathbb{R} \mid P \geq 0,\ \binom{1}{0}, \binom{-\psi}{2p} \notin \ker P\Big\} \quad \text{if } p \neq 0,$$

$$\begin{aligned}\mathbb{P}_0 := &\Big\{(P,\psi)\in\mathbb{R}^{2\times 2}\times\mathbb{R} \mid P\geq 0,\ \ker P=\{0\},\ \psi=0\Big\}\\ &\cup\Big\{(P,\psi)\in\mathbb{R}^{2\times 2}\times\mathbb{R} \mid P\geq 0,\ \binom{1}{0}\notin\ker P,\ \psi\neq 0\Big\}.\end{aligned}$$

The conditions occurring in this definition stem from Lemma 2.5. Also note that $\mathbb{P}_p\subseteq\mathbb{P}$ for all $p>-\frac{1}{2}$.

6.2 Theorem. *Let $\nu>-1$ and set $p:=\nu+\frac{1}{2}$. Then the following statements hold.*

(i) *Let $(P,\psi)\in\mathbb{P}_p$. Then*

- ▷ *$\widehat{\Xi}_p(P,\psi)$ is a Hermite-Biehler function with value 1 at the origin,*
- ▷ *the structure Hamiltonian of $\widehat{\Xi}_p(P,\psi)$ is a reparameterisation of $H_{P,\psi}|_{(0,1]}$ (prolongued with an indivisible interval of type $\frac{\pi}{2}$ up to $-\infty$ if $-\frac{1}{2}<p<\frac{1}{2}$),*
- ▷ *$\mathcal{H}(\widehat{\Xi}_p(P,\psi))$ is homogeneous of order ν.*

(ii) *Let $E\in\mathcal{HB}$ with $E(0)=1$ be such that $\mathcal{H}(E)$ is homogeneous of order ν. Then there exists $(P,\psi)\in\mathbb{P}_p$, such that $E=\widehat{\Xi}_p(P,\psi)$ and that the structure Hamiltonian H_E is a reparameterisation of $H_{P,\psi}|_{(0,1]}$ (prolongued by an indivisible interval if necessary).*

(iii) *Let $(P,\psi),(\tilde{P},\tilde{\psi})\in\mathbb{P}_p$, and write $P=\begin{pmatrix}\kappa_1&\kappa_3\\ \kappa_3&\kappa_2\end{pmatrix}$ and $\tilde{P}=\begin{pmatrix}\tilde\kappa_1&\tilde\kappa_3\\ \tilde\kappa_3&\tilde\kappa_2\end{pmatrix}$. Then $\mathcal{H}(\widehat{\Xi}_p(P,\psi))=\mathcal{H}(\widehat{\Xi}_p(\tilde{P},\tilde{\psi}))$ isometrically, if and only if*

- ▷ $\kappa_1=\tilde\kappa_1$,
- ▷ $\det P=\det\tilde{P}$,
- ▷ $\psi-\tilde\psi=\frac{2p}{\kappa_1}\big[\kappa_3-\tilde\kappa_3\big]$.

The proofs of items (i) and (iii) of Theorem 6.2 are rather easy: the first is obtained by plugging together what we have already shown, and the latter by calculation. The essential part of the proof is to establish the existence result stated in (ii).

Proof of Theorem 6.2(i). We know from Corollary 3.5 that $\widehat{\Xi}_p(P,\psi)\in\mathcal{HB}$. Let us observe some properties of $H_{P,\psi}$. It is clear that $H_{P,\psi}(a)\geq 0$ for all $a>0$ and that $H_{P,\psi}$ is locally integrable. The definition of the class $\mathbb{P}_p$ and Lemma 2.6 imply that $H_{P,\psi}\notin L^1((1,\infty),\mathbb{R}^{2\times 2})$ and that $H_{P,\psi}\in L^1((0,1),\mathbb{R}^{2\times 2})$ if and only if $p\in(-\frac{1}{2},\frac{1}{2})$. Further $H_{P,\psi}$ contains no indivisible intervals: if $\operatorname{rank}P=2$ this is trivial, and if $\operatorname{rank}P=1$ it follows from Lemma 2.5 by the definition of $\mathbb{P}_p$.

Let $A(a,z)$ and $B(a,z)$ be as in (3.1). Theorem 3.1 implies that for all $a\in(0,1]$ the space $\mathcal{H}(A(a,z)-iB(a,z))$ is isometrically contained in $\mathcal{H}(\widehat{\Xi}_p(P,\psi))$. Note here that changing the functions A,B with a matrix from $\mathrm{SL}(2,\mathbb{R})$ does not change the generated de Branges space. However, by the definition (3.1) we have $\mathcal{H}(A(a,z)-iB(a,z))=\mathcal{H}(a\odot_p\widehat{\Xi}_p(P,\psi))$ isometrically. Moreover, the function

$$\begin{pmatrix}1\\0\end{pmatrix}^* H_{P,\psi}(a)\begin{pmatrix}1\\0\end{pmatrix}=a^{2p}\begin{pmatrix}1\\0\end{pmatrix}^* P\begin{pmatrix}1\\0\end{pmatrix},$$

is integrable at 0. Altogether we see that the structure Hamiltonian of $\widehat{\Xi}_p(P,\psi)$ is a reparameterisation of $H_{P,\psi}|_{(0,1]}$ (extended by an indivisible interval if necessary), and that $\mathcal{H}(\widehat{\Xi}_p(P,\psi))$ is homogeneous of order ν. ☐

Proof of Theorem 6.2(ii). Let $\mathcal{H}$ be a de Branges space which is homogeneous of order ν, and choose $E\in\mathcal{HB}$ with $E(0)=1$ such that $\mathcal{H}=\mathcal{H}(E)$. As usual, let H_E denote the structure Hamiltonian of E, let $W_E(t,s,z)$ for $-\infty<t\leq s\leq 0$ be the corresponding family of transfer matrices, and $E(t,z)$ for $t\in(-\infty,0]$ be the corresponding family of Hermite-Biehler functions. Moreover, set

$$t_-:=\sup\bigl\{t\in(-\infty,0]\mid(-\infty,t)\ H_E\text{-indivisible of type }\frac{\pi}{2}\bigr\}.$$

① By Corollary 5.9 the chain of de Branges subspaces of $\mathcal{H}(E)$ is $\{\mathcal{H}(a\odot_p E)\mid a\in(0,1]\}$, and by Lemma 5.8 this chain has no one-dimensional gaps. Hence, the Hamiltonian $H_E|_{(t_-,0)}$ contains no indivisible intervals. In particular, the function

$$\alpha(t):=\Biggl(\frac{1}{K_E(0,0)}\int\limits_{-\infty}^{t}\begin{pmatrix}1\\0\end{pmatrix}^* H_E(t)\begin{pmatrix}1\\0\end{pmatrix}\,\mathrm{d}t\Biggr)^{\frac{1}{2p+1}}\tag{6.1}$$

is an absolutely continuous increasing bijection of $[t_-,0]$ onto $[0,1]$. Its inverse function is thus a continuous increasing bijection of $[0,1]$ onto $[t_-,0]$. Let us point out that at the present stage we do not know that α^{-1} is absolutely continuous.

Remembering (2.2), we see that

$$\forall t\in(t_-,0]\colon\ \ \mathcal{H}(E(t,.))=\mathcal{H}(\alpha(t)\odot_p E).$$

Now define a function $\Psi\colon(0,1]\to\mathrm{SL}(2,\mathbb{R})$ by letting $\Psi(a)$ be the unique matrix with

$$(a\odot_p A,a\odot_p B)=\bigl(A(\alpha^{-1}(a),.),B(\alpha^{-1}(a),.)\bigr)\Psi(a),\tag{6.2}$$

cf. Theorem 1.4. Clearly, $\Psi(1)=I$. Evaluating (6.2) and the relation obtained by differentiating (6.2) w.r.t. z at the point $z=0$, yields that $\Psi(a)$ is explicitly given as (here and in the following a prime denotes differentiation w.r.t. z)

$$\Psi(a)=\begin{pmatrix}1&0\\A'(\alpha^{-1}(a),0)&B'(\alpha^{-1}(a),0)\end{pmatrix}^{-1}\begin{pmatrix}a^p&0\\a^{p+1}A'(0)&a^{p+1}B'(0)\end{pmatrix}.$$

This formula together with the fact that $\Psi(a)\in\mathrm{SL}(2,\mathbb{R})$ shows that Ψ is continuous and triangular of the form

$$\Psi(a)=\begin{pmatrix}a^p&0\\❄&a^{-p}\end{pmatrix}.\tag{6.3}$$

② In this step we consider the function

$$W(a,b,z) := W_E(\alpha^{-1}(a),\alpha^{-1}(b),z) \quad \text{for } 0<a\le b\le 1,$$

and show the central relation

$$\forall 0<a\le b\le 1, 0<c\le \frac{1}{b}:$$
$$\Psi(a)^{-1}W(a,b,cz)\Psi(b) = \Psi(ca)^{-1}W(ca,cb,z)\Psi(cb). \tag{6.4}$$

Let $0<a\le b\le 1$. Then

$$\begin{aligned}\big([a\odot_p A](z),[a\odot_p B](z)\big)&\cdot\Psi(a)^{-1}W(a,b,z)\Psi(b) \qquad (6.5)\\ &= \big(A(\alpha^{-1}(a),z),B(\alpha^{-1}(a),z)\big)W_E(\alpha^{-1}(a),\alpha^{-1}(b),z)\Psi(b)\\ &= \big(A(\alpha^{-1}(b),z),B(\alpha^{-1}(b),z)\big)\Psi(b) = \big([b\odot_p A](z),[b\odot_p B](z)\big).\end{aligned}$$

Let additionally $0<c\le\frac{1}{b}$, then we can compute

$$\begin{aligned}\big([ca\odot_p A](z),&[ca\odot_p B](z)\big)\cdot\Psi(a)^{-1}W(a,b,cz)\Psi(b)\\ &= c\odot_p\Big[\big([a\odot_p A](z),[a\odot_p B](z)\big)\Psi(a)^{-1}W(a,b,z)\Psi(b)\Big]\\ &= c\odot_p\Big[\big([b\odot_p A](z),[b\odot_p B](z)\big)\Big] = \big([cb\odot_p A](z),[cb\odot_p B](z)\big)\\ &= \big([ca\odot_p A](z),[ca\odot_p B](z)\big)\cdot\Psi(ca)^{-1}W(ca,cb,z)\Psi(cb).\end{aligned}$$

Uniqueness of the transfer matrix [Bra68, Problem 100] implies (6.4).

③ We exploit (6.4) to determine Ψ. Using (6.4) with $b=1$ and evaluating at $z=0$ leads to

$$\forall a,c\in(0,1]:\quad \Psi(ca)=\Psi(c)\Psi(a).$$

The function

$$\begin{cases} [0,\infty) & \to \quad \mathbb{C}^{2\times 2}\\ \quad x & \mapsto \quad \Psi(e^{-x})\end{cases}$$

is a continuous semigroup of matrices, and hence is represented as the exponential of its infinitesimal generator:

$$\Psi(e^{-x}) = \exp\big(Gx\big) \quad \text{where} \quad G:=\lim_{x\downarrow 0}\frac{\Psi(e^{-x})-I}{x}.$$

Since $\Psi(a)$ is of the form (6.3) the generator G is of the form

$$G=\begin{pmatrix}-p & 0\\ -\psi & p\end{pmatrix},$$

with some $\psi\in\mathbb{R}$. Now we obtain

$$\Psi(a)=\Psi\big(e^{-(-\log a)}\big)=\exp\Big(\Big(\begin{smallmatrix}-p & 0\\ -\psi & p\end{smallmatrix}\Big)(-\log a)\Big)$$

$$=\begin{cases}\begin{pmatrix}1 & 0\\ \frac{\psi}{2p} & 1\end{pmatrix}\begin{pmatrix}a^p & 0\\ 0 & a^{-p}\end{pmatrix}\begin{pmatrix}1 & 0\\ -\frac{\psi}{2p} & 1\end{pmatrix} & \text{if } p\neq 0,\\[2ex] \begin{pmatrix}1 & 0\\ \psi\log a & 1\end{pmatrix} & \text{if } p=0,\end{cases}$$

i.e., $\Psi(a)=\mathfrak{D}_\psi(a)$.

④ We exploit (6.4) to show that $W(a,b,z)$ satisfies an integral equation and determine the Hamiltonian. Using that relation with $c = \frac{1}{b}$ and evaluating derivatives w.r.t. z at $z = 0$ leads to

$$\forall 0 < a \leq b \leq 1\colon\quad \mathfrak{D}_\psi(a)^{-1}\Big(\frac{1}{b}W'(a,b,0)\Big)\mathfrak{D}_\psi(b) = \mathfrak{D}_\psi\big(\frac{a}{b}\big)^{-1}W'\big(\frac{a}{b},1,0\big).$$

Since $\mathfrak{D}_\psi(b)J\mathfrak{D}_\psi(b)^T = J$, we obtain

$$\forall 0 < a < b \leq 1\colon\quad \frac{W'(a,b,0)J}{b-a} = \mathfrak{D}_\psi(b)\frac{W'(\frac{a}{b},1,0)J}{1-\frac{a}{b}}\mathfrak{D}_\psi(b)^T. \tag{6.6}$$

The function $M : a \mapsto -W'(a,1,0)J$ takes nonpositive matrices as values, is nondecreasing in the matrix sense, and $M(1) = 0$. Thus its diagonal entries are nondecreasing and its off-diagonal entry is of bounded variation. Since $W'(a,1,0) = W_E'(\alpha^{-1}(a),0,0)$, the function M is also continuous. Hence, M is differentiable almost everywhere. In particular, there exists $b \in (0,1]$ such that the limit

$$\lim_{a\uparrow b}\frac{W'(a,b,0)J}{b-a} = \lim_{a\uparrow b}\frac{M(b)-M(a)}{b-a}$$

exists (recall here the multiplicativity property of fundamental solutions). Reading (6.6) from left to right, and remembering that $\mathfrak{D}_\psi$ is continuous, yields that the limit

$$P := \lim_{c\uparrow 1}\frac{W'(c,1,0)J}{1-c}$$

exists. Clearly, P is positive semidefinite. Reading (6.6) from right to left yields that M is everywhere differentiable and has the continuous derivative

$$\frac{dM}{da}(a) = \mathfrak{D}_\psi(a)P\mathfrak{D}_\psi(a)^T = H_{P,\psi}(a) \quad \text{for } a \in (0,1].$$

In particular, M is absolutely continuous and can be written as the integral of its derivative. Since M is not constant, we must have $P \neq 0$.

Let us make a change of variable with the absolutely continuous function $\alpha : (t_-,0] \to (0,1]$ from (6.1). This gives, for each $a \in (0,1]$,

$$\begin{aligned}\int_{\alpha^{-1}(a)}^0 H_E(t)\,\mathrm{d}t &= W_E'(\alpha^{-1}(a),0,0)J = W'(a,1,0)J\\ &= M(1)-M(a) = \int_a^1 H_{P,\psi}(c)\,\mathrm{d}c = \int_{\alpha^{-1}(a)}^0 H_{P,\psi}(\alpha(t))\alpha'(t)\,\mathrm{d}t.\end{aligned}$$

It follows that

$$H_E(t) = H_{P,\psi}(\alpha(t))\alpha'(t) \quad \text{for } t \in (t_-,0) \text{ a.e..} \tag{6.7}$$

Here is the point where we see that α^{-1} is absolutely continuous, since the above relation shows $\alpha'(t) > 0$ a.e. Thus, we may say that $H_E|_{(t_-,0)}$ is a reparameterisation of $H_{P,\psi}$. Since $H_E|_{(t_-,0)}$ does not contain any indivisible intervals, also $H_{P,\psi}|_{(0,1)}$ does not contain any such intervals. Lemma 2.5 implies that $(P,\psi) \in \mathbb{P}_p$.

⑤ Using the same change of variable that led to (6.7) gives the integral equation for $W(a,b,z)$: for all $0 < a \leq b \leq 1$, we have

$$\begin{aligned} z\int_a^b W(a,c,z)\,H_{P,\psi}(c)\,\mathrm{d}c &= z\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} W(a,\alpha(t),z)H_{P,\psi}(\alpha(t))\alpha'(t)\,\mathrm{d}t \\ &= z\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} W_E(\alpha^{-1}(a),t,z)H_E(t)\,\mathrm{d}t \\ &= W_E(\alpha^{-1}(a),\alpha^{-1}(b),z)J - J = W(a,b,z)J - J. \end{aligned}$$

Using this relation for $b = 1$, multiplying from the left with $(a \odot_p A, a \odot_p B)\mathscr{D}_\psi(a)^{-1}$, and remembering (6.5), yields

$$\begin{aligned} &\big(A(z),B(z)\big)J - \big([a\odot_p A](z),[a\odot_p B](z)\big)\mathscr{D}_\psi(a)^{-1}J \\ &\qquad = z\int_a^1 \big([c\odot_p A](z),[c\odot_p B](z)\big)\mathscr{D}_\psi(c)^{-1}H_{P,\psi}(c)\,\mathrm{d}c. \end{aligned}$$

Theorem 3.1 now implies that (2.3) holds, i.e., $E = \widehat{\Xi}_p(P,\psi)$.

□

Proof of Theorem 6.2 (iii). For any two functions $E, \tilde{E} \in \mathcal{HB}$ with $E(0) = \tilde{E}(0) = 1$ the spaces $\mathcal{H}(E)$ and $\mathcal{H}(\tilde{E})$ are equal isometrically if and only if there exists $\gamma \in \mathbb{R}$ such that the structure Hamiltonian $H_{\tilde{E}}$ of $\tilde{E}$ is a reparameterisation of $\left(\begin{smallmatrix}1&0\\ \gamma&1\end{smallmatrix}\right)H_E\left(\begin{smallmatrix}1&\gamma\\ 0&1\end{smallmatrix}\right)$ where H_E is the structure Hamiltonian of E. Using the already proven statement Theorem 6.2 (i), this leads to

$$\begin{aligned} &\mathcal{H}(\widehat{\Xi}_p(P,\psi)) = \mathcal{H}(\widehat{\Xi}_p(\tilde{P},\tilde{\psi}))\ \text{isometrically} \iff \\ &\quad \exists\gamma\in\mathbb{R}\colon\ \ H_{\tilde{P},\tilde{\psi}}|_{(0,1)}\ \text{is a reparameterisation of}\ \begin{pmatrix}1&0\\ \gamma&1\end{pmatrix}H_{P,\psi}|_{(0,1)}\begin{pmatrix}1&\gamma\\ 0&1\end{pmatrix}. \end{aligned}$$

The condition on the right means that there exists an absolutely continuous bijection $\varphi\colon (0,1) \to (0,1)$ with $\varphi' > 0$ a.e., such that

$$\mathscr{D}_{\tilde{\psi}}(a)\tilde{P}\mathscr{D}_{\tilde{\psi}}(a)^T = \begin{pmatrix}1&0\\ \gamma&1\end{pmatrix}\Big[\mathscr{D}_\psi P\mathscr{D}_\psi^T\Big](\varphi(a))\begin{pmatrix}1&\gamma\\ 0&1\end{pmatrix}\cdot\varphi'(a). \tag{6.8}$$

We write $P = \left(\begin{smallmatrix}\kappa_1&\kappa_3\\ \kappa_3&\kappa_2\end{smallmatrix}\right)$ and $\tilde{P} = \left(\begin{smallmatrix}\tilde{\kappa}_1&\tilde{\kappa}_3\\ \tilde{\kappa}_3&\tilde{\kappa}_2\end{smallmatrix}\right)$. Comparing the left upper entries, shows that (6.8) implies

$$\forall a\in(0,1)\colon\ \ a^{2p}\tilde{\kappa}_1 = a^{2p}\kappa_1\cdot\varphi'(a),$$

and hence $\varphi = \mathrm{id}_{(0,1)}$. We see that

$$\begin{aligned} &\mathcal{H}(\widehat{\Xi}_p(P,\psi)) = \mathcal{H}(\widehat{\Xi}_p(\tilde{P},\tilde{\psi}))\ \text{isometrically} \iff \\ &\quad \exists\gamma\in\mathbb{R}\ \forall a\in(0,1)\colon\ \ \mathscr{D}_{\tilde{\psi}}(a)\tilde{P}\mathscr{D}_{\tilde{\psi}}(a)^T = \begin{pmatrix}1&0\\ \gamma&1\end{pmatrix}\mathscr{D}_\psi(a)P\mathscr{D}_\psi(a)^T\begin{pmatrix}1&\gamma\\ 0&1\end{pmatrix}. \end{aligned} \tag{6.9}$$

$\triangleright$ *Case $p = 0$:* We plug the definition of $\mathscr{D}_\psi(a)$ and $\mathscr{D}_{\tilde\psi}(a)$ in (6.9). Remembering that $\kappa_1 \neq 0$, this leads to

$$\begin{aligned}
&\mathcal{H}(\widehat{\Xi}_p(P,\psi)) = \mathcal{H}(\widehat{\Xi}_p(\tilde P,\tilde\psi)) \text{ isometrically}\\
&\qquad\iff\quad \exists\gamma\in\mathbb{R}\ \forall a\in(0,1)\colon\\
&\qquad\qquad\qquad \tilde P = \begin{pmatrix} 1 & 0\\ (\psi-\tilde\psi)\log a+\gamma & 1\end{pmatrix}\cdot P\cdot\begin{pmatrix} 1 & (\psi-\tilde\psi)\log a+\gamma\\ 0 & 1\end{pmatrix}\\
&\qquad\iff\quad \exists\gamma\in\mathbb{R}\ \forall a\in(0,1)\colon\ \ \tilde\kappa_1=\kappa_1\wedge\det\tilde P=\det P\\
&\qquad\qquad\qquad\qquad \wedge\,\tilde\kappa_3=\kappa_3+\kappa_1\big[(\psi-\tilde\psi)\log a+\gamma\big]\\
&\qquad\iff\quad \tilde\kappa_1=\kappa_1\wedge\det\tilde P=\det P\wedge\psi-\tilde\psi=0\\
&\qquad\iff\quad \tilde\kappa_1=\kappa_1\wedge\det\tilde P=\det P\wedge\psi-\tilde\psi=\frac{2p}{\kappa_1}(\kappa_3-\tilde\kappa_3).
\end{aligned}$$

$\triangleright$ *Case $p \neq 0$:* In this case the computation is a bit more complicated; it is based on the formula

$$\begin{pmatrix}1&0\\ \beta&1\end{pmatrix}\cdot\begin{pmatrix}\alpha&0\\0&\frac1\alpha\end{pmatrix}=\begin{pmatrix}\alpha&0\\0&\frac1\alpha\end{pmatrix}\cdot\begin{pmatrix}1&0\\ \alpha^2\beta&1\end{pmatrix}\tag{6.10}$$

which holds for all $\alpha,\beta\in\mathbb{R}$.

Plugging the formulae for $\mathscr{D}_\psi(a)$ and $\mathscr{D}_{\tilde\psi}(a)$, and using (6.10), yields

$$\begin{aligned}
&\mathcal{H}(\widehat{\Xi}_p(P,\psi)) = \mathcal{H}(\widehat{\Xi}_p(\tilde P,\tilde\psi)) \text{ isometrically}\\
&\qquad\iff\quad \exists\gamma\in\mathbb{R}\ \forall a\in(0,1)\colon\\
&\qquad \begin{pmatrix}a^p&0\\0&a^{-p}\end{pmatrix}\begin{pmatrix}1&0\\-\frac{\tilde\psi}{2p}&1\end{pmatrix}\cdot\tilde P\cdot\begin{pmatrix}1&-\frac{\tilde\psi}{2p}\\0&1\end{pmatrix}\begin{pmatrix}a^p&0\\0&a^{-p}\end{pmatrix}=\\
&\qquad \begin{pmatrix}a^p&0\\0&a^{-p}\end{pmatrix}\begin{pmatrix}1&0\\ a^{2p}\big(\gamma+\frac{\psi-\tilde\psi}{2p}\big)-\frac{\psi}{2p}&1\end{pmatrix}\cdot P\cdot\begin{pmatrix}1&a^{2p}\big(\gamma+\frac{\psi-\tilde\psi}{2p}\big)-\frac{\psi}{2p}\\0&1\end{pmatrix}\begin{pmatrix}a^p&0\\0&a^{-p}\end{pmatrix}\\
&\qquad\iff\quad \exists\gamma\in\mathbb{R}\ \forall a\in(0,1)\colon\\
&\qquad\qquad \tilde P=\begin{pmatrix}1&0\\ a^{2p}\big(\gamma+\frac{\psi-\tilde\psi}{2p}\big)-\frac{\psi-\tilde\psi}{2p}&1\end{pmatrix}\cdot P\cdot\begin{pmatrix}1&a^{2p}\big(\gamma+\frac{\psi-\tilde\psi}{2p}\big)-\frac{\psi-\tilde\psi}{2p}\\0&1\end{pmatrix}\\
&\qquad\iff\quad \exists\gamma\in\mathbb{R}\ \forall a\in(0,1]\colon\ \ \tilde\kappa_1=\kappa_1\wedge\det\tilde P=\det P\\
&\qquad\qquad\qquad\qquad \wedge\,\tilde\kappa_3=\kappa_3+\kappa_1\Big[a^{2p}\Big(\gamma+\frac{\psi-\tilde\psi}{2p}\Big)-\frac{\psi-\tilde\psi}{2p}\Big]\\
&\qquad\iff\quad \tilde\kappa_1=\kappa_1\wedge\det\tilde P=\det P\wedge\psi-\tilde\psi=\frac{2p}{\kappa_1}(\kappa_3-\tilde\kappa_3).
\end{aligned}$$

$\square$

Items (i) and (ii) of Theorem 6.2 say that the map $(P,\psi)\mapsto\mathcal{H}(\widehat{\Xi}_p(P,\psi))$ is a surjection of the parameter space $\mathbb{P}_p$ onto the set of all homogeneous de Branges spaces of order $\nu := p-\frac12$, and in item (iii) of the theorem the kernel of this map is described. We write $\approx$ for that kernel, i.e.,

$$\begin{aligned}
(P,\psi)\approx(\tilde P,\tilde\psi) :\Leftrightarrow{}& \binom{1}{0}^{*}P\binom{1}{0}=\binom{1}{0}^{*}\tilde P\binom{1}{0}\ \wedge\ \det P=\det\tilde P\\
&\wedge\ \psi-\tilde\psi=\frac{2p}{\binom{1}{0}^{*}P\binom{1}{0}}\Big[\binom{0}{1}^{*}P\binom{1}{0}-\binom{0}{1}^{*}\tilde P\binom{1}{0}\Big].
\end{aligned}$$

Naturally, we are interested in having at hand complete systems of representatives of our parameter space $\mathbb{P}_p$ modulo $\approx$.

6.3 Lemma. *Let $p > -\frac{1}{2}$.*

(i) *The set*

$$\Big\{(P,\psi) \in \mathbb{P}_p \mid \begin{pmatrix}1\\0\end{pmatrix}^* P \begin{pmatrix}0\\1\end{pmatrix} = 0\Big\} \tag{6.11}$$

is a complete system of representatives of $\mathbb{P}_p$ modulo $\approx$.

(ii) *If $p \neq 0$, then also*

$$\big\{(P,\psi) \in \mathbb{P}_p \mid \psi = 0\big\} \tag{6.12}$$

is a complete system of representatives of $\mathbb{P}_p$ modulo $\approx$.

Proof. Let $(P,\psi) \in \mathbb{P}_p$ be given and write, as usual, $P = \begin{pmatrix}\kappa_1 & \kappa_3\\ \kappa_3 & \kappa_2\end{pmatrix}$. We set

$$\tilde{P} := \begin{pmatrix}\kappa_1 & 0\\ 0 & \kappa_2 - \frac{\kappa_3^2}{\kappa_1}\end{pmatrix}, \quad \tilde{\psi} := \psi - \frac{2p}{\kappa_1}\kappa_3.$$

It is clear that $\tilde{P} \geq 0$, that $\det \tilde{P} = \det P$, $\tilde{\kappa}_1 = \kappa_1$, and that $\psi - \tilde{\psi} = \frac{2p}{\kappa_1}(\kappa_3 - \tilde{\kappa}_3)$. We certainly have $\binom{1}{0} \notin \ker \tilde{P}$. If $p \neq 0$ and $\tilde{P}\binom{-\tilde{\psi}}{2p} = 0$, then also $P\binom{-\psi}{2p} = 0$ which is not possible. If $p = 0$ and $\tilde{\psi} = 0$, then also $\psi = 0$ and hence $\det P \neq 0$, which implies that $\det \tilde{P} \neq 0$. Hence, in all cases, $(\tilde{P},\tilde{\psi}) \in \mathbb{P}_p$, and we see that $(\tilde{P},\tilde{\psi}) \approx (P,\psi)$. If we have $(P,\psi),(\tilde{P},\tilde{\psi}) \in \mathbb{P}_p$ with $\kappa_3 = \tilde{\kappa}_3 = 0$ which are in relation $\approx$, then the definition of $\approx$ immediately implies that $(P,\psi) = (\tilde{P},\tilde{\psi})$. We see that (6.11) is a complete system of representatives.

Assume now that $p \neq 0$. Given $(P,\psi) \in \mathbb{P}_p$, we set

$$\tilde{\kappa}_1 := \kappa_1,\ \tilde{\kappa}_3 := \kappa_3 - \frac{\kappa_1}{2p}\psi,\ \tilde{\kappa}_2 := \frac{1}{\kappa_1}\big(\det P + \tilde{\kappa}_3^2\big), \quad \tilde{\psi} := 0.$$

Then, clearly, $\det \tilde{P} = \det P$, $\tilde{P} \geq 0$, and $\binom{1}{0} \notin \ker \tilde{P}$. Assume towards a contradiction that $\tilde{P}\binom{-\tilde{\psi}}{2p} = 0$. Then $\tilde{\kappa}_2 = 0$ and $\det P = 0$. The first implies that $\kappa_3 = \frac{\kappa_1}{2p}\psi$ and the second that $\kappa_2 = \frac{\kappa_3^2}{\kappa_1}$. From this we see that $P\binom{-\psi}{2p} = 0$, which is a contradiction. Thus $(\tilde{P},\tilde{\psi}) \in \mathbb{P}_p$, and of course the definition of $(\tilde{P},\tilde{\psi})$ is made such that $(P,\psi) \approx (\tilde{P},\tilde{\psi})$. If we have $(P,\psi),(\tilde{P},\tilde{\psi}) \in \mathbb{P}_p$ with $\psi = \tilde{\psi} = 0$ which are in relation $\approx$, then immediately $(P,\psi) = (\tilde{P},\tilde{\psi})$. Thus, (6.12) is a complete system of representatives. ☐

6.4 Remark. Let us discuss the case that $\nu = -\frac{1}{2}$ (i.e., $p = 0$) in some more detail. The de Branges spaces which are homogeneous of order $-\frac{1}{2}$ are in one-to-one correspondence to the parameters

$$(\kappa_1,\kappa_2,\psi) \in \Big[(0,\infty) \times (0,\infty) \times \{0\}\Big] \cup \Big[(0,\infty) \times [0,\infty) \times \big(\mathbb{R} \setminus \{0\}\big)\Big].$$

Set $P := \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}$. If $\psi = 0$, and hence $\kappa_2 > 0$, we find by inspecting the formulae (4.2) and (4.3) that

$$\widehat{\Xi}_0(P, 0) = \cos(\sqrt{\kappa_1 \kappa_2} \cdot z) - i\sqrt{\frac{\kappa_1}{\kappa_2}} \sin(\sqrt{\kappa_1 \kappa_2} \cdot z).$$

This family includes the Paley-Wiener spaces, namely for $\kappa_1 = \kappa_2$. If $\psi \neq 0$, the formulae are much more complicated and involve Kummer functions whose first argument is purely imaginary and nonzero, cf. Theorem 4.1.

In [Bra62; Bra68] the false argument occurs that *for every* $\nu > -\frac{1}{2}$ the totality of homogeneous spaces is obtained with parameter $\psi = 0$. This is true for $p \neq 0$, but for $p = 0$ the whole family of spaces occurring from parameters $(\kappa_1, \kappa_2, \psi) \in (0, \infty) \times [0, \infty) \times (\mathbb{R} \setminus \{0\})$ was lost.

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7 Measures induced by homogeneous spaces

Let $\nu > -1$ and set, as usual, $p := \nu + \frac{1}{2}$. If $\mathcal{H}(E)$ is a de Branges space which is homogeneous of order ν, Lemma 5.2 (i) shows that

$$\big\{\mathcal{H}(a \odot_p E) \mid a \in (0, \infty)\big\} \tag{7.1}$$

is a chain of de Branges spaces with isometric inclusions $\mathcal{H}(a \odot_p E) \subseteq \mathcal{H}(b \odot_p E)$ when $a \leq b$.

The chain (7.1) has no gaps or jumps, i.e.,

$$\forall b \in (0, \infty)\colon \quad \bigcap_{a>b} \mathcal{H}(a \odot_p E) = \mathcal{H}(b \odot_p E) \wedge \operatorname{Clos} \bigcup_{a<b} \mathcal{H}(a \odot_p E) = \mathcal{H}(b \odot_p E),$$

cf. Lemma 5.8. Moreover, due to the kernel relation (2.2), it is exhaustive in the sense that

$$\lim_{a \downarrow 0} K_{a \odot_p E}(z, w) = 0, \quad \lim_{a \uparrow \infty} K_{a \odot_p E}(0, 0) = \infty.$$

These properties show that $\{\mathcal{H}(a \odot_p E) \mid a > 0\}$ is an unbounded chain in the sense of Definition 1.10. By Theorem 1.11 it determines a positive Borel measure on the real line.

7.1 Definition. Let $\nu > -1$ and let $\mathcal{H}$ be a de Branges space which is homogeneous of order ν, and choose $E \in \mathcal{HB}$ such that $\mathcal{H} = \mathcal{H}(E)$. Then we denote by $\mu_{\mathcal{H}}$ the unique positive Borel measure on the real line, such that $L^2(\mu_{\mathcal{H}})$ contains $\bigcup_{a \in (0,\infty)} \mathcal{H}(a \odot_p E)$ isometrically as a dense subspace.

Morever, let us introduce the following abbreviation: given $(P, \psi) \in \mathbb{P}_p$, we denote $\mu_{P,\psi} := \mu_{\mathcal{H}(\hat{\Xi}_p(P,\psi))}$.

The measures occuring as $\mu_{\mathcal{H}}$ for some homogeneous space have a very particular form. We formulate this in terms of the parameter class $\mathbb{P}_p$.

7.2 Theorem. *Let $\nu > -1$ and set $p := \nu + \frac{1}{2}$. Moreover, let λ be the Lebesgue measure on $\mathbb{R}$. Then the following statements hold.*

(i) *Let* $(P,\psi)\in\mathbb{P}_p$ *and write* $P=\begin{pmatrix}\kappa_1&\kappa_3\\\kappa_3&\kappa_2\end{pmatrix}$ *and* $\sigma:=2p\kappa_3-\psi\kappa_1$. *Moreover, let* κ *be the nonnegative square root of* $\det P$. *Then* $\mu_{P,\psi}\ll\lambda$ *and*

$$\frac{d\mu_{P,\psi}}{\mathrm{d}\lambda}(x)=\begin{cases}\mu_+(P,\psi)\cdot|x|^{2p} & \text{if } x>0,\\ \mu_-(P,\psi)\cdot|x|^{2p} & \text{if } x<0,\end{cases} \tag{7.2}$$

where

$$\mu_+(P,\psi):=\begin{cases}\frac{2^{2p}\kappa^{2p+1}|\Gamma(\frac{\sigma}{2i\kappa}+p+1)|^2}{\kappa_1\Gamma(2p+1)^2}\cdot e^{\pi\frac{\sigma}{2\kappa}} & \text{if } \det P\neq 0,\\ \frac{\pi\sigma^{2p+1}}{\kappa_1\Gamma(2p+1)^2} & \text{if } \det P=0,\sigma>0,\\ 0 & \text{if } \det P=0,\sigma<0,\end{cases} \tag{7.3}$$

$$\mu_-(P,\psi):=\begin{cases}\frac{2^{2p}\kappa^{2p+1}|\Gamma(\frac{\sigma}{2i\kappa}+p+1)|^2}{\kappa_1\Gamma(2p+1)^2}\cdot e^{-\pi\frac{\sigma}{2\kappa}} & \text{if } \det P\neq 0,\\ 0 & \text{if } \det P=0,\sigma>0,\\ \frac{\pi|\sigma|^{2p+1}}{\kappa_1\Gamma(2p+1)^2} & \text{if } \det P=0,\sigma<0.\end{cases} \tag{7.4}$$

(ii) *Let* $(\mu_+,\mu_-)\in[0,\infty)^2\setminus\{(0,0)\}$. *Then there exists* $(P,\psi)\in\mathbb{P}_p$ *such that* $\mu_+=\mu_+(P,\psi)$ *and* $\mu_-=\mu_-(P,\psi)$.

(iii) *Let* $(P,\psi),(\tilde P,\tilde\psi)\in\mathbb{P}_p$, *and write* $P=\begin{pmatrix}\kappa_1&\kappa_3\\\kappa_3&\kappa_2\end{pmatrix}$ *and* $P=\begin{pmatrix}\tilde\kappa_1&\tilde\kappa_3\\\tilde\kappa_3&\tilde\kappa_2\end{pmatrix}$. *Then* $\mu_{P,\psi}=\mu_{\tilde P,\tilde\psi}$, *if and only if*

- $\triangleright$ $\kappa_1^{-\frac{2}{1+2p}}\det P=\tilde\kappa_1^{-\frac{2}{1+2p}}\det\tilde P$,
- $\triangleright$ $\kappa_1^{\frac{2p}{1+2p}}\psi-\tilde\kappa_1^{\frac{2p}{1+2p}}\tilde\psi=2p\Big(\kappa_1^{-\frac{1}{1+2p}}\kappa_3-\tilde\kappa_1^{-\frac{1}{1+2p}}\tilde\kappa_3\Big)$.

Concerning item (i) of the theorem, in [Bra62; Bra68] the following less precise statement is shown.

7.3 Proposition. *Let* $\nu>-1$, *set* $p:=\nu+\frac{1}{2}$, *and let* $(P,\psi)\in\mathbb{P}_p$. *Denote by* λ *the Lebesgue measure on* $\mathbb{R}$. *Then* $\mu_{P,\psi}\ll\lambda$ *and its derivative is of the form (7.2) with some numbers* $\mu_\pm(P,\psi)\geq 0$.

Proof. Let $c>0$. Then the map $F(x)\mapsto[c\odot_{p+\frac{1}{2}}F](x)$ is an isometric bijection of $\bigcup_{a>0}\mathcal{H}(a\odot_p E)$ onto itself. Thus it has an extension to a unitary operator of $L^2(\mu_{P,\psi})$ onto itself. Since L^2-convergence implies pointwise a.e. convergence of a subsequence, this extension acts again as $f(x)\mapsto[c\odot_{p+\frac{1}{2}}f](x)$ (for $x\in\mathbb{R}$ a.e.).

We have, for every $c>0$,

$$\mu_{P,\psi}\big((0,c)\big)=\|\mathbb{1}_{(0,c)}\|^2_{L^2(\mu_{P,\psi})}=\|\underbrace{c\odot_{p+\frac{1}{2}}\mathbb{1}_{(0,c)}}_{=c^{p+\frac{1}{2}}\mathbb{1}_{(0,1)}}\|^2_{L^2(\mu_{P,\psi})}=c^{2p+1}\mu_{P,\psi}\big((0,1)\big).$$

Analogously, we find

$$\mu_{P,\psi}\big((-c,0)\big)=c^{2p+1}\mu_{P,\psi}\big((-1,0)\big)\text{ and }\mu_{P,\psi}\big((-c,c)\big)=c^{2p+1}\mu_{P,\psi}\big((-1,1)\big).$$

The first relation shows that $\mathbb{1}_{(0,\infty)}\,\mathrm{d}\mu_{P,\psi}$ is absolutely continuous w.r.t. $\mathrm{d}\lambda$, and

$$\frac{\mathbb{1}_{(0,\infty)}(x)\,\mathrm{d}\mu_{P,\psi}(x)}{\mathrm{d}\lambda(x)} = \mathbb{1}_{(0,\infty)}(x)(2p+1)x^{2p}\mu_{P,\psi}\big((0,1)\big),$$

the second that $\mathbb{1}_{(-\infty,0)}\,\mathrm{d}\mu_{P,\psi}$ is absolutely continuous w.r.t. $\mathrm{d}x$, and

$$\frac{\mathbb{1}_{(-\infty,0)}(x)\,\mathrm{d}\mu_{P,\psi}(x)}{\mathrm{d}\lambda(x)} = \mathbb{1}_{(-\infty,0)}(x)(2p+1)|x|^{2p}\mu_{P,\psi}\big((-1,0)\big),$$

and letting $c\downarrow 0$ in the third relation yields

$$\mu_{P,\psi}(\{0\}) = \lim_{c\downarrow 0}\mu_{P,\psi}\big((-c,c)\big) = 0.$$

$\square$

The explicit formulae (7.3) and (7.4) for $\mu_\pm(P,\psi)$ are stated without a proof in [Bra62, p.210]. Obtaining those values requires knowing the explicit formulae for $\hat{\Xi}_p(P,\psi)$ in terms of Kummer functions. The argument rests on the following fact about asymptotics.

7.4 Lemma. *Let $\delta\in\mathbb{R}$ and $p\in\mathbb{R}\setminus(-\frac{1}{2}(\mathbb{N}_0+1))$. Then there exist bounded functions $R_\pm\colon[1,\infty)\to\mathbb{R}$, such that*

$$\begin{aligned}\Big|\frac{1}{2}M(i\delta+p,2p+1,-iy)+\frac{1}{2}M(i\delta+p+1,2p+1,-iy)\\ -\frac{i}{2(2p+1)}yM(i\delta+p+1,2p+2,-iy)\Big|\\ =\frac{\Gamma(2p+1)}{|\Gamma(i\delta+p+1)|}e^{\pm\frac{\pi}{2}\delta}\cdot\frac{1}{|y|^p}\cdot\Big(1+\frac{R_\pm(|y|)}{|y|}\Big),\end{aligned}$$

where the "+"-sign holds if $y\geq 1$ and the "−"-sign when $y\leq -1$.

Proof. We use the asymptotic expansion of the Kummer function given in [AS64, 13.5.1], see also [Olv+10, 13.7.2]. For a purely imaginary argument $z=iy$, this reads as (for parameters $a\in\mathbb{C}$ and $b\in\mathbb{C}\setminus(-\mathbb{N}_0)$)

$$\begin{aligned}M(a,b,iy)=&\frac{\Gamma(b)}{\Gamma(b-a)}e^{\pm i\frac{\pi}{2}a}e^{-i(\operatorname{Im}a)\log|y|}\cdot|y|^{-\operatorname{Re}a}\cdot\big(1+\mathrm{O}(\tfrac{1}{|y|})\big)\\ &+\frac{\Gamma(b)}{\Gamma(a)}e^{\pm i\frac{\pi}{2}(a-b)}e^{i(y+(\operatorname{Im}(a-b))\log|y|)}\cdot|y|^{\operatorname{Re}(a-b)}\cdot\big(1+\mathrm{O}(\tfrac{1}{|y|})\big),\end{aligned}$$

where the "+"-sign holds if $y>0$ and the "−"-sign if $y<0$, and where the $\mathrm{O}(\frac{1}{|y|})$ is understood for $|y|\to\infty$.

Let $\gamma\in\mathbb{C}$, then these formulae give

$$\begin{aligned}&\frac{1}{2}M(i\delta+p,2p+1,-iy)+\frac{1}{2}M(i\delta+p+1,2p+1,-iy)\\ &\qquad-\gamma yM(i\delta+p+1,2p+2,-iy)\\ &=\pm\tfrac{\Gamma(2p+2)}{\Gamma(-i\delta+p+1)}e^{\mp i\frac{\pi}{2}(i\delta+p+1)}e^{-i\delta\log|y|}\cdot\frac{1}{|y|^p}\cdot\big[\tfrac{i}{2(2p+1)}-\gamma\big]\cdot\big(1+\mathrm{O}(\tfrac{1}{|y|})\big)\\ &\pm\tfrac{\Gamma(2p+2)}{\Gamma(i\delta+p+1)}e^{\pm i\frac{\pi}{2}(-i\delta+p+1)}e^{i(y+\delta\log|y|)}\cdot\frac{1}{|y|^p}\cdot\big[\tfrac{-i}{2(2p+1)}-\gamma\big]\cdot\big(1+\mathrm{O}(\tfrac{1}{|y|})\big)\end{aligned}$$

where the upper sign holds if $y > 0$ and the lower sign if $y < 0$.

Using $\gamma := \frac{i}{2(2p+1)}$ in this formula yields the assertion of the lemma. $\square$

Proof of Theorem 7.2(i); Case $\det P \neq 0$. As usual we write $\Xi_p(P,\psi) = (A, B)$ and $\hat{\Xi}_p(P,\psi) = E$.

① For $a > 0$ and τ in the open upper half-plane $\mathbb{C}_+$ consider the function

$$q_{a,\tau}(z) := \frac{[a \odot_p A](z)\tau + [a \odot_p B](z)}{-[a \odot_p B](z)\tau + [a \odot_p A](z)}, \quad z \in \mathbb{C}_+.$$

Then $q_{a,\tau}$ is a Nevanlinna function, as computing the Nevanlinna kernel shows. Since A and B have no common real zeroes and $\operatorname{Im} \tau > 0$, it has an analytic continuation to some domain containing the closed upper half-plane $\mathbb{C}_+ \cup \mathbb{R}$. Hence, the measure $\nu_{a,\tau}$ in the Nevanlinna integral representation of $q_{a,\tau}$ is absolutely continuous w.r.t. the Lebesgue measure and

$$\frac{\mathrm{d}\nu_{a,\tau}}{\mathrm{d}\lambda}(x) = \frac{1}{\pi} \operatorname{Im} q_{a,\tau}(x), \quad x \in \mathbb{R} \text{ a.e.}$$

The imaginary part of $q_{a,\tau}$ computes as

$$\operatorname{Im} q_{a,\tau}(x) = \frac{(\operatorname{Im} \tau) \cdot |[a \odot_p E](x)|^2}{|-[a \odot_p B](x)\tau + [a \odot_p A](x)|^2}.$$

By [Bra68, Theorem 32] the space $\mathcal{H}(a \odot_p E)$ is contained contractively in $L^2(\mu_{a,\tau})$ where $\mu_{a,\tau}$ is the measure which is mutually absolutely continuous with $\nu_{a,\tau}$ and has derivative

$$\frac{\mathrm{d}\mu_{a,\tau}}{\mathrm{d}\nu_{a,\tau}}(x) = \frac{\pi}{|[a \odot_p E](x)|^2}.$$

Since the chain (7.1) contains no one-dimensional gaps, this inclusion is actually always isometric.

Let us note explicitly that from the above $\mathrm{d}\mu_{a,\tau} \ll \mathrm{d}\lambda$ and

$$\frac{\mathrm{d}\mu_{a,\tau}}{\mathrm{d}\lambda}(x) = \frac{\operatorname{Im} \tau}{|-[a \odot_p B](x)\tau + [a \odot_p A](x)|^2}.$$

② We choose τ such that Lemma 7.4 and Theorem 4.1 is applicable:

$$\tau := -\frac{\kappa_3}{\kappa_1} + i\frac{\kappa}{\kappa_1}.$$

Then Lemma 7.4 yields

$$\big|-[a \odot_p B](x)\tau + [a \odot_p A](x)\big| = \frac{\Gamma(2p+1)}{|\Gamma(-\frac{\sigma}{2i\kappa} + p + 1)|} \cdot e^{\mp \frac{\pi}{2} \frac{\sigma}{2\kappa}} \cdot \frac{1}{|2\kappa x|^p} \cdot \big(1 + \tfrac{R_\pm(|ax|)}{|ax|}\big),$$

and in turn

$$\begin{aligned}
\frac{\mathrm{d}\mu_{a,\tau}}{\mathrm{d}\lambda}(x) &= \frac{\frac{\kappa}{\kappa_1}}{\frac{\Gamma(2p+1)^2}{|\Gamma(-\frac{\sigma}{2i\kappa}+p+1)|^2}\cdot e^{\mp\pi\frac{\sigma}{2\kappa}}\cdot\frac{1}{|2\kappa x|^{2p}}}\cdot\big(1+\tfrac{R_\pm(|ax|)}{|ax|}\big)\\
&= \frac{2^{2p}\kappa^{2p+1}|\Gamma(\frac{\sigma}{2i\kappa}+p+1)|^2}{\kappa_1\Gamma(2p+1)^2}\cdot e^{\pm\pi\frac{\sigma}{2\kappa}}\cdot|x|^{2p}\cdot\big(1+\tfrac{R_\pm(|ax|)}{|ax|}\big)\\
&= \begin{cases}\mu_+(P,\psi)\cdot x^{2p}\cdot\big(1+\frac{R_+(|ax|)}{|ax|}\big) & \text{if } x>0,\\ \mu_-(P,\psi)\cdot|x|^{2p}\cdot\big(1+\frac{R_-(|ax|)}{|ax|}\big) & \text{if } x<0.\end{cases}
\end{aligned}$$

③ By Theorem 1.11 we have $\lim_{a\to\infty}\mu_{a,\tau}=\mu_{P,\psi}$ vaguely. Since

$$\lim_{a\to\infty}\frac{\mathrm{d}\mu_{a,\tau}}{\mathrm{d}\lambda}(x)=\begin{cases}\mu_+(P,\psi)\cdot x^{2p} & \text{if } x>0,\\ \mu_-(P,\psi)\cdot|x|^{2p} & \text{if } x<0,\end{cases}$$

and the convergence is uniform on compact subsets of $\mathbb{R}\setminus\{0\}$, we obtain that $\mathbb{1}_{\mathbb{R}\setminus\{0\}}\cdot\mu_{P,\psi}\ll\lambda$ and

$$\frac{\mathrm{d}(\mathbb{1}_{\mathbb{R}\setminus\{0\}}\cdot\mu_{P,\psi})}{\mathrm{d}\lambda}(x)=\begin{cases}\mu_+(P,\psi)\cdot x^{2p} & \text{if } x>0,\\ \mu_-(P,\psi)\cdot|x|^{2p} & \text{if } x<0.\end{cases}$$

As we saw in Proposition 7.3, $\mu_{P,\psi}$ has no point mass at 0, and (7.2) follows.

□

Proof of Theorem 7.2(i); Case $\det P=0$. We use a continuity argument. Note that, clearly, the set of all $(P,\psi)\in\mathbb{P}_p$ with $\det P\neq 0$ is dense in $\mathbb{P}_p$. The function $\hat{\Xi}_p(P,\psi)$, as well as the Hamiltonian $H_{P,\psi}$, depend continuously on (P,ψ) w.r.t. locally uniform convergence (on $\mathbb{C}$ for the first, and on $(0,\infty)$ for the latter). Hence, the measure $\mu_{P,\psi}$ depends continuously on (P,ψ) w.r.t. vague convergence of measures, cf. Lemma 1.13.

Let us show that $\det P=0$ implies $\sigma\neq 0$. Consider the case that $p\neq 0$. We can write $\sigma=(\kappa_1,\kappa_3)\cdot\binom{-\psi}{2p}$, and since the rows of P are linearly dependent, $\sigma=0$ implies that $\binom{-\psi}{2p}\in\ker P$. This is excluded by the definition of $\mathbb{P}_p$. If $p=0$, then we must have $\psi\neq 0$ by the definition of $\mathbb{P}_p$, and also in this case it follows that $\sigma\neq 0$.

In order to proof the second and third lines in (7.3) and (7.4), we thus have to evaluate the limit of the constants in the respective first lines when $\frac{\sigma}{\kappa}\to\pm\infty$. This is easy using the relation

$$|\Gamma(x+iy)|\sim\sqrt{2\pi}\cdot|y|^{x-\frac12}e^{-\pi\frac{|y|}{2}}\quad\text{for } y\to\pm\infty,$$

cf. [Olv+10, 5.11.9]. Namely, we obtain that (for $\kappa\to 0$)

$$\frac{2^{2p}\kappa^{2p+1}|\Gamma(\frac{\sigma}{2i\kappa}+p+1)|^2)}{\kappa_1\Gamma(2p+1)^2}\cdot e^{\pi\frac{\sigma}{2\kappa}}\sim\frac{\pi|\sigma|^{2p+1}}{\kappa_1\Gamma(2p+1)^2}e^{\frac{\pi}{2\kappa}(\sigma-|\sigma|)},$$

and this yields the second and third lines in (7.3). The relations in (7.4) follow analogously. □

Proof of Theorem 7.2(ii). Let $(\mu_+,\mu_-)\in[0,\infty)^2\setminus\{(0,0)\}$ be given. If $\mu_-=0$ or $\mu_+=0$, we use $P:=\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$ and

$$\psi:=-\Big(\frac{\mu_+}{\pi}\Gamma(2p+1)^2\Big)^{\frac{1}{2p+1}}\quad\text{or}\quad\psi:=\Big(\frac{\mu_-}{\pi}\Gamma(2p+1)^2\Big)^{\frac{1}{2p+1}},$$

respectively.

If $\mu_+,\mu_->0$, we set

$$\kappa_1:=1,\ \kappa_2:=\bigg(\frac{\Gamma(2p+1)^2\sqrt{\mu_+\mu_-}}{2^{2p}\big|\Gamma\big(\frac{i}{2\pi}\log\frac{\mu_-}{\mu_+}+p+1\big)\big|^2}\bigg)^{\frac{1}{p+\frac12}},\ \kappa_3:=0,\ \psi:=\sqrt{\kappa_2}\cdot\frac{1}{\pi}\log\frac{\mu_-}{\mu_+}.$$

Then $(P,\psi)\in\mathbb{P}_p$, and plugging in the definitions yields that $\mu_\pm(P,\psi)=\mu_\pm$. $\square$

Proof of Theorem 7.2(iii). Set $E:=\hat{\Xi}_p(P,\psi)$ and $\tilde{E}:=\hat{\Xi}_p(\tilde{P},\tilde{\psi})$.

① We show that

$$\mu_{P,\psi}=\mu_{\tilde{P},\tilde{\psi}}\quad\Longleftrightarrow\quad\exists c\in\mathbb{R}\colon\ \mathcal{H}(\tilde{E})=\mathcal{H}(c\odot_p E)\text{ isometrically.}$$

The implication "$\Leftarrow$" follows since $\mathcal{H}(\tilde{E})=\mathcal{H}(c\odot_p E)$ implies

$$\big\{\mathcal{H}(a\odot_p\tilde{E})\mid a>0\big\}=\big\{\mathcal{H}(a\odot_p E)\mid a>0\big\},$$

and thus $L^2(\mu_{P,\psi})=L^2(\mu_{\tilde{P},\tilde{\psi}})$. To show the implication "$\Rightarrow$", assume that $\mu_{P,\psi}=\mu_{\tilde{P},\tilde{\psi}}$. The functions E and $\tilde{E}$ are of bounded type in the upper half-plane and have no real zeroes, cf. Corollary 3.2 and Corollary 4.3. Hence, the Ordering Theorem [Bra68, Theorem 35] applies and yields

$$\forall a>0\colon\ \ \mathcal{H}(\tilde{E})\subseteq\mathcal{H}(a\odot_p E)\vee\mathcal{H}(a\odot_p E)\subseteq\mathcal{H}(\tilde{E}).$$

Lemma 5.8 implies $\mathcal{H}(\tilde{E})=\mathcal{H}(c\odot_p E)$ with $c:=\big(\frac{K_{\tilde{E}}(0,0)}{K_E(0,0)}\big)^{\frac{1}{2p+1}}$.

② Given $c>0$, we construct $(P_c,\psi_c)\in\mathbb{P}_p$ such that $\mathcal{H}(\hat{\Xi}_p(P_c,\psi_c))=\mathcal{H}(c\odot_p E)$. Set

$$(A_c,B_c):=(c\odot_p A,c\odot_p B)\begin{pmatrix}c^{-p}&0\\0&c^p\end{pmatrix},$$

then $E_c:=A_c-iB_c$ satisfies $\mathcal{H}(E_c)=\mathcal{H}(c\odot_p E)$ by Theorem 1.4. Write $A(z)=\sum_{n=0}^\infty\alpha_n z^n$ and $B(z)=\sum_{n=0}^\infty\beta_n z^n$, then

$$A(cz)=\sum_{n=0}^\infty\alpha_n c^n\cdot z^n,\quad c^{2p}B(cz)=\sum_{n=0}^\infty\beta_n c^{2p+n}\cdot z^n.$$

A short computation using (6.10) shows that

$$\begin{aligned}(\alpha_{n+1}c^{n+1},\beta_{n+1}c^{2p+n+1})=&-\frac{1}{(n+1)(2p+n+1)}(\alpha_n c^n,\beta_n c^{2p+n})\\&\cdot\begin{pmatrix}c^{\frac12+p}&0\\0&c^{\frac12-p}\end{pmatrix}P\begin{pmatrix}c^{\frac12+p}&0\\0&c^{\frac12-p}\end{pmatrix}\cdot J\begin{pmatrix}2p+n+1&0\\c^{-2p}\psi&n+1\end{pmatrix}\end{aligned}$$

for all $n \in \mathbb{N}_0$.

The pair

$$(P_c, \psi_c) := \left(\begin{pmatrix} c^{\frac{1}{2}+p} & 0 \\ 0 & c^{\frac{1}{2}-p} \end{pmatrix} P \begin{pmatrix} c^{\frac{1}{2}+p} & 0 \\ 0 & c^{\frac{1}{2}-p} \end{pmatrix}, c^{-2p}\psi \right) \tag{7.5}$$

belongs to $\mathbb{P}_p$, and by the above computation $\hat{\Xi}_p(P_c, \psi_c) = E_c$.

③ We use Theorem 6.2 (iii) to complete the proof. This theorem tells us that $\mathcal{H}(\hat{\Xi}_p(P_c, \psi_c)) = \mathcal{H}(\hat{\Xi}_p(\tilde{P}, \tilde{\psi}))$ isometrically, if and only if

▷ $c^{1+2p}\kappa_1 = \tilde{\kappa}_1$,

▷ $c^2 \det P = \det \tilde{P}$,

▷ $c^{-2p}\psi - \tilde{\psi} = \frac{2p}{c^{1+2p}\kappa_1}\big(c\kappa_3 - \tilde{\kappa}_3\big)$.

The first relation gives

$$c = \Big(\frac{\tilde{\kappa}_1}{\kappa_1}\Big)^{\frac{1}{1+2p}}.$$

Plugging this into the second and third relations, leads to the stated formulae.

□

Items (i) and (ii) of Theorem 7.2 say that the map $(P, \psi) \mapsto \mu_{P,\psi}$ is a surjection of the parameter space $\mathbb{P}_p$ onto the set of all measures $\mu \ll \lambda$ whose derivative is of the form

$$\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}(x) = \begin{cases} \mu_+ \cdot x^{2p} & \text{if } x > 0, \\ \mu_- \cdot |x|^{2p} & \text{if } x < 0, \end{cases}$$

with some $(\mu_+, \mu_-) \in [0, \infty)^2 \setminus \{(0,0)\}$. In item (iii) of the theorem the kernel of this map is described (we write $\simeq$ for that kernel). In the next lemma, we provide a complete systems of representatives of our parameter space $\mathbb{P}_p$ modulo $\simeq$.

7.5 Lemma. *Let $p > -\frac{1}{2}$.*

(i) *The set*

$$\Big\{ (P, \psi) \in \mathbb{P}_p \mid \begin{pmatrix} 1 \\ 0 \end{pmatrix}^* P \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 \wedge \begin{pmatrix} 1 \\ 0 \end{pmatrix}^* P \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \Big\} \tag{7.6}$$

is a complete system of representatives of $\mathbb{P}_p$ modulo $\simeq$.

(ii) *If $p \neq 0$, then also*

$$\big\{ (P, \psi) \in \mathbb{P}_p \mid \psi = 0 \wedge \begin{pmatrix} 1 \\ 0 \end{pmatrix}^* P \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \big\} \tag{7.7}$$

is a complete system of representatives of $\mathbb{P}_p$ modulo $\simeq$.

Proof. Let $(P,\psi)\in\mathbb{P}_p$ be given. Set $(\tilde{P},\tilde{\psi}):=(P_c,\psi_c)$, cf. (7.5), with $c:=\kappa_1^{-\frac{1}{2p+1}}$. Then $(P,\psi)\simeq(\tilde{P},\tilde{\psi})$ and $\tilde{\kappa}_1=1$. By the definition in (7.5) it is clear that $\kappa_3=0$ implies $\tilde{\kappa}_3=0$ and $\psi=0$ implies $\tilde{\psi}=0$.

Since $\simeq\supseteq\approx$, it follows from Lemma 6.3 that we can always reduce modulo $\simeq$ to an element of the form written in (7.6) or (7.7), respectively. Assume we have two element $(P,\psi),(\tilde{P},\tilde{\psi})\in\mathbb{P}_p$ with $(P,\psi)\simeq(\tilde{P},\tilde{\psi})$ and $\kappa_1=\tilde{\kappa}_1=1$ and $\kappa_3=\tilde{\kappa}_3=0$ (or $\psi=\tilde{\psi}=0$). Then $\kappa_2=\tilde{\kappa}_2$ from the first relation in Theorem 7.2 (*iii*) and $\psi=\tilde{\psi}$ from the second (or $\kappa_3=\tilde{\kappa}_3$ from the second and then $\kappa_2=\tilde{\kappa}_2$ from the first, respectively). ☐

Combining Theorem 7.2 with Theorem 4.1 we can directly connect measures (7.2) with their corresponding chains (7.1).

7.6 Corollary. *Let $p>-\frac{1}{2}$ and $(\mu_+,\mu_-)\in[0,\infty)^2\setminus\{(0,0)\}$. Let μ be the measure with $\mu\ll\lambda$ and derivative (7.2). We define functions A,B by distinguishing cases.*

(i) *Assume that $\mu_+,\mu_->0$. Define*

$$\alpha:=\frac{i}{2\pi}\log\frac{\mu_-}{\mu_+}+p,\qquad \kappa:=\frac{1}{2}\bigg(\frac{2\Gamma(2p+2)^2\sqrt{\mu_+\mu_-}}{(2p+1)|\Gamma(\alpha+1)|^2}\bigg)^{\frac{1}{2p+1}},$$

$$A(z):=e^{i\kappa z}\frac{1}{2}\Big[M(\alpha,2p+1,-2i\kappa z)+M(\alpha+1,2p+1,-2i\kappa z)\Big],$$

$$B(z):=z\cdot e^{i\kappa z}M(\alpha+1,2p+2,-2i\kappa z).$$

Then $\mu_{\mathcal{H}(E)}=\mu$.

(ii) *Assume that $\mu_+=0$ or $\mu_-=0$. Define*

$$\sigma:=\begin{cases}\Big(\frac{\mu_+}{\pi(2p+1)}\Gamma(2p+2)^2\Big)^{\frac{1}{2p+1}} & \text{if } \mu_+>0,\\ -\Big(\frac{\mu_-}{\pi(2p+1)}\Gamma(2p+2)^2\Big)^{\frac{1}{2p+1}} & \text{if } \mu_->0,\end{cases}$$

$$A(z):={}_0F_1(2p+1,-\sigma z),\qquad B(z)=z\cdot{}_0F_1(2p+2,-\sigma z).$$

Then $\mu_{\mathcal{H}(E)}=\mu$.

Of course there are many choices for the function E generating the chain of μ. The choice in this corollary is made in such a way that $E(0)=1$ and $K_E(0,0)=1$.

Proof of Corollary 7.6. In the proof of Theorem 7.2 (*ii*) we have already exhibited a pair (P,ψ) such that the corresponding measure is equal to μ. Theorem 7.2 (*iii*) allows us to modify this pair; and we use this freedom to obtain that $K_E(0,0)=1$.

▷ If $\mu_+,\mu_->0$, set

$$\kappa_1:=2p+1,\quad \kappa_2:=\Big(\frac{1}{2p+1}\Big)^{1-\frac{1}{p+\frac{1}{2}}}\bigg(\frac{\Gamma(2p+1)^2\sqrt{\mu_+\mu_-}}{2^{2p}|\Gamma(\frac{i}{2\pi}\log\frac{\mu_-}{\mu_+}+p+1|^2}\bigg)^{\frac{1}{p+\frac{1}{2}}},$$

$$\kappa_3:=0,\quad \psi:=\sqrt{\frac{\kappa_2}{2p+1}}\,\frac{1}{\pi}\log\frac{\mu_-}{\mu_+}.$$

▹ If $\mu_+ = 0$ or $\mu_- = 0$, set

$$\kappa_1 := 2p+1, \quad \kappa_2 := 0, \quad \kappa_3 := 0,$$

$$\psi := \Big(\frac{1}{2p+1}\Big)^{1-\frac{1}{2p+1}} \begin{cases} -\Big(\frac{\mu_+}{\pi}\Gamma(2p+1)^2\Big)^{\frac{1}{2p+1}} & \text{if } \mu_+ > 0, \\ \Big(\frac{\mu_-}{\pi}\Gamma(2p+1)^2\Big)^{\frac{1}{2p+1}} & \text{if } \mu_- > 0. \end{cases}$$

Plugging this data into the formulas of Theorem 4.1 leads to the stated assertion. □

7.7 Remark. If one is interested only in the reproducing kernel K_E and not in the function E itself, the formulae from Corollary 7.6 (i) can be written in a slightly different form. Namely, set

$$F(z) := e^{i\kappa z} M(\alpha+1, 2p+1, -2i\kappa z), \quad G(z) := e^{i\kappa z} M(\alpha, 2p+1, -2i\kappa z),$$

then obviously $A(z) = \frac{1}{2}[F(z) + G(z)]$, and [AS64, 13.4.4] shows that

$$B(z) = \frac{i(2p+1)}{2\kappa}\big[F(z) - G(z)\big].$$

We see that

$$K_E(z,w) = \frac{i(2p+1)}{2\kappa} \cdot \frac{F(z)G(\overline{w}) - G(z)F(\overline{w})}{z - \overline{w}}.$$

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