

# Limit behaviour of Nevanlinna functions

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**Abstract:** We study the sets of radial or nontangential limit points towards  $i\infty$  of a Nevanlinna function  $q$ . It is shown that a subset  $\mathcal{L}$  of  $\overline{\mathbb{C}_+}$  is the set of radial limit points of some  $q$ , if and only if it is closed, nonempty, and connected. Given  $\mathcal{L}$ , we explicitly construct a Hamiltonian  $H$  such that  $\mathcal{L}$  is the set of radial and nontangential limit points of the Weyl coefficient  $q_H$  of the canonical system with Hamiltonian  $H$ . Our method is based on a study of the continuous group action of rescaling operators on the set of all Hamiltonians.

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## 1 Introduction

A Nevanlinna function is an analytic function in the open upper half-plane  $\mathbb{C}_+$  whose values lie in  $\mathbb{C}_+ \cup \mathbb{R}$ . Such functions are intensively studied for various reasons; we mention two of them.

- ▷ In complex analysis they occur as regularised Cauchy-transforms of positive Poisson integrable measures, e.g. [Lev80; KK68; GT00]. Namely, a function  $q$  is a Nevanlinna function if and only if it is of the form

$$q(z) = a + bz + \int_{\mathbb{R}} \left( \frac{1}{x-z} - \frac{x}{1+x^2} \right) d\mu(x), \quad z \in \mathbb{C}_+, \quad (1.1)$$

where  $a \in \mathbb{R}$ ,  $b \geq 0$ , and  $\mu$  is a positive Borel measure on the real line with  $\int_{\mathbb{R}} \frac{d\mu(x)}{1+x^2} < \infty$ .

- ▷ In spectral theory of differential operators they occur as Weyl coefficients whenever H.Weyl's nested disks method is applicable, e.g. [Wey10; Tit46; Atk64].

The connection between these two instances is that (for simplicity we suppress some technical issues and exceptional cases) the measure  $\mu$  in the integral representation (1.1) of the Weyl coefficient of an equation is a spectral measure for the corresponding selfadjoint model operator.

The natural context of Weyl's method is the framework of two-dimensional canonical systems

$$y'(t) = zJH(t)y(t), \quad t \in (0, \infty), \quad (1.2)$$

where  $z \in \mathbb{C}$  is the eigenvalue parameter,  $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ , and the Hamiltonian  $H$  of the system is assumed to satisfy  $H(t) \geq 0$  and  $\text{tr } H(t) = 1$  a.e., e.g. [Bra68; HSW00; Rom14; Rem18].

It is a deep theorem due to L.de Branges that the map assigning to each Hamiltonian  $H$  the Weyl coefficient  $q_H$  of the equation (1.2) is a bijection between the set of all Hamiltonians

$$\mathbb{H} := \{H : (0, \infty) \rightarrow \mathbb{R}^{2 \times 2} \mid H \text{ measurable}, H(t) \geq 0, \text{tr } H(t) = 1 \text{ a.e.}\} \quad (1.3)$$

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up to equality a.e., and the set of all Nevanlinna functions including the function identically equal to  $\infty$

$$\mathcal{N} := \{q: \mathbb{C}_+ \rightarrow \overline{\mathbb{C}} \mid q \text{ analytic}, q(\mathbb{C}_+) \subseteq \overline{\mathbb{C}_+}\}.$$

Here  $\overline{\mathbb{C}}$  denotes the Riemann sphere  $\mathbb{C} \cup \{\infty\}$  considered as a Riemann surface in the usual way, and  $\overline{\mathbb{C}_+}$  denotes the closure of  $\mathbb{C}_+$  in the sphere, explicitly,  $\overline{\mathbb{C}_+} = \mathbb{C}_+ \cup \mathbb{R} \cup \{\infty\}$ . The assignment  $H \mapsto q_H$  is also called de Branges' correspondence.

Having available this bijection, it is a natural task to relate properties of  $H$  to properties of  $q_H$ . For many properties of Hamiltonians or Nevanlinna functions it turns out to be quite involved (or even quite impossible) to find their counterpart on the other side of de Branges' correspondence. One type of properties where some explicit relations are known is the high-energy behaviour of  $q_H$ , i.e., its behaviour towards  $i\infty$ . It is a frequently instantiated intuition that the high-energy behaviour of  $q_H$  corresponds to the local behaviour of  $H$  at 0. For example it is shown in [EKT18] that the nontangential limit<sup>1</sup>  $\lim_{z \xrightarrow{*} i\infty} q_H(z)$  exists in  $\overline{\mathbb{C}}$ , if and only if the limit  $\lim_{t \searrow 0} \frac{1}{t} \int_0^t H(s) ds$  exists in  $\mathbb{R}^{2 \times 2}$ . Moreover, if these limits exist, they are related by simple formulae.

In this paper we investigate the situation when the Weyl coefficient does not necessarily have a limit. Substitutes for the limit value are the set of all nontangential limit points of a Nevanlinna function  $q$ , i.e.

$$\text{LP}[q(z)]_* := \{\zeta \in \overline{\mathbb{C}} \mid \exists z_n \in \mathbb{C}_+. \ z_n \xrightarrow{*} i\infty \wedge \lim_{n \rightarrow \infty} q(z_n) = \zeta\}, \quad (1.4)$$

and, for  $\theta \in (0, \pi)$ , the set of all radial limit points of  $q$ , i.e.

$$\text{LP}[q(e^{i\theta}r)]_{r \geq 1} := \{\zeta \in \overline{\mathbb{C}} \mid \exists r_n \geq 1. \ r_n \rightarrow \infty \wedge \lim_{n \rightarrow \infty} q(e^{i\theta}r_n) = \zeta\}.$$

These sets are always nonempty by compactness of  $\overline{\mathbb{C}_+}$ , and connected by continuity of  $q$ . Moreover, a set of radial limit points is always closed.

Our main contribution is a converse construction: given a closed nonempty connected subset  $\mathcal{L}$  of  $\overline{\mathbb{C}_+}$ , we explicitly construct a Hamiltonian  $H$ , such that

$$\text{LP}[q_H(z)]_* = \text{LP}[q(e^{i\theta}r)]_{r \geq 1} = \mathcal{L}, \quad \theta \in (0, \pi).$$

As a corollary, we obtain that a subset  $\mathcal{L} \subseteq \overline{\mathbb{C}_+}$  is the set of radial limit points of some Nevanlinna function if and only if it is closed, nonempty, and connected. This fact is invariant under composition with fractional linear transformations, for instance it immediately transfers to bounded analytic functions in the unit disc, or Caratheodory functions (functions analytic in the unit disk with nonnegative real part). It also can be transferred to generalised Nevanlinna functions in the sense of [KL77] by means of their multiplicative decomposition [Dij+00].

Our method of proof is based on a rescaling trick which goes back at least to Y.Kasahara [Kas76], who applied it on the level of Krein strings, and which

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<sup>1</sup>We write  $z_n \xrightarrow{*} i\infty$  for:  $|z_n| \rightarrow \infty$  while  $\arg z_n \in [\alpha, \pi - \alpha]$  for some  $\alpha \in (0, \frac{\pi}{2}]$ . And we write  $\lim_{z \xrightarrow{*} i\infty} q(z) = \zeta$ , if  $\lim_{n \rightarrow \infty} q(z_n) = \zeta$  for every sequence  $z_n \xrightarrow{*} i\infty$ . Convergence on the Riemann sphere is understood w.r.t. the chordal metric.

was exploited further in [KW10] and [EKT18]. Namely, given a Hamiltonian  $H \in \mathbb{H}$ , one considers rescaled Hamiltonians

$$(\mathcal{A}_r H)(t) := H\left(\frac{t}{r}\right), \quad t \in (0, \infty), \quad r > 0. \quad (1.5)$$

The operators  $\mathcal{A}_r$  blow up the scale and thereby zoom into the vicinity of 0. We will see that limit points of  $q_H$  are related to limit points of the family  $(\mathcal{A}_r H)_{r \geq 1}$ . In fact, one may say that the continuous group action of rescaling operators on  $\mathbb{H}$  is responsible for the fact that high-energy behaviour of  $q_H$  relates to local behaviour of  $H$  at 0.

In [EKT18; Kas76; KW10] a simple continuity property of de Branges' correspondence was sufficient to obtain the desired conclusions. This property goes back at least to [Bra61], where it formed the basis of the existence part in the inverse spectral theorem. Despite being used in the literature ever since, an explicit presentation was given only recently in [Rem18]. In the presently considered general situation, when limits do not necessarily exist, finer arguments and a thorough understanding of the topology on  $\mathbb{H}$  are necessary.

After this introduction the article is structured in three more sections. In Section 2 we study the appropriate topology on  $\mathbb{H}$ ; this section is to a certain extent of expository nature. Contrasting the presentation in [Rem18], we introduce the topology from a higher level viewpoint. Namely, as an inverse limit of weak topologies on sets of Hamiltonians defined on finite intervals ( $T \in (0, \infty)$ )

$$\mathbb{H}_T := \{H: (0, T) \rightarrow \mathbb{R}^{2 \times 2} \mid H \text{ measurable}, H(t) \geq 0, \operatorname{tr} H(t) = 1 \text{ a.e.}\}. \quad (1.6)$$

By this approach the most important features, namely compactness and metrisability, are readily built into the construction. Besides offering structural clarity, it also simplifies matters by avoiding the unnecessary passage from  $L^1$  to the space of complex Borel measures made in [Bra61; Rem18]. For the convenience of the non-specialist reader, we include a complete and concise derivation of the required continuity of de Branges' correspondence  $H \leftrightarrow q_H$ .

Section 3 is the technical core of our work. We study the group action of rescaling operators  $\{\mathcal{A}_r \mid r > 0\}$  on  $\mathbb{H}$ , and relate limit points of  $q_H$  with limit points of  $(\mathcal{A}_r H)_{r \geq 1}$ , cf. Propositions 3.14 and 3.15. The case that limits exist, which has been studied in [EKT18], is revisited.

Section 4 is devoted to the proof of the main result of the paper. In Theorem 4.1 we give the afore mentioned explicit construction of Hamiltonians whose Weyl coefficient has a prescribed set of limit points. From this we deduce by some elementary topological facts that every set subject to the obvious necessary conditions (closed, nonempty, and connected) is the set of radial and nontangential limit points of the Weyl coefficient of an (explicitly known) Hamiltonian, cf. Corollary 4.3.

## 2 Topologising the set of Hamiltonians

Thoroughly understanding convergence of Hamiltonians is crucial for our present investigation. We shall first consider Hamiltonians defined on a finite interval and then pass to Hamiltonians on the half-line by a limiting process.

## 2.1 Hamiltonians on a finite interval

Recall the notation (1.6):

**2.1 Definition.** For  $T > 0$  we denote the set of all Hamiltonians on the interval  $(0, T)$  by  $\mathbb{H}_T$ , i.e.,

$$\mathbb{H}_T := \{H: (0, T) \rightarrow \mathbb{R}^{2 \times 2} \mid H \text{ measureable}, H(t) \geq 0, \text{tr } H(t) = 1 \text{ a.e.}\}.$$

We shall always tacitly identify two Hamiltonians which coincide almost everywhere.  $\diamond$

Let  $\|\cdot\|$  denote the  $\ell^1$ -norm on  $\mathbb{C}^{2 \times 2}$ . For every positive semidefinite matrix  $A = (a_{ij})_{i,j=1}^2$  it holds that  $|a_{ij}| \leq \|A\| \leq 2 \text{tr } A$ . This yields that all  $H \in \mathbb{H}_T$  are entrywise (equivalently, w.r.t.  $\|\cdot\|$ ) essentially bounded by 2. In particular, we have

$$\mathbb{H}_T \subseteq L^1((0, T), \mathbb{C}^{2 \times 2}).$$

The space  $L^1((0, T), \mathbb{C}^{2 \times 2})$ , and with it its subset  $\mathbb{H}_T$ , carries (at least) three natural topologies. As a Banach space  $L^1((0, T), \mathbb{C}^{2 \times 2})$  has its norm and weak topologies  $\mathcal{T}_{\|\cdot\|_1}$  and  $\mathcal{T}_w$ . As an  $L^1$ -space it can be endowed with convergence in measure which gives rise to a metrisable topology  $\mathcal{T}_{\text{meas}}$ , e.g. [Bog07, Remark 2.2.7].

*2.2 Remark.* Every  $\mathcal{T}_{\text{meas}}$ -convergent sequence contains a subsequence which converges pointwise a.e., and hence  $\mathbb{H}_T$  is  $\mathcal{T}_{\text{meas}}$ -closed. Since the norm topology is finer than  $\mathcal{T}_{\text{meas}}$ , it follows that  $\mathbb{H}_T$  is  $\|\cdot\|_1$ -closed. In turn, since  $\mathbb{H}_T$  is convex, it is also weakly closed.

As we already observed, the set  $\mathbb{H}_T$  is uniformly bounded. Hence, every sequence  $(H_n)_{n \in \mathbb{N}}$  in  $\mathbb{H}_T$  which converges pointwise a.e., already converges w.r.t.  $\|\cdot\|_1$ . Moreover, we have  $\mathcal{T}_{\|\cdot\|_1}|_{\mathbb{H}_T} = \mathcal{T}_{\text{meas}}|_{\mathbb{H}_T}$ .  $\diamond$

*2.3 Remark.* In order to work with the weak topology, we recall the following representation of continuous functionals. We have (linearly and homeomorphically)

$$\begin{aligned} L^1((0, T), \mathbb{C}^{2 \times 2})' &\cong [L^1(0, T)^4]' \\ &\cong [L^1(0, T)]^4 \cong [L^\infty(0, T)]^4 \cong L^\infty((0, T), \mathbb{C}^{2 \times 2}). \end{aligned}$$

A linear homeomorphism is given by the assignment

$$\left\{ \begin{array}{ll} L^\infty((0, T), \mathbb{C}^{2 \times 2}) & \rightarrow L^1((0, T), \mathbb{C}^{2 \times 2})' \\ (f_{ij})_{i,j=1}^2 & \mapsto \left[ (h_{ij})_{i,j=1}^2 \mapsto \sum_{i,j=1}^2 \int_0^T h_{ij}(t) f_{ij}(t) dt \right] \end{array} \right.$$

Sometimes it is practical to note that  $L^1((0, T), \mathbb{C}^{2 \times 2})'$  is spanned by the set of functionals

$$\left\{ H \mapsto \int_0^T e_1^* H(t) e_2 \cdot f(t) dt \mid e_1, e_2 \in \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}, f \in L^\infty(0, T) \right\}.$$

$\diamond$

The weak topology on  $\mathbb{H}_T$  has striking properties.

**2.4 Lemma.** *Let  $T > 0$ . The weak topology  $\mathcal{T}_w|_{\mathbb{H}_T}$  is compact and metrisable.*

*Proof.* Since  $\mathbb{H}_T$  is uniformly bounded, it is also uniformly integrable. The Dunford-Pettis Theorem (see, e.g., [Bog07, Theorem 4.7.18]) yields that  $\mathbb{H}_T$  is relatively compact in the weak topology of  $L^1((0, T), \mathbb{C}^{2 \times 2})$ . We already know that  $\mathbb{H}_T$  is weakly closed, and conclude that it is weakly compact.

Since  $L^1((0, T), \mathbb{C}^{2 \times 2})$  is  $\|\cdot\|_1$ -separable, the weak topology on a weakly compact subset is metrisable (see, e.g., [Fab+01, Proposition 3.2.9]).  $\square$

We come to a variant of continuity in de Branges' correspondence for Hamiltonians on finite intervals. To this end, we need some notation. First, denote by  $\mathcal{E}$  the set of all entire  $2 \times 2$ -matrix functions endowed with the topology  $\mathcal{T}_{\text{lu}}$  of locally uniform convergence. Second, we introduce a notation for the fundamental solution of the canonical system.

**2.5 Definition.** Let  $T > 0$ . For  $H \in \mathbb{H}_T$  we denote by  $W(H; t, z)$  the unique solution of the initial value problem

$$\begin{cases} \frac{\partial}{\partial t} W(H; t, z) J = z W(H; t, z) H(t), & t \in [0, T], \\ W(H; 0, z) = I, \end{cases} \quad (2.1)$$

where  $I$  is the  $2 \times 2$ -identity matrix.  $\diamond$

For every fixed  $t \in [0, T]$ , the matrix  $W(H; t, \cdot)$  is an entire function, i.e.,  $W(H; t, \cdot)$  belongs to  $\mathcal{E}$ .

**2.6 Definition.** Let  $T > 0$ . Then we denote by  $\Psi_T$  the map

$$\Psi_T: \begin{cases} \mathbb{H}_T & \rightarrow \mathcal{E} \\ H & \mapsto W(H; T, \cdot) \end{cases}$$

$\diamond$

The above announced continuity result now reads as follows.

**2.7 Theorem (Continuity; fundamental solution).**

*Let  $T > 0$ . Then  $\Psi_T$  is  $\mathcal{T}_w$ -to- $\mathcal{T}_{\text{lu}}$ -continuous.*

This theorem is implicit in [Bra61], and, up to identification of topologies, explicit in [Rem18, Theorem 5.7]. For convenience of the reader we give a complete proof.

*Proof of Theorem 2.7.* Let

$$W(H; t, z) = \sum_{l=0}^{\infty} W_l(H; t) z^l \quad (2.2)$$

be the power series expansion of  $W(H; t, \cdot)$ . Plugging this in the equation (2.1), we obtain that the coefficients  $W_l(H; t)$  satisfy the recurrence

$$W_0(H, t) = I, \quad W_{l+1}(H; t) = - \int_0^t W_l(H; s) H(s) J \, ds, \quad l \in \mathbb{N}.$$

From this one inductively obtains

$$\|W_l(H; t)\| \leq \frac{(2t)^l}{l!}, \quad H \in \mathbb{H}_T, t \in [0, T], l \in \mathbb{N}. \quad (2.3)$$

Therefore, for each compact set  $K \subseteq \mathbb{C}$ , the series (2.2) converges uniformly on  $\mathbb{H}_T \times [0, T] \times K$ , and we have the global growth estimate

$$\|W(H; t, z)\| \leq e^{2t|z|}, \quad (H, t, z) \in \mathbb{H}_T \times [0, T] \times K.$$

Now let  $(H_n)_{n \in \mathbb{N}}$  be a sequence in  $\mathbb{H}_T$  which converges weakly to some  $H \in \mathbb{H}_T$ . By uniformity in  $H$  of the convergence of the series (2.2), it suffices to show that

$$\forall l \in \mathbb{N}. \quad \lim_{n \rightarrow \infty} W_l(H_n; T) = W_l(H; T)$$

in order to conclude that  $\lim_{n \rightarrow \infty} W(H_n; T, \cdot) = W(H; T, \cdot)$  locally uniformly on  $\mathbb{C}$ . We use induction to show the stronger statement

$$\forall l \in \mathbb{N}. \quad \lim_{n \rightarrow \infty} W_l(H_n; t) = W_l(H; t) \text{ uniformly for } t \in [0, T].$$

For  $l = 0$  this is trivial. Assume that it has already been established for some  $l \in \mathbb{N}$ . Using the recurrence gives

$$\begin{aligned} & \|W_{l+1}(H_n; t) - W_{l+1}(H; t)\|_\infty \\ &= \left\| \int_0^t W_l(H_n; s) H_n(s) J \, ds - \int_0^t W_l(H; s) H(s) J \, ds \right\|_\infty \\ &\leq \left\| \int_0^t (W_l(H_n; s) - W_l(H; s)) H_n(s) J \, ds \right\|_\infty \\ &\quad + \underbrace{\left\| \int_0^t W_l(H; s) (H_n(s) - H(s)) J \, ds \right\|_\infty}_{=: g_n(t)}. \end{aligned} \quad (2.4)$$

The first summand is estimated as

$$\left\| \int_0^t (W_l(H_n; s) - W_l(H; s)) H_n(s) J \, ds \right\|_\infty \leq T \cdot \|W_l(H_n; s) - W_l(H; s)\|_\infty \cdot 2,$$

and tends to 0 by the inductive hypothesis. The functions  $g_n$  tend to 0 pointwise on  $[0, T]$  since

$$\|g_n(t)\| = \left\| \int_0^T \mathbb{1}_{(0,t)}(s) W_l(H; s) \cdot (H_n(s) - H(s)) \cdot J \, ds \right\|$$

and  $\lim_{n \rightarrow \infty}^w H_n = H$ . It holds that

$$g_n(0) = 0, \quad \|g_n(t) - g_n(t')\| \leq |t - t'| \cdot \frac{(2T)^l}{l!} \cdot 4,$$

and by the Arzela-Ascoli Theorem the family  $\{g_n \mid n \in \mathbb{N}\}$  is relatively compact in  $C([0, T], \mathbb{C}^{2 \times 2})$ . Thus pointwise convergence upgrades to uniform convergence, and we obtain that also the second summand in (2.4) tends to 0.  $\square$

## 2.2 Hamiltonians on the half-line

We turn to Hamiltonians defined on the whole half-line. Recall the notation (1.3):

**2.8 Definition.** We denote the set of all Hamiltonians on the half-line  $(0, \infty)$  by  $\mathbb{H}$ , i.e.,

$$\mathbb{H} := \{H: (0, \infty) \rightarrow \mathbb{R}^{2 \times 2} \mid H \text{ measurable}, H(t) \geq 0, \text{tr } H(t) = 1 \text{ a.e.}\}.$$

Again we tacitly identify two Hamiltonians which coincide almost everywhere.  $\diamond$

Let us furthermore introduce a notation for restriction maps between various spaces.

**2.9 Definition.** For  $0 < T \leq T'$  we denote (slightly overloading notation by not distinguishing the first two notationally)

$$\begin{aligned} \rho_{T'}^{T'}: & \begin{cases} L^1((0, T'), \mathbb{C}^{2 \times 2}) & \rightarrow & L^1((0, T), \mathbb{C}^{2 \times 2}) \\ H & \mapsto & H|_{(0, T)} \end{cases} \\ \rho_T^{T'}: & \begin{cases} \mathbb{H}_{T'} & \rightarrow & \mathbb{H}_T \\ H & \mapsto & H|_{(0, T)} \end{cases} \\ \rho_T: & \begin{cases} \mathbb{H} & \rightarrow & \mathbb{H}_T \\ H & \mapsto & H|_{(0, T)} \end{cases} \end{aligned}$$

$\diamond$

*2.10 Remark.* Let  $0 < T \leq T'$ . Then the restriction map  $\rho_T^{T'}$  is linear and (norm-)contractive. In particular,  $\rho_T^{T'}$  is continuous, when domain and codomain are either both endowed with the norm topology or both with the weak topology.  $\diamond$

We have

$$\rho_T^T = \text{id}_{\mathbb{H}_T}, \quad 0 < T, \quad \rho_T^{T'} \circ \rho_{T'}^{T''} = \rho_T^{T''}, \quad 0 < T \leq T' \leq T''.$$

This means that the diagram  $\langle (\langle \mathbb{H}_T, \mathcal{T}_w \rangle)_{T>0}, (\rho_T^{T'})_{0 < T \leq T'} \rangle$  is an inverse system of compact Hausdorff (even metrisable) spaces. Its limit is thus a compact Hausdorff space, e.g. [Bou66, Chp.I §9.6].

A Hamiltonian  $H$  can (as every function can) be identified with the family  $(\rho_T H)_{T>0}$  of its restrictions and we have  $\rho_T^{T'} \circ \rho_{T'}^{T''} = \rho_T^{T''}$ ,  $0 < T \leq T' \leq T''$ . This means that the cone  $\langle \mathbb{H}, (\rho_T)_{T>0} \rangle$  is the limit of the diagram  $\langle (\langle \mathbb{H}_T, \mathcal{T}_w \rangle)_{T>0}, (\rho_T^{T'})_{0 < T \leq T'} \rangle$ .

*2.11 Remark.* Because of its importance, let us make this construction more explicit. Let  $\mathcal{T}$  be the initial topology on  $\mathbb{H}$  induced by the restriction maps  $\rho_T: \mathbb{H} \rightarrow \langle \mathbb{H}_T, \mathcal{T}_w \rangle$ ,  $T > 0$ . The family  $\{\rho_T \mid T > 0\}$  is point separating, and hence the evaluation map

$$e: \begin{cases} \langle \mathbb{H}, \mathcal{T} \rangle & \rightarrow & \prod_{T>0} \langle \mathbb{H}_T, \mathcal{T}_w \rangle \\ H & \mapsto & (\rho_T(H))_{T>0} \end{cases}$$

is a homeomorphism onto its image. The image of  $e$  equals

$$e(\mathbb{H}) = \left\{ (H_T)_{T>0} \in \prod_{T>0} \mathbb{H}_T \mid \forall 0 < T \leq T'. \rho_T^{T'}(H_{T'}) = H_T \right\},$$

and, by continuity of  $\rho_T^{T'}$ , is closed. Tychonoff's Theorem yields compactness of the product space, and thus also of  $\langle \mathbb{H}, \mathcal{T} \rangle$ . Clearly, the Hausdorff property is also inherited.  $\diamond$

The diagram  $\langle (\langle \mathbb{H}_T, \mathcal{T}_w \rangle)_{T>0}, (\rho_T^{T'})_{0 < T \leq T'} \rangle$  has a countable cofinal subdiagram, e.g.,  $\langle (\langle \mathbb{H}_n, \mathcal{T}_w \rangle)_{n \in \mathbb{N}}, (\rho_n^m)_{0 < n \leq m} \rangle$ . This implies that also metrisability is inherited by its limit.

*2.12 Remark.* Let us also make this argument explicit. Let  $\tilde{\mathcal{T}}$  be the initial topology on  $\mathbb{H}$  induced by the restriction maps  $\rho_n: \mathbb{H} \rightarrow \langle \mathbb{H}_n, \mathcal{T}_w \rangle$ ,  $n \in \mathbb{N}$ . The family  $\{\rho_n \mid n \in \mathbb{N}\}$  is point separating, and hence the evaluation map

$$\tilde{e}: \begin{cases} \langle \mathbb{H}, \tilde{\mathcal{T}} \rangle & \rightarrow \prod_{n \in \mathbb{N}} \langle \mathbb{H}_n, \mathcal{T}_w \rangle \\ H & \mapsto (\rho_n(H))_{n \in \mathbb{N}} \end{cases}$$

is a homeomorphism onto its image. Being homeomorphic to a subspace of a countable product of metrisable spaces,  $\langle \mathbb{H}, \tilde{\mathcal{T}} \rangle$  is metrisable.

The identity map  $\text{id}_{\mathbb{H}}: \langle \mathbb{H}, \mathcal{T} \rangle \rightarrow \langle \mathbb{H}, \tilde{\mathcal{T}} \rangle$  is bijective and continuous. Its domain is compact and its codomain is Hausdorff, thus it is a homeomorphism. We see that  $\mathcal{T} = \tilde{\mathcal{T}}$ .  $\diamond$

The countable cofinal subdiagram specified above has the additional property that for each of its elements there exist only finitely many smaller ones. Thus we can choose metrics on  $\mathbb{H}_n$  which induce  $\mathcal{T}_w$  on  $\mathbb{H}_n$  and make the restriction maps  $\rho_m^n$ ,  $0 < m \leq n$  contractive.

*2.13 Remark.* Again we make the argument explicit. Choose arbitrary metrics  $\tilde{d}_n$  on  $\mathbb{H}_n$  which induce  $\mathcal{T}_w$  on  $\mathbb{H}_n$ , and set

$$d_n(H_1, H_2) := \sum_{k=1}^n \tilde{d}_k(\rho_k^n(H_1), \rho_k^n(H_2)), \quad H_1, H_2 \in \mathbb{H}_n.$$

Since  $\rho_k^n$  are continuous,  $d_n$  induces the same topology as  $\tilde{d}_n$ . Moreover, for  $m \leq n$  we have

$$\begin{aligned} d_m(\rho_m^n(H_1), \rho_m^n(H_2)) &= \sum_{k=1}^m \tilde{d}_k(\rho_k^m(\rho_m^n(H_1)), \rho_k^m(\rho_m^n(H_2))) \\ &= \sum_{k=1}^m \tilde{d}_k(\rho_k^n(H_1), \rho_k^n(H_2)) \leq \sum_{k=1}^n \tilde{d}_k(\rho_k^n(H_1), \rho_k^n(H_2)) = d_n(H_1, H_2). \end{aligned}$$

$\diamond$

For later use, recall the following property of a limit of metric spaces with contractions.

**2.14 Lemma.** *Let  $d_n$ ,  $n \in \mathbb{N}$ , be metrics on  $\mathbb{H}_n$  which induce the topology  $\mathcal{T}_w$  on  $\mathbb{H}_n$ , and are such that all restriction maps  $\rho_m^n$  are contractive. Let  $(H_k)_{k \in \mathbb{N}}$*



be a sequence in  $\mathbb{H}$ , and  $H \in \mathbb{H}$ . Assume that there exists a sequence  $(l_k)_{k \in \mathbb{N}}$  such that

$$\lim_{k \rightarrow \infty} l_k = \infty, \quad \lim_{k \rightarrow \infty} d_{l_k}(\rho_{l_k}(H_k), \rho_{l_k}(H)) = 0.$$

Then  $\lim_{k \rightarrow \infty} H_k = H$  in  $\langle \mathbb{H}, \mathcal{T} \rangle$ .

*Proof.* Let  $n \in \mathbb{N}$  and  $\varepsilon > 0$ . Choose  $k_0 \in \mathbb{N}$  such that

$$l_k \geq n, \quad d_{l_k}(\rho_{l_k}(H_k), \rho_{l_k}(H)) \leq \varepsilon, \quad k \geq k_0.$$

Then

$$\begin{aligned} d_n(\rho_n(H_k), \rho_n(H)) &= d_{l_k}((\rho_n^{l_k} \circ \rho_{l_k})(H_k), (\rho_n^{l_k} \circ \rho_{l_k})(H)) \\ &\leq d_{l_k}(\rho_{l_k}(H_k), \rho_{l_k}(H)) \leq \varepsilon, \quad k \geq k_0. \end{aligned}$$

We see that  $\lim_{k \rightarrow \infty} \rho_n(H_k) = \rho_n(H)$  in  $\langle \mathbb{H}_n, \mathcal{T}_w \rangle$  for each fixed  $n \in \mathbb{N}$ .  $\square$

Now we turn to continuity of de Branges' correspondence. Recall that  $\mathcal{N}$ , as a subset of the space of all analytic functions of  $\mathbb{C}_+$  into the Riemann sphere, naturally carries the topology  $\mathcal{T}_{\text{lu}}$  of locally uniform convergence.

**2.15 Definition.** We denote by  $\Psi$  the map

$$\Psi: \begin{cases} \mathbb{H} & \rightarrow \mathcal{N} \\ H & \mapsto q_H \end{cases}$$

$\diamond$

### 2.16 Theorem (Continuity; Weyl coefficients).

The map  $\Psi$  is  $\mathcal{T}$ -to- $\mathcal{T}_{\text{lu}}$ -homeomorphic.

Also this theorem is implicit in [Bra61] and explicit in [Rem18, Theorem 5.7], and we provide a complete derivation for the convenience of the reader.

The proof of the “finite interval variant” Theorem 2.7 relied on the uniform estimate (2.3) of power series coefficients. The proof of the present “half-line variant” will follow from a uniform estimate of the size of Weyl disks.

Recall that for  $H \in \mathbb{H}$  and  $T > 0$  the Weyl disk  $\Omega_{T,z}(H)$  at  $z \in \mathbb{C}_+$  is the image of  $\overline{\mathbb{C}_+}$  under the fractional linear transformation with coefficient matrix  $W(H; T, z)$ . Moreover, recall that the inverse stereographical projection is Lipschitz continuous. In fact, considering the Riemann sphere as the unit sphere whose south pole lies at the origin of the complex plane, the chordal distance  $\chi$  of two points  $\zeta, \xi \in \mathbb{C}$  (suppressing explicit notation of the stereographical projection) is

$$\chi(\zeta, \xi) = \frac{2|\zeta - \xi|}{\sqrt{1 + |\zeta|^2} \sqrt{1 + |\xi|^2}},$$

and hence  $\chi(\zeta, \xi) \leq 2|\zeta - \xi|$ ,  $\zeta, \xi \in \mathbb{C} \subseteq \overline{\mathbb{C}}$ .

**2.17 Lemma.** Let  $H \in \mathbb{H}$ ,  $T > 0$ , and  $z \in \mathbb{C}_+$ . The diameter of the Weyl disk  $\Omega_{T,z}$  w.r.t. the chordal metric can be estimated as

$$\text{diam}_\chi \Omega_{T,z}(H) \leq \frac{8}{T \cdot \text{Im } z}.$$

*Proof.* Write  $H = \begin{pmatrix} h_1 & h_3 \\ h_3 & h_2 \end{pmatrix}$ , and assume first that  $\int_0^T h_2(s) ds \geq \frac{T}{2}$ . Then  $\infty \notin \Omega_{T,z}(H)$ . By the usual formula for the euclidean radius of  $\Omega_{T,z}(H)$ , see e.g. [Rem18, Lemma 3.11], the monotonicity result [Bra61, Lemma 4], and the differential equation (2.1), we find

$$\text{diam}_\chi \Omega_{T,z}(H) \leq 2 \text{diam}_{|\cdot|} \Omega_{T,z}(H) \leq 2 \cdot \frac{2}{\text{Im } z \cdot \int_0^T h_2(t) dt} \leq \frac{8}{T \cdot \text{Im } z}.$$

Now consider the case that  $\int_0^T h_2(s) ds < \frac{T}{2}$ . Then we must have  $\int_0^T h_1(s) ds \geq \frac{T}{2}$ , and the already established estimate applies to  $\tilde{H} := -JHJ$ . A computation shows that  $W(\tilde{H}; T, z) = -JW(H; T, z)J$ , and hence the Weyl disk  $\Omega_{T,z}(\tilde{H})$  is the image of  $\Omega_{T,z}(H)$  under the fractional linear transformation with coefficient matrix  $J$ . Since  $J$  is unitary, this is a rotation of the sphere, and hence isometric w.r.t. the chordal metric. We obtain

$$\text{diam}_\chi \Omega_{T,z}(H) = \text{diam}_\chi \Omega_{T,z}(\tilde{H}) \leq \frac{8}{T \cdot \text{Im } z}.$$

□

*Proof of Theorem 2.16.* Let  $(H_n)_{n \in \mathbb{H}}$  be a sequence in  $\mathbb{H}$  which converges to some  $H \in \mathbb{H}$ . By the definition of the topology of  $\mathbb{H}$ , this means that  $\lim_{n \rightarrow \infty}^w \rho_T(H_n) = \rho_T(H)$  for every  $T > 0$ .

Write  $W(H; T, z) = (w_{ij}(H; t, z))_{i,j=1}^2$ , and denote

$$Q_{n,T}(z) := \frac{w_{12}(H_n; T, z)}{w_{22}(H_n; T, z)}, \quad Q_T(z) := \frac{w_{12}(H; T, z)}{w_{22}(H; T, z)}, \quad z \in \mathbb{C}_+.$$

Throughout the following all limits of complex numbers are understood w.r.t. the chordal metric  $\chi$ .

Let  $K \subseteq \mathbb{C}_+$  satisfy  $\inf_{z \in K} \text{Im } z > 0$ . Lemma 2.17 shows that the limit

$$q_{\tilde{H}}(z) = \lim_{T \rightarrow \infty} \frac{w_{12}(\tilde{H}; T, z)}{w_{22}(\tilde{H}; T, z)}$$

defining the Weyl coefficient of a Hamiltonian  $\tilde{H}$  is attained uniformly for  $(\tilde{H}, z) \in \mathbb{H} \times K$ . This implies

$$\triangleright \lim_{T \rightarrow \infty} Q_{n,T}(z) = q_{H_n}(z) \text{ uniformly for } (n, z) \in \mathbb{N} \times K;$$

$$\triangleright \lim_{T \rightarrow \infty} Q_T(z) = q_H(z) \text{ uniformly for } z \in K.$$

Theorem 2.7 says that

$$\triangleright \text{For each } T > 0 \text{ we have } \lim_{n \rightarrow \infty} Q_{n,T}(z) = Q_T(z) \text{ locally uniformly for } z \in \mathbb{C}_+.$$

Together we obtain

$$q_H(z) = \lim_{T \rightarrow \infty} \lim_{n \rightarrow \infty} Q_{n,T}(z) = \lim_{n \rightarrow \infty} \lim_{T \rightarrow \infty} Q_{n,T}(z) = \lim_{n \rightarrow \infty} q_{H_n}(z)$$

locally uniformly for  $z \in \mathbb{C}_+$ .

Being a continuous bijection of a compact space onto a Hausdorff space,  $\Psi$  is a homeomorphism. □

We often use continuity of  $\Psi$  in another form.

**2.18 Definition.** We denote by  $\Phi$  the map

$$\Phi: \begin{cases} \mathbb{H} \times \mathbb{C}_+ & \rightarrow \overline{\mathbb{C}_+} \\ (H, w) & \mapsto q_H(w) \end{cases}$$

◇

The following reformulations of continuity of  $\Psi$  are obtained by elementary arguments.

**2.19 Corollary (Continuity; Weyl coefficients / variant).**

*Each of the below properties (i) and (ii) is equivalent to  $\mathcal{T}$ -to- $\mathcal{T}_{\text{lu}}$ -continuity of  $\Psi$ , and hence holds true.*

- (i) *The map  $\Phi$  is continuous when  $\mathbb{H} \times \mathbb{C}_+$  is endowed with the product topology of  $\mathcal{T}$  and the euclidean topology.*
- (ii) *For every compact set  $K \subseteq \mathbb{C}_+$  the family  $\{\Phi(\cdot, w) \mid w \in K\}$  is equicontinuous.*

*Proof.* Denote by  $d$  a metric on  $\mathbb{H}$  which induces  $\mathcal{T}$ .

① To see that  $\mathcal{T}$ -to- $\mathcal{T}_{\text{lu}}$ -continuity of  $\Psi$  is equivalent to (ii), it is enough to write out the definitions. Since  $\mathbb{H}$  is compact,  $\Psi$  is continuous if and only if it is uniformly continuous. Uniform continuity of  $\Psi$  means

$$\forall K \subseteq \mathbb{C}_+ \text{ compact } \forall \varepsilon > 0 \exists \delta > 0 \forall H_1, H_2 \in \mathbb{H}$$

$$d(H_1, H_2) \leq \delta \Rightarrow \sup_{w \in K} \chi(\Psi(H_1)(w), \Psi(H_2)(w)) \leq \varepsilon$$

On the other hand, (ii) means

$$\forall K \subseteq \mathbb{C}_+ \text{ compact } \forall \varepsilon > 0 \exists \delta > 0 \forall H_1, H_2 \in \mathbb{H} \forall w \in K.$$

$$d(H_1, H_2) \leq \delta \Rightarrow \chi(\Phi(H_1, w), \Phi(H_2, w)) \leq \varepsilon$$

② We show that (ii) implies (i). Assume that  $H_n \rightarrow H$  in  $\mathbb{H}$  and  $w_n \rightarrow w$  in  $\mathbb{C}_+$ . Then  $\{w_n \mid n \in \mathbb{N}\}$  is a compact subset of  $\mathbb{C}_+$ . We have

$$\chi(\Phi(H_n, w_n), \Phi(H, w)) \leq \chi(\Phi(H_n, w_n), \Phi(H, w_n)) + \chi(\Phi(H, w_n), \Phi(H, w)).$$

The family  $\{\Phi(\cdot, w_n) \mid n \in \mathbb{N}\}$  is equicontinuous, and hence the first summand tends to 0 when  $n \rightarrow \infty$ . The function  $\Phi(H, \cdot)$  is continuous, and hence also the second summand tends to 0.

③ We show that (i) implies (ii). Let  $K \subseteq \mathbb{C}_+$  be compact. Then  $\mathbb{H} \times K$  is compact, and hence  $\Phi|_{\mathbb{H} \times K}$  is uniformly continuous. This means that

$$\forall \varepsilon > 0 \exists \delta > 0 \forall H_1, H_2 \in \mathbb{H} \forall w_1, w_2 \in K.$$

$$d(H_1, H_2) + |w_1 - w_2| \leq \delta \Rightarrow \chi(\Phi(H_1, w_1), \Phi(H_2, w_2)) \leq \varepsilon$$

Using  $w_1 = w_2$  gives equicontinuity of  $\{\Phi(\cdot, w) \mid w \in K\}$ .

□

*2.20 Remark.* The topology  $\mathcal{T}$  constructed above coincides with the topology defined in [Rem18, Chapter 5.2]. This follows by writing out our definition and the argument which gave metrisability of  $\mathcal{T}$ .

In [EKT18, Proposition 2.3] convergence of Hamiltonians is introduced in yet another form. To see that this form coincides with convergence w.r.t.  $\mathcal{T}$ , one only has to note that step functions are dense in  $L^1$ .  $\diamond$

### 2.3 Constant Hamiltonians

A particular role is played by Hamiltonians  $H \in \mathbb{H}$  which are constant a.e. on  $(0, \infty)$ . We denote the set of all such as  $\mathbb{CH}$ .

Constant Hamiltonians can be identified with the points of  $\overline{\mathbb{C}_+}$ .

**2.21 Definition.** Let  $\Theta: \overline{\mathbb{C}_+} \rightarrow \mathbb{CH}$  be the map acting as

$$\Theta(\zeta) := \begin{pmatrix} h_1 & h_3 \\ h_3 & h_2 \end{pmatrix},$$

where

$$h_1 := \frac{|\zeta|^2}{|\zeta|^2 + 1}, \quad h_2 := \frac{1}{|\zeta|^2 + 1}, \quad h_3 := \frac{-\operatorname{Re} \zeta}{|\zeta|^2 + 1},$$

if  $\zeta \neq \infty$ , and

$$\Theta(\infty) := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

$\diamond$

The map  $\Theta$  is bijective. Its inverse  $\Theta^{-1}: \mathbb{CH} \rightarrow \overline{\mathbb{C}_+}$  is given as

$$\Theta^{-1} \begin{pmatrix} h_1 & h_3 \\ h_3 & h_2 \end{pmatrix} = \frac{-h_1}{h_2 + i\sqrt{h_1 h_2 - h_3^2}},$$

if  $h_2 \neq 0$ , and

$$\Theta^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \infty.$$

Note that  $\det \Theta(\zeta) = 0$  if and only if  $\zeta \in \overline{\mathbb{R}}$ , and that  $\Theta(\zeta)$  is diagonal if and only if  $\zeta \in i\overline{\mathbb{R}_+}$ .

From the defining formulae it is obvious that for each  $T > 0$  the map  $\rho_T \circ \Theta: \overline{\mathbb{C}_+} \rightarrow \langle \mathbb{H}_T, \mathcal{T}_{\|\cdot\|_1} \rangle$  is continuous. Thus  $\rho_T \circ \Theta$  is also continuous into  $\mathcal{T}_w$ , and hence  $\Theta$  is continuous into  $\langle \mathbb{H}, \mathcal{T} \rangle$ . Since  $\overline{\mathbb{C}_+}$  is compact, each of

$$\langle \rho_T(\mathbb{CH}), \mathcal{T}_{\|\cdot\|_1} \rangle, \quad \langle \rho_T(\mathbb{CH}), \mathcal{T}_w \rangle, \quad \langle \mathbb{CH}, \mathcal{T} \rangle$$

is homeomorphic to  $\overline{\mathbb{C}_+}$ . In particular, these spaces are all compact.

*2.22 Remark.* The definition of  $\Theta$  is made in such a way that

$$q_{\Theta(\zeta)}(z) \equiv \zeta, \quad z \in \overline{\mathbb{C}_+},$$

in other words that  $\Phi(\Theta(\zeta), w) = \zeta$ ,  $w \in \mathbb{C}_+$ . This is shown by a simple calculation, e.g. [EKT18, §2.2, Example 1].  $\diamond$

For later use we introduce a separate notation for constant Hamiltonians corresponding to boundary points of  $\overline{\mathbb{C}_+}$ , namely,

$$\mathbb{CH}_0 := \Theta(\overline{\mathbb{R}}) = \{H \in \mathbb{CH} \mid \det H = 0\}.$$

### 3 The rescaling method

#### 3.1 The group of rescaling operators

We have already mentioned the rescaling operation on Hamiltonians in (1.5). Now we put this in an appropriate framework. The formula (1.5) actually defines maps between various spaces. We use three instances.

**3.1 Definition.** Let  $r > 0$ . Depending on the context, and without distinguishing notationally, we consider the assignment

$$\mathcal{A}_r: H(\omega) \mapsto H(\frac{\omega}{r})$$

as a map

$$\triangleright \mathcal{A}_r: L^1((0, T), \mathbb{C}^{2 \times 2}) \rightarrow L^1((0, rT), \mathbb{C}^{2 \times 2}) \quad (\text{where } T > 0),$$

$$\triangleright \mathcal{A}_r: \mathbb{H}_T \rightarrow \mathbb{H}_{rT} \quad (\text{where } T > 0),$$

$$\triangleright \mathcal{A}_r: \mathbb{H} \rightarrow \mathbb{H}.$$

◇

In the following lemmata we prove some basic properties of the operators  $\mathcal{A}_r$ .

#### 3.2 Lemma.

(i) *We have the computation rules*

$$\mathcal{A}_1 = \text{id}, \quad \mathcal{A}_r \circ \mathcal{A}_s = \mathcal{A}_s \circ \mathcal{A}_r = \mathcal{A}_{rs}, \quad r, s > 0, \quad (3.1)$$

$$\mathcal{A}_r \circ \rho_T = \rho_{rT} \circ \mathcal{A}_r, \quad \mathcal{A}_r \circ \rho_T^{T'} = \rho_{rT}^{T'} \circ \mathcal{A}_r, \quad r > 0, T' \geq T > 0, \quad (3.2)$$

where operators are understood on domains such that all compositions are well-defined.

(ii) *The operator  $\mathcal{A}_r$  is in each instance mentioned in Definition 3.1 a homeomorphism, where in the first and second instance domain and codomain may either both carry the norm topology or both the weak topology.*

*Proof.* The computation rules (3.1) and (3.2) are obvious. To show the stated continuity properties, note that

$$\|\mathcal{A}_r H\|_1 = \int_0^{rT} \|H(\frac{t}{r})\| dt = r \|H\|_1, \quad H \in L^1((0, T), \mathbb{C}^{2 \times 2}).$$

As a bounded linear operator,  $\mathcal{A}_r$  is also weak-to-weak-continuous. Continuity follows in the first two instances (using either topology). By the first rule in (3.2) we obtain continuity in the third instance. Since  $\mathcal{A}_r^{-1} = \mathcal{A}_{\frac{1}{r}}$  by (3.1), all  $\mathcal{A}_r$  are homeomorphisms.  $\square$

**3.3 Lemma.** *Let  $r_0, T > 0$ , and  $H \in L^1((0, \frac{T}{r_0}), \mathbb{C}^{2 \times 2})$ . Then the map*

$$\begin{cases} [r_0, \infty) & \rightarrow L^1((0, T), \mathbb{C}^{2 \times 2}) \\ r & \mapsto \rho_T^{\frac{r}{r_0}} \mathcal{A}_r H \end{cases}$$

*is  $\|\cdot\|_1$ -continuous.*

*Proof.* Let  $q, r \geq r_0$  with  $r \leq 2q$ , and let  $\varepsilon > 0$ . Choose  $F \in C([0, \frac{q}{r_0}T], \mathbb{C}^{2 \times 2})$  with

$$\int_0^{\frac{q}{r_0}T} \|(\mathcal{A}_q H)(t) - F(t)\| dt \leq \varepsilon.$$

Then

$$\begin{aligned} \int_0^T \|(\rho_{\frac{r}{r_0}T} \mathcal{A}_r H)(t) - (\rho_{\frac{q}{r_0}T} \mathcal{A}_q H)(t)\| dt &= \\ &= \int_0^T \|(\mathcal{A}_q H)(\frac{q}{r}t) - (\mathcal{A}_q H)(t)\| dt \leq \underbrace{\int_0^T \|(\mathcal{A}_q H)(\frac{q}{r}t) - F(\frac{q}{r}t)\| dt}_{\leq \frac{r}{q}\varepsilon \leq 2\varepsilon} \\ &\quad + \int_0^T \|F(\frac{q}{r}t) - F(t)\| dt + \underbrace{\int_0^T \|F(t) - (\mathcal{A}_q H)(t)\| dt}_{\leq \varepsilon}. \end{aligned}$$

The middle summand tends to 0 when  $r \rightarrow q$  by bounded convergence.  $\square$

**3.4 Proposition.** *The map*

$$\begin{cases} \mathbb{R}_+ \times \mathbb{H} & \rightarrow \mathbb{H} \\ (r, H) & \mapsto \mathcal{A}_r H \end{cases} \quad (3.3)$$

*is a continuous group action of  $\mathbb{R}_+$  on  $\mathbb{H}$ .*

*Proof.* The fact that (3.3) defines a group action is clear from the computation rules for  $\mathcal{A}_r$ . We have to show continuity, i.e., that for given  $H_n \rightarrow H$ ,  $r_n \rightarrow r$ , and  $T > 0$ ,

$$\lim_{n \rightarrow \infty}^w \rho_T \mathcal{A}_{r_n} H_n = \rho_T \mathcal{A}_r H.$$

Let  $e_1, e_2 \in \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$  and  $f \in L^\infty(0, T)$  be given. For later use denote by  $\tilde{f}$  the extension of  $f$  to  $L^\infty(0, \infty)$  with  $\tilde{f}(t) = 0$ ,  $t \geq T$ . Moreover, assume w.l.o.g. that  $\frac{r}{2} \leq r_n \leq 2r$  for all  $n$ .

First, we note that

$$\begin{aligned} \int_0^T e_1^* ((\rho_T \mathcal{A}_{r_n} H_n)(t) - (\rho_T \mathcal{A}_r H)(t)) e_2 \cdot f(t) dt &= \\ &= \int_0^T e_1^* ((\mathcal{A}_{r_n} H_n)(t) - (\mathcal{A}_{r_n} H)(t)) e_2 \cdot f(t) dt \\ &\quad + \int_0^T e_1^* ((\mathcal{A}_{r_n} H)(t) - (\mathcal{A}_r H)(t)) e_2 \cdot f(t) dt. \end{aligned}$$

The second summand tends to 0 when  $n \rightarrow \infty$  by Lemma 3.3. The first summand rewrites as

$$\begin{aligned} \int_0^T e_1^* (H_n(\frac{t}{r_n}) - H(\frac{t}{r_n})) e_2 \cdot f(t) dt &= \\ &= r_n \int_0^{\frac{T}{r_n}} e_1^* (H_n(s) - H(s)) e_2 \cdot f(r_n s) ds \\ &= r_n \int_0^{\frac{2}{r}T} e_1^* (H_n(s) - H(s)) e_2 \cdot \tilde{f}(r_n s) ds. \end{aligned}$$

The integral on the right side is estimated as

$$\begin{aligned} & \left| \int_0^{\frac{2}{r}T} e_1^*(H_n(s) - H(s)) e_2 \cdot \tilde{f}(r_n s) \, ds \right| \\ & \leq \left| \int_0^{\frac{2}{r}T} e_1^*(H_n(s) - H(s)) e_2 \cdot (\tilde{f}(r_n s) - \tilde{f}(rs)) \, ds \right| \\ & \quad + \left| \int_0^{\frac{2}{r}T} e_1^*(H_n(s) - H(s)) e_2 \cdot \tilde{f}(rs) \, ds \right| \end{aligned}$$

The first summand tends to 0 since  $\|e_1^*(H_n(s) - H(s)) e_2\|_\infty \leq 2$  and  $\|\tilde{f}(r_n s) - \tilde{f}(rs)\|_1 \rightarrow 0$ , and the second summand tends to 0 since  $H_n \rightarrow H$  in  $\mathbb{H}$ .  $\square$

*3.5 Remark.* The stabiliser of an element  $H \in \mathbb{H}$  under the group action (3.3), i.e.  $\{r \in \mathbb{R}^+ \mid \mathcal{A}_r H = H\}$ , is a closed subgroup of  $\mathbb{R}_+$ . Therefore it is either equal to  $\{1\}$  or  $\mathbb{R}_+$ , or it is a non-trivial cyclic subgroup.

▷ An element  $H$  remains fixed under the whole group, i.e. the stabiliser of  $H$  is  $\mathbb{R}^+$ , if and only if it is constant.

▷  $H$  has non-trivial stabiliser (i.e.,  $\neq \{1\}, \mathbb{R}_+$ ), if and only if it is nonconstant and multiplicatively periodic. This means that there exists  $p > 1$  with  $H(pt) = H(t)$  for a.a.  $t > 0$ , but that this does not hold for every  $p$ . The generator of the stabiliser is then the smallest period  $p > 1$ .

◇

Rescaling operators have a rescaling effect on fundamental solutions. This is a particular case of [EKT18, Lemma 2.7]. For the convenience of the reader we recall the argument.

**3.6 Lemma.** *Let  $H \in \mathbb{H}$  and  $r > 0$ . Then the fundamental solutions, Weyl disks, and Weyl coefficients, of  $H$  and  $\mathcal{A}_r H$  are related as ( $t \geq 0, z \in \mathbb{C}_+$ )*

$$W(\mathcal{A}_r H; t, z) = W(H; \frac{t}{r}, rz), \quad \Omega_{t,z}(\mathcal{A}_r H) = \Omega_{\frac{t}{r}, rz}(H), \quad q_{\mathcal{A}_r H}(z) = q_H(rz).$$

Using the notation  $\Phi$  from Definition 2.18, the relation between Weyl coefficients writes as

$$\Phi(\mathcal{A}_r H, z) = \Phi(H, rz), \quad H \in \mathbb{H}, r > 0, z \in \mathbb{C}_+. \quad (3.4)$$

*Proof.* Set  $\tilde{W}(t, z) := W(H; \frac{t}{r}, rz)$ . Then

$$\begin{aligned} \frac{\partial}{\partial t} \tilde{W}(t, z) J &= \frac{1}{r} \frac{\partial}{\partial t} W(H; \frac{t}{r}, rz) = \frac{1}{r} \cdot rz \cdot W(H; \frac{t}{r}, rz) H\left(\frac{t}{r}\right) \\ &= z \tilde{W}(t, z) (\mathcal{A}_r H)(t). \end{aligned}$$

Thus  $\tilde{W}(t, z)$  is the fundamental solution corresponding to  $\mathcal{A}_r H$ .

The relation between Weyl disks follows immediately, and the relation between Weyl coefficients follows by letting  $t \rightarrow \infty$ .  $\square$

## 3.2 Sets of limit points

We have already mentioned nontangential and radial limit points of Nevanlinna functions in (1.4). Recall:

**3.7 Definition.** Let  $q \in \mathcal{N}$ . Then we denote

$$\text{LP}[q(z)]_\star := \{\zeta \in \overline{\mathbb{C}} \mid \exists z_n \in \mathbb{C}_+. \ z_n \xrightarrow{\star} i\infty \wedge \lim_{n \rightarrow \infty} q(z_n) = \zeta\}.$$

If, moreover,  $\theta \in (0, \pi)$ , set

$$\text{LP}[q(e^{i\theta}r)]_{r \geq 1} := \{\zeta \in \overline{\mathbb{C}} \mid \exists r_n \geq 1. \ r_n \rightarrow \infty \wedge \lim_{n \rightarrow \infty} q(e^{i\theta}r_n) = \zeta\}.$$

◇

For every  $\theta$  the set  $\text{LP}[q(e^{i\theta}r)]_{r \geq 1}$  is nonempty since  $\overline{\mathbb{C}}$  is compact. Moreover, we clearly have the inclusion

$$\bigcup_{\theta \in (0, \pi)} \text{LP}[q(e^{i\theta}r)]_{r \geq 1} \subseteq \text{LP}[q(z)]_\star.$$

**3.8 Remark.** It is elementary to see that the sets  $\text{LP}[q(e^{i\theta}r)]_{r \geq 1}$  and  $\text{LP}[q(z)]_\star$  are connected, and that  $\text{LP}[q(e^{i\theta}r)]_{r \geq 1}$  is closed. ◇

Let us provide the necessary argument. In order to reduce the technical effort we impose some countability assumptions which will always be satisfied in our present context.

**Lemma.** Let  $X, Y$  be metric spaces, and assume that  $X$  is a directed set with relation  $\preceq$  such that for each  $x_0 \in X$  the set  $\{x \in X \mid x_0 \preceq x\}$  is path-connected, and that  $\langle X, \preceq \rangle$  has a countable cofinal subset.

For a function  $f: X \rightarrow Y$ , denote

$$\text{LP}[f]_\preceq := \{y \in Y \mid \exists x_n \in X. \ y = \lim_{n \rightarrow \infty} f(x_n) \wedge \{x_n \mid n \in \mathbb{N}\} \text{ cofinal in } X\}.$$

Then  $\text{LP}[f]_\preceq$  is closed. If, moreover,  $f$  is continuous and  $Y$  is compact, then  $\text{LP}[f]_\preceq$  is connected.

To clarify our choice of notation, note that  $\text{LP}[f]_\preceq$  is the set of limit points of the net  $f$ , in the sense that

$$\text{LP}[f]_\preceq = \{y \in Y \mid \exists \text{ subnet } f \circ \iota: J \rightarrow Y. \ y = \lim_{j \in J} (f \circ \iota)(j)\}.$$

*Proof of Lemma.* Fix a cofinal sequence  $\xi_1 \preceq \xi_2 \preceq \dots$

① Assume that  $y \in \overline{\text{LP}[f]_\preceq}$ . Choose  $y_n \in \text{LP}[f]_\preceq$  with  $y_n \rightarrow y$ , and  $(x_{n,l})_{l \in \mathbb{N}}$  cofinal in  $X$  with  $\lim_{l \rightarrow \infty} f(x_{n,l}) = y_n$ . Given  $k \in \mathbb{N}$ , choose  $n(k)$  such that  $d_Y(y_{n(k)}, y) \leq \frac{1}{k}$ . Now choose  $l(k)$  such that  $\xi_k \preceq x_{n(k), l(k)}$  and  $d_Y(f(x_{n(k), l(k)}), y_{n(k)}) \leq \frac{1}{k}$ . Then  $(x_{n(k), l(k)})_{k=1}^\infty$  is cofinal in  $X$  and  $\lim_{k \rightarrow \infty} f(x_{n(k), l(k)}) = y$ .

② Assume towards a contradiction that  $\text{LP}[f]_\preceq = A_1 \cup A_2$  where  $A_1, A_2$  are nonempty disjoint relatively closed subsets of  $\text{LP}[f]_\preceq$ . Since  $\text{LP}[f]_\preceq$  is closed,



$A_1$  and  $A_2$  are closed in  $Y$ . As a metric space  $Y$  is a normal topological space. Hence, we find disjoint open sets  $O_1, O_2$  in  $Y$  with  $O_1 \supseteq A_1, O_2 \supseteq A_2$ .

For  $n \in \mathbb{N}$  consider the set  $M_n := \{x \in X \mid \xi_n \preceq x\}$ . Since  $O_i \cap \text{LP}[f]_{\preceq} \neq \emptyset$ ,  $i = 1, 2$ , and  $f$  is continuous, we find  $x_{i,n} \in M_n$  with  $f(x_{i,n}) \in O_i$ . Choose a path  $\gamma_n$  in  $M_n$  with  $\gamma_n(0) = x_{1,n}$  and  $\gamma_n(1) = x_{2,n}$ . Then  $(f \circ \gamma_n)^{-1}(O_i)$  are nonempty open disjoint subsets of  $[0, 1]$ , and hence cannot cover  $[0, 1]$ . Choose

$$t_n \in [0, 1] \setminus \left( (f \circ \gamma_n)^{-1}(O_1) \cup (f \circ \gamma_n)^{-1}(O_2) \right),$$

then  $\{\gamma_n(t_n) \mid n \in \mathbb{N}\}$  is cofinal in  $X$ . Since  $Y$  is compact, we find a convergent subsequence, say  $y = \lim_{n \rightarrow \infty} f(\gamma_n(t_n))$ . Then we have

$$y \in \text{LP}[f]_{\preceq} \setminus (O_1 \cup O_2),$$

and this is a contradiction. □

We apply this lemma with different choices of  $X, Y, f$ .

- ▷ Let  $\theta \in (0, \pi)$ . We use  $X := [1, \infty)$  with the euclidean metric and the usual order, and  $Y := \overline{\mathbb{C}}$ , and  $f(r) := q(re^{i\theta})$  with some  $q \in \mathcal{N}$ . This yields that  $\text{LP}[q(re^{i\theta})]_{r \geq 1}$  is closed and connected.
- ▷ Let  $\alpha \in (0, \frac{\pi}{2})$ . We use  $X := \{z \in \mathbb{C}_+ \mid |z| \geq 1, \alpha \leq \arg z \leq \pi - \alpha\}$  with the euclidean metric and the preorder  $z_1 \preceq z_2 \Leftrightarrow |z_1| \leq |z_2|$ , with  $Y := \overline{\mathbb{C}}$  and  $f(z) := q(z)$  with some  $q \in \mathcal{N}$ . This yields that each of

$$\mathcal{L}_\alpha := \left\{ \zeta \in \overline{\mathbb{C}} \mid \exists z_n \in \mathbb{C}_+, |z_n| \rightarrow \infty, \alpha \leq \arg z_n \leq \pi - \alpha, \lim_{n \rightarrow \infty} q(z_n) = \zeta \right\}$$

is closed and connected. By compactness of  $Y$  the sets  $\mathcal{L}_\alpha$  are all nonempty. Since  $\mathcal{L}_\alpha \subseteq \mathcal{L}_\beta$  if  $\alpha \geq \beta$ , and

$$\text{LP}[q(z)]_{\star} = \bigcup_{\alpha \in (0, \frac{\pi}{2})} \mathcal{L}_\alpha,$$

we conclude that  $\text{LP}[q(z)]_{\star}$  is connected.

Let us introduce corresponding notation on the side of Hamiltonians.

**3.9 Definition.** Let  $H \in \mathbb{H}$ . Then we denote

$$\text{LP}[\mathcal{A}_r H]_{r \geq 1} := \left\{ \tilde{H} \in \mathbb{H} \mid \exists r_n \geq 1. r_n \rightarrow \infty \wedge \lim_{n \rightarrow \infty} \mathcal{A}_{r_n} H = \tilde{H} \right\}.$$

If, moreover,  $T > 0$ , we set

$$\begin{aligned} \text{LP}_{\|\cdot\|_1}[\rho_T \mathcal{A}_r H]_{r \geq 1} &:= \\ &\left\{ \tilde{H} \in \mathbb{H}_T \mid \exists r_n \geq 1. r_n \rightarrow \infty \wedge \lim_{n \rightarrow \infty}^{\|\cdot\|_1} \rho_T \mathcal{A}_{r_n} H = \tilde{H} \right\}, \\ \text{LP}_w[\rho_T \mathcal{A}_r H]_{r \geq 1} &:= \\ &\left\{ \tilde{H} \in \mathbb{H}_T \mid \exists r_n \geq 1. r_n \rightarrow \infty \wedge \lim_{n \rightarrow \infty}^w \rho_T \mathcal{A}_{r_n} H = \tilde{H} \right\}. \end{aligned}$$

◇

3.10 Remark. If  $H_1, H_2 \in \mathbb{H}$  satisfy  $\rho_\epsilon H_1 = \rho_\epsilon H_2$  for some  $\epsilon > 0$ , then  $H_1$  and  $H_2$  have exactly the same limit points, in all three meanings defined above.

In particular it is easy to formulate a meaningful generalisation of Definition 3.9 for  $H \in \mathbb{H}_\epsilon$ .  $\diamond$

We have the obvious inclusions

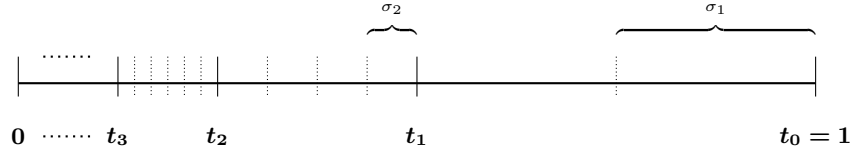
$$\rho_T(\text{LP}[\mathcal{A}_r H]_{r \geq 1}) \subseteq \text{LP}_w[\rho_T \mathcal{A}_r H]_{r \geq 1} \supseteq \text{LP}_{\|\cdot\|_1}[\rho_T \mathcal{A}_r H]_{r \geq 1}. \quad (3.5)$$

The sets  $\text{LP}[\mathcal{A}_r H]_{r \geq 1}$  and  $\text{LP}_w[\rho_T \mathcal{A}_r H]_{r \geq 1}$  are nonempty by compactness, while  $\text{LP}_{\|\cdot\|_1}[\rho_T \mathcal{A}_r H]_{r \geq 1}$  may be empty. An example can be constructed easily using a Hamiltonian which oscillates with increasing frequency towards 0.

*Example.* We are going to define a Hamiltonian  $H$  on  $(0, 1]$  which is constant on intervals accumulating only at 0 and which takes only the two values

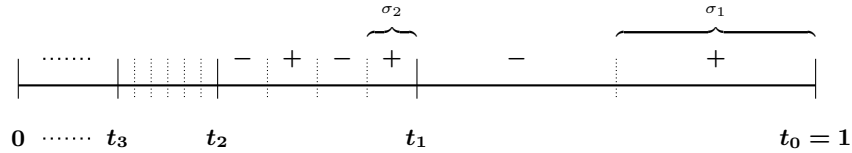
$$H_+ := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad H_- := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

For  $n \geq 0$  set  $t_n := 2^{-n}$  and  $\sigma_n := \frac{t_n}{2n} = \frac{1}{n}2^{-n-1}$ , and consider intervals as follows:



That is, take the intervals  $(t_n, t_{n-1}]$ , and further divide this interval into  $2n$  equal pieces of length  $\sigma_n$ .

We assign to each interval of the above partition a sign “+” or “−”, by starting with “+” on the rightmost interval, and proceeding to the left with alternating signs:



Now we define  $H$  by setting  $H(x) := H_+$  when  $x$  belongs to a “+”-interval, and  $H(x) := H_-$  when  $x$  belongs to a “−”-interval. Note that the rescaled Hamiltonian  $\mathcal{A}_r H$  is of the same form (constant on intervals accumulating only at 0 and taking the values  $H_\pm$ ). Its “+”- and “−”-intervals are exactly those of  $H$  multiplied by  $r$ .

For  $r \geq 2$ , let  $n(r)$  be the smallest nonnegative integer such that  $rt_{n(r)} \leq 1$ . Since the ratio  $\frac{t_{n+1}}{t_n}$  is constant equal to  $\frac{1}{2}$ , we have

$$rt_{n(r)+1} \leq \frac{1}{2} < rt_{n(r)} \leq 1 < rt_{n(r)-1}.$$

The length of a “+”- or “−”-interval of  $\mathcal{A}_r H$  in  $(rt_{n(r)}, rt_{n(r)-1}]$  is

$$r\sigma_{n(r)} = \frac{rt_{n(r)}}{2n(r)} \leq \frac{1}{2n(r)},$$

and the length of a “+”- or “-”-interval in  $(rt_{n(r)+1}, rt_{n(r)}]$  is even smaller. Set

$$M_+(r) := \{x \in (0, \frac{1}{2}) \mid \mathcal{A}_r H(x) = H_+\},$$

$$M_-(r) := \{x \in (0, \frac{1}{2}) \mid \mathcal{A}_r H(x) = H_-\}.$$

Both  $M_+(r)$  and  $M_-(r)$  are unions of finitely many intervals of length at most  $r\sigma_{n(r)} \leq \frac{1}{2n(r)}$ .

Let  $I \subseteq (\frac{1}{2}, 1)$  be a disjoint union of finitely many, say  $N$ , arbitrary intervals, and let  $r > 0$ . Two adjacent intervals of  $\mathcal{A}_r H$ , such that the right one is a “+”-interval and that both are contained in  $I$ , contribute the same to  $\lambda(M_+(r) \cap I)$  and  $\lambda(M_-(r) \cap I)$ . Grouping the intervals of  $\mathcal{A}_r H$  into such pairs and using a worst-case estimate at all  $2N$  endpoints yields  $|\lambda(M_+(r) \cap I) - \lambda(M_-(r) \cap I)| \leq \frac{1}{n(r)}$ . Together with  $\lambda(M_+(r) \cap I) + \lambda(M_-(r) \cap I) = \lambda(I)$  we get

$$\lambda(M_\pm(r) \cap I) \geq \frac{1}{2} \left( \lambda(I) - \frac{N}{n(r)} \right).$$

Now choose  $r > 0$  large enough such that  $n(r) \geq 4$ . Then the above inequality applied with  $I = (\frac{1}{2}, 1)$  gives  $\lambda(M_+(r)) \geq \frac{1}{2} (\frac{1}{2} - \frac{1}{4}) = \frac{1}{8}$ .

Next we use the same inequality with  $I = M_+(r)$ , which is a union of finitely many, say  $N(r)$ , intervals. By choosing  $s > r$  large enough such that  $n(s) \geq 16N(r)$ , we conclude

$$\lambda(M_-(s) \cap M_+(r)) \geq \frac{1}{2} \left( \lambda(M_+(r)) - \frac{N(r)}{n(s)} \right) \geq \frac{1}{2} \left( \frac{1}{8} - \frac{1}{16} \right) = \frac{1}{32}.$$

In particular

$$\|\rho_1 \mathcal{A}_r H - \rho_1 \mathcal{A}_s H\|_1 \geq \lambda(M_-(s) \cap M_+(r)) \geq \frac{1}{32},$$

which means that there exists no  $\|\cdot\|_1$ -convergent sequence  $(\rho_1 \mathcal{A}_{r_n} H)_{n \in \mathbb{N}}$  where  $r_n \rightarrow \infty$ .  $\diamond$

Moreover, each of  $\text{LP}[\mathcal{A}_r H]_{r \geq 1}$ ,  $\text{LP}_{\|\cdot\|_1}[\rho_T \mathcal{A}_r H]_{r \geq 1}$ , and  $\text{LP}_w[\rho_T \mathcal{A}_r H]_{r \geq 1}$  is closed and connected.

To see this, let  $H \in \mathbb{H}$  and apply the topological lemma shown above with  $X := [1, \infty)$  carrying the euclidean metric and the usual order, and with

$$\triangleright Y := \mathbb{H} \text{ and } f(r) := \mathcal{A}_r H,$$

$$\triangleright Y := \mathbb{H}_T, \text{ once endowed with the weak and once with the norm topology, and } f(r) := \rho_T \mathcal{A}_r H.$$

Continuity of the group action (3.3) yields the following noteworthy fact.

*3.11 Remark.* Let  $H \in \mathbb{H}$  and  $s > 0$ . Then  $\mathcal{A}_s(\text{LP}[\mathcal{A}_r H]_{r \geq 1}) = \text{LP}[\mathcal{A}_r H]_{r \geq 1}$ . Hence, the action (3.3) of  $\mathbb{R}_+$  on  $\mathbb{H}$  restricts to a continuous group action on  $\text{LP}[\mathcal{A}_r H]_{r \geq 1}$ .

If the limit  $\tilde{H} := \lim_{r \rightarrow \infty} \mathcal{A}_r H$  exists, i.e., the set  $\text{LP}[\mathcal{A}_r H]_{r \geq 1}$  contains only one element, then  $\tilde{H}$  is constant (remember Remark 3.5).  $\diamond$

It turns out that constant limit points have very particular properties. At this point let us show two lemmata; more will be seen in Proposition 3.15.

First, one reverse inclusion in (3.5) holds for constant limit points. In fact, we show a slightly stronger statement.

**3.12 Lemma.** *Let  $H \in \mathbb{H}$  and  $\zeta \in \overline{\mathbb{C}_+}$ . Then*

$$\left( \exists T > 0. \rho_T \Theta(\zeta) \in \text{LP}_w[\rho_T \mathcal{A}_r H]_{r \geq 1} \right) \Rightarrow \Theta(\zeta) \in \text{LP}[\mathcal{A}_r H]_{r \geq 1}$$

*Proof.* Let  $\zeta \in \overline{\mathbb{C}_+}$  and assume that  $\rho_T \Theta(\zeta) = \lim_{n \rightarrow \infty}^w \rho_T \mathcal{A}_{r_n} H$  for some  $T > 0$  and  $r_n \rightarrow \infty$ . For  $s > 0$  we have

$$\rho_{sT}(\mathcal{A}_{sr_n} H) = (\rho_{sT} \circ \mathcal{A}_s \circ \mathcal{A}_{r_n}) H = \mathcal{A}_s(\rho_T \mathcal{A}_{r_n} H).$$

Since  $\Theta(\zeta)$  is a fixed point of every rescaling operator,

$$(\mathcal{A}_s \circ \rho_T) \Theta(\zeta) = (\rho_{sT} \circ \mathcal{A}_s) \Theta(\zeta) = \rho_{sT} \Theta(\zeta).$$

Now we obtain from continuity of  $\mathcal{A}_s$  that

$$\lim_{n \rightarrow \infty}^w \rho_{sT}(\mathcal{A}_{sr_n} H) = \lim_{n \rightarrow \infty}^w \mathcal{A}_s(\rho_T \mathcal{A}_{r_n} H) = \mathcal{A}_s(\rho_T \Theta(\zeta)) = \rho_{sT} \Theta(\zeta). \quad (3.6)$$

We are going to apply Lemma 2.14 to extract a subsequence which converges in  $\mathbb{H}$ . For  $k \in \mathbb{N}$  set  $l_k := k$ , and use (3.6) with  $s := \frac{k}{T}$  to obtain  $n_k \in \mathbb{N}$  with

$$d_k(\rho_k(\mathcal{A}_{\frac{kr_{n_k}}{T}} H), \rho_k \Theta(\zeta)) \leq \frac{1}{k},$$

where  $d_k$  is the metric on  $\mathbb{H}_k$  used in Lemma 2.14. Applying this lemma, it follows that

$$\lim_{k \rightarrow \infty} \mathcal{A}_{\frac{kr_{n_k}}{T}} H = \Theta(\zeta),$$

and hence  $\Theta(\zeta) \in \text{LP}[\mathcal{A}_r H]_{r \geq 1}$ .  $\square$

The previous lemma ensures existence of constant limit points of  $(\mathcal{A}_r H)_{r \geq 1}$  from constant limit points of restrictions. However, nonconstant limit points still may exist. The next lemma shows that this is not anymore possible when all limit points of restrictions are constant.

**3.13 Lemma.** *Let  $H \in \mathbb{H}$ , then*

$$\left( \exists T > 0. \rho_T(\text{LP}[\mathcal{A}_r H]_{r \geq 1}) \subseteq \rho_T(\mathbb{CH}) \right) \Rightarrow \text{LP}[\mathcal{A}_r H]_{r \geq 1} \subseteq \mathbb{CH}$$

*Proof.* Let  $\tilde{H} \in \text{LP}[\mathcal{A}_r H]_{r \geq 1}$ . By Remark 3.11, for each  $s > 0$  also  $\mathcal{A}_s \tilde{H} \in \text{LP}[\mathcal{A}_r H]_{r \geq 1}$ . Our present assumption provides us with  $\zeta_s \in \overline{\mathbb{C}_+}$  such that  $\rho_T(\mathcal{A}_s \tilde{H}) = \rho_T(\Theta(\zeta_s))$ . We have

$$\rho_{\frac{T}{s}} \tilde{H} = \mathcal{A}_{\frac{1}{s}} \rho_T \mathcal{A}_s \tilde{H} = \mathcal{A}_{\frac{1}{s}} \rho_T \Theta(\zeta_s) = \rho_{\frac{T}{s}} \mathcal{A}_{\frac{1}{s}} \Theta(\zeta_s) = \rho_{\frac{T}{s}} \Theta(\zeta_s),$$

from which, for  $s_1, s_2 > 0$ , we see  $(q := \max\{s_1, s_2\}^{-1})$

$$\rho_{qT} \Theta(\zeta_{s_1}) = \rho_{qT} \tilde{H} = \rho_{qT} \Theta(\zeta_{s_2}).$$

This shows that  $\zeta_{s_1} = \zeta_{s_2}$ , and it follows that  $\tilde{H} = \Theta(\zeta)$  where  $\zeta := \zeta_s$ .  $\square$

### 3.3 Relating limit points of functions and Hamiltonians

Now we show that limit points of  $q_H(z)$  for  $z$  tending towards  $i\infty$  and limit points of  $\mathcal{A}_r H$  for  $r$  tending to  $\infty$  are related. First we give a surjective assignment from limit points of Hamiltonians to such of functions.

**3.14 Proposition.** *Let  $H \in \mathbb{H}$ .*

(i) *For each  $w \in \mathbb{C}_+$  it holds that*

$$\text{LP}[q_H(e^{i\theta}r)]_{r \geq 1} = \Phi(\text{LP}[\mathcal{A}_r H]_{r \geq 1}, w), \quad (3.7)$$

*where  $\theta := \arg w$ .*

(ii) *We have*

$$\text{LP}[q_H(z)]_{\ast} = \bigcup_{\theta \in (0, \pi)} \text{LP}[q_H(e^{i\theta}r)]_{r \geq 1} = \Phi(\text{LP}[\mathcal{A}_r H]_{r \geq 1}, \mathbb{C}_+). \quad (3.8)$$

*Proof.* Assume that  $\tilde{H} = \lim_{n \rightarrow \infty} \mathcal{A}_{r_n} H$  with some  $r_n \rightarrow \infty$ . Then

$$\lim_{n \rightarrow \infty} q_H(r_n |w| e^{i\theta}) = \lim_{n \rightarrow \infty} \Phi(H, r_n |w| e^{i\theta}) = \lim_{n \rightarrow \infty} \Phi(\mathcal{A}_{r_n} H, |w| e^{i\theta}) = q_{\tilde{H}}(w).$$

This shows the inclusion “ $\supseteq$ ” in (3.7) and in the second equality in (3.8). The inclusion “ $\supseteq$ ” of the first equality in (3.8) is trivial.

Consider a sequence  $(z_n)_{n \in \mathbb{N}}$  with  $z_n \xrightarrow{\text{f}} i\infty$  such that the limit  $\zeta := \lim_{n \rightarrow \infty} q_H(z_n)$  exists. Choose a subsequence  $(n_k)_{k \in \mathbb{N}}$  such that both limits

$$\tilde{H} := \lim_{k \rightarrow \infty} \mathcal{A}_{|z_{n_k}|} H, \quad \tilde{\theta} := \lim_{k \rightarrow \infty} \arg z_{n_k},$$

exist. Since  $z_n$  tends to  $i\infty$  nontangentially, we have  $\tilde{\theta} \in (0, \pi)$ . Continuity of  $\Phi$  yields

$$\zeta = \lim_{k \rightarrow \infty} \Phi(H, z_{n_k}) = \lim_{k \rightarrow \infty} \Phi(\mathcal{A}_{|z_{n_k}|} H, e^{i \arg z_{n_k}}) = \Phi(\tilde{H}, e^{i\tilde{\theta}}).$$

This shows that the first term in (3.8) is contained in the last. Moreover, if all points  $z_n$  lie on a certain ray  $\{re^{i\theta} \mid r > 0\}$ , then the limit angle  $\tilde{\theta}$  in the above argument equals  $\theta$ , and we find  $\zeta = \Phi(H, e^{i\theta})$ . From Remark 3.11 we obtain  $\Phi(\text{LP}[\mathcal{A}_r H]_{r \geq 1}, w) = \Phi(\text{LP}[\mathcal{A}_r H]_{r \geq 1}, e^{i\theta})$ , and “ $\subseteq$ ” in (3.7) follows.  $\square$

Second, we show some properties of constant limit points. In particular, we shall see that limit points of  $\mathcal{A}_r H$  in  $\mathbb{CH}_0$  bijectively correspond to limit points of  $q_H$  on the boundary  $\overline{\mathbb{R}}$ .

**3.15 Proposition.** *Let  $H \in \mathbb{H}$ .*

(i)  $\Theta^{-1}(\text{LP}[\mathcal{A}_r H]_{r \geq 1} \cap \mathbb{CH}) \subseteq \bigcap_{\theta \in (0, \pi)} \text{LP}[q_H(e^{i\theta}r)]_{r \geq 1}.$

(ii) *Assume that  $\text{LP}[\mathcal{A}_r H]_{r \geq 1} \subseteq \mathbb{CH}$ . Then for every  $\theta \in (0, \pi)$*

$$\text{LP}[q_H(z)]_{\ast} = \text{LP}[q_H(e^{i\theta}r)]_{r \geq 1} = \Theta^{-1}(\text{LP}[\mathcal{A}_r H]_{r \geq 1}).$$

(iii) For each  $\theta \in (0, \pi)$

$$\text{LP}[q_H(e^{i\theta}r)]_{r \geq 1} \cap \overline{\mathbb{R}} = \Theta^{-1}(\text{LP}[\mathcal{A}_r H]_{r \geq 1} \cap \mathbb{CH}_0).$$

(iv)  $\Phi(\text{LP}[\mathcal{A}_r H]_{r \geq 1} \setminus \mathbb{CH}_0, \mathbb{C}_+) \cap \overline{\mathbb{R}} = \emptyset$ .

*Proof.* If  $\zeta \in \overline{\mathbb{C}_+}$  with  $\Theta(\zeta) \in \text{LP}[\mathcal{A}_r H]_{r \geq 1}$ , then by Remark 2.22 and (3.7)

$$\zeta = \Phi(\Theta(\zeta), e^{i\theta}) \in \text{LP}[q_H(e^{i\theta}r)]_{r \geq 1}, \quad \theta \in (0, \pi).$$

This shows (i) and the inclusion “ $\supseteq$ ” in (iii).

If  $\text{LP}[\mathcal{A}_r H]_{r \geq 1} \subseteq \mathbb{CH}$ , Proposition 3.14(i) and Remark 2.22 imply

$$\text{LP}[q_H(e^{i\theta}r)]_{r \geq 1} = \Phi(\text{LP}[\mathcal{A}_r H]_{r \geq 1}, e^{i\theta}) = \Theta^{-1}(\text{LP}[\mathcal{A}_r H]_{r \geq 1}).$$

In particular, the radial limit points do not depend on  $\theta \in (0, \pi)$ , and Proposition 3.14(ii) implies the present assertion (ii).

Let  $\zeta \in \text{LP}[q_H(e^{i\theta}r)]_{r \geq 1} \cap \overline{\mathbb{R}}$  and pick  $\tilde{H} \in \text{LP}[\mathcal{A}_r H]_{r \geq 1}$  with  $\zeta = \Phi(\tilde{H}, e^{i\theta})$ , which is possible due to Proposition 3.14(i). Since  $\zeta \in \overline{\mathbb{R}}$ , it follows that  $q_{\tilde{H}}$  is constant equal to  $\zeta$ , and in turn that  $\tilde{H} = \Theta(\zeta)$ . This shows the inclusion “ $\subseteq$ ” in (iii).

To see (iv), it is enough to note that for  $\tilde{H} \in \text{LP}[\mathcal{A}_r H]_{r \geq 1} \setminus \mathbb{CH}_0$  the Weyl coefficient  $q_{\tilde{H}}$  is not constant equal to some value from  $\overline{\mathbb{R}}$ , and hence cannot assume any value from  $\overline{\mathbb{R}}$  in  $\mathbb{C}_+$ .  $\square$

## Existence of limits

The situation that the nontangential limit  $\lim_{z \rightarrow i\infty} q_H(z)$  exists in  $\mathbb{C}$  was settled in [EKT18, Theorem 3.1]. There it is shown that for every  $\zeta \in \mathbb{C}_+ \cup \mathbb{R}$ , the conditions  $\lim_{r \rightarrow \infty} \mathcal{A}_r H = \Theta(\zeta)$  and  $\lim_{z \rightarrow i\infty} q_H(z) = \zeta$  are equivalent.

We repeat the proof given there using our present language, and thereby include a minor improvement in the formulation. Namely, in (ii) it is not assumed a priori that the limit is constant. Moreover, we include the case that  $\zeta = \infty$  and assume only existence of the radial limit in (i) (the complex analysis argument to conclude existence of the nontangential limit from existence of the radial limit is not needed in our proof).

**Theorem.** *Let  $H \in \mathbb{H}$ . Then the following statements are equivalent.*

- (i) *The limit  $\lim_{r \rightarrow \infty} q_H(ir)$  exists in  $\overline{\mathbb{C}}$ .*
- (ii) *The limit  $\lim_{r \rightarrow \infty} \mathcal{A}_r H$  exists in  $\mathbb{H}$ .*
- (iii) *The limits  $\zeta := \lim_{z \rightarrow i\infty} q_H(z)$  and  $\tilde{H} := \lim_{r \rightarrow \infty} \mathcal{A}_r H$  exist in  $\overline{\mathbb{C}}$  and  $\mathbb{H}$ , respectively, and  $\tilde{H} = \Theta(\zeta)$ .*

*Proof.*

① We show “(i) $\Rightarrow$ (ii)”: Assume that  $\zeta := \lim_{r \rightarrow \infty} q_H(ir)$  exists in  $\overline{\mathbb{C}}$ . Let  $r_n > 0$  with  $r_n \rightarrow \infty$ . By compactness we find a sequence  $(n_k)_{k \in \mathbb{N}}$  such that the limit  $\tilde{H} := \lim_{k \rightarrow \infty} \mathcal{A}_{r_{n_k}} H$  exists in  $\mathbb{H}$ . Then, by continuity of  $\Phi$  and (3.4), for each  $z \in \mathbb{C}_+$

$$q_{\tilde{H}}(z) = \Phi(\tilde{H}, z) = \lim_{k \rightarrow \infty} \Phi(\mathcal{A}_{r_{n_k}} H, z) = \lim_{k \rightarrow \infty} \Phi(H, r_{n_k} z) = \lim_{k \rightarrow \infty} q_H(r_{n_k} z).$$

The right side can be evaluated if  $z$  lies on the imaginary axis; in this case it equals  $\zeta$ . By analyticity, the function  $q_{\tilde{H}}$  is constant equal to  $\zeta$  on the whole half-plane  $\mathbb{C}_+$ . Injectivity of de Branges' correspondence implies that  $\tilde{H} = \Theta(\zeta)$ . We conclude that the limit  $\lim_{r \rightarrow \infty} \mathcal{A}_r H$  exists and equals  $\Theta(\zeta)$ .

② We show “(ii) $\Rightarrow$ (iii)”: Assume that the limit  $\tilde{H} := \lim_{r \rightarrow \infty} \mathcal{A}_r H$  exists. By Remark 3.11,  $\tilde{H}$  is constant, and we can write  $\tilde{H} = \Theta(\zeta)$ . Consider a sequence  $z_n \xrightarrow{*} i\infty$  with, say,  $\arg z_n \in [\alpha, \pi - \alpha]$ , and set  $K := \{e^{i\theta} \mid \theta \in [\alpha, \pi - \alpha]\}$ . Since  $\Phi|_{\mathbb{H} \times K}$  is uniformly continuous, we obtain

$$\lim_{n \rightarrow \infty} \chi\left(\Phi(\mathcal{A}_{|z_n|} H, \frac{z_n}{|z_n|}), \Phi(\Theta(\zeta), \frac{z_n}{|z_n|})\right) = 0.$$

However,  $\Phi(\mathcal{A}_{|z_n|} H, \frac{z_n}{|z_n|}) = \Phi(H, z_n) = q_H(z_n)$  and  $\Phi(\Theta(\zeta), \frac{z_n}{|z_n|}) = \zeta$ . Thus the limit  $\lim_{z \xrightarrow{*} i\infty} q_H(z)$  exists and equals  $\zeta$ .

③ The implication “(iii) $\Rightarrow$ (i)” is trivial. □

*Remark.* For  $H_0 \in \mathbb{C}\mathbb{H}$  note that  $\lim_{r \rightarrow \infty} \mathcal{A}_r H = H_0$  is equivalent to  $\lim_{x \rightarrow 0} \frac{1}{x} \int_0^x H(t) dt = H_0$ . ◇

## 4 Weyl coefficients with prescribed limits

The below theorem is our main result, where we construct Hamiltonians whose Weyl coefficient has a prescribed set of limit points.

For a sequence  $(\zeta_n)_{n \in \mathbb{N}}$  in  $\overline{\mathbb{C}}$  we use the notation

$$\text{LP}[\zeta_n]_{n \in \mathbb{N}} := \{\zeta \in \overline{\mathbb{C}} \mid \exists n_k \in \mathbb{N}. n_k \rightarrow \infty \wedge \lim_{k \rightarrow \infty} \zeta_{n_k} = \zeta\}.$$

**4.1 Theorem.** *Let  $(t_n)_{n \in \mathbb{N}}$  be a sequence of positive numbers with*

$$1 = t_1 > t_2 > t_3 > \dots, \quad \lim_{n \rightarrow \infty} t_n = 0, \quad \lim_{n \rightarrow \infty} \frac{t_{n+1}}{t_n} = 0,$$

*and let  $(\zeta_n)_{n \in \mathbb{N}}$  be a sequence of points on  $\overline{\mathbb{C}}_+$  with*

$$\lim_{n \rightarrow \infty} \chi(\zeta_{n+1}, \zeta_n) = 0.$$

*Define  $H$  to be the piecewise constant Hamiltonian*

$$H(t) := \begin{cases} \Theta(\zeta_n), & t \in (t_{n+1}, t_n], n \in \mathbb{N}, \\ \Theta(0), & t \in (1, \infty). \end{cases}$$

*Then, for every  $\theta \in (0, \pi)$ ,*

$$\text{LP}[q_H(z)]_* = \text{LP}[q_H(e^{i\theta})r]_{r \geq 1} = \text{LP}[\zeta_n]_{n \in \mathbb{N}}.$$

The proof of this theorem is based on Proposition 3.15(ii). To verify the necessary hypothesis and to compute the set of limit points, we use the following fact.

**4.2 Lemma.** Let  $H \in \mathbb{H}$  and  $T > 0$ , and assume that  $(d_{\|\cdot\|_1})$  denotes the metric induced by  $\|\cdot\|_1$

$$\lim_{r \rightarrow \infty} d_{\|\cdot\|_1}(\rho_T \mathcal{A}_r H, \rho_T(\mathbb{CH})) = 0. \quad (4.1)$$

Then

$$\rho_T(\text{LP}[\mathcal{A}_r H]_{r \geq 1}) = \text{LP}_w[\rho_T \mathcal{A}_r H]_{r \geq 1} = \text{LP}_{\|\cdot\|_1}[\rho_T \mathcal{A}_r H]_{r \geq 1} \subseteq \rho_T(\mathbb{CH}).$$

*Proof.* Since  $\rho_T(\mathbb{CH})$  is closed w.r.t.  $\|\cdot\|_1$ , the present assumption (4.1) clearly implies that

$$\text{LP}_{\|\cdot\|_1}[\rho_T \mathcal{A}_r H]_{r \geq 1} \subseteq \rho_T(\mathbb{CH}).$$

In view of (3.5) it remains to show that

$$\rho_T(\text{LP}[\mathcal{A}_r H]_{r \geq 1}) \supseteq \text{LP}_w[\rho_T \mathcal{A}_r H]_{r \geq 1} \subseteq \text{LP}_{\|\cdot\|_1}[\rho_T \mathcal{A}_r H]_{r \geq 1}. \quad (4.2)$$

Let  $\tilde{H} \in \text{LP}_w[\rho_T \mathcal{A}_r H]_{r \geq 1}$ , say  $\tilde{H} = \lim_{n \rightarrow \infty}^w \rho_T \mathcal{A}_{r_n} H$  with some sequence  $r_n \rightarrow \infty$ . For each  $n \in \mathbb{N}$  choose  $\xi_n \in \overline{\mathbb{C}_+}$  with

$$\|\rho_T \mathcal{A}_{r_n} H - \rho_T \Theta(\xi_n)\|_1 \leq 2d_{\|\cdot\|_1}(\rho_T \mathcal{A}_{r_n} H, \rho_T(\mathbb{CH})).$$

Now choose a subsequence  $(n_k)_{k \in \mathbb{N}}$  such that  $\xi = \lim_{k \rightarrow \infty} \xi_{n_k}$  exists. Then, in view of (4.1), the limit  $\lim_{k \rightarrow \infty}^{\|\cdot\|_1} \rho_T \mathcal{A}_{r_{n_k}} H$  exists, in fact, it is equal to  $\rho_T \Theta(\xi)$ . From this, we obtain

$$\tilde{H} = \lim_{k \rightarrow \infty}^w \rho_T \mathcal{A}_{r_{n_k}} H = \lim_{k \rightarrow \infty}^{\|\cdot\|_1} \rho_T \mathcal{A}_{r_{n_k}} H \in \text{LP}_{\|\cdot\|_1}[\rho_T \mathcal{A}_r H]_{r \geq 1}.$$

We see that the second inclusion in (4.2) holds. In particular, we know that  $\text{LP}_w[\rho_T \mathcal{A}_r H]_{r \geq 1} \subseteq \rho_T(\mathbb{CH})$ . Lemma 3.12 yields the first inclusion in (4.2).  $\square$

*Proof of Theorem 4.1.*

① The first step in the proof is to check the hypothesis (4.1) of Lemma 4.2. To this end, let  $\varepsilon > 0$  be given. The map  $\rho_1 \circ \Theta: \overline{\mathbb{C}_+} \rightarrow \langle \mathbb{H}_1, \|\cdot\|_1 \rangle$  is continuous, and by compactness hence uniformly continuous. This provides  $\delta > 0$  such that

$$\forall \zeta, \xi \in \overline{\mathbb{C}_+}. \quad \chi(\zeta, \xi) \leq \delta \Rightarrow \|\rho_1 \Theta(\zeta) - \rho_1 \Theta(\xi)\|_1 \leq \varepsilon.$$

By the properties required from  $(t_n)_{n \in \mathbb{N}}$  and  $(\zeta_n)_{n \in \mathbb{N}}$  we find  $n_0$  such that

$$\frac{t_{n+1}}{t_n} \leq \varepsilon, \quad \chi(\zeta_{n+1}, \zeta_n) \leq \delta, \quad n \geq n_0.$$

Set  $r_0 := \frac{1}{t_{n_0}}$ , and consider  $r \geq r_0$ . The Hamiltonian  $\mathcal{A}_r H$  is given as

$$\mathcal{A}_r H(t) = \begin{cases} \Theta(\zeta_n), & t \in (rt_{n+1}, rt_n], n \in \mathbb{N}, \\ \Theta(0), & t \in (r, \infty). \end{cases}$$

Let  $m$  be the unique integer, such that  $rt_{m+1} < 1 \leq rt_m$ . Since  $r \geq r_0$ , we have  $m \geq n_0$ .

▷ It holds that  $rt_{m+2} \leq \frac{rt_{m+2}}{rt_{m+1}} \leq \varepsilon$ , and hence

$$\int_0^{rt_{m+2}} \|(\mathcal{A}_r H)(t) - \Theta(\zeta_m)\| dt \leq 4\varepsilon.$$



▷ We have

$$\begin{aligned} & \int_{rt_{m+2}}^{rt_{m+1}} \|(\mathcal{A}_r H)(t) - \Theta(\zeta_m)\| dt \\ &= \int_{rt_{m+2}}^{rt_{m+1}} \|\Theta(\zeta_{m+1}) - \Theta(\zeta_m)\| dt \leq \|\rho_1 \Theta(\zeta_{m+1}) - \rho_1 \Theta(\zeta_m)\|_1 \leq \varepsilon. \end{aligned}$$

$$\triangleright \int_{rt_{m+1}}^1 \|(\mathcal{A}_r H)(t) - \Theta(\zeta_m)\| dt = 0.$$

Together it follows that  $\|\rho_1 \mathcal{A}_r H - \rho_1 \Theta(\zeta_m)\|_1 \leq 5\varepsilon$ .

② Lemma 4.2 applies and yields that

$$\rho_1(\text{LP}[\mathcal{A}_r H]_{r \geq 1}) = \text{LP}_w[\rho_1 \mathcal{A}_r H]_{r \geq 1} = \text{LP}_{\|\cdot\|_1}[\rho_1 \mathcal{A}_r H]_{r \geq 1} \subseteq \rho_1(\mathbb{CH}). \quad (4.3)$$

We obtain from Lemma 3.13 that  $\text{LP}[\mathcal{A}_r H]_{r \geq 1} \subseteq \mathbb{CH}$ , and now Proposition 3.15(ii) and (4.3) yield

$$\text{LP}[q_H(z)]_* = \text{LP}[q_H(e^{i\theta} r)]_{r \geq 1} = (\rho_1 \Theta)^{-1}(\text{LP}_{\|\cdot\|_1}[\rho_1 \mathcal{A}_r H]_{r \geq 1}).$$

③ It remains to evaluate  $\text{LP}_{\|\cdot\|_1}[\rho_1 \mathcal{A}_r H]_{r \geq 1}$ .

First assume that  $\tilde{H} \in \text{LP}_{\|\cdot\|_1}[\rho_1 \mathcal{A}_r H]_{r \geq 1}$ , say  $\tilde{H} = \lim_{n \rightarrow \infty}^{\|\cdot\|_1} \rho_1 \mathcal{A}_{r_n} H$  with some  $r_n \rightarrow \infty$ . For  $l \in \mathbb{N}$  let  $r_0(l)$  be as in the first step of the proof for  $\varepsilon := \frac{1}{l}$ . Choose  $n \in \mathbb{N}$  such that

$$r_n \geq \max\{r_0, l\}, \quad \|\rho_1 \mathcal{A}_{r_n} H - \tilde{H}\|_1 \leq \frac{1}{l},$$

and let  $m_l$  be the unique integer with  $r_n t_{m_l+1} < 1 \leq r_n t_{m_l}$ . Then

$$\|\tilde{H} - \rho_1 \Theta(\zeta_{m_l})\|_1 \leq \|\tilde{H} - \rho_1 \mathcal{A}_{r_n} H\|_1 + \|\rho_1 \mathcal{A}_{r_n} H - \rho_1 \Theta(\zeta_{m_l})\|_1 \leq \frac{6}{l}.$$

We see that  $\tilde{H} = \lim_{l \rightarrow \infty}^{\|\cdot\|_1} \rho_1 \Theta(\zeta_{m_l})$ . Since  $\rho_1 \Theta$  is a homeomorphism onto its image  $\rho_1(\mathbb{CH})$  and this image is closed, we infer that  $\tilde{H} \in \rho_1(\mathbb{CH})$  and that  $(\rho_1 \Theta)^{-1} \tilde{H} = \lim_{l \rightarrow \infty} \zeta_{m_l} \in \text{LP}[\zeta_n]_{n \in \mathbb{N}}$ .

Conversely, assume that  $\zeta \in \text{LP}[\zeta_n]_{n \in \mathbb{N}}$ , say  $\zeta = \lim_{k \rightarrow \infty} \zeta_{n_k}$  with some  $n_k \rightarrow \infty$ . Set  $r_k := \frac{1}{t_{n_k}}$ , then

$$\|\rho_1 \mathcal{A}_{r_k} H - \rho_1 \Theta(\zeta)\|_1 \leq \underbrace{\|\rho_1 \mathcal{A}_{r_k} H - \rho_1 \Theta(\zeta_{n_k})\|_1}_{\leq 4 \frac{t_{n_k}+1}{t_{n_k}} \rightarrow 0} + \underbrace{\|\rho_1 \Theta(\zeta_{n_k}) - \rho_1 \Theta(\zeta)\|_1}_{\rightarrow 0}.$$

We see that  $\rho_1 \Theta(\zeta) \in \text{LP}_{\|\cdot\|_1}[\rho_1 \mathcal{A}_r H]_{r \geq 1}$ .

□

Theorem 4.1 has the following consequence.

**4.3 Corollary.** *Let  $\mathcal{L} \subseteq \overline{\mathbb{C}_+}$ . Then the following statements are equivalent.*

- (i) There exists  $\theta \in (0, \pi)$  and a function  $q \in \mathcal{N}$  with  $\mathcal{L} = \text{LP}[q(e^{i\theta}r)]_{r \geq 1}$ .
- (ii) There exists a function  $q \in \mathcal{N}$  with  $\mathcal{L} = \overline{\text{LP}[q(z)]_\star}$ .
- (iii) There exists a function  $q \in \mathcal{N}$  such that  $\mathcal{L} = \text{LP}[q(e^{i\theta}r)]_{r \geq 1} = \text{LP}[q(z)]_\star$  for every  $\theta \in (0, \pi)$ .
- (iv)  $\mathcal{L}$  is closed and connected.

In the proof of this result we use the following elementary fact.

**4.4 Lemma.** Let  $\langle X, d \rangle$  be a metric space, and let  $\mathcal{L} \subseteq X$  be compact and connected. Then there exists a sequence  $(x_n)_{n \in \mathbb{N}}$  in  $X$  with

$$\lim_{n \rightarrow \infty} d(x_{n+1}, x_n) = 0, \quad \text{LP}[x_n]_{n \in \mathbb{N}} = \mathcal{L}.$$

*Proof.* Let  $l \in \mathbb{N}$ . By compactness we can choose finitely many pairwise different open balls with radius  $\frac{1}{l}$ , say  $O_{l,1}, \dots, O_{l,n(l)}$ , such that

$$\bigcup_{i=1}^{n(l)} O_{l,i} \supseteq \mathcal{L}, \quad O_{l,i} \cap \mathcal{L} \neq \emptyset, i = 1, \dots, n(l).$$

Choose  $x_{l,i} \in O_{l,i} \cap \mathcal{L}$ .

Let  $\mathcal{G}_l$  be the graph with vertices  $\{O_{l,i} \mid i = 1, \dots, n(l)\}$  and edges  $\{(O_{l,i}, O_{l,j}) \mid O_{l,i} \cap O_{l,j} \neq \emptyset\}$ . If  $\mathcal{C}$  is a connected component of  $\mathcal{G}_l$ , then  $\bigcup \mathcal{C}$  is an open set which has nonempty intersection with  $\mathcal{L}$ . If  $\mathcal{C}_1, \mathcal{C}_2$  are different connected components of  $\mathcal{G}_l$ , then  $\bigcup \mathcal{C}_1$  and  $\bigcup \mathcal{C}_2$  are disjoint. Since  $\mathcal{L}$  is connected,  $\mathcal{G}_l$  is connected.

Now we inductively construct for each  $l \in \mathbb{N}$  a finite sequence of points.

- ▷  $l = 1$ : Choose a path  $(O_{1,i_{1,1}}, \dots, O_{1,i_{1,m(1)}})$  in  $\mathcal{G}_1$  which contains all vertices. As finite sequence of points for  $l = 1$  now take  $(x_{1,i_{1,1}}, \dots, x_{1,i_{1,m(1)}})$
- ▷ Let  $l > 1$  and assume we have already constructed a sequence for  $l - 1$ . Choose a path  $(O_{l,i_{l,1}}, \dots, O_{l,i_{l,m(l)}})$  in  $\mathcal{G}_l$  which contains all vertices and satisfies  $x_{l-1,i_{l-1,m(l-1)}} \in O_{l,i_{l,1}}$ . As finite sequence of points for  $l$  now take  $(x_{l,i_{l,1}}, \dots, x_{l,i_{l,m(l)}})$

A sequence  $(x_n)_{n \in \mathbb{N}}$  is defined by plugging together the constructed sequences for  $l = 1, 2, 3, \dots$ . We have

$$d(x_{l,i_{l,j}}, x_{l,i_{l,j+1}}) \leq \text{diam}(O_{l,i_{l,j}} \cup O_{l,i_{l,j+1}}) \leq \frac{4}{l},$$

since  $O_{l,i_{l,j}} \cap O_{l,i_{l,j+1}} \neq \emptyset$ . Moreover,

$$d(x_{l,i_{l,m(l)}}, x_{l+1,i_{l+1,1}}) \leq \text{diam}(O_{l+1,i_{l+1,1}}) = \frac{2}{l},$$

since both points belong to this ball. Thus  $\lim_{n \rightarrow \infty} d(x_{n+1}, x_n) = 0$ . Since  $x_n \in \mathcal{L}$  and  $\mathcal{L}$  is closed, we have  $\text{LP}[x_n]_{n \in \mathbb{N}} \subseteq \mathcal{L}$ . Conversely, let  $x \in \mathcal{L}$ . For each  $l \in \mathbb{N}$  we can choose an index  $i_{l,k(l)}$  with  $x \in O_{l,i_{l,k(l)}}$ . Then  $\lim_{l \rightarrow \infty} x_{l,i_{l,k(l)}} = x$ , and we see that  $x \in \text{LP}[x_n]_{n \in \mathbb{N}}$ .  $\square$

*Proof of Corollary 4.3.* The implications

$$(iii) \Rightarrow ((i) \wedge (ii) \wedge (iv)), \quad ((i) \vee (ii)) \Rightarrow (iv),$$

are obvious (remember here Remark 3.8).

To show “(iv) $\Rightarrow$ (iii)”, assume  $\mathcal{L}$  is given and apply Lemma 4.4 and Theorem 4.1.  $\square$

## Sets of nontangential limit points

It is an open problem to describe those subsets  $\mathcal{L}$  of  $\overline{\mathbb{C}_+}$  which are the set of nontangential limit points for some Nevanlinna function  $q$ . We have the necessary condition that  $\mathcal{L}$  must be nonempty and connected, and we have the sufficient condition that  $\mathcal{L}$  is closed, nonempty, and connected (this condition is not necessary, cf. Example 4.5 below).

Let us briefly discuss this issue, and in particular explain that this question is probably closely related to understanding nonconstant limit points of  $(\mathcal{A}_r H)_{r \geq 1}$ .

The Hamiltonians  $H$  constructed in Theorem 4.1 (and used in Corollary 4.3) have the property that

$$\text{LP}[\mathcal{A}_r H]_{r \geq 1} \subseteq \mathbb{CH}. \quad (4.4)$$

Whenever  $H \in \mathbb{H}$  has this property, Proposition 3.15(ii) gives

$$\text{LP}[q_H(z)]_{\star} = \text{LP}[q_H(e^{i\theta} r)]_{r \geq 1}, \quad \theta \in (0, \pi).$$

In particular, (4.4) implies that  $\text{LP}[q_H(z)]_{\star}$  is closed.

In the other extreme case concerning constant limit points, namely that

$$\text{LP}[\mathcal{A}_r H]_{r \geq 1} \cap \mathbb{CH} = \emptyset, \quad (4.5)$$

$\text{LP}[q_H(z)]_{\star}$  is open. This follows from (3.7) since the image of a nonconstant analytic function is open.

An easily understood example where the extreme case (4.5) takes place, is given by multiplicatively periodic Hamiltonians, cf. Remark 3.5.

*4.5 Example.* A Hamiltonian  $H \in \mathbb{H}$  is multiplicatively periodic if and only if  $q_H$  has this property. In fact, for any  $p > 1$  it holds that  $H(pt) = H(t)$ ,  $t > 0$ , if and only if  $q_H(pz) = q_H(z)$ ,  $z \in \mathbb{C}_+$ . Moreover, note that  $H$  is constant if and only if  $q_H$  is constant.

Now consider a nonconstant and multiplicatively periodic Hamiltonian, say with period  $p > 1$ . Then

$$\text{LP}[\mathcal{A}_r H]_{r \geq 1} = \{\mathcal{A}_r H \mid r > 0\}. \quad (4.6)$$

First,  $\{\mathcal{A}_r H \mid r > 0\} = \{\mathcal{A}_r H \mid 1 \leq r \leq p\}$ , and hence this orbit is compact. This yields “ $\subseteq$ ”. To see the reverse inclusion, note that  $\mathcal{A}_{sp^n} H = \mathcal{A}_s H$ ,  $n \in \mathbb{N}$ , and hence  $\mathcal{A}_s H = \lim_{n \rightarrow \infty} \mathcal{A}_{sp^n} H \in \text{LP}[\mathcal{A}_r H]_{r \geq 1}$ . From (4.6) we also obtain that

▷  $\text{LP}[\mathcal{A}_r H]_{r \geq 1}$  contains no constant Hamiltonians;

▷ for every  $T > 0$  we have

$$\text{LP}_w[\rho_T \mathcal{A}_r H]_{r \geq 1} = \text{LP}_{\|\cdot\|_1}[\rho_T \mathcal{A}_r H]_{r \geq 1} = \{\rho_T \mathcal{A}_r H \mid r > 0\}.$$

In the same way as above, periodicity of  $q_H$  implies that

$$\begin{aligned} \text{LP}[q(re^{i\theta})]_{r \geq 1} &= q(e^{i\theta}[1, p]), \\ \text{LP}[q(z)]_{\star} &= \bigcup_{\alpha \in (0, \frac{\pi}{2})} q(\{re^{i\theta} \mid 1 \leq r \leq p, \alpha \leq \theta \leq \pi - \alpha\}) = q(\mathbb{C}_+). \end{aligned}$$

◇

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