# An upper bound for the Nevanlinna matrix of an indeterminate moment sequence

RAPHAEL PRUCKNER \* JAKOB REIFFENSTEIN \* HARALD WORACEK<sup>‡</sup>

**Abstract:** The solutions of an indeterminate Hamburger moment problem can be parameterised using the Nevanlinna matrix of the problem. The entries of this matrix are entire functions of minimal exponential type, and any growth less than that can occur.

An indeterminate moment problem can be considered as a canonical system in limit circle case by rewriting the three-term recurrence of the problem to a first order vector-valued recurrence. We give a bound for the growth of the Nevanlinna matrix in terms of the parameters of this canonical system. In most situations this bound can be evaluated explicitly. It is sharp in the sense that for well-behaved parameters it coincides with the actual growth of the Nevanlinna matrix up to multiplicative constants.

**AMS MSC 2020:** 44A60, 47B36, 34L20, 30D15, 37J99 **Keywords:** Indeterminate moment problem, Nevanlinna matrix, order of entire function, canonical system, distribution of eigenvalues

## 1 Introduction

Let  $(s_n)_{n=0}^{\infty}$  be a sequence of real numbers. The Hamburger moment problem is the task of describing the set

$$\mathcal{M}((s_n)_{n=0}^{\infty}) := \left\{ \mu \mid \begin{array}{c} \mu \text{ positive Borel measure on } \mathbb{R} \\ s_n = \int_{\mathbb{R}} t^n \, \mathrm{d}\mu(t) \text{ for } n = 0, 1, 2, \dots \end{array} \right\}$$

This is a classical problem of analysis and was treated extensively in work of H. Hamburger, M. Riesz, R. Nevanlinna, and many others. Standard references are, e.g., [ST43; Akh61; Sch17]. It is known that one of the following alternatives takes place for the set  $\mathcal{M}((s_n)_{n=0}^{\infty})$ : it is either empty, or contains exactly one element, or contains infinitely many elements. In the last case  $(s_n)_{n=0}^{\infty}$  is called an *indeterminate moment sequence*, and this is the case we are concerned with in the present paper.

For an indeterminate moment sequence  $(s_n)_{n=0}^{\infty}$  the set  $\mathcal{M}((s_n)_{n=0}^{\infty})$  can be parameterised: There exist four entire functions A, B, C, D, such that the formula

$$\int_{\mathbb{R}} \frac{1}{t-z} \, \mathrm{d}\mu(t) = \frac{A(z)\tau(z) + B(z)}{C(z)\tau(z) + D(z)}$$
(1.1)

establishes a bijection between  $\mathcal{M}((s_n)_{n=0}^{\infty})$  and the set of all functions  $\tau$  that are analytic in the open upper half-plane  $\mathbb{C}^+$  and have nonnegative imaginary part (formally including the function constant equal to  $\infty$ ). The matrix

$$W(z) := \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}$$

<sup>&</sup>lt;sup> $\ddagger$ </sup>This work was supported by the project P 30715-N35 of the Austrian Science Fund (FWF). The second and third authors were supported by the joint project I 4600 of the Austrian Science Fund (FWF) and the Russian foundation of basic research (RFBR).

is called the Nevanlinna matrix of the sequence  $(s_n)_{n=0}^{\infty}$ .

The Nevanlinna matrix admits an operator theoretic interpretation (and this viewpoint could be used to prove (1.1)). To the moment sequence there is an associated sequence  $(p_n)_{n=0}^{\infty}$  of orthogonal polynomials that satisfies a three-term recurrence of the form

$$zp_n(z) = b_n p_{n+1}(z) + a_n p_n(z) + b_{n-1} p_{n-1}(z), \qquad n = 0, 1, 2, \dots$$
(1.2)

with certain parameters  $a_n \in \mathbb{R}$  and  $b_n > 0$  for n = 0, 1, 2, ... (and formally  $b_{-1} = 0$ ). The *Jacobi operator* is the closure of the operator

$$(Ju)_n = \begin{cases} b_0 u_1 + a_0 u_0 & \text{if } n = 0, \\ b_n u_{n+1} + a_n u_n + b_{n-1} u_{n-1} & \text{if } n \ge 1 \end{cases}$$

defined on  $\mathcal{D} := \{u \in \ell^2 \mid u_n = 0 \text{ for almost all } n\}$ , and (1.2) is the formal eigenvalue equation for J. The Jacobi operator is closed and symmetric, and has deficiency index (1, 1). Hence, the self-adjoint extensions of the Jacobi operator (we write again J for simplicity) are described by M.G. Krein's resolvent formula. The Nevanlinna matrix of  $(s_n)_{n=0}^{\infty}$  is precisely the *u*-resolvent matrix of J for a certain generating element u.

The entries A, B, C, D of the Nevanlinna matrix all have the same growth [BP94], and a classical theorem of M. Riesz in [Rie23] states that they are of minimal exponential type. Moreover, it is known that any growth smaller than that may occur, e.g. [BS98]. Revealing more refined information about the growth of the Nevanlinna matrix is an intricate problem. It is of great interest also for spectral theoretic reasons: due to the above interpretation of W as a resolvent matrix, the set of zeros of D coincides with the spectrum of a particular self-adjoint extension of J. Hence, if the growth of W is known, information about the distribution of eigenvalues can be obtained using standard tools from complex analysis.

In the present paper we give an upper bound for

$$M(r) := \max_{|z|=r} \|W(z)\|$$

of which there are two versions: First, a general formulation where any (possibly rough) data is admitted, and second, a much more explicit bound for data satisfying mild regularity conditions. If, in addition, the data decays sufficiently fast, then the upper bound coincides with M(r) up to multiplicative constants.

So far, we talked about two different (equivalent) objects, i.e., the moment sequence and the Jacobi operator. Our method actually relies on a third object – a *canonical system* of differential equations. It is obtained from rewriting the three-term recurrence (1.2) as a first order vector difference equation and interpreting this as a discrete differential equation. From the fundamental solution of the canonical system, we obtain its *monodromy matrix* which again coincides with the Nevanlinna matrix W.

The canonical systems occurring in the context of moment problems are represented by a *Hamiltonian* which reflects the discrete nature of the difference equation. It is determined by two sequences, its *lengths*  $l_j > 0$  and *angles*  $\phi_j \in \mathbb{R}$ , which we call the *Hamiltonian parameters* (details are given below, cf. Definition 1.1). This model for an indeterminate moment sequence is well suited for the study of various properties. For example, the moment sequence is indeterminate if and only if  $(l_j)_{j=1}^{\infty}$  is summable. Our results will almost exclusively be formulated in terms of the Hamiltonian parameters.

Let us briefly review some earlier results on the growth of Nevanlinna matrices. The history of the subject starts probably with a theorem of M.S. Livšic in [Liv39] that gives a lower bound for M(r) in terms of the moment sequence itself. This bound is easy to handle, but will usually not give the correct size. In recent work of C. Berg and R. Szwarc [BS14] it is shown that the order and type of W

$$\rho \coloneqq \limsup_{r \to \infty} \frac{\log \log M(r)}{\log r}, \qquad \tau \coloneqq \limsup_{r \to \infty} \frac{\log M(r)}{r^{\rho}},$$

coincide with those of a certain entire function built in a complicated way from the coefficients of the orthonormal polynomials. A theorem which takes the Jacobi parameters as input is due to Yu.M. Berezanskii [Ber56] and was generalised in [BS14]. It states that the order of W coincides with the convergence exponent of the sequence  $(b_n)_{n=0}^{\infty}$ , but the assumptions are very restrictive, involving regularity of  $b_n$  and smallness of  $a_n$ . Bounds for the order of W in terms of the Hamiltonian parameters, which under certain conditions give the correct value, are obtained in [PRW17].

In our present theorems we start with the Hamiltonian parameters, and give bounds for the type of W with respect to a general comparison function (e.g., a proximate order in the sense of Valiron). This improves significantly upon [PRW17] in several ways: we work on a much more refined scale of measuring growth, we obtain type estimates, and in some situations our bound improves the earlier results even on the rough scale of order. Several cases occur, which are presented in Theorem 4.6. The path that leads to this result is divided into two main sections: First, we prove a very general, albeit complicated, upper bound in Theorem 2.2. Second, in Theorem 4.1 we use J. Karamata's theory of regularly varying functions to evaluate this general bound. For the convenience of the reader, the significantly simpler case of usual order and type is covered separately in Corollary 2.5. We show in Theorem 5.3 that the upper bound is attained if the lengths and angle differences are themselves close to regularly varying and decay sufficiently fast (corresponding to order less than  $\frac{1}{2}$ ). In this case the growth of log M(r) is fully determined up to multiplicative constants.

The proof of our foundational theorem is based on a somewhat tricky application of a recent result from [PW22], approximating the target Hamiltonian with a finite dimensional one. Building upon that, a detailed (and partly tedious) analysis of functions follows. We recommend the reader to get an overall picture before diving into the actual estimates.

### Canonical systems with Hamburger Hamiltonian

A two-dimensional canonical system is an equation of the form

$$y'(t) = zJH(t)y(t), \qquad t \in (a,b) \text{ a.e.},$$

where

 $\triangleright \ -\infty \le a < b \le \infty,$ 

 $\succ \ H \in L^1_{\text{loc}}((a,b), \mathbb{R}^{2 \times 2}), \text{ and } H(t) \ge 0 \text{ and } H(t) \ne 0 \text{ for } t \in (a,b) \text{ a.e.},$  $\triangleright \ J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } z \in \mathbb{C},$ 

and the solution  $y: (a, b) \to \mathbb{C}^2$  is required to be locally absolutely continuous. The function H is called the *Hamiltonian* of the system.

Canonical systems that occur from indeterminate moment sequences are those whose Hamiltonian has the following particular – discrete – form. Here we denote

$$\xi_{\phi} := \begin{pmatrix} \cos \phi \\ \sin \phi \end{pmatrix}, \qquad \phi \in \mathbb{R}.$$

**1.1 Definition.** Let  $(l_j)_{j=1}^{\infty}$  be a summable sequence of positive numbers and  $(\phi_j)_{j=1}^{\infty}$  be a sequence of real numbers. Set  $x_0 := 0, x_j := \sum_{i=1}^{j} l_i$  for  $j \ge 1$ , and  $L := \sum_{j=1}^{\infty} l_j < \infty$ . The Hamburger Hamiltonian with lengths  $l_j$  and angles  $\phi_j$  is the function H defined on the interval (0, L) as

$$H(t) := \xi_{\phi_j} \xi_{\phi_j}^T \quad \text{for } j \in \mathbb{N} \text{ and } x_{j-1} \le t < x_j$$

A Hamburger Hamiltonian H thus can be pictured as

Since a Hamburger Hamiltonian in the sense of the above definition is even integrable on the whole interval (0, L), there exists a unique  $2 \times 2$  matrix-valued solution  $W: [0, L] \times \mathbb{C} \to \mathbb{C}^{2 \times 2}$  of the initial value problem

$$\left\{ \begin{array}{ll} \frac{\partial}{\partial t}W(t;z)J=zW(t;z)H(t),\qquad t\in(0,L) \text{ a.e.},\\ W(0;z)=I. \end{array} \right.$$

We refer to W as the fundamental solution of H, and to the matrix  $W_H(z) := W(L; z)$  as its monodromy matrix.

#### A notational convention

We frequently compare functions up to multiplicative constants or asymptotically, and throughout the paper use the following notation.

**1.2 Notation.** Let X be a set and  $f, g: X \to (0, \infty)$ .

- (i) We write " $f \leq g$ " (or " $f(x) \leq g(x)$ ") to say that there exists a constant C > 0 such that  $f(x) \leq C \cdot g(x)$  for all  $x \in X$ . We write " $f \gtrsim g$ " if  $g \leq f$ , and " $f \approx g$ " if  $f \leq g$  and  $f \gtrsim g$ .
- (ii) Assume that X is directed. We say that " $f \leq g$  for sufficiently large x", if there exists  $x_0 \in X$  such that  $f|_Y \leq g|_Y$  where  $Y := \{x \in X \mid x \succeq x_0\}$ .

- (iii) Assume again that X is directed. We write " $f \sim g$ " if  $\lim_{x \in X} \frac{f(x)}{g(x)} = 1$ , and " $f \ll g$ " if  $\lim_{x \in X} \frac{f(x)}{g(x)} = 0$ , and " $f \approx g$ " if  $\lim_{x \in X} \frac{f(x)}{g(x)}$  exists in  $(0, \infty)$ .
- (iv) Assume that X is a subset of a topological space. We say that " $f \leq g$  locally", if  $f|_K \leq g|_K$  for every compact subset K of X. Analogous wording applies to " $\asymp$ ".

For the convenience of the reader, we include an appendix where the definition and some basic results about regularly varying functions in Karamata sense are recalled.

## 2 An upper bound for the monodromy matrix

In this section we give a generic upper bound for  $\log ||W_H(z)||$ . The functions  $d_l, d_{\phi}$  used below play the role of well-behaved comparison functions for the lengths and for the differences of consecutive angles of the Hamiltonian.

**2.1 Definition.** Let  $d_l: [1, \infty) \to (0, \infty)$  and  $d_{\phi}: [1, \infty) \to (0, 1]$  be measurable with  $d_l \simeq 1 \simeq d_{\phi}$  locally. Then we denote

$$\hbar(R) := \sup \Big\{ t \in [1,\infty) \mid \sup_{1 \le s \le t} \frac{2}{R(d_l d_\phi)(s)} \le 1 \Big\},$$
(2.1)

$$\hbar(R) := \sup\left\{t \in [1,\infty) \mid \sup_{1 \le s \le t} \frac{d_{\phi}(s)}{Rd_l(s)} \le 1\right\},\tag{2.2}$$

where  $R \in [\frac{2}{(d_l d_{\phi})(1)}, \infty)$  and  $R \in [\frac{d_{\phi}(1)}{d_l(1)}, \infty)$ , respectively. Further, we set

$$g(t,R) := \begin{cases} \log\left(R(d_l d_{\phi})(t)\right) & \text{if } 1 \le t < \hbar(R), \\ R^{\frac{1}{2}}(d_l d_{\phi})^{\frac{1}{2}}(t) & \text{if } \hbar(R) \le t < \hbar(R) \\ R d_l(t) & \text{if } \hbar(R) \le t, \end{cases}$$

where  $(t, R) \in [1, \infty) \times \left[\frac{2}{(d_l d_{\phi})(1)}, \infty\right)$ , and

$$g(t,R) = \int_{1}^{t} g(s,R) \,\mathrm{d}s. \tag{2.3}$$

Note here that  $d_{\phi}(t) \leq 1$ , and hence  $\hbar(R) \leq \hbar(R)$  for all  $R \geq \frac{2}{(d_1 d_{\phi})(1)}$ .

**2.2 Theorem.** Let  $(l_j)_{j=1}^{\infty}$  be a summable sequence of positive numbers, and  $(\phi_j)_{j=1}^{\infty}$  a sequence of real numbers. Denote by H the Hamburger Hamiltonian with these lengths and angles, and let  $W_H$  be its monodromy matrix.

Let  $\psi \in \mathbb{R}$  and let  $d_l, d_{\phi}, c_l, c_{\phi} \colon [1, \infty) \to (0, \infty)$  be measurable and nonincreasing, with  $d_{\phi} \leq 1$  and  $c_{\phi} \leq c_l$ . Assume that

$$\forall j \in \mathbb{N}: \ l_j \le d_l(j) \land \ |\sin(\phi_{j+1} - \phi_j)| \le d_\phi(j), \tag{2.4}$$

$$\forall N \in \mathbb{N}: \sum_{j=N+1}^{\infty} l_j \le c_l(N) \land \sum_{j=N+1}^{\infty} l_j \sin^2(\phi_j - \psi) \le c_\phi(N).$$
(2.5)

Let g(t, R) be as in (2.3), and set

$$L(t,R) := 1 + \log^{+} R + \log^{+} \frac{c_{l}(\lceil t \rceil)}{c_{\phi}(\lceil t \rceil)} + \log^{+} \frac{d_{l}(1)}{d_{\phi}(1)}$$
(2.6)

$$+\log^{+}\frac{d_{l}(\min\{\lceil t\rceil,\lfloor\hbar(R)\rfloor\})}{d_{\phi}(\min\{\lceil t\rceil,\lfloor\hbar(R)\rfloor\})} + \sum_{j=1}^{\min\{\lceil t\rceil,\lfloor\hbar(R)\rfloor\}-1} \left|\log\left(\frac{d_{\phi}(j)}{d_{l}(j)}\Big/\frac{d_{\phi}(j+1)}{d_{l}(j+1)}\right)\right|$$

Then we have, for all  $R \geq \frac{2}{(d_l d_{\phi})(1)}$ ,

$$\log\left(\max_{|z|=R} \|W_H(z)\|\right) \le 9 \cdot \inf_{t\ge 1} \left(\max\left\{g\left(t,R\right), R(c_l c_{\phi})^{\frac{1}{2}}(t)\right\} + L(t,R)\right).$$
(2.7)

2.3 Remark. The terms appearing in the upper bound in (2.7) have the following meaning:

- (i) q(t, R) estimates the contribution of the first [t] intervals of H;
- (ii)  $R(c_l c_{\phi})^{\frac{1}{2}}(t)$  estimates the contribution of the remaining intervals of H;
- (iii) L(t, R) is usually a remainder term.

We mention two possible scenarios. If lengths and angle differences are nonincreasing but decay slowly, choosing  $d_l(j)$ ,  $d_{\phi}(j)$  such that equality holds in (2.4) makes q(t, R) a rather precise bound for the contribution of the first  $\lceil t \rceil$ intervals. On the other hand, if lengths and angle differences behave irregularly, then q(t, R) likely overestimates the contribution of the first  $\lceil t \rceil$  intervals. However, choosing  $c_l$ ,  $c_{\phi}$  such that equality holds in (2.5), the bound (2.7) is still a good estimate for log  $||W_H(z)||$  as long as the decay is sufficiently fast.

If  $\lim_{t\to\infty} (c_l c_{\phi})(t) > 0$ , the bound (2.7) is  $\gtrsim R$  and thus trivial: we know that  $W_H$  is of minimal exponential type by the classical Krein-de Branges formula. Hence we may safely assume that  $\lim_{t\to\infty} (c_l c_{\phi})(t) = 0$  whenever this is convenient.

**2.4 Lemma.** Let  $d_l: [1, \infty) \to (0, \infty)$  and  $d_{\phi}: [1, \infty) \to (0, 1]$  be measurable with  $d_l \simeq 1 \simeq d_{\phi}$  locally, and let  $c_l, c_{\phi}$  be continuous and nonincreasing with  $\lim_{t \to \infty} (c_l c_{\phi})(t) = 0$ . Then the equation

$$q(t,R) = R(c_l c_{\phi})^{\frac{1}{2}}(t), \qquad (2.8)$$

has a unique solution T(R), and

$$\min_{t \ge 1} \max\left\{ g(t, R), R(c_l c_{\phi})^{\frac{1}{2}}(t) \right\} = g(T(R), R) = R(c_l c_{\phi})^{\frac{1}{2}}(T(R)).$$

In addition,  $\lim_{R\to\infty} T(R) = \infty$ .

The following lemma hints at a way to evaluate the upper bound in Theorem 2.2.

*Proof.* For each fixed R the function  $t \mapsto q(t, R)$  is continuous and increasing while  $t \mapsto R(c_l c_{\phi})^{\frac{1}{2}}(t)$  is continuous and nonincreasing. Moreover,  $q(1, R) = 0 < R(c_l c_{\phi})^{\frac{1}{2}}(1)$  and  $\lim_{t\to\infty} R(c_l c_{\phi})^{\frac{1}{2}}(t) = 0$ . By the intermediate value theorem, (2.8) has a unique solution T(R).

If  $\liminf_{R\to\infty} T(R) < \infty$ , choose a sequence  $(R_n)_{n\in\mathbb{N}}$  such that  $(T(R_n))_{n\in\mathbb{N}}$  is bounded. Using the crude estimate

$$g(t,R) \le t \log \left( R(d_l d_{\phi})(1) \right) + \sqrt{2t} + t \lesssim t \log R,$$

this leads to the contradiction

$$R(c_l c_{\phi})^{\frac{1}{2}}(T(R_n)) = q(T(R_n), R) \lesssim T(R_n) \cdot \log R \ll R \asymp R(c_l c_{\phi})^{\frac{1}{2}}(T(R_n)).$$

In many situations, we are going to first determine T(R), and then show that L(T(R), R) is small. The bound (2.7) is then asymptotically equal to  $R(c_l c_{\phi})^{\frac{1}{2}}(T(R))$ .

Before we go into the proof of Theorem 2.2, let us discuss a particular situation.

**2.5 Corollary.** Let  $(l_j)_{j=1}^{\infty}$  be a summable sequence of positive numbers, and  $(\phi_j)_{j=1}^{\infty}$  a sequence of real numbers. Denote by H the Hamburger Hamiltonian with these lengths and angles, and let  $W_H$  be its monodromy matrix.

Assume that we have  $\delta_l, \delta_{\phi}, \gamma_l, \gamma_{\phi} \ge 0$ , such that

$$\begin{aligned} \forall j \in \mathbb{N}: \ l_j \lesssim j^{-\delta_l} & \wedge \ |\sin(\phi_{j+1} - \phi_j)| \lesssim j^{-\delta_{\phi}}, \\ \forall N \in \mathbb{N}: \ \sum_{j=N+1}^{\infty} l_j \lesssim N^{-\gamma_l} & \wedge \ \sum_{j=N+1}^{\infty} l_j \sin^2(\phi_j - \psi) \lesssim N^{-\gamma_{\phi}}, \end{aligned}$$

i.e., we are in the situation of Theorem 2.2 with

$$d_l(t) = ct^{-\delta_l}, \ d_{\phi}(t) = ct^{-\delta_{\phi}}, \ c_l(t) = ct^{-\gamma_l}, \ c_{\phi}(t) = ct^{-\gamma_{\phi}},$$

where c is some positive constant. Set  $\delta := \delta_l + \delta_{\phi}$ ,  $\gamma := \frac{1}{2}(\gamma_l + \gamma_{\phi})$ , and assume that  $\delta > 0$  and  $\gamma > 0$ . Then we have, for sufficiently large R, the following bounds for T(R) (cf. Lemma 2.4),  $\log \max_{|z|=R} ||W_H(z)||$  and the order  $\rho_H$  of  $W_H$ .

Data satisf	les		$T(R) \asymp$	$\log \max_{ z =R} \ W_H(z)\  \lesssim$	$\rho_H \leq$
$\delta < 1 + \gamma$			$\left(\frac{R}{\log R}\right)^{\frac{1}{1+\gamma}}$	$R^{\frac{1}{1+\gamma}}(\log R)^{\frac{\gamma}{1+\gamma}}$	$\frac{1}{1+\gamma}$
$\delta = 1 + \gamma$			$R^{\frac{1}{\delta}}$	$R^{rac{1}{\delta}}$	$\frac{1}{\delta}$
$1+\gamma < \delta,$	$\delta > 2,$		$R^{\frac{\delta-1}{\gamma\delta}}$	$R^{rac{1}{\delta}}$	$\frac{1}{\delta}$
	$\delta = 2,$		$\left(\frac{R^{\frac{1}{2}}}{\log R}\right)^{\frac{1}{\gamma}}$	$R^{\frac{1}{2}}\log R$	$\frac{1}{2}$
	$\delta < 2,$	$\delta_l \leq 1 + \gamma$	$R^{\frac{1}{2-\delta+2\gamma}}$	$R^{\frac{2-\delta+\gamma}{2-\delta+2\gamma}}$	$\tfrac{2-\delta+\gamma}{2-\delta+2\gamma}$
		$1+\gamma < \delta_l$	$R^{\frac{\delta_l - 1}{\gamma(\delta_l - \delta_\phi)}}$	$R^{\frac{1-\delta_\phi}{\delta_l-\delta_\phi}}$	$\frac{1-\delta_\phi}{\delta_l-\delta_\phi}$

All these formulae are checked by elementary calculations which we do not elaborate here; the necessary computations can be found in the preprint version of this article [PRW23].

Corollary 2.5 already shows that our present results may improve drastically upon our previous work [PRW17]. The main feature that enables this is the use of  $c_{\phi}$ : the present results exploit convergence of angles more efficiently.

2.6 Example. Let  $\alpha > 1$  and  $\beta \ge 0$ , and let H be the Hamburger Hamiltonian with lengths and angles

$$l_j := j^{-\alpha}, \quad \phi_j := (-1)^j j^{-\beta}.$$

Considering the expressions (2.4) and (2.5), we see that Corollary 2.5 can be applied with

$$\delta_l := \alpha, \ \delta_\phi := \beta, \ \gamma_l := \alpha - 1, \ \gamma_\phi := \alpha + 2\beta - 1.$$

Since  $\delta = \alpha + \beta$  and  $\gamma = \alpha + \beta - 1$ , this yields

$$\log\left(\max|z| = R \|W_H(z)\|\right) \lesssim R^{\frac{1}{\alpha+\beta}}$$

and hence  $\rho_H \leq \frac{1}{\alpha + \beta}$ . Now recall [PRW17, Example 2.23], where we saw that

the order  $\rho_H$  is at least  $\frac{1}{\alpha+\beta}$ . Thus,  $\rho_H = \frac{1}{\alpha+\beta}$ . Let us compare this with what we can obtain from [PRW17]. For  $\alpha+\beta \geq 2$ , the formula  $\rho_H = \frac{1}{\alpha+\beta}$  was already established in [PRW17, Example 2.23]. For  $\alpha + \beta < 2$ , we had obtained the upper bound  $\frac{1-\beta}{\alpha-\beta}$  in that place. However, this did not take into account that angles converge. If we want to take convergence into account that angles converge. 2 of The mention of the mentio into account, we should use [PRW17, Corollary 2.9]. The quantities used there identify as

$$\Delta_l^+ = \alpha, \quad \Delta_\phi^* = \Lambda^* = \beta_l$$

and the bound for order so obtained thus is  $\frac{1-\frac{\beta}{2}}{\alpha}$ . If  $\beta > 0$ , this lies strictly between  $\frac{1}{\alpha+\beta}$  and  $\frac{1-\beta}{\alpha-\beta}$ , since

$$\frac{1-\beta}{\alpha-\beta} - \frac{1-\frac{\beta}{2}}{\alpha} = \frac{\frac{\beta}{2}(2-\alpha-\beta)}{(\alpha-\beta)\alpha} > 0,$$
$$\frac{1-\frac{\beta}{2}}{\alpha} - \frac{1}{\alpha+\beta} = \frac{\frac{\beta}{2}(2-\alpha-\beta)}{\alpha(\alpha+\beta)} > 0.$$

#### 2.1Proof of Theorem 2.2

Denote by W(s,t;z) the unique solution of the initial value problem

$$\left\{ \begin{array}{ll} \frac{\partial}{\partial t}W(s,t;z)J=zW(s,t;z)H(t),\qquad t\in(0,L) \text{ a.e.},\\ W(s,s;z)=I. \end{array} \right.$$

and observe the multiplicativity property

$$W(s,t;z)W(t,u;z) = W(s,u;z).$$

Our method for the proof of Theorem 2.2 is to fix  $t \ge 1$  and estimate  $||W(0, x_{\lceil t \rceil}; z)||$  and  $||W(x_{\lceil t \rceil}, L; z)||$  separately. The contribution of the first  $\lceil t \rceil$  intervals is estimated using the explicit form of the fundamental solution on indivisible intervals. On the remaining part, we use Grönwall's Lemma. This method is a refined version of an idea from [Rom17] and its improved variant from [PW22].

Set  $W_j(z) := W(x_{j-1}, x_j; z)$ , then by a direct computation

$$W_j(z) = I + z l_j \xi_{\phi_j} \xi_{\phi_j}^T J$$

Moreover, multiplicativity yields

$$W(0, x_N; z) = W_1(z) \cdot \ldots \cdot W_N(z), \qquad N \in \mathbb{N}.$$
(2.9)

We often use the matrices introduced in [PW22, Definition 4.4]: for a > 0 and  $\psi \in \mathbb{R}$ , denote

$$\Omega(a,\psi) := \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \exp(-\psi J) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix}.$$

Recall the following properties from [PW22, Lemma 4.6].

**2.7 Lemma.** Let a, b > 0 and  $\phi, \psi \in \mathbb{R}$ .

- (i)  $\|\Omega(a,\psi)\| = \|\Omega(a,\psi)^{-1}\| = \max\{a,a^{-1}\};$
- (ii)  $\|\Omega(a,\psi)\xi_{\phi}\xi_{\phi}^{T}J\Omega(a,\psi)^{-1}\| = a^{2}\cos^{2}(\phi-\psi) + \frac{1}{a^{2}}\sin^{2}(\phi-\psi);$
- (iii)  $\|\Omega(a,\psi)\Omega(b,\phi)^{-1}\| \le \max\left\{\frac{a}{b},\frac{b}{a}\right\} |\cos(\phi-\psi)| + \max\left\{ab,\frac{1}{ab}\right\} |\sin(\phi-\psi)|.$

We use  $\Omega(a_j, \psi_j)$  to rotate and dilate  $W_j(z)$  such that the norm becomes small. In the following lemma, we make the obvious choice  $\psi_j = \phi_j$ , but retain the freedom of choosing suitable parameters  $a_j$ .

**2.8 Lemma.** Let  $N \in \mathbb{N}$  and let numbers  $a_j \in (0,1]$ ,  $j = 1, \ldots, N$ , be given. Then, for each  $z \in \mathbb{C}$ , we have the estimate

$$\|W(0, x_N; z)\| \le \frac{1}{a_1 a_N} \cdot \prod_{j=1}^N \left(1 + |z| l_j a_j^2\right)$$

$$+ \prod_{j=1}^{N-1} \left( \max\left\{\frac{a_j}{a_{j+1}}, \frac{a_{j+1}}{a_j}\right\} \cdot \left|\cos\left(\phi_j - \phi_{j+1}\right)\right| + \frac{|\sin(\phi_j - \phi_{j+1})|}{a_j a_{j+1}} \right).$$

$$(2.10)$$

*Proof.* With  $\Omega_j := \Omega(a_j, \phi_j)$  we have

$$\Omega_j W_j(z) \Omega_j^{-1} = I + z l_j \Omega_j \xi_{\phi_j} \xi_{\phi_j}^T J \Omega_j^{-1},$$

and we can use Lemma 2.7, (ii), to estimate

$$\|\Omega_j W_j(z)\Omega_j^{-1}\| \le 1 + |z|l_j a_j^2.$$

Expanding on (2.9),

$$W(0, x_N; z) = \Omega_1^{-1} \cdot (\Omega_1 W_1(z) \Omega_1^{-1}) \cdot \Omega_1 \Omega_2^{-1} \cdot \ldots \cdot (\Omega_N W_N(z) \Omega_N^{-1}) \cdot \Omega_N.$$

Using submultiplicativity of the norm and Lemma 2.7, (i), (iii), we arrive at the desired estimate.

The next lemma is the ingredient of Theorem 2.2 that accounts for the term q(t, R) in (2.7).

**2.9 Lemma.** Let  $(l_j)_{j=1}^{\infty}$  be a summable sequence of positive numbers, and  $(\phi_j)_{j=1}^{\infty}$  a sequence of real numbers. Let H be the Hamburger Hamiltonian with these lengths and angles. Let  $d_l: [1, \infty) \to (0, \infty)$  and  $d_{\phi}: [1, \infty) \to (0, 1]$  be measurable and nonincreasing, and assume that

$$\forall j \in \mathbb{N}: \ l_j \le d_l(j) \land \ |\sin(\phi_{j+1} - \phi_j)| \le d_{\phi}(j).$$

Then, for every  $(N, R) \in \mathbb{N} \times \left[\frac{2}{(d_l d_{\phi})(1)}, \infty\right)$ ,

$$\log\left(\max_{|z|=R} \|W(0,x_N;z)\|\right) \le \frac{7}{2} \cdot \left[g(N,R) + \lambda\left(\min\{N,\hbar(R)\},R\right)\right]$$

where

$$\lambda(t,R) := 1 + \log^+ R + \log^+ \frac{d_l(1)}{d_\phi(1)} + \log^+ \frac{d_l(\lfloor t \rfloor)}{d_\phi(\lfloor t \rfloor)} + \sum_{j=1}^{\lfloor t \rfloor - 1} \bigg| \log \frac{d_\phi(j)d_l(j+1)}{d_l(j)d_\phi(j+1)} \bigg|$$

*Proof.* Fix z with |z| = R. In the first step we estimate  $\log ||W(0, x_N; z)||$  when  $N \leq \hbar(R)$ . This is done by an application of Lemma 2.8 with

$$a_j := \left(\frac{d_{\phi}(j)}{Rd_l(j)}\right)^{\frac{1}{4}}, \qquad j = 1, \dots, N.$$

Since  $N \leq \hbar(R)$  we have  $a_j \in (0, 1]$ . The factors in (2.10) are treated separately:

$$\begin{split} \frac{1}{a_1 a_N} &= R^{\frac{1}{2}} \cdot \left(\frac{d_l(1)}{d_{\phi}(1)}\right)^{\frac{1}{4}} \cdot \left(\frac{d_l(N)}{d_{\phi}(N)}\right)^{\frac{1}{4}},\\ 1 + |z| l_j a_j^2 &\leq 1 + R d_l(j) a_j^2 = 1 + \left(R(d_l d_{\phi})(j)\right)^{\frac{1}{2}},\\ \max\left\{\frac{a_j}{a_{j+1}}, \frac{a_{j+1}}{a_j}\right\} \left|\cos\left(\psi_j - \psi_{j+1}\right)\right| + \frac{|\sin(\psi_j - \psi_{j+1})|}{a_j a_{j+1}}\\ &= \max\left\{\frac{a_j}{a_{j+1}}, \frac{a_{j+1}}{a_j}\right\} \cdot \left(|\cos(\phi_{j+1} - \phi_j)| + \frac{|\sin(\phi_{j+1} - \phi_j)|}{\max\{a_j^2, a_{j+1}^2\}}\right) \end{split}$$

$$\leq \max\left\{\frac{a_j}{a_{j+1}}, \frac{a_{j+1}}{a_j}\right\} \cdot \left(1 + \frac{d_{\phi}(j)}{a_j^2}\right) \\ = \left[\max\left\{\frac{d_{\phi}(j)d_l(j+1)}{d_l(j)d_{\phi}(j+1)}, \frac{d_l(j)d_{\phi}(j+1)}{d_{\phi}(j)d_l(j+1)}\right\}\right]^{\frac{1}{4}} \cdot \left(1 + \left(R(d_ld_{\phi})(j)\right)^{\frac{1}{2}}\right).$$

Now (2.10) yields

$$\log \|W(0, x_N; z)\| \le \frac{1}{2} \log^+ R + \frac{1}{4} \log^+ \frac{d_l(1)}{d_{\phi}(1)} + \frac{1}{4} \log^+ \frac{d_l(N)}{d_{\phi}(N)} + \sum_{j=1}^N \log \left( 1 + \left( R(d_l d_{\phi})(j) \right)^{\frac{1}{2}} \right)$$

$$+\frac{1}{4}\sum_{j=1}^{N-1} \left|\log\frac{d_{\phi}(j)d_{l}(j+1)}{d_{l}(j)d_{\phi}(j+1)}\right| + \sum_{j=1}^{N-1}\log\left(1 + \left(R(d_{l}d_{\phi})(j)\right)^{\frac{1}{2}}\right).$$

Since  $d_l, d_\phi$  are nonincreasing, we can estimate sums by integrals as

$$\begin{split} \sum_{j=2}^{N} \log \left( 1 + \left( R(d_{l}d_{\phi})(j) \right)^{\frac{1}{2}} \right) &\leq \int_{1}^{N} \log \left( 1 + \left( R(d_{l}d_{\phi})(s) \right)^{\frac{1}{2}} \right) \mathrm{d}s \\ &\leq \frac{3}{2} \cdot \int_{1}^{\min\{N, \hbar(R)\}} \log \left( R(d_{l}d_{\phi})(s) \right) \mathrm{d}s + \int_{\min\{N, \hbar(R)\}}^{N} \left( R(d_{l}d_{\phi})(s) \right)^{\frac{1}{2}} \mathrm{d}s. \end{split}$$

Noting that

$$\begin{split} \log \left( 1 + \left( R(d_l d_{\phi})(1) \right)^{\frac{1}{2}} \right) &\leq \frac{3}{2} \Big( \log^+ R + \log^+ (d_l d_{\phi})(1) \Big) \\ &\leq \frac{3}{2} \Big( \log^+ R + \log^+ \frac{d_l(1)}{d_{\phi}(1)} \Big), \end{split}$$

leads to

$$\log \|W(0, x_N; z)\| \leq \frac{7}{2} \log^+ R + \frac{13}{4} \log^+ \frac{d_l(1)}{d_{\phi}(1)} + \frac{1}{4} \log^+ \frac{d_l(N)}{d_{\phi}(N)} \\ + \frac{1}{4} \sum_{j=1}^{N-1} \left| \log \frac{d_{\phi}(j) d_l(j+1)}{d_l(j) d_{\phi}(j+1)} \right| \\ \sup_{\substack{\min\{N, \& (R)\}\\1}} \left| \log \left( R(d_l d_{\phi})(s) \right) ds + 2 \cdot \int_{\min\{N, \& (R)\}}^{N} \left( R(d_l d_{\phi})(s) \right)^{\frac{1}{2}} ds.$$
(2.11)

We see that

$$\log \|W(0, x_N; z)\| \leq \frac{7}{2} \cdot \left[g(N, R) + \lambda(N, R)\right].$$

Consider now the case that  $N > \hbar(R)$ . Since  $\|\xi_{\phi_j}\xi_{\phi_j}^T J\| = 1$ , we see that

$$\|W(x_{\lfloor \hbar(R) \rfloor}, x_N; z)\| = \left\| \prod_{j=\lfloor \hbar(R) \rfloor+1}^N W(x_{j-1}, x_j; z) \right\|$$
$$\leq \prod_{j=\lfloor \hbar(R) \rfloor+1}^N \left\| I + z l_j \xi_{\phi_j} \xi_{\phi_j}^T J \right\| \leq \prod_{j=\lfloor \hbar(R) \rfloor+1}^N \left( 1 + R d_l(j) \right).$$

Noting that

$$Rd_l(\lfloor \hbar(R) \rfloor + 1) < 1,$$

and estimating the sum by an integral we arrive at

$$\log \|W(x_{\lfloor \hbar(R) \rfloor}, x_N; z)\| \leq \sum_{j=\lfloor \hbar(R) \rfloor + 1}^N \log \left(1 + Rd_l(j)\right)$$

$$< \log 2 + \int_{\lfloor \hbar(R) \rfloor + 1}^{N} R d_l(s) \,\mathrm{d}s.$$

We combine this with (2.11) to obtain

$$\log \|W(0, x_N; z)\| \leq \log \|W(0, x_{\lfloor \hbar(R) \rfloor}; z)\| + \log \|W(x_{\lfloor \hbar(R) \rfloor}, x_N; z)\|$$
  
$$\leq \frac{7}{2} \cdot \left[ g(N, R) + \lambda(\hbar(R), R) \right].$$

The second lemma accounts for the contribution of the remaining intervals.

**2.10 Lemma.** Let  $(l_j)_{j=1}^{\infty}$  be a summable sequence of positive numbers, and  $(\phi_j)_{j=1}^{\infty}$  a sequence of real numbers. Denote by H the Hamburger Hamiltonian with these lengths and angles, and let  $W_H$  be its monodromy matrix.

Let  $c_l, c_{\phi} \colon [1, \infty) \to (0, \infty)$  be two functions with  $c_{\phi} \leq c_l$ . Choose  $\psi \in \mathbb{R}$ and assume that

$$\forall N \in \mathbb{N}: \sum_{j=N+1}^{\infty} l_j \le c_l(N) \land \sum_{j=N+1}^{\infty} l_j \sin^2(\phi_j - \psi) \le c_{\phi}(N).$$

Then, for any  $N \in \mathbb{N}$  and  $z \in \mathbb{C}$ ,

$$||W(x_N, L; z)|| \le \left(\frac{c_l(N)}{c_{\phi}(N)}\right)^{\frac{1}{2}} \exp\left(2|z|(c_l c_{\phi})^{\frac{1}{2}}(N)\right).$$

Proof. Let  $\Omega \in GL(2, \mathbb{R})$  and consider the function  $\tilde{W}(t) := \Omega W(x_N, t; z) \Omega^{-1}$ defined on  $[x_N, L]$ . This function satisfies  $\tilde{W}(x_N) = I$  and

$$\tilde{W}'(t) = -z\tilde{W}(t)\Omega H(t)J\Omega^{-1}, \qquad t \in [x_N, L].$$

By Grönwall's Lemma,

$$\|\tilde{W}(t)\| \le \exp\left(|z| \int_{x_N}^t \|\Omega H(s) J \Omega^{-1}\| \,\mathrm{d}s\right).$$

Set  $a := \left(\frac{c_{\phi}(N)}{c_l(N)}\right)^{\frac{1}{4}}$  and  $\Omega = \Omega(a, \psi)$ . Using Lemma 2.7, (i), (ii), we obtain

$$\begin{split} \|W(x_N, L; z)\| &\leq \|\Omega^{-1}\| \|\tilde{W}(L)\| \|\Omega\| \leq \frac{1}{a^2} \exp\left(|z| \int_{x_N}^L \|\Omega H(s) J \Omega^{-1}\| \, \mathrm{d}s\right) \\ &= \left(\frac{c_l(N)}{c_{\phi}(N)}\right)^{\frac{1}{2}} \exp\left(|z| \sum_{j=N+1}^{\infty} l_j \cdot \|\Omega \xi_{\phi_j} \xi_{\phi_j}^T J \Omega^{-1}\|\right) \\ &= \left(\frac{c_l(N)}{c_{\phi}(N)}\right)^{\frac{1}{2}} \exp\left(|z| \sum_{j=N+1}^{\infty} l_j [a^2 \cos^2(\phi_j - \psi) + \frac{1}{a^2} \sin^2(\phi_j - \psi)]\right) \\ &\leq \left(\frac{c_l(N)}{c_{\phi}(N)}\right)^{\frac{1}{2}} \exp\left(|z| \left[a^2 \sum_{j=N+1}^{\infty} l_j + \frac{1}{a^2} \sum_{j=N+1}^{\infty} l_j \sin^2(\phi_j - \psi)\right]\right) \\ &\leq \left(\frac{c_l(N)}{c_{\phi}(N)}\right)^{\frac{1}{2}} \exp\left(2|z| (c_l c_{\phi})^{\frac{1}{2}}(N)\right) \end{split}$$

The proof of Theorem 2.2 is now easily completed.

Proof of Theorem 2.2. Let  $(t, R) \in [1, \infty) \times \left[\frac{2}{(d_l d_{\phi})(1)}\right)$  and set  $N := \lceil t \rceil$ . We use Lemma 2.9 and Lemma 2.10 to estimate, for |z| = R,

$$\begin{split} \log \|W_{H}(z)\| &\leq \log \|W(0, x_{N}; z)\| + \log \|W(x_{N}, L; z)\| \\ &\leq \frac{7}{2} \cdot \left[g(N, R) + \lambda \left(\min\{N, \hbar(R)\}, R\right)\right] \\ &+ \left[\frac{1}{2} \log \frac{c_{l}(N)}{c_{\phi}(N)} + 2R (c_{l}c_{\phi})^{\frac{1}{2}}(N)\right] \\ &\leq \frac{7}{2} \cdot \left[g(t, R) + R (c_{l}c_{\phi})^{\frac{1}{2}}(t)\right] + \frac{7}{2} \cdot \left|g(N, R) - g(t, R)\right| \\ &+ \frac{7}{2} \cdot \lambda \left(\min\{N, \hbar(R)\}, R\right) + \frac{1}{2} \log \frac{c_{l}(N)}{c_{\phi}(N)}. \end{split}$$

Since  $0 \le N - t \le 1$  and

$$g(s,R) \leq \begin{cases} \frac{3}{2} \left( \log^+ R + \log^+ \frac{d_l(1)}{d_{\phi}(1)} \right) & \text{if } 1 \leq s < \hbar(R), \\ \sqrt{2} & \text{if } \hbar(R) \leq s < \hbar(R), \\ 1 & \text{if } \hbar(R) \leq s, \end{cases}$$
$$\leq \frac{3}{2} \cdot \lambda \Big( \min\{N, \hbar(R)\}, R \Big), \end{cases}$$

we have

$$\left|g(N,R)-g(t,R)\right| \leq \frac{3}{2} \cdot \lambda (\min\{N,\hbar(R)\},R),$$

and obtain

$$\log \|W_H(z)\| \le 9 \cdot \left[ \max \left\{ g(t, R), R(c_l c_{\phi})^{\frac{1}{2}}(t) \right\} + L(t, R) \right].$$

Since t was arbitrary, we can pass to the infimum on the right side.

## 3 Properties of the upper bound

In this section we study the expression on the right side of (2.7) independently of its meaning in the context of Theorem 2.2.

**3.1 Definition.** Let  $d_l: [1,\infty) \to (0,\infty)$  and  $d_\phi: [1,\infty) \to (0,1]$  be measurable with  $d_l \simeq 1 \simeq d_\phi$  locally. Then we denote

$$B(R) := \inf_{t \ge 1} \left( \max\left\{ q\left(t, R\right), R(c_l c_{\phi})^{\frac{1}{2}}(t) \right\} + L(t, R) \right).$$
(3.1)

We consider the questions

- $\triangleright$  Is L(t, R) small compared to max  $\{q(t, R), R(c_l c_{\phi})^{\frac{1}{2}}(t)\}$ ?
- $\triangleright \text{ Does } \inf_{t\geq 1} \max\left\{ q\left(t,R\right), R(c_l c_{\phi})^{\frac{1}{2}}(t) \right\} \text{ depend monotonically on the data } d_l, d_{\phi}, c_l, c_{\phi} ?$

It turns out that already very weak assumptions on the data, concerning monotonicity, continuity, variation, and power boundedness, are sufficient to ensure that the answers are affirmative. This tells us that usually L(t, R) can be dropped from (3.1), and hence B(R) can be evaluated much more easily, and that usually the natural intuition which stems from the context of Theorem 2.2, that faster decay means smaller growth, is indeed reflected in B(R).

#### 3.1 About monotonicity in the data

The essence is the following result.

**3.2 Proposition.** Let  $d_l$ ,  $\hat{d}_l$ :  $[1, \infty) \to (0, \infty)$  and  $d_{\phi}$ ,  $\hat{d}_{\phi}$ :  $[1, \infty) \to (0, 1]$  be measurable with  $d_l \simeq d_{\phi} \simeq \hat{d}_l \simeq \hat{d}_{\phi} \simeq 1$  locally, and assume that  $\hat{d}_l \hat{d}_{\phi}$  is  $\simeq$  to some nonincreasing function. If  $\hat{d}_l \lesssim d_l$  and  $\hat{d}_{\phi} \lesssim d_{\phi}$  on  $[1, \infty)$ , then (quantities  $\hat{q}(t, R)$  etc. are defined correspondingly for  $\hat{d}_l$ ,  $\hat{d}_{\phi}$ )

$$\widehat{g}(t,R) \lesssim g(t,R) \tag{3.2}$$

for  $R \geq \frac{2}{(d_l d_{\phi})(1)}$  and a.e. (t, R) with  $t < \widehat{\hbar}(R)$  or  $t > \hbar(R)$ . If additionally  $\frac{\widehat{d}_{\phi}}{\widehat{d}_l}$  is bounded or  $\asymp$  on  $[1, \infty)$  to a nondecreasing function, then (3.2) holds for R sufficiently large and a.e.  $t \in [1, \infty)$ .

The constant that is implicit in the relation (3.2) depends on the constants implicit in the assumptions, but not on the actual functions  $d_l, d_{\phi}, \hat{d}_l, \hat{d}_{\phi}$ .

We start with a preparatory lemma.

**3.3 Lemma.** Let  $\Phi, \Psi: [1, \infty) \to (0, \infty)$  with  $\Phi$  nondecreasing, and assume that  $\alpha, \alpha' > 0$  are such that

$$\forall t \in [1,\infty): \ \alpha \cdot \Phi(t) \le \Psi(t) \le \alpha' \cdot \Phi(t)$$

Set

$$t_0 := \sup\left(\left\{t \in [1,\infty) \mid \sup_{1 \le s \le t} \Psi(s) \le 1\right\} \cup \{1\}\right) \in [1,\infty].$$

Then, if  $t_0 < \infty$ ,

$$\forall t \in (t_0, \infty): \ \Psi(t) > \frac{\alpha}{\alpha'}$$

*Proof.* Since  $\Phi$  is nondecreasing, we have

$$\begin{split} \left\{t \in [1,\infty) \mid \, \alpha'\Phi(t) \leq 1\right\} \cup \left\{1\right\} &= \left\{t \in [1,\infty) \mid \, \sup_{1 \leq s \leq t} \alpha'\Phi(s) \leq 1\right\} \cup \{1\}\\ &\subseteq \left\{t \in [1,\infty) \mid \, \sup_{1 \leq s \leq t} \Psi(s) \leq 1\right\} \cup \{1\}. \end{split}$$

Set

$$t_1 := \sup\left(\left\{t \in [1,\infty) \mid \alpha'\Phi(t) \le 1\right\} \cup \{1\}\right) \in [1,\infty],$$

then  $t_1 \leq t_0$ . Again by monotonicity,  $\alpha' \Phi(t) > 1$  for all  $t > t_1$ . and this yields that for  $t > t_1$ 

$$\Psi(t) \ge \alpha \Phi(t) > \frac{\alpha}{\alpha'}.$$

Proof of Proposition 3.2. We have to trace constants, and thus make the constants from the assumptions explicit: choose a nondecreasing function u and  $\alpha, \alpha' > 0$ , such that

$$lpha u(t) \leq rac{2}{(\widehat{d}_l \widehat{d}_{\phi})(t)} \leq lpha' u(t), \qquad t \in [1,\infty),$$

and choose  $\kappa_l, \kappa_\phi > 0$ , such that

$$\widehat{d}_l(t) \le \kappa_l d_l(t), \quad \widehat{d}_{\phi}(t) \le \kappa_{\phi} d_{\phi}(t), \qquad t \in [1, \infty).$$

Moreover, set

$$\lambda := \max_{t \ge 2} \frac{\log t}{\sqrt{t}},$$

and note that

$$x \le \gamma y \implies \log x \le \left(1 + \frac{\log^+ \gamma}{\log 2}\right) \cdot \log y \quad \text{for } x > 0, y \ge 2, \gamma > 0.$$

Next, observe that

$$\left[R(\hat{d}_l\hat{d}_{\phi})(t)\right]^{\frac{1}{2}} < \left(\frac{2\alpha'}{\alpha}\right)^{\frac{1}{2}}, \qquad t > \hat{k}(R).$$
(3.3)

In order to see this, let R be fixed and apply Lemma 3.3 with  $\Phi(t) = \frac{u(t)}{R}$  and  $\Psi(t) = \frac{2}{R(\hat{d}_{t}\hat{d}_{\phi})(t)}$ . Then  $t_{0} = \hat{k}(R)$ , and hence for  $t > \hat{k}(R)$  we have  $\Psi(t) > \frac{\alpha}{\alpha'}$ , which implies (3.3).

In order to establish that  $\widehat{g}(t,R) \leq g(t,R)$  we first assume  $t < \widehat{\hbar}(R)$  and distinguish the following six cases according to the definitions of  $\widehat{g}(t,R)$  and g(t,R).

 $>> \ 1 \leq t < \widehat{k}(R) \ \land \ 1 \leq t < k(R):$ 

$$\log\left[R(\widehat{d}_{l}\widehat{d}_{\phi})(t)\right] \leq \left(1 + \frac{\log^{+}(\kappa_{l}\kappa_{\phi})}{\log 2}\right)\log\left[R(d_{l}d_{\phi})(t)\right].$$

 $\vartriangleright \ 1 \leq t < \widehat{k}(R) \ \land \ k(R) < t < \hbar(R) :$ 

$$\log\left[R(\widehat{d}_{l}\widehat{d}_{\phi})(t)\right] \leq \lambda\left[R(\widehat{d}_{l}\widehat{d}_{\phi})(t)\right]^{\frac{1}{2}} \leq \lambda(\kappa_{l}\kappa_{\phi})^{\frac{1}{2}}\left[R(d_{l}d_{\phi})(t)\right]^{\frac{1}{2}}.$$

 $\vartriangleright 1 \leq t < \widehat{k}(R) \land \ \hbar(R) < t :$ 

$$\log \left[ R(\hat{d}_l \hat{d}_{\phi})(t) \right] \leq \lambda \left[ R(\hat{d}_l \hat{d}_{\phi})(t) \right]^{\frac{1}{2}} \leq \lambda \left[ R(\hat{d}_l \hat{d}_{\phi})(t) \right]$$
$$\leq \lambda \left[ R \hat{d}_l(t) \right] \leq \lambda \kappa_l \left[ R d_l(t) \right].$$

 $\vartriangleright \ \widehat{k}(R) < t < \widehat{k}(R) \ \land \ 1 \leq t < k(R). \ \text{By (3.3)},$ 

$$\left[R(\widehat{d}_l\widehat{d}_{\phi})(t)\right]^{\frac{1}{2}} < \left(\frac{2\alpha'}{\alpha}\right)^{\frac{1}{2}} \le \frac{1}{\log 2} \left(\frac{2\alpha'}{\alpha}\right)^{\frac{1}{2}} \log\left[R(d_ld_{\phi})(t)\right].$$

$$\begin{split} & \triangleright \triangleright \ \widehat{k}(R) < t < \widehat{\hbar}(R) \ \land \ \widehat{\kappa}(R) < t < \hbar(R): \\ & \left[ R(\widehat{d}_l \widehat{d}_{\phi})(t) \right]^{\frac{1}{2}} \leq (\kappa_l \kappa_{\phi})^{\frac{1}{2}} \Big[ R(d_l d_{\phi})(t) \Big]^{\frac{1}{2}} \\ & \triangleright \ \widehat{k}(R) < t < \widehat{\hbar}(R) \ \land \ \hbar(R) < t: \\ & \left[ R(\widehat{d}_l \widehat{d}_{\phi})(t) \right]^{\frac{1}{2}} \leq R \widehat{d}_l(t) \leq \kappa_l R d_l(t). \end{split}$$

Since (t, R) is assumed to satisfy  $t < \hat{\hbar}(R) \lor \hbar(R) < t$ , and  $t < \hat{\hbar}(R)$  was already covered, the only case left to treat is

$$\begin{split} & \triangleright \triangleright \ \widehat{\hbar}(R) < t \ \land \ \hbar(R) < t : \\ & R \widehat{d}_l(t) \leq \kappa_l R d_l(t). \end{split}$$

Now we assume the additional condition on  $\frac{\hat{d}_{\phi}}{\hat{d}_{l}}$ . If this quotient is bounded, then  $\hat{\hbar}(R) = \infty$  for sufficiently large R, and there is nothing to show. Otherwise, choose a nondecreasing function v and  $\beta, \beta' > 0$  such that

$$\beta v(t) \le \frac{\hat{d}_{\phi}}{\hat{d}_l} \le \beta' v(t), \qquad t \in [1, \infty)$$

We claim that

$$R\widehat{d}_{l}(t) \leq \frac{\beta'}{\beta}\widehat{d}_{\phi}(t), \qquad t > \widehat{\hbar}(R).$$
(3.4)

In fact, if we fix R and apply Lemma 3.3 with  $\Phi(t) = \frac{v(t)}{R}$  and  $\Psi(t) = \frac{\hat{d}_{\phi}(t)}{R\hat{d}_{l}(t)}$ ,

we find that  $t_0 = \hat{\hbar}(R)$  and  $\Psi(t) > \frac{\beta}{\beta'}$  for  $t > \hat{\hbar}(R)$ . This is equivalent to (3.4). With these preparations, the assertion follows from the following simple inequalities:

$$\begin{split} & \triangleright \widehat{\hbar}(R) < t \ \land \ 1 \leq t < \hbar(R) : \\ & R \widehat{d}_l(t) \leq \frac{\beta'}{\beta} \widehat{d}_{\phi}(t) \leq \frac{\beta'}{\beta} \leq \frac{\beta'}{\beta \log 2} \log \left[ R(d_l d_{\phi})(t) \right]. \end{split}$$

 ${\rm de} \ \widehat{\hbar}(R) < t \ \land \ \hbar(R) < t < \hbar(R) \colon$ 

$$R\widehat{d}_{l}(t) \leq \frac{\beta'}{\beta}\widehat{d}_{\phi}(t) \leq \frac{\beta'}{\beta}\kappa_{\phi}d_{\phi}(t) \leq \frac{\beta'}{\beta}\kappa_{\phi}\left[R(d_{l}d_{\phi})(t)\right]^{\frac{1}{2}}.$$

**3.4 Corollary.** Let  $d_l, d_{\phi}, \hat{d}_l, \hat{d}_{\phi}$  be as in Proposition 3.2 and subject to the additional condition of this proposition. Further, let  $c_l, c_{\phi}, \hat{c}_l, \hat{c}_{\phi} \colon [1, \infty) \to (0, \infty)$ . If

$$\widehat{d}_l \lesssim d_l, \; \widehat{d}_\phi \lesssim d_\phi, \; \widehat{c}_l \lesssim c_l, \; \widehat{c}_\phi \lesssim c_\phi$$

on  $[1,\infty)$ , then

$$\inf_{t\geq 1} \max\left\{\widehat{g}(t,R), R(\widehat{c}_l \widehat{c}_{\phi})^{\frac{1}{2}}(t)\right\} \lesssim \inf_{t\geq 1} \max\left\{g(t,R), R(c_l c_{\phi})^{\frac{1}{2}}(t)\right\}$$

for sufficiently large R. The constant that is implicit in the assertion depends on the constants implicit in the assumptions, but not on the actual functions  $d_l, \hat{d_l}$  etc.

#### 3.2About smallness of L(t, R)

To start with, let us observe that q(t, R) must grow at least logarithmically.

**3.5 Lemma.** Let  $d_l: [1, \infty) \to (0, \infty)$  and  $d_{\phi}: [1, \infty) \to (0, 1]$  be measurable with  $d_l \simeq 1 \simeq d_{\phi}$  locally. Then, for each  $\varepsilon > 0$ , there exists a positive constant c > 0 such that, for  $t \ge 1 + \varepsilon$  and  $R \ge \sup_{1 \le s \le 1 + \varepsilon} \frac{2}{(d_l d_{\phi})(s)}$  we have

 $q(t,R) \ge c \log R.$ 

*Proof.* Let  $\eta := \sup_{1 \le s \le 1+\varepsilon} \frac{2}{(d_l d_{\phi})(s)}$ . Then  $R \ge \eta$  is equivalent to  $1+\varepsilon \le \Re(R)$ . Hence

$$g(t,R) \ge \int_{1}^{1+\varepsilon} \log \left( R(d_{l}d_{\phi})(s) \right) ds$$
$$= \left( \varepsilon + \frac{1}{\log R} \int_{1}^{1+\varepsilon} \log(d_{l}d_{\phi})(s) ds \right) \cdot \log R.$$

The term in the bracket tends to  $\varepsilon$  for  $R \to \infty$ , and hence we find  $R_0 \ge \eta$  such that  $g(t,R) \geq \frac{\varepsilon}{2} \cdot \log R$  for all  $t \geq 1 + \varepsilon$  and  $R \geq R_0$ . The function g is nonzero, and hence bounded away from zero, on the compact set  $\{1 + \varepsilon\} \times [\eta, R_0]$ . By monotonicity, it is therefore also bounded away from zero on  $[1 + \varepsilon, \infty) \times [\eta, R_0]$ . Together, we see that a constant c > 0 can be chosen as required. 

The most cumbersome part of L(t, R) is the sum written as the last of the six summands in (2.6). Assuming monotonicity of the quotient  $\frac{d_{\phi}}{d_{\ell}}$ , this sum turns into a telescoping sum and can be estimated.

**3.6 Lemma.** Let  $d_l, d_{\phi}: [1, \infty) \to (0, \infty)$  and let  $\hbar(R)$  be as in (2.2). Assume that the quotient  $\frac{d_{\phi}}{d_l}$  is eventually monotone but not eventually constant. Then

$$\sum_{j=1}^{\min\{\lceil t\rceil, \lfloor\hbar(R)\rfloor\}-1} \left|\log\left(\frac{d_{\phi}(j)}{d_{l}(j)} \middle/ \frac{d_{\phi}(j+1)}{d_{l}(j+1)}\right)\right| \asymp 1 + \left|\log\frac{d_{\phi}(\min\{\lceil t\rceil, \lfloor\hbar(R)\rfloor\})}{d_{l}(\min\{\lceil t\rceil, \lfloor\hbar(R)\rfloor\})}\right|$$

for  $(t, R) \in [1, \infty) \times \left[\frac{d_{\phi}(1)}{d_l(1)}, \infty\right)$ .

*Proof.* We abbreviate  $T := \min\{\lceil t \rceil, \lfloor \hbar(R) \rfloor\}$ . Choose  $M \in \mathbb{N}$  such that  $\frac{d_{\phi}}{d_l}$  is monotone on  $[M, \infty)$  and  $\frac{d_{\phi}(M)}{d_l(M)} \neq \frac{d_{\phi}(M+1)}{d_l(M+1)}$ . For all (t, R) such that  $T \geq M+1$ we have

$$\begin{split} \sum_{j=M}^{T-1} \left| \log \left( \frac{d_{\phi}(j)}{d_l(j)} \middle/ \frac{d_{\phi}(j+1)}{d_l(j+1)} \right) \right| &= \left| \sum_{j=M}^{T-1} \log \left( \frac{d_{\phi}(j)}{d_l(j)} \middle/ \frac{d_{\phi}(j+1)}{d_l(j+1)} \right) \right| \\ &= \left| \log \frac{d_{\phi}(M)}{d_l(M)} - \log \frac{d_{\phi}(T)}{d_l(T)} \right|. \end{split}$$

Clearly, this value is nondecreasing in T and positive for  $T \ge M + 1$ . If  $\left| \log \frac{d_{\phi}(T)}{d_{l}(T)} \right|$  is bounded, it is  $\asymp 1$ . If  $\left| \log \frac{d_{\phi}(T)}{d_{l}(T)} \right|$  is unbounded, it is  $\asymp \left| \log \frac{d_{\phi}(T)}{d_{l}(T)} \right|$ . The beginning part of the sum which we cut off, i.e.,

$$\sum_{j=1}^{M-1} \left| \log \left( \frac{d_{\phi}(j)}{d_l(j)} \middle/ \frac{d_{\phi}(j+1)}{d_l(j+1)} \right) \right|$$

is some nonnegative number independent of t and R.

**3.7 Corollary.** Let  $d_l, d_{\phi}: [1, \infty) \to (0, \infty)$  and let  $\hbar(R)$  be as in (2.2). Assume that one of the following two assumptions holds.

- (i) The quotient  $\frac{d_{\phi}}{d_l}$  is eventually nondecreasing and  $\frac{d_{\phi}}{d_l}(t) \lesssim \sup_{1 \le s < t} \frac{d_{\phi}}{d_l}(s)$  for sufficiently large t.
- (ii) The quotient  $\frac{d_{\phi}}{d_l}$  is eventually nonincreasing and power bounded from below (i.e. there exists  $\alpha > 0$  such that  $\frac{d_{\phi}(t)}{d_l(t)} \gtrsim t^{-\alpha}$  for sufficiently large t).

Then

$$\sum_{j=1}^{\min\{\lceil t\rceil,\lfloor\hbar(R)\rfloor\}-1} \left|\log\left(\frac{d_{\phi}(j)}{d_{l}(j)} \middle/ \frac{d_{\phi}(j+1)}{d_{l}(j+1)}\right)\right| \lesssim \begin{cases} 1 + \log^{+} R & \text{if (i) holds,} \\ 1 + \log t & \text{if (ii) holds,} \end{cases}$$

for  $(t, R) \in [1, \infty) \times \left[\frac{d_{\phi}(1)}{d_l(1)}, \infty\right)$ .

*Proof.* Again abbreviate  $T := \min\{\lceil t \rceil, \lfloor \hbar(R) \rfloor\}$ , and let  $M \in \mathbb{N}$  be such that  $\frac{d_{\phi}}{d_I}$  is monotone on  $[M, \infty)$ . Under the assumption (i) we have

$$rac{d_{\phi}(T)}{d_l(T)} \leq rac{d_{\phi}(\hbar(R))}{d_l(\hbar(R))} \lesssim R_{*}$$

and hence

$$\left|\log \frac{d_{\phi}(T)}{d_l(T)}\right| \lesssim 1 + \log R.$$

Under the assumption (ii) we have

$$t^{-lpha} \lesssim T^{-lpha} \lesssim rac{d_{\phi}(T)}{d_l(T)} \lesssim 1,$$

and hence

$$\left|\log \frac{d_{\phi}(T)}{d_l(T)}\right| \lesssim 1 + \log t.$$

Another type of condition on the quotient  $\frac{d_{\phi}}{d_l}$  that ensures that the sum can be estimated, is that its variation is not too fast.

**3.8 Lemma.** Let  $d_l, d_{\phi}: [1, \infty) \to (0, \infty)$  and let  $\hbar(R)$  be as in (2.2). Assume that the quotient  $\frac{d_{\phi}}{d_l}$  can be represented as

$$\frac{d_{\phi}(t)}{d_{l}(t)} = c(t) \cdot \exp\left(\int_{1}^{t} \epsilon(u) \frac{\mathrm{d}u}{u}\right), \qquad t \in [1, \infty),$$

where  $\epsilon: [1,\infty) \to \mathbb{R}$  is locally integrable and eventually bounded, and where  $c: [1,\infty) \to (0,\infty)$  is eventually constant. Then

$$\sum_{j=1}^{\min\{\lceil t\rceil, \lfloor \hbar(R) \rfloor\}-1} \left| \log\left(\frac{d_{\phi}(j)}{d_l(j)} \middle/ \frac{d_{\phi}(j+1)}{d_l(j+1)}\right) \right| \lesssim 1 + \log t$$

for  $(t, R) \in [1, \infty) \times \left[\frac{d_{\phi}(1)}{d_l(1)}, \infty\right)$ .

m

*Proof.* We again abbreviate  $T := \min\{\lceil t \rceil, \lfloor \hbar(R) \rfloor\}$ . For  $M \in \mathbb{N}$  such that  $\epsilon(t)$  is bounded and c(t) is constant on  $[M, \infty)$ , we can estimate, for all (t, R) such that T > M,

$$\sum_{j=M}^{T-1} \left| \log \left( \frac{d_{\phi}(j)}{d_l(j)} \middle/ \frac{d_{\phi}(j+1)}{d_l(j+1)} \right) \right| = \sum_{j=M}^{T-1} \left| \int_j^{j+1} \epsilon(u) \frac{\mathrm{d}u}{u} \right|$$
$$\leq \sum_{j=M}^{T-1} \int_j^{j+1} |\epsilon(u)| \frac{\mathrm{d}u}{u} = \int_M^T |\epsilon(u)| \frac{\mathrm{d}u}{u} \lesssim \log t.$$

Now we give three sets of conditions, each of which ensures that L(t, R) can be neglected.

**3.9 Proposition.** Let  $d_l: [1, \infty) \to (0, \infty)$  and  $d_{\phi}: [1, \infty) \to (0, 1]$  be measurable with  $d_l \simeq 1 \simeq d_{\phi}$  locally, let  $c_l, c_{\phi}$  be continuous and nonincreasing with  $\lim_{t \to \infty} (c_l c_{\phi})(t) = 0$ , and let T(R) be the unique solution of (2.8). Assume that one of the following three sets of assumptions holds.

- (i) The quotient  $\frac{d_{\phi}}{d_l}$  is eventually nondecreasing and  $\frac{d_{\phi}}{d_l}(t) \lesssim \sup_{1 \leq s < t} \frac{d_{\phi}}{d_l}(s)$ for sufficiently large t. We have  $c_{\phi}(t) \lesssim c_{\phi}(t+1)$  for sufficiently large t.
- (ii) The quotient  $\frac{d_{\phi}}{d_l}$  is eventually nonincreasing and power bounded from below. There exists  $\alpha > 0$  such that  $(c_l c_{\phi})(t) \lesssim t^{-\alpha}$  for sufficiently large t. The quotient  $\frac{c_l}{c_{\phi}}$  is power bounded from above or  $c_{\phi}(t) \lesssim c_{\phi}(t+1)$  for sufficiently large t.
- (iii) The quotient  $\frac{d_{\phi}}{d_l}$  can be represented as

$$\frac{d_{\phi}(t)}{d_{l}(t)} = c(t) \cdot \exp\left(\int_{1}^{t} \epsilon(u) \frac{\mathrm{d}u}{u}\right), \qquad t \in [1, \infty),$$

where  $\epsilon \colon [1,\infty) \to \mathbb{R}$  is locally integrable and eventually bounded, and where  $c \colon [1,\infty) \to (0,\infty)$  is eventually constant. There exists  $\alpha > 0$  such that  $(c_l c_{\phi})(t) \lesssim t^{-\alpha}$  for sufficiently large t. The quotient  $\frac{c_l}{c_{\phi}}$  is power bounded from above or  $c_{\phi}(t) \lesssim c_{\phi}(t+1)$  for sufficiently large t.

Let B(R) be as in (3.1). Then  $L(T(R), R) \leq 1 + \log^+ R$ , and

$$B(R) \asymp \mathfrak{g}(T(R), R) = R(\mathfrak{c}_l \mathfrak{c}_\phi)^{\frac{1}{2}}(T(R)).$$

*Proof.* Assume (i). The first sentence of (i) is nothing but the assumption (i) in Corollary 3.7, and hence the sum in the definition of L(t, R) is  $\leq 1 + \log^+ R$ independently of t. The last but one summand in (2.6) is  $\leq 1$  by monotonicity of  $\frac{d_{\phi}}{d_{i}}$ . It remains to estimate the third summand in (2.6), and here we use the second sentence in the present assumption: using  $\lim_{R\to\infty} T(R) = \infty$  and Lemma 3.5,

$$\log^{+} \frac{c_{l}(\lceil T(R) \rceil)}{c_{\phi}(\lceil T(R) \rceil)} \lesssim 1 + \log^{+} \frac{c_{l}(T(R))}{c_{\phi}(T(R))} \lesssim 1 + \log^{+} \frac{1}{(c_{l}c_{\phi})(T(R))}$$

$$= 1 + \log^{+} \frac{R^{2}}{q(T(R), R)^{2}} \lesssim 1 + \log^{+} R.$$
(3.5)

Assume (ii). Let  $\alpha > 0$  be as in the second sentence of (ii), then  $R(c_l c_{\phi})^{\frac{1}{2}}(R^{\frac{2}{\alpha}}) \lesssim 1 \ll q(R^{\frac{2}{\alpha}}, R)$ , and hence  $T(R) \lesssim R^{\frac{2}{\alpha}}$ . The first sentence of (ii) is nothing but assumption (ii) in Corollary 3.7. This yields that the sum in the definition of L(t, R) is  $\leq 1 + \log^+ t$ , and hence the sum in L(T(R), R) is  $\lesssim 1 + \log^+ R$ . The same holds for the last but one summand in (2.6) by power boundedness of  $\frac{d_{\phi}}{d_l}$ , and for the third summand (2.6) in case that  $\frac{c_l}{c_{\phi}}$  is power bounded. If  $c_{\phi}(t) \lesssim c_{\phi}(t+1)$  for sufficiently large t, then the third summand in (2.6) is  $\leq 1 + \log^+ R$  by (3.5).

Assume (iii). The second sentence again ensures that T(R) is power bounded from above. The first sentence is the assumption of Lemma 3.8, and it follows that the sum in L(T(R), R) is  $\leq 1 + \log^+ R$ . The form of  $\frac{d_{\phi}}{d_l}$  implies that this quotient is power bounded from below, and hence the last but one summand (2.6) is  $\lesssim 1 + \log^+ R$ . Concerning the third summand, just argue as above. 

To finish the proof it remains to refer to Lemma 3.5.

#### Regularly varying decay 4

In the preceding section, we studied properties of the function B(R), in particular, we saw that the term L(t, R) is usually neglectable. Our next goal is to explicitly evaluate

$$\min_{t\geq 1}\left(\max\left\{g(t,R),R(c_lc_{\phi})^{\frac{1}{2}}(t)\right\}\right)$$

in the situation that  $d_l, d_{\phi}, c_l, c_{\phi}$  are all regularly varying, cf. Theorem 4.1. As in the preceding section, this is a pure analysis of functions, and independent of the meaning in the context of Theorem 2.2. However, after having completed the proof of Theorem 4.1, we will return to the study of Hamburger Hamiltonians and combine Theorem 4.1 with Theorem 2.2. This establishes an explicit upper bound for the growth of the monodromy matrix, cf. Theorem 4.6.

The setup for Theorem 4.1 is as follows:

- (i) We are given regularly varying functions  $d_l, d_{\phi}, c_l, c_{\phi} \colon [1, \infty) \to (0, \infty)$ with nonpositive indices,  $d_{\phi} \leq 1$ , and  $d_l \approx 1 \approx d_{\phi}$  locally.
- (ii) We denote

$$\delta_l := -\operatorname{ind} d_l, \quad \delta_\phi := -\operatorname{ind} d_\phi, \quad \gamma_l := -\operatorname{ind} c_l, \quad \gamma_\phi := -\operatorname{ind} c_\phi,$$

and further

$$\mathfrak{D}(t) := \frac{1}{(d_l d_{\phi})(t)}, \quad \delta := \operatorname{ind} \mathfrak{D} = \delta_l + \delta_{\phi}, \tag{4.1}$$

$$\mathscr{C}(t) := \frac{1}{(c_l c_{\phi})^{\frac{1}{2}}(t)}, \quad \gamma := \operatorname{ind} \mathscr{C} = \frac{\gamma_l + \gamma_{\phi}}{2}.$$
(4.2)

(iii) The quantities of interest are

$$\mathfrak{B}(t,R) := \max\left\{ \mathfrak{g}(t,R), \frac{R}{\mathfrak{C}(t)} \right\},$$
  
$$\mathfrak{B}(R) := \min_{t \ge 1} \mathfrak{B}(t,R).$$
  
(4.3)

Four fundamentally different cases occur. They depend on the relative size of  $\mathfrak{D}(t)$  vs.  $\mathscr{C}(t)$ , and on the absolute size of  $\mathfrak{D}(t)$ . Moreover, there are a few exceptional cases that have to be ruled out.

**4.1 Theorem.** Assume we are given data as in (i) above, and let notation be as in (ii) and (iii). Assume further that  $\delta > 0$ , that  $\mathfrak{C}$  is  $\asymp$  on  $[1,\infty)$  to a nondecreasing function, and that  $\lim_{t\to\infty} \mathfrak{C}(t) = \infty$ .

Then the following estimates for  $\mathfrak{B}(R)$  hold for sufficiently large R.

A Case  $\mathfrak{D}(t) \lesssim t \mathscr{C}(t)$ :

Choose  $\alpha \geq 4 \cdot \sup_{t \geq 1} \frac{\mathfrak{D}(t)}{t\mathfrak{C}(t)}$ , and set  $f(t) := t\mathfrak{C}(t) \log \left[\alpha \frac{t\mathfrak{C}(t)}{\mathfrak{D}(t)}\right]$ . Then an

asymptotic inverse of f exists, and with  $\tau(R) := f^-(\frac{1}{\alpha}R)$  we have

$$\mathfrak{B}(R) \asymp \mathfrak{B}\left(\tau(R), R\right) \asymp \frac{R}{\mathscr{C}(\boldsymbol{f}^{-}(R))}.$$
(4.4)

The function on the right side is regularly varying with index  $\frac{1}{1+\gamma}$ .

If  $\delta < 1 + \gamma$  we have  $f(t) \simeq t \mathfrak{C}(t) \log t$ , and  $\mathfrak{B}(R)$  is (up to  $\simeq$ ) independent of  $d_l$  and  $d_{\phi}$ .

B Case  $t\mathscr{C}(t) \lesssim \mathfrak{D}(t)$ ,  $\int_{1}^{\infty} \mathfrak{D}(s)^{-\frac{1}{2}} ds < \infty$ , and if  $(\delta_l, \delta_{\phi}, \gamma) = (1, 1, 0)$  then  $\frac{d_{\phi}}{d_l}$  is bounded or  $\asymp$  to a monotone function: Then

 $\hbar(R) \lesssim \mathfrak{B}(R) \lesssim R^{\frac{1}{2}} \int_{\hbar(R)}^{\infty} \mathfrak{D}(s)^{-\frac{1}{2}} \,\mathrm{d}s.$ (4.5)

Both bounds in (4.5) are regularly varying with index  $\frac{1}{\delta}$ .

If  $\delta > 2$ , then  $\approx$  holds throughout (4.5) and  $\mathfrak{B}(R)$  is (up to  $\approx$ ) independent of  $c_l$  and  $c_{\phi}$ .

C Case 
$$\frac{1}{d_l(t)} \lesssim t \mathscr{C}(t) \lesssim \mathfrak{D}(t), \int_{1}^{\infty} \mathfrak{D}(s)^{-\frac{1}{2}} ds = \infty, \text{ and } (\delta, \gamma) \neq (2, 0)$$
:

Set  $f_0(t) := \frac{t^2 \mathfrak{C}(t)^2}{\mathfrak{D}(t)}$  and  $f_1(t) := \left(\mathfrak{C}(t) \int_1^t \mathfrak{D}(s)^{-\frac{1}{2}} ds\right)^2$ . Then asymptotic inverses  $f_0^-, f_1^-$  exist, and with

$$\tau(R) \mathrel{\mathop:}= \min\big\{\max\big\{f_1^-(R), \hbar(R)\big\}, \hbar(R)\big\}$$

we have

$$\frac{R}{\mathscr{C}(\boldsymbol{f}_0^-(R))} \lesssim \mathfrak{B}(R) \lesssim \mathfrak{B}(\tau(R), R) \lesssim \frac{R}{\mathscr{C}(\boldsymbol{f}_1^-(R))}.$$
(4.6)

Both bounds in (4.6) are regularly varying with index  $\frac{2-\delta+\gamma}{2-\delta+2\gamma}$ . If  $\delta < 2$ , then  $\asymp$  holds throughout (4.6).

$$\begin{array}{|c|c|c|c|c|} \hline \textbf{D} & \text{Case } t \mathscr{C}(t) \lesssim \frac{1}{d_l(t)}, \ \int\limits_1^\infty \mathfrak{D}(s)^{-\frac{1}{2}} \, \mathrm{d}s = \infty, \ \text{and} \ \int_1^\infty d_l(s) \, \mathrm{d}s < \infty \\ & \text{Then } \frac{d_\phi}{d_l} \ is \ unbounded \ and \end{array}$$

$$R\hbar(R)d_l(\hbar(R)) \lesssim \mathfrak{B}(R) \lesssim R^{\frac{1}{2}} \int_{1}^{\hbar(R)} \mathfrak{D}(s)^{-\frac{1}{2}} \,\mathrm{d}s + R \int_{\hbar(R)}^{\infty} d_l(s) \,\mathrm{d}s. \tag{4.7}$$

If  $\delta_l > \delta_{\phi}$ , then both bounds in (4.7) are regularly varying with index  $\frac{1-\delta_{\phi}}{\delta_l-\delta_{\phi}}$ . If  $\delta < 2$  and  $\delta_l > 1$ , then  $\approx$  holds throughout (4.7).

## 4.1 Proof of Theorem 4.1

We start with two lemmata that determine the function g(t, R) on the intervals  $[1, \mathcal{K}(R)]$  and  $[\mathcal{K}(R), \mathcal{K}(R)]$ , respectively.

**4.2 Lemma.** Let  $\mathfrak{D}: [1,\infty) \to (0,\infty)$  be regularly varying with  $\operatorname{ind} \mathfrak{D} > 0$  and  $\mathfrak{D} \simeq 1$  locally. Set

$$\hbar(R) := \sup\left\{t \in [1,\infty) \mid \sup_{1 \le s \le t} \frac{2\mathfrak{D}(s)}{R} \le 1\right\}, \qquad R \ge 2\mathfrak{D}(1).$$

$$(4.8)$$

Then we have

$$\int_{1}^{t} \log \frac{R}{\mathfrak{D}(s)} \, \mathrm{d}s \asymp t \log \frac{R}{\mathfrak{D}(t)} \tag{4.9}$$

for  $R \in [2\mathfrak{D}(1), \infty)$  and  $t \in \left[2 \log \frac{R}{\mathfrak{D}(1)}, \mathfrak{K}(R)\right]$ .

*Proof.* In the first part of the proof we establish the assertion under the additional assumption that  $\mathfrak{D}$  is continuously differentiable with  $\mathfrak{D}'(t) > 0$  for all  $t \geq 1$ . Let  $R \in [2\mathfrak{D}(1), \infty)$  and  $t \in [2\log \frac{R}{\mathfrak{D}(1)}, \mathfrak{K}(R)]$ . Integration by parts yields that

$$\int_{1}^{t} \log \frac{R}{\mathfrak{D}(s)} \, \mathrm{d}s = t \log \frac{R}{\mathfrak{D}(t)} - \log \frac{R}{\mathfrak{D}(1)} + \int_{1}^{t} s \frac{\mathfrak{D}'(s)}{\mathfrak{D}(s)} \, \mathrm{d}s.$$
(4.10)

The relation " $\gtrsim$ " in (4.9) readily follows: The integral on the right side is positive, and by the definition of  $\Re(R)$  we have  $\frac{R}{\Im(t)} \ge 2$ . For  $t \ge 2 \log \frac{R}{\Im(1)}$  it follows that

$$\int_{1}^{t} \log \frac{R}{\mathfrak{D}(s)} \, \mathrm{d}s \ge \left(\frac{1}{2} \cdot t \log \frac{R}{\mathfrak{D}(t)} + \frac{1}{2} \cdot 2 \log \frac{R}{\mathfrak{D}(1)} \cdot \log 2\right) - \log \frac{R}{\mathfrak{D}(1)}$$
$$\ge \frac{1}{2} \cdot t \log \frac{R}{\mathfrak{D}(t)}.$$

To show that " $\leq$ " holds in (4.9) we make a change of variable and use Karamata's theorem. The inverse function  $\mathfrak{D}^{-1}$  exists and is continuously differentiable with positive derivative, and is regularly varying with positive index. We obtain that, for  $v \to \infty$ ,

$$\int_{1}^{v} s \frac{\mathfrak{D}'(s)}{\mathfrak{D}(s)} \, \mathrm{d}s = \int_{\mathfrak{D}(1)}^{\mathfrak{D}(v)} \mathfrak{D}^{-1}(u) \frac{\mathrm{d}u}{u} \sim (\mathrm{ind}\,\mathfrak{D}) \cdot v.$$

As a function of  $v \in [2 \log 2, \infty)$ , both v and the integral above are continuous and nonzero. Hence

$$\int_{1}^{v} s \frac{\mathfrak{D}'(s)}{\mathfrak{D}(s)} \, \mathrm{d}s \asymp v, \qquad v \in [2\log 2, \infty)$$

Referring to (4.10) and again using that  $\frac{R}{\mathfrak{D}(t)} \geq 2$ , we obtain

$$\int_{1}^{t} \log \frac{R}{\mathfrak{D}(s)} \, \mathrm{d}s \leq t \log \frac{R}{\mathfrak{D}(t)} + \int_{1}^{t} s \frac{\mathfrak{D}'(s)}{\mathfrak{D}(s)} \, \mathrm{d}s \lesssim t \log \frac{R}{\mathfrak{D}(t)}.$$

The second part of the proof is to reduce to the above settled special case. Hence, assume that  $\mathfrak{D}$  is given as in the statement of the lemma. Lemma A.9 gives a regularly varying function  $\mathfrak{D}_1$  which is continuously differentiable with  $\mathfrak{D}'_1 > 0$ , such that  $\mathfrak{D} \simeq \mathfrak{D}_1$  on  $[1, \infty)$ . Dividing  $\mathfrak{D}_1$  by a sufficiently large constant, we obtain a function  $\mathfrak{D}$  with the properties listed above and

$$\frac{1}{\alpha}\mathfrak{D}(t) \le \tilde{\mathfrak{D}}(t) \le \mathfrak{D}(t) \quad \text{for } t \ge 1$$

for some  $\alpha \geq 1$ .

Clearly,  $\hbar(R) \leq \tilde{\hbar}(R)$  (where  $\tilde{\hbar}(R)$  is defined by (4.8) with  $\tilde{\mathfrak{D}}$  in place of  $\mathfrak{D}$ ). It follows that for  $t < \hbar(R)$ 

$$\log \frac{R}{\mathfrak{D}(t)} \le \log \frac{R}{\mathfrak{\tilde{D}}(t)} \le \log \left( \alpha \frac{R}{\mathfrak{D}(t)} \right) \le \log \frac{R}{\mathfrak{D}(t)} + \log \alpha \le \left( 1 + \frac{\log \alpha}{\log 2} \right) \log \frac{R}{\mathfrak{D}(t)}.$$

We see that

$$\int_{1}^{t} \log \frac{R}{\mathfrak{D}(s)} \, \mathrm{d}s \asymp \int_{1}^{t} \log \frac{R}{\mathfrak{D}(s)} \, \mathrm{d}s \asymp t \log \frac{R}{\mathfrak{D}(t)} \asymp t \log \frac{R}{\mathfrak{D}(t)}.$$

**4.3 Lemma.** Let  $\mathfrak{D}: [1,\infty) \to (0,\infty)$  be regularly varying with  $\operatorname{ind} \mathfrak{D} > 0$  and  $\mathfrak{D} \asymp 1$  locally. Then we have

$$R^{\frac{1}{2}}t\mathfrak{D}(t)^{-\frac{1}{2}} \lesssim \mathfrak{k}(R) + R^{\frac{1}{2}} \int_{\mathfrak{k}(R)}^{t} \mathfrak{D}(s)^{-\frac{1}{2}}(s) \,\mathrm{d}s \lesssim R^{\frac{1}{2}} \int_{1}^{t} \mathfrak{D}(s)^{-\frac{1}{2}} \,\mathrm{d}s$$

for  $R \in [2\mathfrak{D}(1), \infty)$  and  $t \in [\mathfrak{K}(R), \infty)$ .

*Proof.* The asserted estimate from above is easy to see: by Karamata's theorem we have

$$\mathfrak{k}(R) \asymp R^{\frac{1}{2}} \mathfrak{k}(R) \mathfrak{D}(\mathfrak{k}(R))^{-\frac{1}{2}} \lesssim R^{\frac{1}{2}} \int_{1}^{\mathfrak{k}(R)} \mathfrak{D}(s)^{-\frac{1}{2}} \, \mathrm{d}s.$$

To obtain an estimate from below, we use Lemma A.9. This gives a regularly varying function  $\mathfrak{D}_1$  which is continuously differentiable with  $\mathfrak{D}'_1 > 0$  and satisfies  $\mathfrak{D} \simeq \mathfrak{D}_1$ . By multiplying  $\mathfrak{D}_1$  with a sufficiently large constant, we obtain a function  $\mathfrak{D}$  with the properties listed above and

$$\mathfrak{D}(t) \leq \mathfrak{D}(t) \leq \alpha \mathfrak{D}(t)$$

with some  $\alpha \geq 1$ .

Consider the functions

$$f_1(t) := R^{\frac{1}{2}} t \tilde{\mathfrak{D}}(t)^{-\frac{1}{2}}, \quad f_2(t) := \sqrt{2} \, k(R) + R^{\frac{1}{2}} \int_{k(R)}^t \tilde{\mathfrak{D}}(s)^{-\frac{1}{2}} \, \mathrm{d}s$$

Then

$$\begin{split} \boldsymbol{f}_1(\boldsymbol{k}(R)) &= R^{\frac{1}{2}} \boldsymbol{k}(R) \tilde{\boldsymbol{\mathfrak{D}}}(\boldsymbol{k}(R))^{-\frac{1}{2}} \\ &\leq R^{\frac{1}{2}} \boldsymbol{k}(R) \boldsymbol{\mathfrak{D}}(\boldsymbol{k}(R))^{-\frac{1}{2}} = \sqrt{2} \, \boldsymbol{k}(R) = \boldsymbol{f}_2(\boldsymbol{k}(R)), \end{split}$$

and, for  $t > \mathcal{K}(R)$ ,

$$f_{1}'(t) = R^{\frac{1}{2}} \tilde{\mathfrak{D}}(t)^{-\frac{1}{2}} + R^{\frac{1}{2}} t \underbrace{\frac{\mathrm{d}}{\mathrm{d}t} \left( \tilde{\mathfrak{D}}(t)^{-\frac{1}{2}} \right)}_{<0} \le R^{\frac{1}{2}} \tilde{\mathfrak{D}}(t)^{-\frac{1}{2}} = f_{2}'(t).$$

We see that

$$f_1(t) \le f_2(t) \quad \text{for } t \ge k(R).$$

It remains to note that

$$f_1(t) \gtrsim R^{\frac{1}{2}} t \mathfrak{D}(t)^{-\frac{1}{2}}, \quad f_2(t) \lesssim \mathfrak{K}(R) + R^{\frac{1}{2}} \int_{\mathfrak{K}(R)}^t \mathfrak{D}(s)^{-\frac{1}{2}} \mathrm{d}s.$$

**4.4 Lemma.** Consider regularly varying functions  $d_l, d_{\phi}, \mathfrak{D}$  as introduced in the discussion preceding Theorem 4.1. Suppose that  $\int_1^{\infty} \mathfrak{D}(s)^{-\frac{1}{2}} ds < \infty$ . If  $\delta_{\phi} = \delta_l = 1$ , we also assume that  $\frac{d_{\phi}}{d_l}$  is bounded or  $\asymp$  to a nondecreasing function. Then for sufficiently large R we have, independently of t, that

$$g(t,R) \lesssim R^{\frac{1}{2}} \int_{\Re(R)}^{\infty} \mathfrak{D}(s)^{-\frac{1}{2}} \mathrm{d}s.$$

Proof. We distinguish cases.

 $\triangleright$  Assume that  $\frac{d_{\phi}}{d_l}$  is bounded, i.e.,  $\hbar(R) = \infty$  for all sufficiently large R. This is surely the case if  $\delta_l < \delta_{\phi}$ .

By Karamata's theorem

$$\mathfrak{k}(R) = \sqrt{2} R^{\frac{1}{2}} \mathfrak{k}(R) \mathfrak{D}(\mathfrak{k}(R))^{-\frac{1}{2}} \lesssim R^{\frac{1}{2}} \int_{\mathfrak{k}(R)}^{\infty} \mathfrak{D}(s)^{-\frac{1}{2}} \mathrm{d}s.$$

Now it follows that, for all  $t \geq \mathcal{K}(R)$ ,

$$g(t,R) \asymp \Re(R) + R^{\frac{1}{2}} \int_{\Re(R)}^{t} \mathfrak{D}(s)^{-\frac{1}{2}} \, \mathrm{d}s \lesssim R^{\frac{1}{2}} \int_{\Re(R)}^{\infty} \mathfrak{D}(s)^{-\frac{1}{2}} \, \mathrm{d}s.$$

▷ Assume that  $\frac{d_{\phi}}{d_l}$  is  $\approx$  to a nondecreasing function. This is surely the case if  $\delta_l > \delta_{\phi}$ .

If R is large enough and  $t > \hbar(R)$ ,

$$Rd_l(t) = \left(\frac{Rd_l(t)}{d_{\phi}(t)}\right)^{\frac{1}{2}} \cdot \left[R(d_l d_{\phi})(t)\right]^{\frac{1}{2}} \lesssim \left[R(d_l d_{\phi})(t)\right]^{\frac{1}{2}}.$$

We see that also in this case,

$$\begin{split} g(t,R) \asymp & \hbar(R) + R^{\frac{1}{2}} \int_{\Re(R)}^{\min\{t,\hbar(R)\}} \mathfrak{D}(s)^{-\frac{1}{2}} \, \mathrm{d}s + R \int_{\min\{t,\hbar(R)\}}^{t} d_{l}(s) \, \mathrm{d}s \\ \lesssim & R^{\frac{1}{2}} \int_{\Re(R)}^{\infty} \mathfrak{D}(s)^{-\frac{1}{2}} \, \mathrm{d}s. \end{split}$$

 $\triangleright$  Assume that  $\delta_l = \delta_{\phi} > 1$ .

By Karamata's theorem,

$$R\int_{\hbar(R)}^{\infty} d_l(s) \, \mathrm{d}s \asymp R\hbar(R) d_l(\hbar(R)) \asymp \hbar(R) d_{\phi}(\hbar(R)).$$

Since  $\delta_{\phi} > 1$ , the function  $t \mapsto t d_{\phi}(t)$  is eventually decreasing. Since it is locally bounded, it is even bounded on  $[1, \infty)$ . Again we see that  $g(t, R) \lesssim R^{\frac{1}{2}} \int_{\Re(R)}^{\infty} \mathfrak{D}(s)^{-\frac{1}{2}} ds$  independently of t.

**4.5 Lemma.** In the situation of Theorem 4.1, the inequality

$$\min\left\{g(s,R),\frac{R}{\mathscr{C}(s)}\right\} \lesssim \mathfrak{B}(R) \le \mathfrak{B}(t,R) = \max\left\{g(t,R),\frac{R}{\mathscr{C}(t)}\right\}$$
(4.11)

holds for  $(s,t,R) \in [1,\infty)^2 \times [\frac{2}{(d_l d_{\phi})(1)},\infty)$ . In particular, if  $\tau(R)$  is a function satisfying  $q(\tau(R),R) \asymp \frac{R}{\mathscr{C}(\tau(R))}$ , then

$$\mathfrak{B}(R) \asymp \mathfrak{g}(\tau(R), R) \asymp \frac{R}{\mathscr{C}(\tau(R))}.$$

*Proof.* The upper bound in (4.11) is obvious from the definition of  $\mathfrak{B}(t, R)$  and  $\mathfrak{B}(R)$ . For the lower bound, we observe that

$$\begin{array}{ll} s \leq t & \Rightarrow & g(s,R) \leq g(t,R), \\ s > t & \Rightarrow & \displaystyle \frac{R}{\mathscr{C}(s)} \lesssim \displaystyle \frac{R}{\mathscr{C}(t)} \end{array}$$

and hence

$$\min\left\{q(s,R),\frac{R}{\mathscr{C}(s)}\right\} \lesssim \mathscr{B}(t,R) \qquad (s,t) \in [1,\infty)^2.$$

Taking the infimum over t proves the lower bound in (4.11).

Now we are ready for the proof of Theorem 4.1.

Proof of Case A. We have ind  $f = 1 + \gamma > 0$ , and hence  $f^-$  exists. Set

$$\tau(R) := \ell^- \left(\frac{1}{\alpha}R\right).$$

Then, for  $R \to \infty$ ,

$$\frac{1}{\alpha}R \sim f(\tau(R)) = \tau(R)\mathscr{C}(\tau(R))\log\Big[\alpha\frac{\tau(R)\mathscr{C}(\tau(R))}{\mathfrak{D}(\tau(R))}\Big],$$

and hence

$$\frac{R}{\mathfrak{D}(\tau(R))} \sim \Big(\alpha \frac{\tau(R) \mathscr{C}(\tau(R))}{\mathfrak{D}(\tau(R))} \Big) \log \Big[ \alpha \frac{\tau(R) \mathscr{C}(\tau(R))}{\mathfrak{D}(\tau(R))} \Big].$$

Our choice of  $\alpha$  implies that  $2\mathfrak{D}(\tau(R)) \leq R$  for all sufficiently large values of R, and therefore that  $\tau(R) \leq \mathfrak{k}(R)$  for such R.

It holds that

$$\frac{t}{\log t} \circ t \log t \asymp t \quad \text{for } t \ge 4,$$

and we further obtain

$$\frac{\frac{R}{\mathfrak{D}(\tau(R))}}{\log\left(\frac{R}{\mathfrak{D}(\tau(R))}\right)} \asymp \frac{\tau(R)\mathfrak{C}(\tau(R))}{\mathfrak{D}(\tau(R))}.$$

Using this and Lemma 4.2, thus

$$\frac{R}{\mathscr{C}(\tau(R))} \asymp \tau(R) \log \left(\frac{R}{\mathfrak{D}(\tau(R))}\right) \asymp g(\tau(R), R).$$

In view of Lemma 4.5, this completes the proof of (4.4). To see the additional statements, note first that

$$\operatorname{ind} \frac{R}{\mathscr{C}(\boldsymbol{\ell}^{-}(R))} = 1 - \gamma \cdot \frac{1}{1 + \gamma} = \frac{1}{1 + \gamma}.$$

Further  $\delta < 1 + \gamma$  implies  $\operatorname{ind} \frac{t^{\mathfrak{C}(t)}}{\mathfrak{D}(t)} > 0$ , and thus  $\log \left(\alpha \frac{t^{\mathfrak{C}(t)}}{\mathfrak{D}(t)}\right) \asymp \log t$ .

*Proof of Case* B. To obtain the bound from below, we use Lemma 4.2 and our assumption that  $t\mathscr{C}(t) \leq \mathfrak{D}(t)$ . Setting  $t = \mathfrak{K}(R) \asymp \mathfrak{D}^-(R)$ , we find that

$$q(k(R),R) \asymp k(R), \qquad \frac{R}{\mathscr{C}(k(R))} \asymp \frac{R}{\mathscr{C}(\mathfrak{D}^-(R))} \gtrsim \mathfrak{D}^-(R) \asymp k(R).$$

Now we prove the upper bound for  $\mathfrak{B}(R)$ . If Lemma 4.4 is applicable, then

$$g(t,R) \lesssim R^{\frac{1}{2}} \int_{\Re(R)}^{\infty} \mathfrak{D}(s)^{-\frac{1}{2}} \mathrm{d}s.$$

Choosing  $\tau(R)$  so large that  $\frac{R}{\mathscr{C}(\tau(R))}$  is less than the right hand side of this relation, we see that

$$\mathfrak{B}(R) \leq \mathfrak{B}(\tau(R), R) \lesssim R^{\frac{1}{2}} \int_{\mathfrak{K}(R)}^{\infty} \mathfrak{D}(s)^{-\frac{1}{2}} \, \mathrm{d}s.$$

If Lemma 4.4 is not applicable, we must have  $\delta_l = \delta_{\phi} = 1$  and  $\gamma > 0$ . Choose  $\rho$  so large that

$$\operatorname{ind} k = \frac{1}{\delta} > 1 - \gamma \rho = \operatorname{ind} \left( \frac{R}{\mathscr{C}(R^{\rho})} \right).$$

Since  $\frac{d_{\phi}}{d_l}$  is slowly varying, we have  $R^{\rho} \lesssim \hbar(R)$ , cf. Lemma A.8. Therefore,

$$g(\hbar(R),R)\gtrsim \hbar(R)\gg rac{R}{\mathscr{C}(R^
ho)}\geq rac{R}{\mathscr{C}(\hbar(R))}$$

We thus obtain

$$\mathfrak{B}(R) \lesssim \mathfrak{B}(\hbar(R), R) \asymp \mathfrak{g}(\hbar(R), R) \lesssim R^{\frac{1}{2}} \int_{\hbar(R)}^{\infty} \mathfrak{D}(s)^{-\frac{1}{2}} \, \mathrm{d}s.$$

To see the additional statements, note first that  $\operatorname{ind} k = \frac{1}{\bar{\delta}}$  and also

$$\operatorname{ind}\left(R^{\frac{1}{2}}\int_{\Re(R)}^{\infty}\mathfrak{D}(s)^{-\frac{1}{2}}\,\mathrm{d}s\right) = \frac{1}{2} + \frac{1}{\delta}\cdot\left(1 - \frac{\delta}{2}\right) = \frac{1}{\delta}.$$

Further,  $\delta > 2$  implies that

$$R^{\frac{1}{2}} \int_{\mathfrak{k}(R)}^{\infty} \mathfrak{D}(s)^{-\frac{1}{2}} \, \mathrm{d}s \asymp R^{\frac{1}{2}} \mathfrak{k}(R) \mathfrak{D}(\mathfrak{k}(R))^{-\frac{1}{2}} \asymp \mathfrak{k}(R).$$

Proof of Case C. We have

ind 
$$f_0 = 2 - \delta + 2\gamma$$
, ind  $f_1 = 2\left(\gamma + \left(1 - \frac{\delta}{2}\right)\right) = 2 - \delta + 2\gamma$ .

Since  $\mathfrak{D}^{-\frac{1}{2}}$  is not integrable, we must have  $\delta \leq 2$ . The case that  $\delta = 2$  and  $\gamma = 0$  is ruled out by assumption, and it follows that  $2 - \delta + 2\gamma > 0$ . Thus, asymptotic inverses  $f_0^-$  and  $f_1^-$  exist.

To prove the bound from below, we use  $f_0^-(R)$  pushed into the interval  $[\hbar(R), \hbar(R)]$ , namely we set

$$\tau(R) := \min\left\{\max\left\{f_0^-(R), \hbar(R)\right\}, \hbar(R)\right\}.$$

Our assumption that  $\frac{1}{d_l(t)} \lesssim t \mathscr{C}(t) \lesssim \mathfrak{D}(t)$  gives

$$rac{d_{\phi}(t)}{d_{l}(t)} \lesssim f_{0}(t) \lesssim \mathfrak{D}(t),$$

and hence  $\hbar(R) \lesssim f_0^-(R) \lesssim \hbar(R)$ . Thus,  $\tau(R) \asymp f_0^-(R)$ . Using Lemma 4.3 we obtain

$$g(\tau(R), R) \gtrsim R^{\frac{1}{2}} \tau(R) \mathfrak{D}(\tau(R))^{-\frac{1}{2}} \asymp R^{\frac{1}{2}} f_0^-(R) \mathfrak{D}(f_0^-(R))^{-\frac{1}{2}}.$$

On the other hand, the definition of  $f_0$  yields for  $R \to \infty$  that

$$R \sim f_0(f_0^-(R)) = \frac{f_0^-(R)^2 \mathscr{C}(f_0^-(R))^2}{\mathfrak{D}(f_0^-(R))},$$

and hence

$$\frac{R^{\frac{1}{2}}}{\mathscr{C}(f_0^-(R))} \sim f_0^-(R) \mathfrak{D}(f_0^-(R))^{-\frac{1}{2}}.$$

Together, thus,

$$\min\left\{\boldsymbol{g}(\tau(R),R),\frac{R}{\mathscr{C}(\tau(R))}\right\}\gtrsim R^{\frac{1}{2}}\boldsymbol{f}_{0}^{-}(R)\mathfrak{D}(\boldsymbol{f}_{0}^{-}(R))^{-\frac{1}{2}}\asymp\frac{R}{\mathscr{C}(\boldsymbol{f}_{0}^{-}(R))}$$

For the proof of the upper bound we use

$$\tau(R) := \min\left\{\max\left\{\ell_1^-(R), \ell(R)\right\}, \ell(R)\right\}.$$

Since  $f_0 \lesssim f_1$  we have  $f_1^- \lesssim f_0^- \lesssim \hbar$ , and therefore  $f_1^-(R) \lesssim \tau(R)$ . This readily implies that

$$\frac{R}{\mathscr{C}(\tau(R))} \lesssim \frac{R}{\mathscr{C}(\boldsymbol{f}_1^-(R))}$$

To estimate  $q(\tau(R), R)$  we distinguish two cases. First assume that  $f_1^-(R) \le k(R)$ . Then  $\tau(R) = k(R)$ , and hence

$$q(\tau(R),R) \asymp \mathfrak{k}(R) \lesssim \frac{R}{\mathscr{C}(\mathfrak{k}(R))} \leq \frac{R}{\mathscr{C}(\mathfrak{f}_1^-(R))}.$$

Second assume that  $f_1^-(R) \ge k(R)$ . Then  $\tau(R) \le f_1^-(R)$ . We use Lemma 4.3 to obtain

$$q(\tau(R), R) \lesssim R^{\frac{1}{2}} \int_{1}^{\tau(R)} \mathfrak{D}(s)^{-\frac{1}{2}} \,\mathrm{d}s \lesssim R^{\frac{1}{2}} \int_{1}^{\ell_{1}^{-}(R)} \mathfrak{D}(s)^{-\frac{1}{2}} \,\mathrm{d}s.$$

The last integral can be evaluated from the definition of  $f_1 \colon \text{for } R \to \infty$ 

$$R \sim f_1(f_1^-(R)) = \left( \mathscr{C}(f_1^-(R)) \int_1^{f_1^-(R)} \mathfrak{D}(s)^{-\frac{1}{2}} \, \mathrm{d}s \right)^2,$$

and hence

$$\int_{1}^{\ell_{1}^{-}(R)} \mathfrak{D}(s)^{-\frac{1}{2}} \, \mathrm{d}s \sim \frac{R^{\frac{1}{2}}}{\mathscr{C}(\ell_{1}^{-}(R))}$$

Altogether we see that always

$$\max\left\{ q\left(\tau(R),R\right),\frac{R}{\mathscr{C}(\tau(R))}\right\}\lesssim \frac{R}{\mathscr{C}(f_{1}^{-}(R))}$$

This completes the proof of (4.6).

To see the additional statements, recall that ind  $f_0 = \inf f_1 = 2 - \delta + 2\gamma$ . This implies that

$$\operatorname{ind} \frac{R}{\mathscr{C}(f_0^-(R))} = \operatorname{ind} \frac{R}{\mathscr{C}(f_1^-(R))} = 1 - \gamma \cdot \frac{1}{2 - \delta + 2\gamma} = \frac{2 - \delta + \gamma}{2 - \delta + 2\gamma}.$$

Further,  $\delta < 2$  implies that  $\int_1^t \mathfrak{D}(s)^{-\frac{1}{2}} ds \asymp t \mathfrak{D}(t)^{-\frac{1}{2}}$ , and hence  $f_0 \asymp f_1$ .  $\Box$ 

*Proof of Case* D. The first thing to show is that  $\frac{d_{\phi}}{d_l}$  is unbounded: if we had  $d_{\phi}(t) \leq d_l(t)$ , it would follow that

$$\mathfrak{D}(t)^{-\frac{1}{2}} = \sqrt{d_l(t)} d_{\phi}(t) \lesssim d_l(t),$$

and this contradicts the fact that  $d_l(t)$  is integrable while  $\mathfrak{D}^{-\frac{1}{2}}(t)$  is not. To show the bound from below asserted in (4.7), we note that

$$q(\hbar(R), R) \gtrsim R^{\frac{1}{2}} \hbar(R) \mathfrak{D}(\hbar(R))^{-\frac{1}{2}}$$

by Lemma 4.3, and that

$$\frac{R}{\mathscr{C}(\hbar(R))} \gtrsim R\hbar(R)d_l(\hbar(R))$$

by our assumption that  $t\mathscr{C}(t) \lesssim \frac{1}{d_l(t)}$ . Since  $R \asymp \frac{d_{\phi}}{d_l}(\hbar(R))$  for  $R \to \infty$ , we obtain

$$R\hbar(R)d_l(\hbar(R)) \asymp R^{\frac{1}{2}}\hbar(R) \Big(\frac{d_\phi(\hbar(R))}{d_l(\hbar(R))}\Big)^{\frac{1}{2}}d_l(\hbar(R)) = R^{\frac{1}{2}}\hbar(R)\mathfrak{D}(\hbar(R))^{-\frac{1}{2}}$$

and see that

$$\min\left\{ q(\hbar(R),R),\frac{R}{\mathscr{C}(\hbar(R))}\right\}\gtrsim R\hbar(R)d_l(\hbar(R))$$

For the proof of the bound from above, we show that q(t, R) is bounded independently of  $t \ge \hbar(R)$ . Using again Lemma 4.3, we find

$$g(t,R) = g(\hbar(R),R) + R \int_{\hbar(R)}^{t} d_l(s) \, \mathrm{d}s$$

$$\lesssim R^{\frac{1}{2}} \int_{1}^{\hbar(R)} \mathfrak{D}(s)^{-\frac{1}{2}} \,\mathrm{d}s + R \int_{\hbar(R)}^{\infty} d_{l}(s) \,\mathrm{d}s.$$

It remains to choose  $\tau(R)$  sufficiently large, so that  $\frac{R}{\mathscr{C}(\tau(R))} \leq \mathfrak{q}(\tau(R)), R)$ , which is clearly possible. This completes the proof of (4.7).

To see the additional statements, assume first that  $\delta_l > \delta_{\phi}$ . Then  $\hbar(R)$  is regularly varying with index  $\frac{1}{\delta_l - \delta_{\phi}}$ . We obtain

$$\operatorname{ind}\left(R\hbar(R)d_{l}(\hbar(R))\right) = 1 + \frac{1}{\delta_{l} - \delta_{\phi}} - \frac{\delta_{l}}{\delta_{l} - \delta_{\phi}} = \frac{1 - \delta_{\phi}}{\delta_{l} - \delta_{\phi}}.$$

By Karamata's theorem

$$\operatorname{ind}\left(R^{\frac{1}{2}}\int_{1}^{\hbar(R)}\mathfrak{D}(s)^{-\frac{1}{2}}\,\mathrm{d}s\right) = \frac{1}{2} + \left(1 - \frac{\delta}{2}\right)\frac{1}{\delta_{l} - \delta_{\phi}} = \frac{1 - \delta_{\phi}}{\delta_{l} - \delta_{\phi}},$$
$$\operatorname{ind}\left(R\int_{\hbar(R)}^{\infty}\mathfrak{d}_{l}(s)\,\mathrm{d}s\right) = 1 + (1 - \delta_{l})\frac{1}{\delta_{l} - \delta_{\phi}} = \frac{1 - \delta_{\phi}}{\delta_{l} - \delta_{\phi}}.$$

Further, again referring to Karamata's theorem,  $\delta < 2$  implies

$$R^{\frac{1}{2}} \int_{1}^{\hbar(R)} \mathfrak{D}(s)^{-\frac{1}{2}} \,\mathrm{d}s \asymp R^{\frac{1}{2}} \hbar(R) \mathfrak{D}(\hbar(R))^{-\frac{1}{2}} \asymp R \hbar(R) d_{l}(\hbar(R)),$$

and  $\delta_l > 1$  implies

$$R \int_{\hbar(R)}^{\infty} d_l(s) \,\mathrm{d}s \asymp R\hbar(R) d_l(\hbar(R)).$$

### 4.2 Bound for the monodromy matrix

We combine Theorem 2.2 with Theorem 4.1 to obtain a bound for the growth of  $W_H$  when the lengths and angle differences of H are bounded by regularly varying functions. This yields a far reaching generalisation of Corollary 2.5.

**4.6 Theorem.** Let  $(l_j)_{j=1}^{\infty}$  be a summable sequence of positive numbers, and  $(\phi_j)_{j=1}^{\infty}$  a sequence of real numbers. Denote by H the Hamburger Hamiltonian with these lengths and angles, and let  $W_H$  be its monodromy matrix.

Let  $\psi \in \mathbb{R}$  and let  $d_l, d_{\phi}, c_l, c_{\phi}$  be regularly varying functions that are  $\approx$  to some nonincreasing functions, such that  $d_l \approx 1 \approx d_{\phi}$  locally,  $c_{\phi}(t) \leq c_l(t)$  for sufficiently large t, ind  $d_l$  + ind  $d_{\phi} < 0$ , and  $\lim_{t\to\infty} (c_l c_{\phi})(t) = 0$ . Assume that

$$l_j \lesssim d_l(j), \qquad |\sin(\phi_{j+1} - \phi_j)| \lesssim d_{\phi}(j), \quad j \text{ sufficiently large,}$$
$$\sum_{j=N+1}^{\infty} l_j \lesssim c_l(N), \quad \sum_{j=N+1}^{\infty} l_j \sin^2(\phi_j - \psi) \lesssim c_{\phi}(N), \quad N \text{ sufficiently large.}$$

Denote

$$\mathfrak{D}(t) := \frac{1}{(d_l d_{\phi})(t)}, \ \delta := \operatorname{ind} \mathfrak{D}, \quad \mathfrak{C}(t) := \frac{1}{(c_l c_{\phi})^{\frac{1}{2}}(t)}, \ \gamma := \operatorname{ind} \mathfrak{C}.$$

Then we have the following bounds for  $W_H$  and its order  $\rho_H$ .

Data satisfies			$\left  \log \left( \max_{ z =R} \  W_H(z) \  \right) \right  $ is $\lesssim$	$\rho_H \leq$
$\mathfrak{D}(t) \lesssim t \mathfrak{C}(t)$			$ \begin{array}{l} \frac{R}{\mathscr{C}(\ell^{-}(R))} & \text{where} \\ f(t) := t\mathscr{C}(t) \log \left[ \alpha \frac{t\mathscr{C}(t)}{\mathfrak{D}(t)} \right] \\ \alpha := 4 \sup_{t \ge 1} \frac{\mathfrak{D}(t)}{t\mathscr{C}(t)} \end{array} $	$\frac{1}{1+\gamma}$
$t\mathscr{C}(t) \lesssim \mathfrak{D}(t),$	$\int_{1}^{\infty} \mathfrak{D}(s)^{-\frac{1}{2}}  \mathrm{d}s < \infty,$ $(\gamma > 0 \text{ or } \frac{d_{\phi}}{d_{l}} \approx \text{ to r}$	nondecreasing)	$R^{\frac{1}{2}} \int_{\mathfrak{k}(R)}^{\infty} \mathfrak{D}(s)^{-\frac{1}{2}}  \mathrm{d}s$	$\frac{1}{\delta}$
	$\int_{1}^{\infty} \mathfrak{D}(s)^{-\frac{1}{2}}  \mathrm{d}s = \infty,$	$\begin{split} & \frac{1}{d_l(t)} \lesssim t \mathscr{C}(t), \\ & (\delta, \gamma) \neq (2, 0) \end{split}$	$ \begin{array}{l} \frac{R}{\mathscr{C}(\ell_1^{-}(R))} & \text{where} \\ \\ f_1(t) := [\mathscr{C}(t) \int_1^t \mathfrak{D}(s)^{-\frac{1}{2}}  \mathrm{d}s]^2 \end{array} $	$\frac{2-\delta+\gamma}{2-\delta+2\gamma}$
		$\begin{split} t \mathcal{C}(t) \lesssim \frac{1}{d_l(t)}, \\ \int \limits_{0}^{\infty} d_l(s)  \mathrm{d}s < \infty, \\ \delta_l > \delta_{\phi} \end{split}$	$R^{\frac{1}{2}} \int_{1}^{\hbar(R)} \mathfrak{D}(s)^{-\frac{1}{2}} ds \\ +R \int_{\hbar(R)}^{\infty} d_{l}(s) ds$	$\frac{1-\delta_{\phi}}{\delta_l-\delta_{\phi}}$

In each case the relation  $\leq$  holds for R sufficiently large.

The main assumptions on the data  $d_l, d_{\phi}, c_l, c_{\phi}$  put in this theorem are that those functions are regularly varying and that ind  $d_l + \text{ind } d_{\phi} < 0$  (i.e.,  $\neq 0$ ). Monotonicity assumptions are only a minor restriction: for example they are automatically fulfilled whenever the function under consideration is not slowly varying. The assumption that  $c_{\phi} \leq c_l$  is no loss in generality, since replacing  $c_{\phi}$  by min $\{c_{\phi}, c_l\}$  does not affect validity of any of the other assumptions.

Proof of Theorem 4.6.

① Observe that neither the assumptions of the theorem nor the case distinction in the assertion of the theorem depends on the equivalence class modulo  $\approx$  of the functions  $d_l, d_{\phi}, c_l, c_{\phi}$ . Further, using Remark A.7, we see that the functions written in the second column of the table change only up to  $\approx$  when we pass to other data equivalent modulo  $\approx$  to  $d_l, d_{\phi}, c_l, c_{\phi}$ .

Hence, it is enough to prove the theorem for suitable modifications  $\hat{d}_l, \hat{d}_{\phi}, \hat{c}_l, \hat{c}_{\phi}$  of  $d_l, d_{\phi}, c_l, c_{\phi}$  which differ only up to  $\approx$ .

<sup>(2)</sup> The task is to define  $\hat{d}_l$ ,  $\hat{d}_{\phi}$ ,  $\hat{c}_l$ ,  $\hat{c}_{\phi}$  in such a way that Theorem 2.2, Proposition 3.9, and Theorem 4.1 become applicable. To this end we use Lemma A.10 and the freedom of choice of nonincreasing smoothenings mentioned in Remark A.11.

We choose  $\mathcal{S}[\boldsymbol{c}_{\phi}]$  such that

$$\forall N \in \mathbb{N}: \sum_{j=N+1}^{\infty} l_j \sin^2(\phi_j - \psi) \le \mathcal{S}[c_{\phi}](N),$$

which is clearly possible by first choosing an arbitrary nonincreasing smoothening of  $c_{\phi}$  and then multiplying it with a sufficiently large positive constant. Next, we choose  $\mathcal{S}[c_l]$  such that

$$\mathcal{S}[c_{\phi}] \leq \mathcal{S}[c_{l}], \qquad \forall N \in \mathbb{N}: \ \sum_{j=N+1}^{\infty} l_{j} \leq \mathcal{S}[c_{l}](N).$$

A suitable modification of  $d_{\phi}$  is found as follows. If  $\liminf_{t\to\infty} d_{\phi}(t) > 0$ , we choose  $\mathcal{S}[d_{\phi}] := 1$  (which corresponds to the choices  $\epsilon[d_{\phi}] := 0$  and  $\kappa[d_{\phi}] := 1$ ). If  $\lim_{t\to\infty} d_{\phi}(t) = 0$ , we choose  $\mathcal{S}[d_{\phi}]$  such that

$$\mathcal{S}[d_{\phi}] \le 1, \qquad \forall j \in \mathbb{N}: |\sin(\phi_{j+1} - \phi_j)| \le \mathcal{S}[d_{\phi}](j),$$

which is possible by choosing an arbitrary nonincreasing smoothening of  $d_{\phi}$ , then multiplying it with a sufficiently large positive constant, and then cutting it off at 1.

It remains to define a modification of  $d_l$ . In the generic case that  $\gamma > 0$  we proceed just the same as above and choose  $\mathcal{S}[c_l]$  such that

$$\forall j \in \mathbb{N}: \ l_j \le \mathcal{S}[d_l](j). \tag{4.12}$$

In the boundary case that  $\gamma = 0$  we make a further case distinction. If we are in the situation of the 1<sup>st</sup> or the 3<sup>rd</sup> row of the table in the theorem, we do just the same as above. If we are in the situation of the 2<sup>nd</sup> or the 4<sup>th</sup> row, the additional assumption ensures that  $\frac{d_l}{d_{\phi}}$  is  $\approx$  to some nonincreasing function, and we choose

$$\mathcal{S}[d_l] \coloneqq \mathcal{S}[d_{\phi}] \cdot \mathcal{S}\Big[rac{d_l}{d_{\phi}}\Big],$$

where  $\delta\left[\frac{d_l}{d_{\phi}}\right]$  is sufficiently large to ensure that (4.12) holds.

Now we set

$$\widehat{d}_l \coloneqq \mathbb{S}[d_l], \ \widehat{d}_\phi \coloneqq \mathbb{S}[d_\phi], \ \widehat{c}_l \coloneqq \mathbb{S}[c_l], \ \widehat{c}_\phi \coloneqq \mathbb{S}[c_\phi].$$

<sup>(3)</sup> We apply our previous results with the data  $\hat{d}_l, \hat{d}_{\phi}, \hat{c}_l, \hat{c}_{\phi}$ . Here we denote by  $\hat{B}(R), \hat{L}(t, R), \hat{\mathcal{B}}(R)$ , etc. the correspondingly defined functions.

Theorem 2.2 implies that

$$\log\left(\max_{|z|=R} \|W_H(z)\|\right) \lesssim \widehat{B}(R),$$

and we face the task to control  $\hat{L}(t, R)$ . In almost all cases Proposition 3.9 takes care of this:

- ▷ Proposition 3.9 (i) applies if  $\delta_{\phi} > \delta_l$ , or if  $\gamma = 0$  and we are in the situation of the 2<sup>nd</sup> row of the table.
- ▷ Proposition 3.9 (ii) applies if  $\gamma > 0$  and  $\delta_{\phi} < \delta_l$ .
- ▷ Proposition 3.9 (iii) applies if  $\gamma > 0$  and  $\delta_{\phi} = \delta_l$ .

Thus, in all of these cases,  $\widehat{B}(R) \simeq \widehat{\mathfrak{B}}(R)$ , and the bounds asserted in the second column of the table follows from Theorem 4.1.

It remains to study the situation that  $\gamma = 0$ ,  $\delta_{\phi} \leq \delta_l$ , and we are in the 1<sup>st</sup> or 3<sup>rd</sup> row of the table. Applying Corollary 3.7 (ii) if  $\delta_{\phi} < \delta_l$  and Lemma 3.8 if  $\delta_{\phi} = \delta_l$ , yields

$$\widehat{L}(t,R) \lesssim 1 + \log t.$$

Let  $\tau(R)$  be the power bounded function used in Theorem 4.1 A C to estimate  $\widehat{\mathscr{B}}(R)$ . Then

$$\widehat{B}(R) \leq \widehat{\mathscr{B}}(\tau(R),R) + \widehat{L}(\tau(R),R) \lesssim \widehat{\mathscr{B}}(\tau(R),R).$$

The bounds asserted in the second column of the table thus follow from Theorem 4.1.

(4) The bounds for  $\rho_H$  arise simply by taking the indices of the regularly varying functions in the second column.

In the following corollary we revisit the setting of Theorem 4.6, except we are not given functions  $c_l, c_{\phi}$ .

**4.7 Corollary.** Let H be a Hamburger Hamiltonian and let  $d_l, d_{\phi}$  be regularly varying. Assume that  $d_{\phi}(t)$  is  $\sim$  to a nonincreasing function as  $t \to \infty$  and that  $d_l \approx 1 \approx d_{\phi}$  locally.

Assume that the lenghts and angles of H are bounded as

$$|l_j \lesssim d_l(j), |\sin(\phi_{j+1} - \phi_j)| \lesssim d_{\phi}(j)$$
 for sufficiently large  $j$ ,

and that  $d_l \in L^1([1,\infty))$ . Then the following statements hold.

 $\triangleright If \delta > 2, then \log \left( \max_{|z|=R} \|W_H(z)\| \right) \lesssim k(R).$ 

- $\triangleright If 0 < \delta < 2, then \log \left( \max_{|z|=R} \|W_H(z)\| \right) \lesssim R \int_{\hbar(R)}^{\infty} d_l(x) \, \mathrm{d}x.$
- $| If 0 < \delta \leq 2, \ \delta_l > 1, \ and \ there \ exists \ \psi \in \mathbb{R} \ such \ that \ |\sin(\phi_j \psi)| \lesssim |\sin(\phi_{j+1} \phi_j)|, \ then \ again \ \log\left(\max_{|z|=R} \|W_H(z)\|\right) \lesssim \hbar(R).$
- $\triangleright$  If  $\delta = 2$  and  $(\delta_l, \delta_{\phi}) \neq (1, 1)$ , then  $\rho_H \leq \frac{1}{2}$ .

*Proof.* Our goal is to apply Theorem 4.6, and in order to do that we need to construct suitable functions  $c_l, c_{\phi}$ . Note that the assumption  $d_l \in L^1([1,\infty))$  implies  $\delta_l \geq 1$ .

① Without any a priori assumption, we can set

$$c_l(t) := c_{\phi}(t) := \int_t^\infty d_l(x) \, \mathrm{d}x \gtrsim t d_l(t).$$

Then  $d_l$ ,  $d_{\phi}$  together with  $c_l$ ,  $c_{\phi}$  satisfy the general assumptions of Theorem 4.6. We have

$$t\mathscr{C}(t) \lesssim \frac{1}{d_l(t)} \lesssim \mathfrak{D}(t)$$

and  $\gamma = \delta_l - 1 \ge 0$ . This choice of  $c_l, c_{\phi}$  is sufficient to prove the asserted bounds in the following cases:

- $arphi \delta > 2 \wedge \delta_l > 1.$  Since  $\gamma > 0$ , Theorem 4.6 gives the upper bound  $R^{\frac{1}{2}} \int_{\mathfrak{K}(R)}^{\infty} \mathfrak{D}(s)^{-\frac{1}{2}} \mathrm{d}s \asymp \mathfrak{K}(R).$
- $\triangleright \delta < 2$ . Then  $\delta_{\phi} = \delta \delta_l < 1 \leq \delta_l$ , hence  $\frac{d_{\phi}}{d_l}$  has positive index and is eventually nondecreasing. Since

$$R^{\frac{1}{2}} \int_{1}^{\hbar(R)} \mathfrak{D}(s)^{-\frac{1}{2}} \,\mathrm{d}s \asymp R^{\frac{1}{2}} \hbar(R) \mathfrak{D}(\hbar(R))^{-\frac{1}{2}} \lesssim R \int_{\hbar(R)}^{\infty} d_l(s) \,\mathrm{d}s,$$

Theorem 4.6 gives the asserted upper bound.

 $\flat \delta = 2 \land \delta_l > 1$ . Then we are either in the second or fourth row of Theorem 4.6, but in both cases the upper bound for the order is equal to  $\frac{1}{2}$ .

② Assume  $\delta > 2$  and  $\delta_{\phi} > 1$ . This implies that  $\sum_{j=1}^{\infty} |\sin(\phi_{j+1} - \phi_j)| < \infty$ . We want to choose  $\psi := \lim_{j\to\infty} \phi_j$ , so we need to prove that this limit actually exists (at least if the angles  $\phi_j$  are all modified by adding integer multiples of  $\pi$  which leaves H unchanged). Start by choosing  $n_0$  so large that  $\sum_{j=n_0}^{\infty} |\sin(\phi_{j+1} - \phi_j)| \leq \frac{\pi}{8}$ . Adding to  $\phi_j$  an integer multiple of  $\pi$ , we may assume  $|\phi_j - \phi_{n_0}| \leq \frac{\pi}{2}$  for all  $j \in \mathbb{N}$ . Since  $|x| \leq 2|\sin(x)|$  for  $|x| \leq \frac{\pi}{2}$ , and  $|\sin(x+y)| \leq |\sin(x)| + |\sin(y)|$ , we have for  $j > k \geq n_0$ 

$$\begin{aligned} |\phi_j - \phi_k| &\le |\phi_j - \phi_{n_0}| + |\phi_{n_0} - \phi_k| \le 2 \left( |\sin(\phi_j - \phi_{n_0})| + |\sin(\phi_{n_0} - \phi_k)| \right) \\ &\le 4 \sum_{n=n_0}^{\infty} |\sin(\phi_{n+1} - \phi_n)| \le \frac{\pi}{2}. \end{aligned}$$

Therefore,

$$|\phi_j - \phi_k| \le 2|\sin(\phi_j - \phi_k)| \le 2\sum_{n=k}^{j-1} |\sin(\phi_{n+1} - \phi_n)|$$

and thus  $(\phi_j)_{j=1}^{\infty}$  is a Cauchy sequence. Let  $\psi$  be its limit. We set

$$c_l(t) := \int_t^\infty d_l(x) \, \mathrm{d}x, \qquad c_\phi(t) := c_l(t) \cdot \left(\int_t^\infty d_\phi(x) \, \mathrm{d}x\right)^2$$

and observe that  $d_l$ ,  $d_{\phi}$  together with  $c_l$ ,  $c_{\phi}$  satisfy the assumptions of Theorem 4.6. A calculation shows that  $\gamma = \delta - 2 > 0$ . Hence  $\operatorname{ind}[t\mathscr{C}(t)] = \delta - 1 < \delta$ and in particular  $t\mathscr{C}(t) \ll \mathfrak{D}(t)$ . Again Theorem 4.6 provides the desired upper bound.

③ Assume that we have  $\psi \in \mathbb{R}$  such that  $|\phi_j - \psi| \lesssim |\phi_{j+1} - \phi_j|$ , and  $\delta_l > 1$ . Set

$$c_l(t) := \int_t^\infty d_l(x) \, \mathrm{d}x \asymp t d_l(t),$$
  
$$c_\phi(t) := \int_t^\infty d_l(x) d_\phi(x)^2 \, \mathrm{d}x \asymp t d_l(t) d_\phi(t)^2.$$

Then  $t\mathscr{C}(t) \simeq \mathfrak{D}(t)$  and we are in the first row of Theorem 4.6. Now we note that the bound given in the theorem is  $\simeq$  to  $\mathfrak{k}(R)$ .

### 5 Additions and examples

#### 5.1 Combining with a bound from below

In the same way that upper bounds for lengths and angle differences lead to upper bounds for the growth of the monodromy matrix, lower bounds lead to lower bounds. We recall a result obtained in [PW22, Corollary 2.5].

**5.1 Proposition.** Let H be a Hamburger Hamiltonian with lengths  $(l_j)_{j=1}^{\infty}$  and angles  $(\phi_j)_{i=1}^{\infty}$ , and let f be regularly varying with positive index. If

$$l_{j+1}l_j\sin^2(\phi_{j+1} - \phi_j) \gtrsim \frac{1}{f(j)}, \qquad j \in \mathbb{N},$$
(5.1)

then

$$\log\left(\max_{|z|=R} \|W_H(z)\|\right) \gtrsim \ell^-(R^2)$$

for sufficiently large R.

Let us translate Proposition 5.1 to the setting where we compare the lengths and the angle differences to regularly varying functions  $d_l, d_{\phi}$ .

**5.2 Corollary.** Let  $(l_j)_{j=1}^{\infty}$  be a summable sequence of positive numbers, and let  $(\phi_j)_{j=1}^{\infty}$  be a sequence of real numbers. Denote by H the Hamburger Hamiltonian with these lengths and angles, and let  $W_H$  be its monodromy matrix. Assume that  $d_l, d_{\phi}$  are regularly varying and satisfy

$$\forall j \in \mathbb{N}. \quad l_j \gtrsim d_l(j) \qquad |\sin(\phi_{j+1} - \phi_j)| \gtrsim d_\phi(j).$$

Then

$$\log\left(\max_{|z|=R} \|W_H(z)\|\right) \gtrsim \left[\frac{1}{d_l d_\phi}\right]^-(R)$$

for sufficiently large R.

In particular, the order of  $W_H$  is at least  $\frac{1}{\delta}$ , where  $\delta := -(\operatorname{ind} d_l + \operatorname{ind} d_{\phi})$ .

Proof. Since  $d_l$  is regularly varying, we have  $d_l(t+1) \sim d_l(t)$ . Setting  $\mathfrak{D}(t) := \frac{1}{(d_l d_{\phi})(t)}$ , we see that (5.1) is satisfied for  $f(t) := \mathfrak{D}(t)^2$ . Since  $f^-(t) = \mathfrak{D}^-(t^{\frac{1}{2}})$ , we obtain

$$\log\left(\max_{|z|=R} \|W_H(z)\|\right) \gtrsim \ell^-(R^2) = \mathfrak{D}^-(R) = \left[\frac{1}{d_l d_\phi}\right]^-(R).$$

If the lengths and angle differences are well-behaved and summable, the growth of  $W_H$  can be determined up to  $\asymp$ . Note that no functions  $c_l, c_{\phi}$  appear in the formulation of the following theorem.

**5.3 Theorem.** Let  $(l_j)_{j=1}^{\infty}$  be a summable sequence of positive numbers, and let  $(\phi_j)_{j=1}^{\infty}$  be a sequence of real numbers. Denote by H the Hamburger Hamiltonian with these lengths and angles. Consider regularly varying functions  $d_l$ ,  $d_{\phi}$  with  $d_l \simeq 1 \simeq d_{\phi}$  locally. If

(i)  $l_j \asymp d_l(j)$  and  $|\sin(\phi_{j+1} - \phi_j)| \asymp d_{\phi}(j)$  for sufficiently large j,

(ii)  $\delta := -(\operatorname{ind} d_l + \operatorname{ind} d_{\phi}) > 2,$ 

then

$$\log\left(\max_{|z|=R} \|W_H(z)\|\right) \asymp \left[\frac{1}{d_l d_{\phi}}\right]^-(R) \quad \text{for sufficiently large } R,$$

and

$$\rho_H = \frac{1}{\delta}.$$

*Proof.* Due to Corollary 5.2, we only need to show that

$$\log\left(\max_{|z|=R} \|W_H(z)\|\right) \lesssim \left[rac{1}{d_l d_\phi}
ight]^-(R).$$

We notice that ind  $d_l \leq -1$  since  $(l_j)_{j=1}^{\infty}$  is summable. Hence  $d_l$  is  $\sim$  to an eventually monotone function, and thus

$$\int_{M}^{\infty} d_{l}(x) \, \mathrm{d}x \lesssim \sum_{j=M}^{\infty} d_{l}(j) \lesssim \sum_{j=M}^{\infty} l_{j} < \infty.$$

Since  $d_l \approx 1$  locally, this shows that  $d_l \in L^1([1,\infty))$ . By Corollary 4.7,

$$\log\left(\max_{|z|=R} \|W_H(z)\|\right) \lesssim k(R) \asymp \left[\frac{1}{d_l d_\phi}\right]^-(R).$$

### 5.2 Power-log-majorisations and exceptional cases

By considering  $d_l$ ,  $d_{\phi}$ ,  $c_l$ ,  $c_{\phi}$  consisting of a power times a power of a logarithm, we can gain some insight into the exceptional cases of our results. For data of this form all functions occurring in our results can in principle be computed explicitly. We do not aim at giving a complete picture, but rather give a couple of illustrative examples. It should also be added that not all phenomena (counterexamples, exceptional cases, or similar) can be illustrated with functions of this form; one would have to admit an additional double-logarithmic factor.

The facts presented below are shown by straightforward, yet tedious and elaborate, computations. For this reason we defer all proofs to the preprint version of this article [PRW23].

It is practical to use the lexicographic order on  $\mathbb{R}^2$ , and we denote it by  $\leq$ . Explicitly, thus

$$(\alpha,\beta) \preceq (\alpha',\beta') :\Leftrightarrow \alpha < \alpha' \lor (\alpha = \alpha' \land \beta \le \beta')$$

and, as usus al,  $\prec$  stands for " $\preceq$  but not =".

5.4 Setting and Notation. Assume we are given parameters

$$(\delta_l, \alpha_l), (\delta_{\phi}, \alpha_{\phi}), (\gamma_l, \beta_l), (\gamma_{\phi}, \beta_{\phi}) \in [0, \infty) \times \mathbb{R},$$

and denote

$$\delta := \delta_l + \delta_{\phi}, \ \alpha := \alpha_l + \alpha_{\phi}, \gamma := \frac{1}{2}(\gamma_l + \gamma_{\phi}), \ \beta := \frac{1}{2}(\beta_l + \beta_{\phi}).$$

Assume that these parameters satisfy

- $\triangleright \ (\delta_l, \alpha_l), (\delta_{\phi}, \alpha_{\phi}), (\gamma_l, \beta_l), (\gamma_{\phi}, \beta_{\phi}) \succeq (0, 0),$
- $\triangleright \ \delta > 0, (\gamma,\beta) \succ (0,0),$
- $\triangleright \ (\gamma_l, \beta_l) \preceq (\gamma_\phi, \beta_\phi).$

Let  $d_l, d_{\phi}, c_l, c_{\phi}$  be continuous and nonincreasing functions, such that  $d_{\phi} \leq 1$ ,  $c_{\phi} \leq c_l$ , and that (for sufficiently large t)

$$d_{l}(t) = t^{-\delta_{l}} (\log t)^{-\alpha_{l}}, \quad d_{\phi}(t) = t^{-\delta_{\phi}} (\log t)^{-\alpha_{\phi}}, c_{l}(t) = t^{-\gamma_{l}} (\log t)^{-\beta_{l}}, \quad c_{\phi}(t) = t^{-\gamma_{\phi}} (\log t)^{-\beta_{\phi}}.$$

We compute some of the basic ingredients. Here, and throughout the following, all formulas are understood to hold for sufficiently large t or R.

#### 5.5 Lemma. We have

(i) 
$$\mathfrak{D}(t) = t^{\delta} (\log t)^{\alpha}, \quad \mathfrak{C}(t) = t^{\gamma} (\log t)^{\beta},$$
  
$$\frac{d_{\phi}(t)}{d_{l}(t)} = t^{\delta_{l} - \delta_{\phi}} (\log t)^{\alpha_{l} - \alpha_{\phi}}, \quad \frac{c_{l}(t)}{c_{\phi}(t)} = t^{\gamma_{\phi} - \gamma_{l}} (\log t)^{\beta_{\phi} - \beta_{l}}.$$

(ii) 
$$\mathfrak{k}(R) \approx R^{\frac{1}{\delta}} (\log R)^{-\frac{\alpha}{\delta}},$$
  
 $\mathfrak{k}(R) = \infty \Leftrightarrow (\delta_l, \alpha_l) \preceq (\delta_{\phi}, \alpha_{\phi}),$ 
  
 $\mathfrak{k}(R) \approx \begin{cases} R^{\frac{1}{\delta_l - \delta_{\phi}}} (\log R)^{-\frac{\alpha_l - \alpha_{\phi}}{\delta_l - \delta_{\phi}}} & \text{if } \delta_l > \delta_{\phi},$ 
  
 $\exp\left(R^{\frac{1}{\alpha_l - \alpha_{\phi}}}\right) & \text{if } \delta_l = \delta_{\phi} \land \alpha_l > \alpha_{\phi}.$ 

5.6 Lemma. We have

$$\int_{1}^{\infty} \mathfrak{D}(s)^{-\frac{1}{2}} ds < \infty \iff (\delta, \alpha) \succ (2, 2)$$

$$\int_{t}^{\infty} \mathfrak{D}(s)^{-\frac{1}{2}} ds = \begin{cases} \frac{1}{\frac{\delta}{2}-1} \cdot t^{1-\frac{\delta}{2}} (\log t)^{-\frac{\alpha}{2}} & \text{if } \delta > 2\\ \frac{1}{\frac{\alpha}{2}-1} \cdot (\log t)^{1-\frac{\alpha}{2}} & \text{if } \delta = 2 \land \alpha > 2 \end{cases}$$

$$\int_{1}^{t} \mathfrak{D}(s)^{-\frac{1}{2}} ds \sim \begin{cases} \frac{1}{1-\frac{\delta}{2}} \cdot t^{1-\frac{\delta}{2}} (\log t)^{-\frac{\alpha}{2}} & \text{if } \delta < 2, \\ \frac{1}{1-\frac{\alpha}{2}} \cdot (\log t)^{1-\frac{\alpha}{2}} & \text{if } \delta = 2 \land \alpha < 2, \\ \log \log t & \text{if } \delta = \alpha = 2. \end{cases}$$

In our first example we discuss the role of the term L(t, R); this fits the context of Proposition 3.9. We show that there are situations where L(t, R) cannot be neglected, but also that the assumptions in Proposition 3.9 are only sufficient for  $B(R) \simeq \mathfrak{B}(R)$ .

5.7 Example. Assume that

$$(\delta_l, \alpha_l) \prec (\delta_\phi, \alpha_\phi), \quad \gamma = 0, \quad (\delta, \alpha) \succ (2, 2).$$
 (5.2)

The first assumption is there to rule out applicability of Proposition 3.9 (i), the second to rule out applicability of Proposition 3.9 (ii),(iii), and the third to reduce computational effort (the facts we want to illustrate occur already under this additional assumption).

For parameters subject to (5.2) it holds that

$$\mathfrak{B}(R) \asymp \begin{cases} R^{\frac{1}{2}} (\log R)^{1-\frac{\alpha}{2}} & \text{if } \delta = 2, \\ R^{\frac{1}{\delta}} (\log R)^{-\frac{\alpha}{\delta}} & \text{if } \delta > 2. \end{cases}$$

$$(5.3)$$

$$B(R) \begin{cases} \approx \mathfrak{B}(R) & \text{if } \delta_l = \delta_\phi \lor \beta > 1 \lor (\beta = \delta - 1 > 1 \land \alpha < 0), \\ \approx R^{\frac{1}{1+\beta}} \gg \mathfrak{B}(R) & \text{otherwise} . \end{cases}$$
(5.4)

In our second example we discuss the bounds from Theorem 4.1 C. We show that they are sharp but need not necessarily be attained.

5.8 Example. Assume that

$$(\delta_l, \alpha_l) \preceq (1 + \gamma, \beta) \preceq (\delta, \alpha), \quad \delta = 2, \alpha \le 2, \quad \gamma > 0.$$
 (5.5)

These assumptions ensure that we are in the situation of Theorem 4.1 C and that in (4.6) we do not automatically have equality.

For parameters subject to (5.5) it holds that

$$\frac{R}{\mathscr{C}(f_0^-(R))} \asymp R^{\frac{1}{2}} (\log R)^{-\frac{\alpha}{2}} \asymp \mathscr{K}(R),$$
(5.6)

$$\frac{R}{\mathscr{C}(f_1^-(R))} \asymp \begin{cases} R^{\frac{1}{2}} (\log R)^{1-\frac{\alpha}{2}} & \text{if } \alpha < 2, \\ R^{\frac{1}{2}} \log \log R & \text{if } \alpha = 2. \end{cases}$$
(5.7)

$$\mathfrak{B}(R) \asymp \begin{cases} R^{\frac{1}{2}} (\log R)^{1-\frac{\alpha}{2}} & \text{if } \gamma < 1, \\ R^{\frac{1}{2}} (\log R)^{-\frac{\alpha}{2}} \log \log R & \text{if } \gamma = 1 \land \alpha > \beta, \\ R^{\frac{1}{2}} (\log R)^{-\frac{\alpha}{2}} & \text{if } \gamma = 1 \land \alpha = \beta. \end{cases}$$
(5.8)

We see that

$$\begin{split} & \rhd \ \mathfrak{B}(R) \asymp \frac{R}{\mathfrak{C}(f_1^-(R))} \ \text{if} \ \gamma < 1 \land \alpha < 2, \\ & \triangleright \ \frac{R}{\mathfrak{C}(f_0^-(R))} \ll \mathfrak{B}(R) \ll \frac{R}{\mathfrak{C}(f_1^-(R))} \ \text{if} \ \gamma < 1 \land \alpha = 2 \ \text{or} \ \gamma = 1 \land \alpha > \beta, \\ & \triangleright \ \mathfrak{B}(R) \asymp \frac{R}{\mathfrak{C}(f_0^-(R))} \ \text{if} \ \gamma = 1 \land \alpha = \beta. \end{split}$$

In our third example we discuss the exceptional case " $(\delta, \gamma) = (2, 0)$ " in the third row of the table in Theorem 4.6. We show that in some cases (interpreting  $f_1^$ appropriately) the written bound still holds and is even attained by B(R), while in others B(R) is strictly larger due to domination of L(t, R).

5.9 Example. Assume that

$$(\delta_l, \alpha_l) \preceq (1, 1+\beta), \quad (\delta, \gamma) = (2, 0), \quad \alpha \le 2.$$
 (5.9)

These assumptions ensure that the exceptional case from the third row of the table in Theorem 4.6 (and also of Theorem 4.1 C) takes place. Note that (5.9) implies that  $\delta_l \leq \delta_{\phi}$ . Moreover,  $\beta > 0$  since  $(\gamma, \beta) \succ (0, 0)$ .

For parameters subject to (5.9) it holds that

$$\mathfrak{B}(R) \asymp \begin{cases} R^{\frac{2-\alpha+\beta}{2-\alpha+2\beta}} & \text{if } \alpha < 2, \\ R^{\frac{1}{2}} \log R & \text{if } \alpha = 2, \end{cases}$$

and

$$B(R) \begin{cases} \asymp \mathfrak{B}(R) & \text{if } \delta_l = \delta_\phi \lor (\delta_l < \delta_\phi \land \alpha \le 1 + \beta), \\ \asymp R^{\frac{1}{1+\beta}} \gg \mathfrak{B}(R) & \text{if } \delta_l < \delta_\phi \land \alpha > 1 + \beta. \end{cases}$$

Observe, moreover, that the bound for order in the third row of the table in Theorem 4.6 has no continuous extension to (2,0); its directional limits vary from  $\frac{1}{2}$  to 1. The above formula shows that the actual order of the bound B(R) has nothing to do with  $(\delta, \gamma)$  (being equal to (2,0)). Yet, it is sometimes given by the same formula, only with the "logarithmic exponents"  $\alpha, \beta$  instead of  $\delta, \gamma$ . Also note that the exponent  $\frac{1}{1+\beta}$  also occurred in Example 5.7.

## 5.3 Two corollaries given in terms of Jacobi parameters

We present two applications of Theorem 5.3 in which we return to the regime of power moment problems. The first is a supplement to a result from [Pru20], and in the second we give examples where the Nevanlinna matrix has prescribed growth.

At this point we need the concrete formulae relating Jacobi parameters with Hamiltonian parameters. They read as

$$\frac{1}{b_n} = \sin(\phi_{n+1} - \phi_n) \sqrt{l_{n+1}l_n},$$

$$a_n = -\frac{1}{l_n} \left[ \cot(\phi_{n+1} - \phi_n) + \cot(\phi_n - \phi_{n-1}) \right],$$
(5.10)

where the angles are chosen such that  $\phi_{n+1} - \phi_n \in [0, \pi)$ , cf. [Kac99]. Given the Jacobi parameters, it is in general hard to solve the equations (5.10) for the Hamiltonian parameters. Under the assumption that  $b_n, a_n$  have a certain power-like asymptotic, the approximate size of  $l_n$  and  $|\sin(\phi_{n+1} - \phi_n)|$  can be determined.

**5.10 Corollary.** Let  $a_n \in \mathbb{R}$  and  $b_n > 0$  be sequences which have asymptotics

$$b_n = n^{\sigma} \left( \frac{|y_0|}{2} + \frac{x_1}{n} + \frac{x_2}{n^2} + O\left(\frac{1}{n^{2+\epsilon}}\right) \right),$$

$$a_n = n^{\sigma} \left( y_0 + \frac{y_1}{n} + \frac{y_2}{n^2} + O\left(\frac{1}{n^{2+\epsilon}}\right) \right),$$

where

$$\sigma > 2, y_0 \neq 0, \quad x_1, x_2, y_1, y_2 \in \mathbb{R}, \quad \epsilon > 0.$$

Assume that the moment problem with these Jacobi parameters is indeterminate, and let W(z) be its Nevanlinna matrix. Then

$$\log\left(\max_{|z|=R} \|W(z)\|\right) \asymp R^{\frac{1}{\sigma}}.$$

Before we come to the proof, let us put this statement in the right context. It deals with the critical situation that off-diagonal and diagonal of the Jacobi matrix are comparable with ratio 2. This setting was considered in [Pru20, Theorem 2]. In that theorem occurrence of limit circle case was characterised in terms of the data of the expansions, and it was shown that W(z) is of order  $\frac{1}{\sigma}$  with positive type. The significance of Corollary 5.10 is that now we know that W(z) is also of finite type.

*Proof.* In the proof of [Pru20, Theorem 2] it was shown that

$$l_n \simeq \lambda(n)^2$$
,  $|\sin(\phi_{n+1} - \phi_n)| \simeq \frac{1}{n^{\sigma}\lambda(n)^2}$ 

where  $\lambda$  is a function of the form  $\lambda(t) = n^{\tau}$  or  $\lambda(t) = n^{\tau} \log t$  with some  $\tau \in [-\frac{\sigma}{2}, -\frac{1}{2})$ . The assumptions of Theorem 5.3 are thus satisfied with  $d_l(t) := \lambda(t)^2$  and  $d_{\phi}(t) := \frac{1}{n^{\sigma}\lambda(n)^2}$ . Applying this theorem yields

$$\log\left(\max_{|z|=R} \|W(z)\|\right) \asymp \left[\frac{1}{d_l d_{\phi}}\right]^-(R) \asymp R^{\frac{1}{\sigma}}.$$

We come to our second corollary, where we produce a variety of examples with prescribed growth of the Nevanlinna matrix (slower than the threshold  $R^{\frac{1}{2}}$ ). Thereby, the speed of growth is always determined by the off-diagonal, and the diagonal can be as large or as small as we please.

**5.11 Corollary.** Let g be regularly varying with  $\operatorname{ind} g \in (0, \frac{1}{2})$ .

(i) Let  $\omega \in [-2, 2]$ . Then there exist Jacobi parameters  $b_n$  and  $a_n$ , such that

$$b_n \sim g^-(n), \ \frac{a_n}{b_n} \to \omega, \quad \log\left(\max_{|z|=R} \|W(z)\|\right) \asymp g(R)$$

(ii) Let  $\omega_n \neq 0$ ,  $\omega_n \to 0$ , be such that  $\lim_{n\to\infty} \frac{\omega_{n-1}}{\omega_n}$  exists in  $(-1,\infty)$ . Then there exist Jacobi parameters  $b_n$  and  $a_n$ , such that

$$b_n \sim g^-(n), \ \frac{a_n}{b_n} \sim \omega_n, \quad \log\left(\max_{|z|=R} \|W(z)\|\right) \asymp g(R).$$

The condition in (i) that  $|\omega| \leq 2$  is no restriction, since otherwise we could not have limit circle case by Wouk's theorem.

Proof. We specify lengths and angles, which is done differently in different cases.

 $\triangleright \text{ Assume that } \omega \in (-2,2). \text{ Let } \psi \in (0,\pi) \text{ be such that } \cos \psi = -\frac{\omega}{2}, \text{ and set}$ 

$$l_n := \frac{1}{\sin \psi \cdot g^{-}(n)}, \qquad \phi_{n+1} := n\psi, \ n \in \mathbb{N},$$

so that  $\phi_{n+1} - \phi_n = \psi$ .

 $\triangleright$  Assume that  $\omega = -2$ . Set

$$l_n := \frac{n}{g^{-}(n)}, \qquad \phi_{n+1} := \sum_{k=1}^n \frac{1}{k}, \ n \in \mathbb{N},$$

so that  $\phi_{n+1} - \phi_n = \frac{1}{n}$ .

 $\triangleright$  Assume that  $\omega = 2$ . Set

$$l_n := \frac{n}{q^{-}(n)}, \qquad \phi_{n+1} := n\pi - \sum_{k=1}^n \frac{1}{k}, \ n \in \mathbb{N},$$

so that  $\phi_{n+1} - \phi_n = \pi - \frac{1}{n}$ .

 $\triangleright$  Assume that  $\omega_n \neq 0$ ,  $\omega_n \to 0$ , and  $\gamma := \lim_{n \to \infty} \frac{\omega_{n-1}}{\omega_n}$  exists in  $(-1, \infty)$ . Set

$$l_n := \frac{1}{q^{-}(n)}, \qquad \phi_{n+1} := n\frac{\pi}{2} + \frac{1}{2(1+\gamma)} \sum_{\substack{k=1\\|\omega_n| < \pi}}^n \omega_n, \ n \in \mathbb{N},$$

so that  $\phi_{n+1} - \phi_n = \frac{\pi}{2} + \frac{\omega_n}{2(1+\gamma)}$  for all sufficiently large n.

Let  $b_n$  and  $a_n$  be the Jacobi parameters given by (5.10). Then, in all cases,  $b_n \sim q^-(n)$ . Multiplying the two equations from (5.10), shows that

$$\frac{a_n}{b_n} = -\sqrt{\frac{l_{n+1}}{l_n}} \cdot \bigg( \cos(\phi_{n+1} - \phi_n) + \cos(\phi_n - \phi_{n-1}) \frac{\sin(\phi_{n+1} - \phi_n)}{\sin(\phi_n - \phi_{n-1})} \bigg),$$

and this implies the asserted property of  $\frac{a_n}{b_n}$ . Finally, we apply Theorem 5.3 with the obvious choices for  $d_l, d_{\phi}$  to obtain that

$$\log\left(\max_{|z|=R} \|W(z)\|\right) \asymp g(R)$$

-

## Appendix A. Regularly varying functions

In complex analysis the growth of an analytic function is compared with functions of the form  $\exp(\alpha(r))$ . The most classical comparison functions are powers  $\alpha(r) = r^{\rho}$ , and this leads to the notions of order and type. A refined comparison scale was introduced already at a very early stage by E.Lindelöf [Lin05] who considered comparison functions behaving for  $r \to \infty$  like

$$r^{\alpha} \cdot \left(\log r\right)^{\beta_1} \cdot \left(\log \log r\right)^{\beta_2} \cdot \ldots \cdot \left(\underbrace{\log \cdots \log r}_{m^{\text{th iterate}}} r\right)^{\beta_m},$$

where  $\alpha > 0$  and  $\beta_1, \ldots, \beta_m \in \mathbb{R}$ . Functions which are nowadays commonly used as comparison functions are regularly varying functions in Karamata sense, cf. [BGT89, Chapter 7] (for other levels of generality see also [Lev80; Rub96]). Lindelöf's comparison functions are examples of functions of that kind.

Let us now recall Karamata's definition of regular variation.

**A.1 Definition.** A function  $a: [1, \infty) \to (0, \infty)$  is called *regularly varying* at  $\infty$  with *index*  $\alpha \in \mathbb{R}$ , if it is measurable and

$$\forall \lambda \in (0,\infty): \lim_{r \to \infty} \frac{a(\lambda r)}{a(r)} = \lambda^{\alpha}.$$

We write ind a for the index of regular variation of function a. A regularly varying function with index 0 is also called *slowly varying*.

Regularly varying function a are used to quantify growth for  $r \to \infty$ , and hence the values of a(r) for small r are irrelevant. This allows to change a on any finite interval without changing the essence of results, and this freedom can often be used to assume a has some additional practical properties.

We cite a number of fundamental theorems on regularly varying functions. Proofs can be found, e.g., in [BGT89] or [Sen76]. We start with an *representation theorem*.

**A.2 Theorem** (Representation theorem). Let  $\alpha \in \mathbb{R}$ . A function  $a : [1, \infty) \to (0, \infty)$  is regularly varying with index  $\alpha$  if and only if it has a representation of the form

$$a(r) = r^{\alpha} \cdot c(r) \exp\left(\int_{1}^{r} \epsilon(u) \frac{\mathrm{d}u}{u}\right), \qquad r \in [1, \infty),$$

where  $c, \epsilon$  are measurable,  $\lim_{r\to\infty} c(r) = c \in (0,\infty)$ , and  $\lim_{r\to\infty} \epsilon(r) = 0$ .

If  $\alpha$  is slowly varying (i.e.,  $\alpha = 0$ ) and eventually nondecreasing (nonincreasing), then  $\epsilon$  may be taken eventually nonnegative (nonpositive).

It is a legitimate intuition that regularly varying functions fill in the scale of powers, and that a regularly varying function with index  $\alpha$  behaves roughly like the power  $r^{\alpha}$ . The following results, which we will use frequently, express this intuition very clearly. The first is a variant of the *Potter bounds*, and the second is the classical *Karamata Theorem* about asymptotic integration.

**A.3 Theorem** (Potter bounds (variant)). Let  $\alpha$  be regularly varying with index  $\alpha \in \mathbb{R}$ .

- (i)  $\forall \epsilon > 0$ :  $r^{\alpha \epsilon} \ll a(r) \ll r^{\alpha + \epsilon}$ ,
- (ii)  $\lim_{r \to \infty} \frac{\log a(r)}{\log r} = \alpha$ ,
- (iii) For all  $\epsilon > 0$  the quotients  $\frac{a(r)}{r^{\alpha-\epsilon}}$  and  $\frac{r^{\alpha+\epsilon}}{a(r)}$  are  $\sim$  to an eventually increasing function.

**A.4 Theorem** (Karamata's Theorem). Let a be regularly varying with index  $\alpha \in \mathbb{R}$ .

(i) Assume that  $\alpha \geq -1$ . Then the function  $x \mapsto \int_1^x a(t) dt$  is regularly varying with index  $\alpha + 1$ , and

$$\lim_{x \to \infty} \left( \frac{xa(x)}{\int_{1}^{x} a(t) \, \mathrm{d}t} \right) = \alpha + 1$$

(ii) Assume that  $\alpha \leq -1$  and  $\int_1^\infty a(t) dt < \infty$ . Then the function  $x \mapsto \int_x^\infty a(t) dt$  is regularly varying with index  $\alpha + 1$ , and

$$\lim_{x \to \infty} \left( \frac{x a(x)}{\int_{x}^{\infty} a(t) dt} \right) = -(\alpha + 1).$$

A regularly varying function a with positive index is – at least asymptotically – invertible. In fact, if

$$a^{-}(x) \coloneqq \sup \left\{ t \in [1, \infty) \mid a(t) < x \right\},$$

we have the following result, cf. [BGT89, Theorem 1.5.12].

**A.5 Theorem.** Let a be regularly varying with index  $\alpha > 0$ . Then  $a^-$  is regularly varying with index  $\frac{1}{\alpha}$ , and

$$(a \circ a^{-})(x) \sim (a^{-} \circ a)(x) \sim x. \tag{A.1}$$

Any regularly varying function  $a^-$  with the property (A.1) is called an *asymptotic inverse* of a, and asymptotic inverses are determined uniquely up to  $\sim$ . We recall a useful formula for computing asymptotic inverses for functions of a particular form.

A.6 Remark. Assume that  $\rho > 0$  and that f is regularly varying. Set  $g := f \circ \log$  (for sufficiently large t). Then

$$a(t) \coloneqq t^{\rho} g(t), \quad a^{-}(t) \coloneqq \rho^{\frac{\operatorname{ind} f}{\rho}} \cdot \left(\frac{t}{g(t)}\right)^{\frac{1}{\rho}}$$

are asymptotic inverses of each other.

Another practical observation is the following.

A.7 Remark. Let f be regularly varying with  $\operatorname{ind} f > 0$ , and assume we have a function g with  $g \approx f$ . Then g is regularly varying with  $\operatorname{ind} g = \operatorname{ind} f$  and  $g^- \approx f^-$ .

By the Potter bounds every regularly varying function  $\alpha$  is bounded and bounded away from zero on every interval  $[r_1, r_2]$  sufficiently far to the right. Sometimes it is needed for technical reasons to assume this property for all compact intervals in the domain of  $\alpha$ . Of course, this is no loss in generality; remember that modification on a finite interval does not change the essence of the function  $\alpha$ .

**A.8 Lemma.** Let  $a : [1, \infty) \to (0, \infty)$  be slowly varying and assume that  $a \leq 1$  locally. For  $R \geq a(1)$  set

$$\label{eq:product} \mathcal{C}(R) \mathrel{\mathop:}= \sup \left\{ t \in [1,\infty) \mid \, \sup_{1 \leq s \leq t} \frac{\alpha(s)}{R} \leq 1 \right\} \, \in [1,\infty].$$

Then  $\mathfrak{G}$  grows faster than any power, i.e.,  $R^{\rho} \leq \mathfrak{G}(R)$  for every  $\rho > 0$  and R sufficiently large.

*Proof.* Let  $\rho > 0$  and set  $\epsilon := \frac{1}{\rho}$ . By Theorem A.3, there exists M > 0 such that  $\alpha(r) \leq r^{\epsilon}$  for all  $r \geq M$ . If  $R \geq \sup_{r \in [1,M]} \alpha(r)$ , this means that

$$\Big\{t\in[1,\infty)\mid \sup_{1\leq s\leq t}\frac{s^{\epsilon}}{R}\leq 1\Big\}\subseteq \Big\{t\in[1,\infty)\mid \sup_{1\leq s\leq t}\frac{\alpha(s)}{R}\leq 1\Big\}.$$

The assertion follows if we take suprema of both sets.

**A.9 Lemma.** Let  $\alpha$  be regularly varying with index  $\alpha > 0$  and assume that  $\alpha \approx 1$  locally. Then there exists a continuously differentiable and regularly varying function s, where s'(t) > 0 for  $t \in [1, \infty)$ ,

$$a(t) \asymp \mathfrak{s}(t), \qquad t \in [1, \infty),$$
  
 $a(t) \sim s(t), \qquad t \to \infty.$ 

*Proof.* We use the smooth variation theorem [BGT89, Theorem 1.8.2]. This gives a function  $\mathfrak{s}$  that is continuously differentiable, whose derivative is positive for all sufficiently large t, and such that  $\mathfrak{a} \sim \mathfrak{s}$  for  $t \to \infty$ . W.l.o.g. we assume  $\mathfrak{s}'(t) > 0$  for all  $t \in [1, \infty)$ .

Choose  $t_0 \geq 1$  such that  $\frac{1}{2}\alpha(t) \leq \mathfrak{s}(t) \leq \frac{3}{2}\alpha(t)$  for all  $t \geq t_0$ . By our assumption,  $\alpha \approx 1$  on  $[1, t_0]$ . Since  $\mathfrak{s}$  is continuous,  $\mathfrak{s} \approx 1 \approx \alpha$  on  $[1, t_0]$ . Summing up, we have  $\mathfrak{s} \approx \alpha$  on  $[1, \infty)$ .

Another lemma in a similar direction is the following. The proof is immediate from the representation theorem and we do not go into details.

**A.10 Lemma.** Let a be regularly varying, and assume that a is  $\approx$  to some nonincreasing function. Then there exist

$$\epsilon[a], \kappa[a] \colon [1, \infty) \to (0, \infty)$$

such that

- $\triangleright \epsilon[a]$  is locally integrable,  $\lim_{t\to\infty} \epsilon[a](t) = 0$ , and  $\epsilon[a] \leq 0$  if ind a = 0,
- $\triangleright \kappa[a]$  is eventually constant,

 $\triangleright$  The function

$$\mathcal{S}[a](t) := \kappa[a](t) \cdot t^{\operatorname{ind} a} \cdot \exp\left(\int_{1}^{r} \epsilon[a](u) \frac{\mathrm{d}u}{u}\right)$$

is nonincreasing, continuous, and  $\approx a$ .

We speak of any function  $\mathcal{S}[\alpha]$  as in the lemma as a *nonincreasing smoothening* of  $\alpha$ .

Of course,  $\kappa[a]$  and  $\epsilon[a]$  are far from unique. We mention two particular instances of the freedom of choice in  $\mathcal{S}[a]$ .

A.11 Remark.

- (i) If  $\mathscr{S}[\alpha]$  is some nonincreasing smoothening of  $\alpha$  and  $\alpha > 0$ , then also  $\alpha \cdot \mathscr{S}[\alpha]$  is a nonincreasing smoothening of  $\alpha$ . This corresponds to multiplying  $\kappa[\alpha]$  by  $\alpha$ .
- (ii) If  $\mathscr{S}[\alpha]$  is some nonincreasing smoothening of  $\alpha$  and  $\alpha > \lim_{t\to\infty} \mathscr{S}[\alpha](t)$ , then also  $\min\{\mathscr{S}[\alpha](t), \alpha\}$  is a nonincreasing smoothening of  $\alpha$ . This is seen by modifying  $\kappa[\alpha]$  on a finite interval.

$$\kappa[a_1a_2] \coloneqq \kappa[a_1] \cdot \kappa[a_2], \quad \epsilon[a_1a_2] \coloneqq \epsilon[a_1] + \epsilon[a_2].$$

## References

- [Akh61] N.I. Akhiezer, Классическая проблема моментов и некоторые вопросы анализа, связанные с нею. Russian, English translation: *The classical moment problem and some related questions in analy*sis, Oliver & Boyd, Edinburgh, 1965, Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1961.
- [Ber56] Yu.M. Berezanskii, "Expansion according to eigenfunction of a partial difference equation of order two", *Trudy Moskov. Mat. Obšč.* 5 (1956), pp. 203–268.
- [BP94] C. Berg and H.L. Pedersen, "On the order and type of the entire functions associated with an indeterminate Hamburger moment problem", Ark. Mat. 32.1 (1994), pp. 1–11.
- [BS14] C. Berg and R. Szwarc, "On the order of indeterminate moment problems", *Adv. Math.* 250 (2014), pp. 105–143.
- [BGT89] N.H. Bingham, C.M. Goldie, and J.L. Teugels, *Regular variation*, vol. 27, Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 1989.

One more property of this construction is as follows. Assume we have two functions  $a_1, a_2$  that are both regularly varying and  $\approx$  to some nonincreasing function. If  $\mathcal{S}[a_1]$  and  $\mathcal{S}[a_2]$  are nonincreasing smoothenings of  $a_1$  and  $a_2$ , respectively, then  $\mathcal{S}[a_1] \cdot \mathcal{S}[a_2]$  is a nonincreasing smoothening of  $a_1 \cdot a_2$ . This corresponds to taking

- [BS98] A. Borichev and M. Sodin, "The Hamburger moment problem and weighted polynomial approximation on discrete subsets of the real line", J. Anal. Math. 76 (1998), pp. 219–264.
- [Kac99] I.S. Kac, "Inclusion of the Hamburger power moment problem in the spectral theory of canonical systems", Russian, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 262.Issled. po Linein. Oper. i Teor. Funkts. 27 (1999), English translation: J. Math. Sci. (New York) 110 (2002), no. 5, 2991–3004, pp. 147–171, 234.
- [Lev80] B.Ja. Levin, Distribution of zeros of entire functions, Revised, vol. 5, Translations of Mathematical Monographs, Translated from the Russian by R. P. Boas, J. M. Danskin, F. M. Goodspeed, J. Korevaar, A. L. Shields and H. P. Thielman, Providence, R.I.: American Mathematical Society, 1980.
- [Lin05] E. Lindelöf, "Sur les fonctions entières d'ordre entier.", French, Ann. Sci. Éc. Norm. Supér. (3) 22 (1905), pp. 369–395.
- [Liv39] M.S. Livšic, "On some questions concerning the determinate case of Hamburger's moment problem", Russian. English summary, *Rec. Math. N. S. [Mat. Sbornik]* 6(48) (1939), pp. 293–306.
- [Pru20] R. Pruckner, "Density of the spectrum of Jacobi matrices with power asymptotics", Asymptot. Anal. 117.3-4 (2020), pp. 199–213.
- [PRW23] R. Pruckner, J. Reiffenstein, and H. Woracek, An upper bound for the Nevanlinna matrix of an indeterminate moment sequence, July 20, 2023, arXiv: 2307.10748v1.
- [PRW17] R. Pruckner, R. Romanov, and H. Woracek, "Bounds on order of indeterminate moment sequences", Constr. Approx. 46 (2017), pp. 199–225.
- [PW22] R. Pruckner and H. Woracek, "A growth estimate for the monodromy matrix of a canonical system", J. Spectr. Theory 12.4 (2022), pp. 1623–1657.
- [Rie23] M. Riesz, "Sur le problème des moments. III.", French, Ark. Mat. Astron. Fys. 17.16 (1923), p. 52.
- [Rom17] R. Romanov, "Order problem for canonical systems and a conjecture of Valent", Trans. Amer. Math. Soc. 369.2 (2017), pp. 1061–1078.
- [Rub96] L.A. Rubel, Entire and meromorphic functions, Universitext, With the assistance of James E. Colliander, New York: Springer-Verlag, 1996.
- [Sch17] K. Schmüdgen, The moment problem, vol. 277, Graduate Texts in Mathematics, Springer, Cham, 2017.
- [Sen76] E. Seneta, Regularly varying functions, Lecture Notes in Mathematics, Vol. 508, Springer-Verlag, Berlin-New York, 1976.
- [ST43] J. A. Shohat and J. D. Tamarkin, *The Problem of Moments*, American Mathematical Society Mathematical surveys, vol. I, New York: American Mathematical Society, 1943.

R. Pruckner

Fachgruppe Patent- und Lizenzmanagement

Vienna University of Technology Resselgasse 3/058 1040 Wien AUSTRIA email: raphael.pruckner@tuwien.ac.at

J. Reiffenstein Department of Mathematics University of Vienna Oskar-Morgenstern-Platz 1 1090 Wien AUSTRIA email: jakob.reiffenstein@univie.ac.at

H. Woracek Institute for Analysis and Scientific Computing Vienna University of Technology Wiedner Hauptstraße 8–10/101 1040 Wien AUSTRIA email: harald.woracek@tuwien.ac.at