Estimates for order of Nevanlinna matrices and a Berezanskii-type theorem

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Abstract: We give an upper estimate for the order of the entire functions in the Nevanlinna parameterisation of the solutions of an indeterminate Hamburger moment problem. Under a regularity condition this estimate becomes explicit and takes the form of a convergence exponent. Proofs are based on transformations of canonical systems and I.S.Kac' formula for the spectral asymptotics of a string. Combining with a lower estimate from previous work, we obtain a class of moment problems for which order can be computed. This generalises a theorem of Yu.M.Berezanskii about spectral asymptotics of a Jacobi matrix (in the case that order is $\leq 1/2$).

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1 Introduction

Let H be a 2×2-matrix valued integrable function on a finite interval $[0,L]$ whose values are almost everywhere real and positive semidefinite matrices. The *canonical system with Hamiltonian H* is the equation

$$y'(x) = zJH(x)y(x), \quad x \in [0,L], \tag{1.1}$$

where J is the symplectic matrix $J := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, and z is a complex parameter. The *fundamental solution* of the system (for practical reasons one passes to transposes) is the solution of the initital value problem

$$\begin{cases} \frac{d}{dx}W(x,z)J = zW(x,z)H(x), & x \in [0,L], \\ W(0,z) = I. \end{cases} \tag{1.2}$$

Classical theory of differential equations says that $W(x,z) = (w_{ij}(x,z))_{i,j=1}^2$ exists, is unique, and depends analytically on z.

For each $x \in [0,L]$ the entries $w_{ij}(x,z)$, $i,j = 1,2$, are entire functions of Cartwright class. Their common exponential type is given by the Krein-de Branges formula

$$\tau(x) = \int_0^x \sqrt{\det H(y)}\, dy, \quad i,j = 1,2,$$

cf. [Kre51], [Bra61]. If $\tau(L) > 0$ it follows that the eigenvalues ω_n of the differential operator associated with (1.1) form a two-sided infinite sequence and have asymptotics (when arranged increasingly)

$$|\omega_n| = \tau(L) \cdot \pi n + \mathrm{o}(n), \quad n \in \mathbb{Z}.$$

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If $\det H(x) = 0$ a.e., the Krein-de Branges formula gives no information about eigenvalues other than $\omega_n = o(n)$, and it is a difficile problem to detect finer asymptotic behaviour. Thinking, for instance, of bounds of the form $\limsup_{n\to\infty} n^{-\frac{1}{\rho}}\omega_n < \infty$, this corresponds to the problem to determine the order of the functions $w_{ij}(x,\cdot)$.

The question to determine order has been studied in particular in the context of Hamburger moment problems where $W(L,z)$ is the Nevanlinna matrix associated with an indeterminate moment problem, see, e.g., [BP94; BP07; BS14], or for Krein-Feller operators $\frac{d^2}{dm\,dx}$ where $W(L,z)$ contains the fundamental solutions of the second order equation and their derivatives, see, e.g., [Kac86b; Fre05].

The connection between moment problems and canonical systems is made as follows: indeterminate Hamburger moment problems correspond to canonical systems whose Hamiltonian is piecewise constant and rank one on a sequence of intervals accumulating only at L, i.e., H being of the form

$$H(x) = \xi_{\phi_n}\xi_{\phi_n}^*, \quad x \in [x_{n-1}, x_n),$$
$$0 = x_0 < x_1 < x_2 < \cdots < x_n < \cdots \to L,$$

where

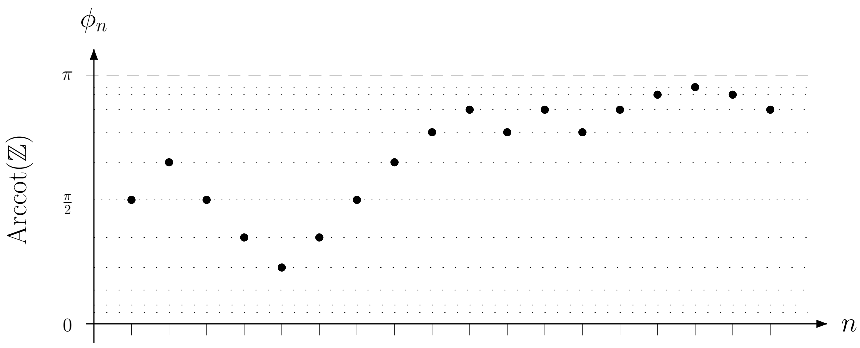
$$\xi_\phi := \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}, \quad \phi \in \mathbb{R},$$

cf. [Kac99]. Following I.S.Kac, we call such Hamiltonians *Hamburger Hamiltonians.*

In the present paper we establish an upper estimate for the order of a Hamburger Hamiltonian; see Theorem 4.1, which is our first main result. The proof is achieved by associating with the given Hamburger Hamiltonian a certain (singular) Krein-string. During this process several different types of arguments come into play. Our method relies on an operator theoretic limiting argument (Proposition 2.5), some purely algebraic computations and transformations (§3), and estimates for canonical products by means of the density of their zeroes. Moreover, on the way, we leave the positive definite scheme and encounter Hamiltonians which may take negative semidefinite matrices as values.

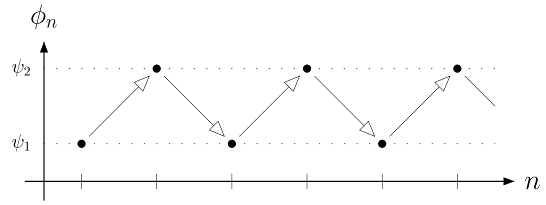
The estimate from Theorem 4.1 is incomparable with the one obtained recently in [PRW16]; in some cases it is better and in some others it is worse, cf. Proposition 4.5 and Example 4.6.

Our second main result is Theorem 4.4 where we discuss a class of Hamiltonians whose order can be determined. We consider a Hamburger Hamiltonian H whose angles ϕ_n (up to a small deviation) walk on the grid $\operatorname{Arccot}(\mathbb{Z})$:



and assume that lengths $l_n := x_n - x_{n-1}$ and angles together decay sufficiently rapidly (the series $\sum_{n=1}^{\infty}[l_n \sin^2 \phi_n]^{\frac{1}{2}} \ln n$ should converge) and regularly (the sequence $l_n \sin^2 \phi_n$ should be nonincreasing). The conclusion then is that the order of $w_{ij}(L, z)$ is equal to the convergence exponent of $([l_n \sin^2 \phi_n]^{-1})_{n=1}^{\infty}$ The proof is obtained by evaluating the upper estimate Theorem 4.1 with help of [Kac90], and combining this with a lower estimate from [PRW16].

Theorem 4.4 can be seen as a generalisation for orders $\leq 1/2$ of a theorem of Yu.M.Berezanskii. In the language of Hamburger Hamiltonians the essence of Berezanskii's theorem can be phrased as follows: Consider a Hamburger Hamiltonian H whose angles alternate between two values:



If lengths decay regularly (the sequence l_{n-2}/l_n should be monotone), then the order of $w_{ij}(L, z)$ is equal to the convergence exponent of $(l_n^{-1})_{n=1}^{\infty}$.

A detailed discussion of the connection with Berezanskii's theorem is given in §4.3, where we shall also see that the present result actually goes far beyond Berezanskii's result, cf. Example 4.8.

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2 Schatten-class properties and order

Let H be a Hamiltonian. An interval (a, b) is called *indivisible* for H, if $H(x) = h(x)\xi_\phi\xi_\phi^*$, $x \in (a, b)$ a.e., with some scalar valued function h. We refer to ϕ as the *angle* of the interval and to $\int_a^b h(x)\, dx$ as its *length*. Obviously, ϕ is determined up to integer multiples of π.

Consider now a positive semidefinite Hamiltonian $H : [0, L) \to \mathbb{R}^{2\times 2}$, which is defined and locally integrable on a finite or infinite interval $[0, L)$. With H there is associated a Hilbert $L^2(H)$ and a linear relation $T_{\max}(H)$ acting in this space, cf. [Kac85; Kac86a] or (in a more accessible form) [HSW00]. The space $L^2(H)$ consists of 2-vector valued measurable functions satisfying a usual L^2-condition and a constancy condition on indivisible intervals[1]. The relation $T_{\max}(H)$ is given by the differential expression $f' = JHg$ on its natural maximal domain in $L^2(H)$.

There is a rich spectral theory for canonical systems. The adjoint $T_{\min}(H) := T_{\max}(H)^*$ is a completely nonselfadjoint symmetry in $L^2(H)$. It has defect index $(2, 2)$ or $(1, 1)$ depending whether the integral $\int_0^L \operatorname{tr} H(x)\, dx$ is finite or infinite.

[1] One word of caution concerning notation: In [HSW00] the space we call $L^2(H)$ is denoted as $L_s^2(H)$, and $L^2(H)$ is used for the space obtained only requiring finiteness of the L^2-integral.

This distinction is known as *limit circle case (lcc)* if the integral is finite, and *limit point case (lpc)* if it is infinite.

Recall the construction of the Titchmarsh-Weyl coefficient associated with a positive semidefinite Hamiltonian in lpc:

2.1. The Weyl-construction: We denote by $\mathcal{N}_0$ the *Nevanlinna class*, i.e., the set of all functions Q which are analytic in $\mathbb{C}\setminus\mathbb{R}$, satisfy $Q(\overline{z})=\overline{Q(z)}$, and have nonnegative imaginary part throughout the upper half-plane $\mathbb{C}^+$.

Let H be a positive semidefinite Hamiltonian in lpc. Then for each parameter $\tau\in\mathcal{N}_0\cup\{\infty\}$ the limit

$$Q_H(z):=\lim_{x\to L}\frac{w_{11}(x,z)\tau(z)+w_{12}(x,z)}{w_{21}(x,z)\tau(z)+w_{22}(x,z)} \tag{2.1}$$

exists locally uniformly on $\mathbb{C}\setminus\mathbb{R}$ and does not depend on τ, cf. [HSW00, Theorem 2.1(2.7)] The function Q_H is called the *Titchmarsh-Weyl coefficient* of H. It belongs to the Nevanlinna class and is the Q-function of the canonical selfadjoint extension of $T_{\min}(H)$ given as

$$A(H):=\big\{(f;g)\in T_{\max}(H):(1,0)f(0)=0\big\},$$

cf. [HSW00, Theorem 4.3]. An inverse theorem holds: Given a function $Q\in\mathcal{N}_0$, there exists a positive semidefinite Hamiltonian H such that $Q=Q_H$, and H is unique up to a normalisation. This result is due to L.de Branges and follows from [Bra68] (an explicit deduction from this source is given in [Win95]). $\Diamond$

Let us turn to the case that H is integrable up to L.

2.2. Limit circle case: In this case the limit $W(L,z):=\lim_{x\nearrow L}W(x,z)$ exists locally uniformly on $\mathbb{C}$, and therefore the right side of (2.1) can be evaluated as

$$Q_{H/\tau}(z):=\frac{w_{11}(L,z)\tau(z)+w_{12}(L,z)}{w_{21}(L,z)\tau(z)+w_{22}(L,z)},\quad z\in\mathbb{C}\setminus\mathbb{R}.$$

When τ runs through $\mathcal{N}_0\cup\{\infty\}$, the functions $Q_{H/\tau}$ parameterise the family of regularised 1-resolvents of all selfadjoint exit space extensions of the symmetric extension of $T_{\min}(H)$ given as

$$S(H):=\big\{(f;g)\in T_{\max}(H):(1,0)f(0)=0,f(L)=0\big\}.$$

Thereby constant parameters correspond to canonical extensions. This is due to the interpretation of $W(L,z)$ as a resolvent matrix in the sense of M.G.Krein, which follows from [HSW00, Proposition 4.4].

If $\tau\in\mathbb{R}\cup\{\infty\}$, the function $Q_{H/\tau}$ is the Q-function of $S(H)$ induced by its extension

$$A_\tau(H):=\big\{(f;g)\in T_{\max}(H):(1,0)f(0)=0,\xi_\phi^*f(L)=0\big\}, \tag{2.2}$$

where $\phi=\operatorname{Arccot}\tau$. This can be shown e.g. by appending an indivisible interval with angle ϕ and infinite length to H, and checking that for the resulting lpc Hamiltonian $\tilde{H}$ in fact $S(H)=T_{\min}(\tilde{H})$ and $A_\tau(H)=A(\tilde{H})$. $\Diamond$

The following notion of order is the central subject of our studies.

2.3 Definition. Let H be a positive semidefinite Hamiltonian defined and locally integrable on the finite or infinite interval $[0, L)$.

If H is lpc and Q_H is not meromorphic throughout $\mathbb{C}$, set $\rho(H) := \infty$. Otherwise, let $(\omega_n)_{n=1,2,...}$ be the sequence of non-zero poles of Q_H (or $Q_{H/0}$ if H is lcc) arranged according to nondecreasing modulus, and define $\rho(H)$ as the convergence exponent of $(\omega_n)_{n=1,2,...}$, i.e.

$$\rho(H) = \inf \Big\{ \alpha > 0 : \sum_{n=1,2,...} \omega_n^{-\alpha} < \infty \Big\}.$$

We call $\rho(H)$ the *order of H*. ♢

Our motivation to introduce order in this way comes from the lcc. In this case the entries $w_{ij}(L, z)$ are entire functions of bounded type in both half-planes $\mathbb{C}^+$ and $\mathbb{C}^-$ and are real along the real axis. Hence, they are canonical products and the convergence exponent of their zeroes equals their order. However, the poles of a function $Q_{H/\tau}$ (where $\tau \in \mathbb{R} \cup \{\infty\}$ is arbitrary) interlace with the zeroes of $w_{21}(L, z)$, and hence have the same convergence exponent.

We use an operator theoretic interpretation of $\rho(H)$. For $p > 0$ denote by $\mathfrak{S}_p$ the Schatten-von Neumann ideal of all compact operators whoes sequence of s-numbers belongs to ℓ^p, see, e.g., [GK69].

2.4 Remark. Let H be a positive semidefinite Hamiltonian which is either lcc or lpc with Q_H meromorphic throughout $\mathbb{C}$. Then the spectrum of $A(H)$ (or $A_\tau(H)$, respectively) coincides with the set of poles of Q_H (or $Q_{H/\tau}$, respectively). Therefore $A(H)$ (or $A_\tau(H)$, respectively) has compact resolvents and, for arbitrary z in the resolvent set of the operator

$$\rho(H) = \inf \big\{ p > 0 : (A(H) - z)^{-1} \in \mathfrak{S}_p \big\},$$

or $\rho(H) = \inf \big\{ p > 0 : (A_\tau(H) - z)^{-1} \in \mathfrak{S}_p \big\}$, respectively.

Assume now that H is lpc with Q_H meromorphic throughout $\mathbb{C}$. There exists a unique canonical selfadjoint extension of $T_{\min}(H)$ having 0 in its spectrum, and hence as an eigenvalue. This means that there exists some constant $\xi_{\phi(H)}$ belonging to $L^2(H)$. Since we are in lpc, the angle $\phi(H)$ is uniquely determined (modulo π). It is related to Q_H by

$$Q_H(0) = -\tan \phi(H), \tag{2.3}$$

cf. [HSW00, Theorem 2.1(2.8)]. ♢

In our present considerations we employ the following result which is interesting on its own right.

2.5 Proposition. *Let $H : [0, L) \to \mathbb{R}^{2\times 2}$ be a positive semidefinite Hamiltonian in lpc such that $(0, L)$ is not indivisible. Assume that Q_H is meromorphic in $\mathbb{C}$, $Q_H(0) = 0$, and $\sum_n \frac{1}{|\omega_n|} < \infty$, where $(\omega_n)_{n=1,2,...}$ is the sequence of poles of Q_H arranged according to nondecreasing modulus. Then the following statements hold.*

(*i*) *Denote by J the set of all points $x \in (0, L)$ such that x is not inner point of an indivisible interval and $(0, x)$ is not indivisible. The limits*

$$b(z) := \lim_{\substack{x \to \sup J \\ x \in J}} w_{12}(x, z), \quad d(z) := \lim_{\substack{x \to \sup J \\ x \in J}} w_{22}(x, z),$$

exist locally uniformly on $\mathbb{C}$.

(*ii*) *The functions* b *and* d *are real along the real axis, have no common zeroes, are of Polya class and of order* $\rho(H)$ *(with zero type if* $\rho(H) = 1$*).*

(*iii*) *For each* $\varepsilon > 0$ *there exists a constant* $C_\varepsilon > 0$ *such that*

$$\forall\, x \in J: \quad |w_{ij}(x,z)| \leq C_\varepsilon \exp\big(|z|^{\rho(H)+\varepsilon}\big), \ z \in \mathbb{C}, (i,j) \in \{(1,2),(2,2)\}.$$

In the proof we exploit the connection Remark 2.4 and use a standard estimate for canonical products[2].

Proof. For $x \in J$ we can consider $L^2(H|_{[0,x]})$ as a subspace of $L^2(H)$, namely by identifying a function f from $L^2(H_{[0,x]})$ with its extension $\hat{f}$ defined by $\hat{f}(y) = 0$, $y \in (x, L)$. We shall always tacitly apply this identification.

Set $P_x : f \mapsto \mathbb{1}_{[0,x]} f$, where $\mathbb{1}_{[0,x]}$ denotes the indicator function of the interval $[0,x]$. Then P_x is the orthogonal projection of $L^2(H)$ onto $L^2(H|_{[0,x]})$. Moreover, set

$$T := A(H)^{-1}, \qquad T_x := A_0(H|_{[0,x]})^{-1}, \ x \in J.$$

Note here that $0 \in \rho(A(H))$ by assumption and $0 \in \rho(A_0(H|_{[0,x]}))$ by the boundary condition in the definition (2.2). The spectrum of T equals $(\omega_n^{-1})_{n=1,2,\dots}$ with all eigenvalues being simple. Hence, $T \in \mathfrak{S}_1$.

The crucial observation is that

$$T_x = P_x T|_{\operatorname{ran} P_x}, \quad x \in J.$$

To see this, let $g \in \operatorname{ran} P_x$ be given and set $f := Tg$. Then $f'(x) = JH(x)g(x)$, $x \in [0,L)$ a.e., and $(1,0)f(0) = 0$. Since $g(y) = 0$, $y \in (x,L)$, the function $f|_{[x,L)}$ is constant. It follows from (2.3) that $f|_{[x,L)} \in \operatorname{span}\{\binom{1}{0}\}$, which implies $(0,1)f(x) = 0$. We see that $T_x g = \mathbb{1}_{[0,x]} f = P_x T g$.

We proceed with establishing the required properties of the right lower entries $w_{22}(x,z)$. Since $P_x \to I$ in the strong operator topology when $x \nearrow \sup J$ and $T \in \mathfrak{S}_1$, we have $P_x T P_x \to T$ in the norm of $\mathfrak{S}_1$. This implies that

$$\lim_{\substack{x \to \sup J \\ x \in J}} \det(I - zP_xTP_x) = \det(I - zT)$$

locally uniformly on $\mathbb{C}$. We have

$$\ker(P_xTP_x - \lambda) = \ker(P_xT|_{\operatorname{ran} P_x} - \lambda), \quad \lambda \neq 0,$$

and hence $\det(I - zP_xTP_x) = \det(I - zP_xT|_{\operatorname{ran} P_x}) = \det(I - zT_x)$.

Let $\omega_n(x)$ be the zeroes of $w_{22}(x,\cdot)$ arranged according to nondecreasing modulus. The spectrum of T_x equals $\{\omega_1(x), \omega_2(x), \dots\}$, and all eigenvalues of T_x are simple. Using that $w_{22}(x,\cdot)$ is of bounded type in $\mathbb{C}^+$ and real along the real axis we obtain

$$w_{22}(x,z) = \prod_n \Big(1 - \frac{z}{\omega_n(x)}\Big) = \det(I - zT_x) = \det(I - zP_xTP_x).$$

[2]Probably an alternative proof could proceed using [Bra68, Theorem 41, Problem 154] and the "reversing direction transformation" [KW11, Definition 2.6]. However, we did not try to work out details of this approach since we believe that the operator theoretic argument is simple and elegant.

Thus the limit in (i) exists, in fact $d(z) = \det(I - zT)$. Since $\det(I - zT) = \prod_n \big(1 - \frac{z}{\omega_n}\big)$, the properties of d listed in (ii) follow.

For the proof of the uniform estimate in (iii) consider the counting functions

$$n(x, r) := \#\{n : |\omega_n(x)| \le r\}, \quad n(r) := \#\{n : |\omega_n| \le r\}.$$

Denote by $s_n(\cdot)$ the n-th s-number of an operator, then

$$|\omega_n(x)|^{-1} = s_n(T_x) = s_n\big(P_x T|_{\operatorname{ran} P_x}\big) = s_n(P_x T P_x) \le s_n(T) = |\omega_n|^{-1},$$

whence $n(x, r) \le n(r)$, $x \in J$, $r > 0$. Using [Lev80, Lemma I.4.3] we obtain the required bound.

We turn to the function $w_{12}(x, z)$. Let $\tilde{\omega}_n(x)$ be the nonzero zeroes of $w_{12}(x, \cdot)$ arranged according to nondecreasing modulus, and let $\tilde{n}(x, r)$ be the counting function for $\tilde{\omega}_1(x), \tilde{\omega}_2(x), \ldots$. Since the zeroes of $w_{12}(x, \cdot)$ interlace with the zeroes of $w_{22}(x, \cdot)$ and $w_{12}(x, 0) = 0$, we have

$$\tilde{n}(x, r) \le n(x, r) \le n(r), \quad x \in J, r > 0.$$

Again [Lev80, Lemma I.4.3] applies and yields a uniform estimate for the canonical product $\prod_n \big(1 - \frac{z}{\tilde{\omega}_n(x)}\big)$. The function $w_{12}(x, \cdot)$ is of bounded type in $\mathbb{C}^+$ and real along the real axis, hence admits the representation (a prime denotes differentiation w.r.t. z)

$$w_{12}(x, z) = w_{12}'(x, 0) \cdot z \prod_n \Big(1 - \frac{z}{\tilde{\omega}_n(x)}\Big).$$

However,

$$w_{12}'(x, 0) = \int_0^x \binom{1}{0}^* H(y) \binom{1}{0} \, dy \le \Big\| \binom{1}{0} \Big\|_{L^2(H)}^2 < \infty,$$

and the bound required in (iii) for $w_{12}(x, \cdot)$ follows.

We have $w_{12}(x, \cdot) w_{22}(x, \cdot)^{-1} \to Q_H$ locally uniformly on $\mathbb{C} \setminus \mathbb{R}$, in particular, $w_{12}(x, z) \to d(z) Q_H(z)$ pointwise on $\mathbb{C} \setminus \mathbb{R}$. Since the functions $w_{12}(x, \cdot)$ form a normal family of entire function and $b := dQ_H$ is entire, this limit is actually assumed locally uniformly on all of $\mathbb{C}$. Using the product representation of Q_H and the fact that the zeroes of d are exactly the poles of Q_H, we obtain

$$b(z) = \Big\| \binom{1}{0} \Big\|_{L^2(H)}^2 \cdot z \cdot \prod_n \Big(1 - \frac{z}{\tilde{\omega}_n}\Big)$$

where $\tilde{\omega}_n$ denote the nonzero zeroes of Q_H. Thus b has all the properties listed in (ii). ❑

In Proposition 2.5 we assume the normalisation $Q_H(0) = 0$, equivalently, that $\phi(H) = 0$. Passing to arbitrary angles $\phi(H)$ is easily possible by performing a rotation (see, e.g., [KW11, Definition 2.4, Lemma 3.29]). Due to the (annoying) fact that different sources of literature use different normalisations, we need the corresponding result obtained after a rotation by $\frac{\pi}{2}$.

2.6 Corollary. *Assume in Proposition* 2.5 *that* Q_H *has a pole at* 0 *instead of the value* 0*. Then the assertion remains true when the functions* $w_{12}(x,z)$ *and* $w_{22}(x,z)$ *are replaced by* $w_{11}(x,z)$ *and* $w_{21}(x,z)$*.*

2.7 Remark. Proposition 2.5 is a natural generalisation of the lcc.

— Assume that H is lcc: The limits $w_{ij}(L,z) = \lim_{x\nearrow L} w_{ij}(x,z)$, $i,j = 1,2$, exist, and the functions $w_{ij}(L,z)$ are real along the real axis and have no common zeroes.

— Assume in addition that $\det H = 0$ a.e.: The functions $w_{ij}(L,z)$, $i,j = 1,2$, are of Polya class and of order $\rho(H)$ (with zero type if $\rho(H) = 1$).

— Uniform estimate: For each $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$\forall\, x \in [0,L]: \quad |w_{ij}(x,z)| \leq C_\varepsilon \exp\left(|z|^{\rho(H)+\varepsilon}\right),\ z \in \mathbb{C},\ i,j = 1,2.$$

To see the uniform estimate just append an indivisible interval of infinite length and type $\frac{\pi}{2}$ (type 0 for the first column), and apply Proposition 2.5 (Corollary 2.6 for the first column)[3]. ◊

3 Signed Hamburger Hamiltonians

For an equation (1.1) where H is not anymore positive semidefinite, no comprehensive theory corresponding to what we mentioned in 2.1 and 2.2 is known. Generalisations to some particular indefinite situations have been undertaken in [KL79; KL80; KL85], [Fle96], [LW98], [KW06; KW11; KW10]. Except of [Fle96] all papers deal with a Pontryagin space situation (i.e., finite negative index).

We deal with a class of possibly indefinite Hamiltonians having the very simple form analogous to Hamburger Hamiltonians.

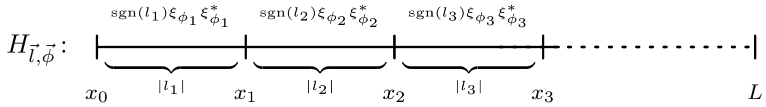
3.1 Definition. Let $\vec{l} = (l_n)_{n=1}^\infty$ and $\vec{\phi} = (\phi_n)_{n=1}^\infty$ be sequences of real numbers with $l_n \neq 0$ and $\phi_{n+1} \not\equiv \phi_n \bmod \pi$, $n \in \mathbb{N}$, and set

$$x_0 := 0, \qquad x_n := \sum_{k=1}^n |l_k|,\ n \in \mathbb{N}, \qquad L := \sum_{k=1}^\infty |l_k| \in (0,\infty]. \tag{3.1}$$

Then we call the function $H_{\vec{l},\vec{\phi}} : [0,L) \to \mathbb{R}^{2\times 2}$ defined as

$$H_{\vec{l},\vec{\phi}}(x) := \operatorname{sgn}(l_n)\xi_{\phi_n}\xi_{\phi_n}^*, \quad x \in [x_{n-1},x_n),\ n \in \mathbb{N},$$

the *signed Hamburger Hamiltonian* with *lengths* $\vec{l}$ and *angles* $\vec{\phi}$. The points x_n are called the *nodes* of $H_{\vec{l},\vec{\phi}}$.



[3] A direct proof can be given repeating some of the arguments from the proof of Proposition 2.5.

$\Diamond$

A signed Hamburger Hamiltonian $H_{\vec{l},\vec{\phi}}$ is a.e. positive semidefinite if and only if all lengths l_n are positive. If $H_{\vec{l},\vec{\phi}}$ is positive semidefinite, then lpc or lcc takes place depending whether $L = \infty$ or $L < \infty$, where L is as in (3.1). A signed Hamburger Hamiltonian is associated with an indefinite power moment problem as in [KL79; KL80] if and only if all but finitely many lengths are positive.

3.2 Remark. The facts mentioned right after (1.2), that a fundamental solution exists and is entire, depend only on local integrability of H and not on definiteness properties.

For a signed Hamburger Hamiltonian $H_{\vec{l},\vec{\phi}}$ the fundamental solution $W_{\vec{l},\vec{\phi}}$ can easily be computed explicitly. Denote

$$w_\phi(x,z) := \begin{pmatrix} 1 - xz\sin\phi\cos\phi & xz\cos^2\phi \\ -xz\sin^2\phi & 1 + xz\sin\phi\cos\phi \end{pmatrix} = I - zx\xi_\phi\xi_\phi^* J, \qquad x \in \mathbb{R},\ z \in \mathbb{C},\ \phi \in \mathbb{R},$$

then

$$W_{\vec{l},\vec{\phi}}(x,z) = w_{\phi_1}(l_1,z) \cdot \ldots \cdot w_{\phi_{n-1}}(l_{n-1},z) \cdot w_{\phi_n}\big(\operatorname{sgn}(l_n)(x - x_{n-1}), z\big), \qquad x \in [x_{n-1}, x_n),\ n \in \mathbb{N}.$$

$\Diamond$

A particular class of systems is given by Hamiltonians which are almost everywhere a diagonal matrix, and we refer to such as *diagonal Hamiltonians.* Observe that a signed Hamburger Hamiltonian is diagonal if and only if $\phi_n \in \{0, \frac{\pi}{2}\}$ (modulo π), $n \in \mathbb{N}$. Diagonal Hamiltonians (in the positive semidefinite situation) are in many ways easier to treat and a variety of symmetry properties is present, see, e.g., [Bra68, Chapter 47].

Square-root and square transform

The *Stieltjes class* $\mathcal{S}$ is the subclass of $\mathcal{N}_0$ consisting of all Nevanlinna functions Q which are analytic in $\mathbb{C} \setminus [0,\infty)$ and satisfy $Q(x) \geq 0$, $x \in (-\infty, 0)$. If $Q \in \mathcal{S}$ the function $Q_d(z) := zQ(z^2)$ also belongs to the Nevanlinna class, cf. [KK68, Lemma S1.5.1]. Hence, for $Q \in \mathcal{S}$, de Branges' inverse theorem gives two positive semidefinite Hamiltonians H and H_d, namely those having Q and Q_d as corresponding Titchmarsh-Weyl coefficients. Since $Q_d(-z) = -Q_d(z)$, H_d is a diagonal Hamiltonian. These two Hamiltonians can be transformed into each other by explicit formulae, see, e.g., [KWW07]. We speak of the *square-root transform* turning H_d into H, and its converse, the *square transform.* These transformations can also be carried out on the level of fundamental solutions. A systematic discussion on this level including certain indefinite cases is given in [KWW06].

For a positive semidefinite Hamburger Hamiltonian the mentioned transformations are established by explicit algebraic formulae. We use the same formulae to define corresponding transforms for signed Hamburger Hamiltonians.

First, let us introduce a practical abbreviation: for two sequences of real numbers $\vec{x} = (x_n)_{n=1}^{\infty}$ and $\vec{y} = (y_n)_{n=1}^{\infty}$, we denote by $\vec{x} : \vec{y}$ the mixed sequence

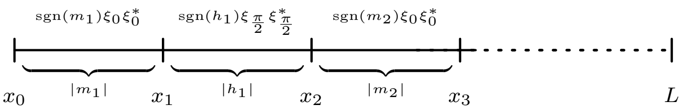
$$\vec{x} : \vec{y} \ := \ (x_1, y_1, x_2, y_2, x_3, \ldots).$$

Moreover, we set

$$\vec{\delta} := \big(0, \frac{\pi}{2}, 0, \frac{\pi}{2}, 0, \ldots\big).$$

3.3 Definition. Let H be a diagonal signed Hamburger Hamiltonian, and assume (for normalisation) that its first angle is equal to 0. Denote by $\vec{m}$ and $\vec{h}$ the sequences of odd and even lengths of H, respectively. That just means that we write H in the form $H = H_{\vec{m}:\vec{h},\vec{\delta}}$:

$H = H_{\vec{m}:\vec{h},\vec{\delta}}$:



Set

$$l_n := h_n\Big(1 + \Big(\sum_{k=1}^{n} m_k\Big)^2\Big), \ \phi_n := \operatorname{Arccot}\Big(\sum_{k=1}^{n} m_k\Big), \quad n \in \mathbb{N}. \tag{3.2}$$

Then we call $H_{\vec{l},\vec{\phi}}$ the *square-root transform* of H. ♢

The converse transformation is obtained by simply inverting the relations (3.2).

3.4 Definition. Let $H_{\vec{l},\vec{\phi}}$ be a signed Hamburger Hamiltonian, and assume that $\phi_n \not\equiv 0 \mod \pi$, $n \in \mathbb{N}$. Set (with $\phi_0 := \frac{\pi}{2}$)

$$m_n := \cot(\phi_n) - \cot(\phi_{n-1}), \ h_n := l_n \sin^2(\phi_n), \quad n \in \mathbb{N}.$$

Then we call $H_{\vec{m}:\vec{h},\vec{\delta}}$ the *square transform* of $H_{\vec{l},\vec{\phi}}$. ♢

Inductively applying the computation [KWW06, Proposition 3.6(i)] yields the following fact.

3.5 Lemma. *Let $H_{\vec{l},\vec{\phi}}$ be a signed Hamburger Hamiltonian with $\phi_n \not\equiv 0 \bmod \pi$, $n \in \mathbb{N}$, and let H_d be its square transform. Denote by $W_{\vec{l},\vec{\phi}}(x,z)$ and $W_d(y,z)$ the corresponding fundamental solutions, and let x_n and y_n be the nodes of $H_{\vec{l},\vec{\phi}}$ and H_d, respectively. Then for all $n \in \mathbb{N}$ (a prime denote differentiation w.r.t. z)*

$$W_{\vec{l},\vec{\phi}}(x_n, z^2) = \begin{pmatrix} w_{d,11}(y_{2n}, z) & \frac{w_{d,12}(y_{2n},z)}{z} - w'_{d,12}(y_{2n},0) w_{d,11}(y_{2n}, z) \\ z w_{d,21}(y_{2n}, z) & w_{d,22}(y_{2n}, z) - w'_{d,12}(y_{2n},0) z w_{d,21}(y_{2n}, z) \end{pmatrix}. \tag{3.3}$$

Let us now state some immediate properties of these transformations.

3.6 Remark.

(*i*) The square-root transform of a diagonal signed Hamburger Hamiltonian H_d is positive semidefinite if and only if all even lengths of H_d are positive. The square transform of a signed Hamburger Hamiltonian $H_{\vec{l},\vec{\phi}}$ is positive semidefinite if and only if $H_{\vec{l},\vec{\phi}}$ itself is positive semidefinite and the sequence of angles is monotonically decreasing when considered modulo π as a sequence in $(0, \pi)$.

(*ii*) Assume that $H_{\vec{l},\vec{\phi}}$ and its square transform H_d are both positive semidefinite. Then

$$\rho(H_d) = 2\rho(H_{\vec{l},\vec{\phi}}).$$

To see this, let Q_d be the function Q_{H_d} or $Q_{H_d/\infty}$ depending whether H_d is lpc or lcc, and let $Q_{\vec{l},\vec{\phi}}$ be defined analogously for $H_{\vec{l},\vec{\phi}}$. Lemma 3.5 shows that $Q_d(z) = zQ_{\vec{l},\vec{\phi}}(z^2)$.

(*iii*) Assume again that $H_{\vec{l},\vec{\phi}}$ and H_d are both positive semidefinite. If $H_{\vec{l},\vec{\phi}}$ is lcc and H_d is lpc, then $\phi(H_d) = \frac{\pi}{2}$. This follows since (denote $\hat{L} := \sum_{n=1}^{\infty}(m_n + h_n)$)

$$\int_0^{\hat{L}} \begin{pmatrix}0\\1\end{pmatrix}^* H_d(y) \begin{pmatrix}0\\1\end{pmatrix} dy = \sum_{n=1}^{\infty} h_n \le \sum_{n=1}^{\infty} l_n < \infty.$$

$\Diamond$

The modulus transform

A signed Hamburger Hamiltonian can be transformed into a positive semidefinite one simply by taking absolute values of its lengths.

For a sequence $\vec{l}$ of real numbers denote

$$|\vec{l}| := (|l_n|)_{n=1}^{\infty}.$$

3.7 Definition. Let $H_{\vec{l},\vec{\phi}}$ be a signed Hamburger Hamiltonian. Then we call $H_{|\vec{l}|,\vec{\phi}}$ the *modulus transform* of $H_{\vec{l},\vec{\phi}}$. $\Diamond$

The next result shows that the fundamental solution of a diagonal signed Hamburger Hamiltonian can be estimated by the fundamental solution of its modulus transform.

3.8 Proposition. *Let $\vec{l}$ be a sequence of nonzero real numbers, and consider the Hamburger Hamiltonians $H_{\vec{l},\vec{\delta}}$ and $H_{|\vec{l}|,\vec{\delta}}$ with corresponding fundamental solutions $W_{\vec{l},\vec{\delta}}$ and $W_{|\vec{l}|,\vec{\delta}}$, respectively. Then (note that the sequences $(x_n)_{n=1}^{\infty}$ defined in* (3.1) *for $\vec{l}$ and $|\vec{l}|$ coincide)*

$$\left|(1,0)W_{\vec{l},\vec{\delta}}(x_{2n},z)\begin{pmatrix}1\\0\end{pmatrix}\right| \le (1,0)W_{|\vec{l}|,\vec{\delta}}(x_{2n},i|z|)\begin{pmatrix}1\\0\end{pmatrix}, \quad n\in\mathbb{N}, z\in\mathbb{C}. \tag{3.4}$$

The proof follows from a purely algebraic and explicit formula for (the first row of) the fundamental solution of a diagonal signed Hamburger Hamiltonian. We define for each $n \in \mathbb{N}$ and $k \in \{0,\dots,n\}$ polynomials $a_{n,k}$ and $b_{n,k}$ in variables $v_1, v_2, \dots$ by the recursions

$$a_{1,0}(\vec{v}) := 1, \qquad a_{1,1}(\vec{v}) := v_1 v_2,$$
$$b_{1,0}(\vec{v}) := v_1, \qquad b_{1,1}(\vec{v}) := 0,$$

$$a_{n+1,k}(\vec{v}) := \begin{cases} 1 & , \quad k=0,\\ a_{n,k}(\vec{v}) + v_{2n+1}v_{2n+2}a_{n,k-1}(\vec{v}) + v_{2n+2}b_{n,k-1}(\vec{v}) , & \quad k=1,\dots,n,\\ v_{2n+1}v_{2n+2}a_{n,n}(\vec{v}) & , \quad k=n+1,\end{cases}$$

$$b_{n+1,k}(\vec{v}) := \begin{cases} b_{n,k}(\vec{v}) + v_{2n+1}a_{n,k}(\vec{v}) , & \quad k=0,\dots,n,\\ 0 & , \quad k=n+1.\end{cases}$$

Observe that $a_{n,k}$ and $b_{n,k}$ have nonnegative integer coefficients. The polynomial $a_{n,k}$ involves only the variables $v_1, \dots, v_{2n}$, and $b_{n,k}$ only the variables $v_1, \dots, v_{2n-1}$. Moreover,

$$a_{n,0}(\vec{v}) = 1,\ a_{n,n}(\vec{v}) = \prod_{k=1}^{2n} v_k, \quad b_{n,0}(\vec{v}) = \sum_{k=1}^{n} v_{2k-1},\ b_{n,n}(\vec{v}) = 0,$$

for all $n \in \mathbb{N}$.

3.9 Lemma. *Let $\vec{l}$ be a sequence of nonzero real numbers, and let $W_{\vec{l},\vec{\delta}}$ be the fundamental solution of $H_{\vec{l},\vec{\delta}}$. Then*

$$(1,0)W_{\vec{l},\vec{\delta}}(x_{2n}, z) = \sum_{k=0}^{n} \Big(\frac{z}{i}\Big)^{2k} \big(a_{n,k}(\vec{l}), z b_{n,k}(\vec{l})\big), \quad n \in \mathbb{N}. \tag{3.5}$$

Proof. We use induction on n where the computation is based on the formula

$$w_0(l,z) w_{\frac{\pi}{2}}(h,z) = \begin{pmatrix} 1 + \big(\frac{z}{i}\big)^2 lh & zl \\ -zh & 1 \end{pmatrix}. \tag{3.6}$$

For $n = 1$ this formula already establishes the required representation of $W_{\vec{l},\delta}(x_2, z)$. Assume (3.5) holds for some $n \in \mathbb{N}$. Then (3.6) yields

$$\begin{aligned}
(1,0)W_{\vec{l},\vec{\delta}}(x_{2n+2}, z) &= (1,0)W_{\vec{l},\vec{\delta}}(x_{2n}, z) \begin{pmatrix} 1 + \big(\frac{z}{i}\big)^2 l_{2n+1} l_{2n+2} & z l_{2n+1} \\ -z l_{2n+2} & 1 \end{pmatrix} \\
&= \sum_{k=0}^{n} \Big(\frac{z}{i}\Big)^{2k} \big(a_{n,k}(\vec{l}) + (\tfrac{z}{i})^2 l_{2n+1} l_{2n+2} a_{n,k}(\vec{l}) - z^2 l_{2n+2} b_{n,k}(\vec{l}), \\
&\qquad\qquad z l_{2n+1} a_{n,k}(\vec{l}) + z b_{n,k}(\vec{l})\big) \\
&= \sum_{k=0}^{n} \Big(\frac{z}{i}\Big)^{2k} \big(a_{n,k}(\vec{l}), z[l_{2n+1} a_{n,k}(\vec{l}) + b_{n,k}(\vec{l})]\big) \\
&\quad + \sum_{k=1}^{n+1} \Big(\frac{z}{i}\Big)^{2k} \big(l_{2n+1} l_{2n+2} a_{n,k-1}(\vec{l}) + l_{2n+2} b_{n,k-1}(\vec{l}), 0\big) \\
&= \sum_{k=0}^{n} \Big(\frac{z}{i}\Big)^{2k} \big(a_{n+1,k}(\vec{l}), z b_{n+1,k}(\vec{l})\big).
\end{aligned}$$

❑

The estimate (3.4) is now nearly obvious.

Proof of Proposition 3.8. We use the representation from Lemma 3.9 and the fact that the polynomials $a_{n,k}$ have nonnegative coefficients to estimate

$$\begin{aligned}
\Big|(1,0)W_{\vec{l},\vec{\delta}}(x_{2n}, z)\begin{pmatrix}1\\0\end{pmatrix}\Big| &\le \sum_{k=0}^{n} |z|^{2k} \big|a_{n,k}(\vec{l})\big| \\
&\le \sum_{k=0}^{n} |z|^{2k} a_{n,k}(|\vec{l}|) = (1,0)W_{|\vec{l}|,\vec{\delta}}(x_{2n}, i|z|)\begin{pmatrix}1\\0\end{pmatrix}.
\end{aligned}$$

❑

4 An estimate for order

4.1 Formulation and proof of our two main theorems

The next statement is the first main theorem. In order not to complicate notation, we include the normalisation that $\phi_n \not\equiv 0 \bmod \pi$ for all $n \in \mathbb{N}$. Note that any Hamburger Hamiltonian can be transformed into one with nonzero angles by adding a certain constant offset to the angles, i.e., by performing a rotation as discussed before the statement of Corollary 2.6. The form of the rotation transformation [KW11, Definition 2.4] ensures $\rho(H_{\vec{l},(\phi_n+\alpha)}) = \rho(H_{\vec{l},\vec{\phi}})$.

4.1 Theorem. *Let $H_{\vec{l},\vec{\phi}}$ be a positive semidefinite Hamburger Hamiltonian in lcc, and assume that $\phi_n \not\equiv 0 \bmod \pi$, $n \in \mathbb{N}$. Set (with $\phi_0 := \frac{\pi}{2}$)*

$$m_n := \cot(\phi_n) - \cot(\phi_{n-1}),\ h_n := l_n \sin^2(\phi_n), \quad n \in \mathbb{N},$$
$$\vec{\delta} := \Big(0, \frac{\pi}{2}, 0, \frac{\pi}{2}, \dots\Big), \qquad |\vec{m} : \vec{h}| := \big(|m_1|, |h_1|, |m_2|, |h_2|, \dots\big).$$

Then

$$\rho(H_{\vec{l},\vec{\phi}}) \leq \frac{1}{2}\rho\big(H_{|\vec{m}:\vec{h}|,\vec{\delta}}\big).$$

The main point here is that the Hamiltonian appearing on the right side is *diagonal*. This implies that $\rho(H_{|\vec{m}:\vec{h}|,\vec{\delta}})$ can *in principle* be determined using Kac's formula [Kac86b, Theorems A–C] for the order of a string (unfortunately, a quite bulky expression).

Proof of Theorem 4.1. Starting from $H := H_{\vec{l},\vec{\phi}}$ build the following successive transforms:

— $H_d = H_{\vec{m}:\vec{h},\vec{\delta}}$ is the square transform of H;

— $H_d^+ = H_{|\vec{m}:\vec{h}|,\vec{\delta}}$ is the modulus transform of H_d;

— H^+ is the square-root transform of H_d^+.

The Hamiltonian H_d will in general carry signs, whereas H_d^+ and H^+ are positive semidefinite, cf. Remark 3.6, (*i*).

Denote by x_n the nodes of H, by y_n the common nodes of H_d and H_d^+, and by x_n^+ the nodes of H^+. Denote by W, W_d, W_d^+, W^+ the fundamental solutions of the respective Hamiltonian H, H_d, H_d^+, H^+, let Q^+ be either the function $Q_{H^+/\infty}$ if H^+ is lcc or the Titchmarsh-Weyl coefficient Q_{H^+} if H^+ is lpc, and let Q_d^+ be defined in the same way for H_d^+, respectively. Then $Q_d^+(z) = zQ^+(z^2)$ and $\rho(H_d) = 2\rho(H_d^+)$, cf. Remark 3.6, (*ii*).

The assertion of the theorem is equivalent to $\rho(H) \leq \rho(H^+)$. This is trivially true when $\rho(H^+) \geq 1$. Hence, assume throughout the following that $\rho(H^+) < 1$. In particular, Q^+ is meromorphic throughout the plane, and the sequence $(\omega_n)_{n=1,2,\dots}$ of its nonzero poles satisfies $\sum_n \frac{1}{|\omega_n|} < \infty$.

If H_d^+ is lcc, the function Q_d^+ has a pole at 0 by its definition. If H_d^+ is lpc, we have (denoting $\hat{L} := \sum_{n=1}^{\infty}(m_n + h_n)$)

$$\int_0^{\hat{L}} \binom{0}{1}^* H_d^+(y) \binom{0}{1}\, dy = \sum_{n=1}^{\infty} h_n \leq \sum_{n=1}^{\infty} l_n < \infty,$$

i.e., $\binom{0}{1} \in L^2(H_d^+)$. Again it follows that Q_d^+ has a pole at 0. From the relation $Q_d^+(z) = zQ^+(z^2)$ we see that also Q^+ has a pole at 0.

Corollary 2.6 and Remark 2.7 provide estimates ($\varepsilon > 0$ arbitrary)

$$\left|w_{11}^+(x_n^+, z)\right| \leq C_\varepsilon \exp\left(|z|^{\rho(H^+)+\varepsilon}\right), \quad n \geq 2, z \in \mathbb{C},$$

and (3.3) and Proposition 3.8 yield

$$\begin{aligned}|w_{11}(x_n, z^2)| &= |w_{d,11}(y_{2n}, z)| \leq w_{d,11}^+(y_{2n}, i|z|) \\ &= w_{11}^+(x_n^+, -|z|^2) \leq C_\varepsilon \exp\left(|z^2|^{\rho(H^+)+\varepsilon}\right), \quad n \geq 2, z \in \mathbb{C}.\end{aligned}$$

Passing to the limit $n \to \infty$ in the leftmost term, which is possible since H is lcc, we obtain that the same estimates hold for $w_{11}(L, z^2)$. We conclude that the order of $w_{11}(L, \cdot)$, which equals $\rho(H)$, does not exceed $\rho(H^+)$. ❑

For the case of a Stieltjes string (translated to the language of Hamiltonians this means for a diagonal Hamburger Hamiltonian) Kac' formula [Kac86b, Theorems A–C] takes the form [Kac90, p.31 (15)]. Still, a complicated expression which hardly allows explicit evaluation. Under some regularity assumptions on the involved data, however, it was shown in [Kac90] that it can be handled. We recall this result in the language of Hamiltonians. The following statement is the direct translation of [Kac90, Theorem 1].

4.2 Theorem ([Kac90], Theorem 1). *Let $\vec{M} = (M_n)_{n=1}^\infty$ and $\vec{L} = (L_n)_{n=1}^\infty$ be sequences of positive real numbers such that $\vec{M}$ is nonincreasing and $\vec{L}$ is nondecreasing. Set*

$$\vec{M} : \vec{L} := \left(M_1, L_1, M_2, L_2, \dots\right), \qquad \vec{\Delta} := \left(\frac{\pi}{2}, 0, \frac{\pi}{2}, 0, \dots\right),$$

and consider the positive definite diagonal Hamburger Hamiltonian $H_{\vec{M}:\vec{L},\vec{\Delta}}$. Then the following statements hold.

(i) *If $\alpha \in (0, \frac{1}{2})$ and $\sum_{n=1}^\infty (L_n M_{n+1})^\alpha < \infty$, then $\rho(H_{\vec{M}:\vec{L},\vec{\Delta}}) \leq 2\alpha$.*

(ii) *If $\sum_{n=1}^\infty (L_n M_{n+1})^{\frac{1}{2}} \ln n < \infty$, then $\rho(H_{\vec{M}:\vec{L},\vec{\Delta}}) \leq 1$.*

(iii) *If $\alpha \in (\frac{1}{2}, 1)$ and $\sum_{n=1}^\infty (L_n M_{n+1})^\alpha n^{2\alpha-1} < \infty$, then $\rho(H_{\vec{M}:\vec{L},\vec{\Delta}}) \leq 2\alpha$.*

Proof. The Hamiltonian $H_{\vec{M}:\vec{L},\vec{\Delta}}$ is related to the Stieltjes string with masses $(M_{n+1})_{n=0}^\infty$ and lengths $(L_n)_{n=1}^\infty$, cf. [KWW07, (4.4),(4.6)].

With the notation from [Kac90], this string is an element of $\mathcal{S}_\alpha$, by definition, if $\rho(H_{\vec{M}:\vec{L},\vec{\Delta}}) \leq 2\alpha$. The statement follows from [Kac90, Theorem 1]. ❑

Concerning [Kac90, Theorem 1] one word of caution is in order. This statement contains the a priori assumption that the string under consideration is of trace class, i.e. that

$$\sum_{n=1}^\infty \left(\sum_{k=n+1}^\infty M_k\right) L_n < \infty$$

or, equivalently, $\sum_{n=1}^\infty \left(\sum_{k=1}^n L_k\right) M_{n+1} < \infty$. It is said without a proof on p.31 right after Theorem 2 that this assumption is superfluous: convergence of this series can be deduced from convergence of the respective series in (i), (ii), or (iii). In the next result – which is our second main theorem – this fact is used for the cases (i) and (ii).Let us give a proof for these cases.

In the subsequent computations we use the following practical notation:

$$f(x) \asymp g(x) \quad :\Longleftrightarrow \quad \exists\, c_1, c_1 > 0 \,\forall\, x : c_1 f(x) \le g(x) \le c_2 f(x).$$

The notation $f(x) \lesssim g(x)$ and $f(x) \gtrsim g(x)$ refers to the corresponding one-sided properties.

Lemma. *Let $\vec{M} = (M_n)_{n=1}^{\infty}$ and $\vec{L} = (L_n)_{n=1}^{\infty}$ be sequences of positive real numbers such that $\vec{M}$ is nonincreasing and $\vec{L}$ is nondecreasing, and let $\alpha \in (0, \frac{1}{2}]$. If*

$$\sum_{n=1}^{\infty} (L_n M_{n+1})^{\alpha} < \infty,$$

then also

$$\sum_{n=1}^{\infty} \Big(\sum_{k=1}^{n} L_k \Big) M_{n+1} < \infty.$$

Proof. Set $r_n := \frac{1}{M_{n+1}}$, $n \in \mathbb{N}$, then r_n is positive, nondecreasing, and unbounded. Let μ be the positive measure (δ_r denotes the unit point mass at r)

$$\mu := \sum_{n=1}^{\infty} L_n \delta_{r_n},$$

and choose a decreasing C^{∞}-function $f : [0, \infty) \to (0, \infty)$ with $f(r_n) = L_n^{\alpha-1}$, $n \in \mathbb{N}$. We have

$$\int_0^{\infty} t^{-\alpha} f(t)\, d\mu(t) = \sum_{n=1}^{\infty} r_n^{-\alpha} f(r_n) L_n = \sum_{n=1}^{\infty} \Big(\frac{L_n}{r_n} \Big)^{\alpha} < \infty.$$

Integrating by parts yields that for each $T > 0$

$$\begin{aligned} \int_0^T t^{-\alpha} f(t)\, d\mu(t) &= T^{-\alpha} f(T) \mu([0,T]) - \int_0^T \underbrace{\frac{d}{dt}\big[t^{-\alpha} f(t) \big]}_{<0} \cdot \mu([0,t])\, dt \\ &\ge T^{-\alpha} f(T) \mu([0,T]), \end{aligned}$$

and, choosing $T = r_n$, we obtain the estimate

$$1 \gtrsim r_n^{-\alpha} f(r_n) \mu([0, r_n]) = r_n^{-\alpha} L_n^{\alpha-1} \Big(\sum_{k=1}^{n} L_k \Big).$$

Since $1 - \alpha \ge \alpha$ and $\frac{L_n}{r_n} \le 1$ for large n, it follows that

$$\frac{1}{r_n} \Big(\sum_{k=1}^{n} L_k \Big) \lesssim \Big(\frac{L_n}{r_n} \Big)^{1-\alpha} \lesssim \Big(\frac{L_n}{r_n} \Big)^{\alpha}.$$

❑

Combining Theorem 4.1 with Theorem 4.2 leads to the following corollary.

4.3 Corollary. *Let $H_{\vec{l},\vec{\phi}}$ be a positive semidefinite Hamburger Hamiltonian in lcc, and let notation $\vec{m}, \vec{h}$, etc. be as in Theorem 4.1. Assume that $|\vec{m}|$ is nondecreasing and $\vec{h}$ is nonincreasing. Then the following statements hold.*

(i) If $\alpha \in (0, \frac{1}{2})$ and $\sum_{n=1}^{\infty} (h_n |m_n|)^\alpha < \infty$, then $\rho(H_{\vec{l},\vec{\phi}}) \leq \alpha$.

(ii) If $\sum_{n=1}^{\infty} (h_n |m_n|)^{\frac{1}{2}} \ln n < \infty$, then $\rho(H_{\vec{l},\vec{\phi}}) \leq \frac{1}{2}$.

(iii) If $\alpha \in (\frac{1}{2}, 1)$ and $\sum_{n=1}^{\infty} (h_n |m_n|)^\alpha n^{2\alpha-1} < \infty$, then $\rho(H_{\vec{l},\vec{\phi}}) \leq \alpha$.

Proof. Theorem 4.1 gives

$$\rho(H_{\vec{l},\vec{\phi}}) \leq \frac{1}{2} \rho(H_{|\vec{m}:\vec{h}|,\vec{\delta}}).$$

Set $\vec{m_1} := (m_{n+1})_{n=1}^{\infty}$. Removing the first interval of a Hamburger Hamiltonian does not change the order, i.e. $\rho(H_{|\vec{m}:\vec{h}|,\vec{\delta}}) = \rho(H_{|\vec{h}:\vec{m}_1|,\vec{\Delta}})$. Apply Theorem 4.2. ❑

4.4 Theorem. *Let $H_{\vec{l},\vec{\phi}}$ be a positive semidefinite Hamburger Hamiltonian in lcc, and assume that $\phi_n \not\equiv 0 \bmod \pi$, $n \in \mathbb{N}$. Set $\phi_0 := \frac{\pi}{2}$, and assume that $(|\cot \phi_n - \cot \phi_{n-1}|)_{n=1}^{\infty}$ is nondecreasing and bounded, $(l_n \sin^2 \phi_n)_{n=1}^{\infty}$ is nonincreasing, and*

$$\sum_{n=1}^{\infty} [l_n \sin^2 \phi_n]^{\frac{1}{2}} \ln n < \infty. \tag{4.1}$$

Then

$$\rho(H_{\vec{l},\vec{\phi}}) = \inf \{\alpha > 0 : (l_n \sin^2 \phi_n)_{n=1}^{\infty} \in l^\alpha\},$$

i.e., the order of $H_{\vec{l},\vec{\phi}}$ equals the convergence exponent of $([l_n \sin^2 \phi_n]^{-1})_{n=1}^{\infty}$.

To make the connection with what we explained in the introduction, observe that angles ϕ_n performing a walk on the grid $\operatorname{Arccot}(\mathbb{Z})$ just means that $|\cot \phi_n - \cot \phi_{n-1}|$ is constant equal to 1.

Proof of Theorem 4.4. Let $\vec{m}$ and $\vec{h}$ be as in Theorem 4.1, set $M_n := \sum_{k=1}^{n} m_k$, and let γ be the convergence exponent of $(h_n^{-1})_{n=1}^{\infty}$. We have to show that $\rho(H_{\vec{l},\vec{\phi}}) = \gamma$.

By our assumptions $\vec{h}$ is nonincreasing, and $|\vec{m}|$ is nondecreasing and convergent (say $m_\infty := \lim_{n\to\infty} |m_n|$) whence $|m_n| \asymp 1$ and $M_n \lesssim n$.

We start with showing $\rho(H_{\vec{l},\vec{\phi}}) \leq \gamma$. Corollary 4.3, *(ii)*, yields $\rho(H_{\vec{l},\vec{\phi}}) \leq \frac{1}{2}$. If $\gamma = \frac{1}{2}$ (note that by (4.1) certainly $\gamma \leq \frac{1}{2}$), we are done. If $\gamma < \frac{1}{2}$, we can apply Corollary 4.3, *(i)*, to obtain the desired inequality.

To establish that actually equality holds, we start from [PRW16, Proposition 2.14], which says that

$$\rho(H_{\vec{l},\vec{\phi}}) \geq \limsup_{n\to\infty} \frac{-n \ln n}{\ln \Big(\sqrt{l_n} \prod_{i=1}^{n-1} l_i |\sin(\phi_{i+1} - \phi_i)| \Big)}.$$

To evaluate the product, remember (3.2), which yields

$$l_i|\sin(\phi_{i+1}-\phi_i)| = h_i\cdot(1+M_i^2)\,\big|\sin(\operatorname{Arccot} M_{i+1}-\operatorname{Arccot} M_i)\big|.$$

Now, $\sup_{i\in\mathbb{N}}|\operatorname{Arccot}(M_{i+1})-\operatorname{Arccot}(M_i)|<\pi$ since $|M_{i+1}-M_i|=|m_i|\le m_\infty$, and hence

$$\sin\big(|\operatorname{Arccot}(M_{i+1})-\operatorname{Arccot}(M_i)|\big)\asymp|\operatorname{Arccot}(M_{i+1})-\operatorname{Arccot}(M_i)|.$$

The mean value theorem provides $\xi_i\in(\min\{M_i,M_{i+1}\},\max\{M_i,M_{i+1}\})$ with

$$|\operatorname{Arccot}(M_{i+1})-\operatorname{Arccot}(M_i)|=\frac{1}{1+\xi_i^2}. \tag{4.2}$$

Since the length of the written interval is at most m_∞, it follows that $1+\xi_i^2\asymp 1+M_i^2$. Together $l_i|\sin(\phi_{i+1}-\phi_i)|\asymp h_i$, whence

$$\rho(H_{\vec{l},\vec{\phi}})\ge\limsup_{n\to\infty}\frac{-n\ln n}{\ln\Big(\sqrt{l_n}\prod_{i=1}^{n-1}h_i\Big)}\ge\limsup_{n\to\infty}\frac{-n\ln n}{\ln\Big(\sqrt{h_n}\prod_{i=1}^{n-1}h_i\Big)}.$$

Denote the rightmost expression by d, and set

$$D:=\big(\sup\{\tau\ge 0: h_n=\mathrm{O}(n^{-\tau})\}\big)^{-1}.$$

Since $\vec{h}$ is nonincreasing, we have $d=D$, cf. [PRW16, Lemma 2.21]. However, $\gamma\le D$ and putting together thus

$$d\le\rho(H_{\vec{l},\vec{\phi}})\le\gamma\le D=d.$$

❑

In this context note the following elementary fact.

Lemma. *Let $\vec{h}=(h_n)_{n=1}^\infty$ be a sequence of positive real numbers, let $\beta\ge 0$, and denote by $\gamma\in[0,\infty]$ the convergence exponent of $(h_n^{-1})_{n=1}^\infty$. Then*

$$\gamma\ge\limsup_{n\to\infty}\frac{-n\ln n}{\ln\Big(h_n^\beta\prod_{i=1}^{n-1}h_i\Big)}.$$

Proof. If $\gamma=\infty$ there is nothing to prove, hence assume $\gamma<\infty$.

Consider $a_n>0$ with $\sum_{i=1}^\infty a_i<\infty$ and let $\epsilon>0$. Then we find $n_0(\epsilon)\in\mathbb{N}$ such that

$$a_n\le 1,\quad a_n^\beta\prod_{i=1}^{n-1}a_i\le 1,\quad \Big(a_n^\beta+\sum_{i=1}^{n-1}a_i\Big)^{1+\epsilon}\le n^\epsilon,\qquad n\ge n_0(\epsilon).$$

Thus

$$\left(\sqrt[n]{a_n^\beta\prod_{i=1}^{n-1}a_i}\right)^{1+\epsilon}\le\left(\frac{1}{n}\Big(a_n^\beta+\sum_{i=1}^{n-1}a_i\Big)\right)^{1+\epsilon}\le\frac{1}{n},\quad n\ge n_0(\epsilon),$$

and taking logarithms yields

$$\frac{-n\ln n}{\ln\big(a_n^\beta\prod_{i=1}^{n-1}a_i\big)}\le 1+\epsilon,\quad n\ge n_0(\epsilon).$$

Now apply this fact with $a_n:=h_n^{\gamma'}$ where $\gamma'>\gamma$, let $\epsilon\searrow 0$ and $\gamma'\searrow\gamma$. ❑

4.2 Relation with the estimate from [PRW16]

In [PRW16, Theorem 2.7] we proved an upper estimate for the order of a Hamburger Hamiltonian $H_{\vec{l},\vec{\phi}}$, which coincides with the order when lengths and angle-differences are regularly behaving, cf. [PRW16, Theorem 2.22]. In Theorem 4.4 we obtained a formula for $\rho(H_{\vec{l},\vec{\phi}})$ when lengths and angles commonly behave regularly, angle-differences are never too large, and order is at most 1/2. This theorem, however, allows that lengths and angles separately are very irregular. In this subsection we show that these two results are incomparable.

First, we show that for a large class of Hamiltonians Theorem 4.4 is applicable whereas the upper estimate [PRW16, Theorem 2.7] does not coincide with the order (and hence order cannot be computed by means of our previous work).

4.5 Proposition. *Let $\vec{h}$ be a nonincreasing sequence of positive real numbers which satisfies*

$$\sum_{n=1}^{\infty} h_n^{\frac{1}{2}} \ln n < \infty, \tag{4.3}$$

and denote by γ the convergence exponent of $(h_n^{-1})_{n=1}^{\infty}$. Let $\delta_\phi^\circ > 0$ and $\delta_l^\circ \geq 1$ be given such that

$$\delta_\phi^\circ < \frac{1}{\gamma} - \delta_l^\circ < 2.$$

Then there exists a sequence of angles $\vec{\phi}$ performing a walk on $\operatorname{Arccot}(\mathbb{Z})$*, such that the Hamburger Hamiltonian $H_{\vec{l},\vec{\phi}}$ with lengths $l_n := h_n \sin^{-2}\phi_n$, $n \in \mathbb{N}$, and angles $\vec{\phi}$ satisfies (quantities $\delta_{l,\phi}(H), \delta_l(H), \delta_\phi(H)$ as in [PRW16, Definitions 2.13/2.16])*

$$\rho(H_{\vec{l},\vec{\phi}}) = \delta_{l,\phi}(H_{\vec{l},\vec{\phi}})^{-1} = \gamma, \quad \delta_l(H_{\vec{l},\vec{\phi}}) = \delta_l^\circ,\ \delta_\phi(H_{\vec{l},\vec{\phi}}) = \delta_\phi^\circ.$$

The proof is based on the following elementary construction.

Lemma. *Let $\beta \in (0,1)$. Then there exists a sequence of signs $\varepsilon_{\beta,n} \in \{+1,-1\}$, such that the partial sums*

$$s_\beta(n) := \sum_{i=1}^{n} \varepsilon_{\beta,i}, \quad n \in \mathbb{N},$$

satisfy

$$\lim_{n\to\infty} \frac{s_\beta(n)}{n^\beta} = 1. \tag{4.4}$$

Proof. We simply make $s_\beta(n)$ oscillating around n^β as close as possible: Define inductively

$$\varepsilon_{\beta,1} := 1, \qquad \varepsilon_{\beta,n+1} := \begin{cases} +1\,, & \frac{s_\beta(n)}{n^\beta} \leq 1 \\ -1\,, & \frac{s_\beta(n)}{n^\beta} > 1 \end{cases}$$

The sequence $\sigma_n := \frac{s_\beta(n)}{n^\beta}$ can be handled easily.

— Monotonicity behaviour: Assume first $\sigma_n \leq 1$. Then (with appropriate $\xi_n \in (n, n+1)$)

$$\sigma_{n+1} - \sigma_n = \frac{-\sigma_n[(n+1)^\beta - n^\beta] + 1}{(n+1)^\beta} = \frac{1 - \sigma_n \beta \xi_n^{\beta-1}}{(n+1)^\beta} \begin{cases} \geq \frac{1-\beta}{(n+1)^\beta} > 0 \\ \leq \frac{1}{(n+1)^\beta} \end{cases}$$

Second, if $\sigma_n > 1$ then $s_\beta(n+1) < s_\beta(n)$ and hence trivially $\sigma_{n+1} < \sigma_n$.

— Convergence: Let n_k be those indices (arranged in increasing order) where $\varepsilon_{\beta,n}$ changes its sign, i.e., where either $\sigma_n \le 1 < \sigma_{n+1}$ or $\sigma_n > 1 \ge \sigma_{n+1}$. Note that the first of these cases occurs for all odd k and the second for all even. Then $\limsup_{n\to\infty} \sigma_n = \limsup_{k\to\infty} \sigma_{n_{2k-1}+1}$ and $\liminf_{n\to\infty} \sigma_n = \liminf_{k\to\infty} \sigma_{n_{2k}+1}$. By the previous estimate,

$$\sigma_{n_{2k-1}+1} \le \sigma_{n_{2k-1}} + \frac{1}{(n+1)^\beta} \le 1 + \frac{1}{(n+1)^\beta} \to 1,$$

whence $\limsup_{k\to\infty} \sigma_n \le 1$.

In particular, $\sigma_n \le 2$ for large n. Now we estimate for all (sufficiently large) n with $\sigma_n > 1$

$$\sigma_n - \sigma_{n+1} = \frac{1 + \sigma_n \beta \xi_n^{\beta-1}}{(n+1)^\beta} \le \frac{1+2\beta}{(n+1)^\beta}.$$

This shows that

$$\sigma_{n_{2k}+1} \ge \sigma_{n_{2k}} - \frac{1+2\beta}{(n+1)^\beta} \ge 1 - \frac{1+2\beta}{(n+1)^\beta} \to 1,$$

whence $\liminf_{k\to\infty} \sigma_n \ge 1$.

❑

Proof of Proposition 4.5. For a sequence $\vec{a} = (a_n)_{n=1}^\infty$ of positive numbers and $\alpha \in [0,1]$ set

$$G(n; \vec{a}, \alpha) := \frac{-1}{n \ln n} \ln \Big(a_n^\alpha \prod_{i=1}^{n-1} a_i \Big), \quad n \in \mathbb{N}.$$

By Stirling's formula

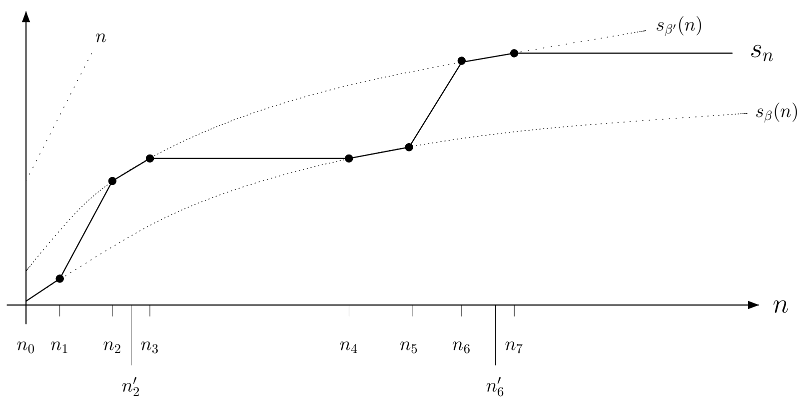
$$\lim_{n\to\infty} G(n; (n^\beta)_{n=1}^\infty, \alpha) = -\beta, \quad \beta \in \mathbb{R},$$

and from (4.4) thus also

$$\lim_{n\to\infty} G(n; \vec{s_\beta}, \alpha) = -\beta.$$

For the proof of the present proposition we construct a sequence of signs whose sequence $\vec{s} = (s_n)_{n=1}^\infty$ of partial sums alternates between $\vec{s_\beta}$ and $\vec{s_{\beta'}}$ where

$$\beta := \frac{1}{2}\delta_\phi^\circ, \quad \beta' := \frac{1}{2}\Big(\frac{1}{\gamma} - \delta_l^\circ\Big).$$



The formula for the sequence $\vec{s}$ is (here $[x \bmod 2]$ denotes the element of $\{0,1\}$ with the same parity as x)

$$s_n := \begin{cases} s_\beta(n) & , \quad 1 \le n \le n_0 \\ s_\beta(n) & , \quad n_k < n \le n_{k+1}, \quad k \equiv 0 \bmod 4 \\ s_{n_k} + (n - n_k) & , \quad n_k < n \le n_{k+1}, \quad k \equiv 1 \bmod 4 \\ s_{\beta'}(n) & , \quad n_k < n \le n_{k+1}, \quad k \equiv 2 \bmod 4 \\ s_{n_k} + \big[(n - n_k) \bmod 2\big], & \quad n_k < n \le n_{k+1}, \quad k \equiv 3 \bmod 4 \end{cases}$$

where the sequence $(n_k)_{k=0}^\infty$ of switching indices will be constructed inductively.

To start with, choose $n_0 > 1$ such that $s_{\beta'}(n) > s_\beta(n)$, $n \ge n_0$, and define s_n, $1 \le n \le n_0$, by the first line of the above formula. Now let $k \in \mathbb{N}_0$ and assume that n_k has already been defined (and with it s_n for $n \le n_k$).

(*i*) $k \equiv 0 \bmod 4$: Consider the auxiliary sequence

$$b_{0,n} := \begin{cases} s_n & , \quad n \le n_k \\ s_\beta(n), & \quad n > n_k \end{cases}$$

Then $G(n; \vec{b_0}, \alpha) = G(n; \vec{s}, \alpha)$, $n \le n_k$, and $\lim_{n\to\infty} G(n; \vec{b_0}, \alpha) = -\beta$. Choose $n_{k+1} > n_k$ such that

$$G(n_{k+1}; \vec{b_0}, \alpha) \ge -\beta - \frac{1}{k}.$$

(*ii*) $k \equiv 1 \bmod 4$: Set

$$n_{k+1} := \min\big\{n > n_k : s_{n_k} + (n - n_k) = s_{\beta'}(n)\big\}.$$

This is well-defined since $s_{n_k} = s_\beta(n_k) < s_{\beta'}(n_k)$ and $s_{\beta'}(n) = \mathrm{o}(n)$.

(*iii*) $k \equiv 2 \bmod 4$: Consider the auxiliary sequence

$$b_{2,n} := \begin{cases} s_n & , \quad n \le n_k \\ s_{\beta'}(n), & \quad n > n_k \end{cases}$$

Then $G(n; \vec{b_2}, \alpha) = G(n; \vec{s}, \alpha)$, $n \le n_k$, and $\lim_{n\to\infty} G(n; \vec{b_2}, \alpha) = -\beta'$. Choose $n_k' > n_k$ such that

$$G(n; \vec{b_2}, \alpha) \le -\beta' + \frac{1}{k}, \quad n \ge n_k'.$$

Since $\vec{h}$ is nonincreasing, we have $\liminf G(n; \vec{h}, \alpha) = \frac{1}{\gamma}$, and hence can choose $n_{k+1} > n_k'$ such that

$$\exists n \in [n_k', n_{k+1}] : \; G(n; \vec{h}, \alpha) \le \frac{1}{\gamma} + \frac{1}{k}. \tag{4.5}$$

(*iv*) $k \equiv 3 \bmod 4$: Set

$$n_{k+1} := \min\big\{n > n_k : s_{n_k} + [(n - n_k) \bmod 2] = s_\beta(n)\big\}.$$

This is well-defined since $s_{n_k} = s_{\beta'}(n_k) > s_\beta(n_k)$ and $\lim_{n\to\infty} s_\beta(n) = \infty$.

Set $\phi_n := \operatorname{Arccot} s_n$ and $l_n := h_n \sin^{-2}\phi_n$. Then Theorem 4.4 is applicable and yields $\rho(H_{\vec{l},\vec{\phi}}) = \gamma = \delta_{\vec{l},\vec{\phi}}(H_{\vec{l},\vec{\phi}})$.

Remembering (4.2) and the formulae before and after, we have

$$|\sin(\phi_{n+1} - \phi_n)| \asymp \frac{1}{s_n^2}$$

and therefore

$$G\big(n; (|\sin(\phi_{n+1} - \phi_n)|)_{n=1}^\infty, \alpha\big) = -2G(n; \vec{s}, \alpha) + \mathrm{o}(1).$$

For $k \equiv 0 \bmod 4$ it holds that

$$G(n_{k+1}; \vec{s}, \alpha) \geq -\beta - \frac{1}{k} = -\frac{1}{2}\delta_\phi^\circ - \frac{1}{k},$$

and we conclude that

$$\delta_\phi(H_{\vec{l},\vec{\phi}}) = \liminf_{n\to\infty} G\big(n; (|\sin(\phi_{n+1} - \phi_n)|)_{n=1}^\infty, \alpha\big) \leq \delta_\phi^\circ.$$

However, $s_n \geq s_\beta(n)$ for all $n \in \mathbb{N}$, whence

$$-2G(n; \vec{s}, \alpha) \geq -2G(n; \vec{s_\beta}, \alpha) \to 2\beta = \delta_\phi^\circ,$$

and this shows $\delta_\phi(H_{\vec{l},\vec{\phi}}) \geq \delta_\phi^\circ$.

Since $\lim_{n\to\infty} s_n = \infty$, we have $\lim_{n\to\infty} \phi_n = 0$ and hence $\sin^2\phi_n \asymp s_n^{-2}$. Thus $l_n \asymp h_n \cdot s_n^2$. Let $k \equiv 2 \bmod 4$ and choose $n \in [n_k', n_{k+1}]$ according to (4.5). Then

$$\begin{aligned} G(n; \vec{l}, \alpha) &= G(n; \vec{h}, \alpha) + 2G(n; \vec{s}, \alpha) + \mathrm{o}(1) \\ &\leq \Big(\frac{1}{\gamma} + \frac{1}{k}\Big) + \Big(-2\beta' + \frac{2}{k}\Big) + \mathrm{o}(1) = \delta_l^\circ + \mathrm{o}(1), \end{aligned}$$

which gives $\delta_l \leq \delta_l^\circ$. However, $s_n \leq s_{\beta'}(n)$ for all $n \in \mathbb{N}$, and hence

$$2G(n; \vec{s}, \alpha) \geq 2G(n; \vec{s_{\beta'}}, \alpha) \to -2\beta' = -\Big(\frac{1}{\gamma} - \delta_l^\circ\Big).$$

This shows that $\delta_l(H_{\vec{l},\vec{\phi}}) \geq \delta_l^\circ$. ❑

Next, we show that (for arbitrary small orders) it might be possible to compute $\rho(H_{\vec{l},\vec{\phi}})$ with help of [PRW16, Theorem 2.22], but $\rho(H_{\vec{l},\vec{\phi}})$ is not equal to the convergence exponent of $([l_n \sin^2\phi_n]^{-1})_{n=1}^\infty$.

4.6 Example. Let $\alpha > -1$ and $\beta > 3 + 2\alpha$, set

$$M_n := \sum_{k=1}^n k^\alpha, \qquad l_n := n^{-\beta}(1 + M_n^2),\ \ \phi_n := \operatorname{Arccot} M_n,$$

and consider the Hamiltonian $H_{\vec{l},\vec{\phi}}$.

Since $\alpha > -1$, we have $M_n \asymp n^{\alpha+1}$ and hence $l_n \asymp n^{2(\alpha+1)-\beta}$. The assumption on β just says that $2(\alpha+1) - \beta < -1$, i.e., that $H_{\vec{l},\vec{\phi}}$ is lcc. From the asymptotics of $\vec{l}$ and [PRW16, Example 2.23] we obtain that

$$\delta_l = \beta - 2(\alpha + 1) \qquad\qquad \text{(exists as a limit)}.$$

In order to compute δ_ϕ, we use the identity

$$\big|\sin(\operatorname{Arccot} x - \operatorname{Arccot} y)\big| = \Big|\sin\Big(\operatorname{Arccot}\Big(\frac{xy+1}{x-y}\Big)\Big)\Big| = \Big[\Big(\frac{xy+1}{x-y}\Big)^2 + 1\Big]^{-\frac{1}{2}},$$

which holds for arbitrary $x, y \in \mathbb{R}$, $x \neq y$. Clearly, $M_{n+1} - M_n = k^\alpha$, and we find

$$|\sin(\phi_{n+1} - \phi_n)| \asymp n^{-(\alpha+2)},$$

whence $\delta_\phi = \alpha + 2$. Since $\delta_l + \delta_\phi = \beta - \alpha > 2$, we can apply [PRW16, Theorem 2.22(A)] to obtain

$$\rho\big(H_{\vec{l},\vec{\phi}}\big) = \frac{1}{\beta - \alpha}.$$

We have $\sin^{-2}\phi_n = 1 + M_n^2$ and hence

$$l_n \sin^2 \phi_n = n^{-\beta}.$$

The convergence exponent of $([l_n \sin^2 \phi_n]^{-1})_{n=1}^\infty$ thus equals $\frac{1}{\beta}$. For $\alpha < 0$ this is larger than the order, for $\alpha > 0$ it is smaller.

It is interesting to observe which hypothesis of Theorem 4.4 are violated in this example. Of course, if $\beta < 2$ already (4.1) fails. If $\alpha \in (-1, 0)$ the sequence $(|\cot\phi_n - \cot\phi_{n-1}|)_{n=1}^\infty$ is decreasing, if $\alpha > 0$ it is increasing but unbounded. $\Diamond$

4.3 Discussion of Berezanskii's theorem

Berezanskii's theorem is formulated in terms of the Jacobi parameters associated with a Hamburger moment sequence.

4.7 Theorem (Berezanskii [Ber56])**.** *Let $\rho_n > 0$, $q_n \in \mathbb{R}$, and let J be the Jacobi matrix with off-diagonal parameters ρ_n and diagonal parameters q_n. Assume that*

$$\sum_{n=1}^\infty \frac{1}{\rho_n} < \infty \qquad \textit{(Carleman condition)}$$

$$\rho_n^2 \geq \rho_{n-1}\rho_{n+1} \text{ or } \rho_n^2 \leq \rho_{n-1}\rho_{n+1} \qquad \textit{(log-concave or log-convex)}$$

$$\Big(\frac{q_n}{\rho_n}\Big)_{n=1}^\infty \in \ell^1 \qquad \textit{(small diagonal)}$$

Then J is of type C, i.e., the corresponding moment problem is indeterminate. The order of the functions in the Nevanlinna matrix of the corresponding moment sequence is equal to the convergence exponent of $(\rho_n)_{n=1}^\infty$.

Berezanskii has treated the case of log-concave ρ_n. Inclusion of the log-convex case was done in [BS14, Theorem 1.4].

In this subsection we explain the connection with our present (and previous) results. To this end we need the explicit formulae connecting the Jacobi parameters with the lengths $\vec{l}$ and angles $\vec{\phi}$ of the Hamburger Hamiltonian whose monodromy matrix coincides with the Nevaninna matrix of the moment sequence.

These are (cf. [Kac99])

$$\frac{1}{\rho_n} = |\sin(\phi_{n+1} - \phi_n)|\sqrt{l_n l_{n+1}},$$
$$q_n = -\frac{1}{l_n}\big[\cot(\phi_{n+1} - \phi_n) + \cot(\phi_n - \phi_{n-1})\big].$$

The essence of Theorem 4.7 is the case of a zero-diagonal; adding a small diagonal can be achieved with a perturbation argument. Let us therefore focus on this case, where the above formulae are easy to handle.

First, we see that $q_n = 0$ for all n if and only if the angles ϕ_n alternate between two fixed values. Due to common normalisation, these are 0 and $\frac{\pi}{2}$. However, multiplying a Jacobi matrix with a positive scalar or adding an offset to the sequence of angles of a Hamburger Hamiltonian does not influence the respective order. Hence, we are free to choose those two values and work with different ones interchangeably.

Plugging the above formula for ρ_n (with alternating angles) log-concavity or convexity means that

$$\frac{l_{n+1}}{l_{n-1}} \leq \frac{l_{n+2}}{l_n} \quad \text{or} \quad \frac{l_{n+1}}{l_{n-1}} \geq \frac{l_{n+2}}{l_n} \text{ resp.,} \tag{4.6}$$

or equivalently,

$$\frac{l_n}{l_{n-1}} \leq \frac{l_{n+2}}{l_{n+1}} \quad \text{or} \quad \frac{l_n}{l_{n-1}} \geq \frac{l_{n+2}}{l_{n+1}} \text{ resp..} \tag{4.7}$$

Monotonicity of the quotients (4.6) leads to the distinction of three cases.

(I) $\frac{l_{n+1}}{l_{n-1}} \geq 1$ *for large* n*:* Then $\rho_n \geq \rho_{n+1}$ for those n, which contradicts Carleman's condition.

(II) $\frac{l_{n+1}}{l_{n-1}} \leq t < 1$ *for large* n*:* Then $l_n, \frac{1}{\rho_n} \lesssim t^n$, whence the convergence exponents of $(\rho_n)_{n=1}^\infty$ and $(l_n^{-1})_{n=1}^\infty$ are zero, and the order is zero either by [PRW16, Example 2.24] ($\Delta_l = \infty$), or [BS14, Theorem 1.2].

(III) $\frac{l_{n+1}}{l_{n-1}} \nearrow 1$*:* This is the nontrivial case concerning order (note that it appears only when ρ_n are log-concave), and requires some further analysis.

First, since $\frac{l_{n+1}}{l_{n-1}} < 1$, the sequence $\vec{l}$ splits into two decreasing subsequences $(l_{2k-1})_{k=1}^\infty$ and $(l_{2k})_{k=1}^\infty$. The quotients $\big(\frac{l_{2k}}{l_{2k-1}}\big)_{k=1}^\infty$ and $\big(\frac{l_{2k+1}}{l_{2k}}\big)_{k=1}^\infty$ are nondecreasing by (4.7), and hence have limits $t_0, t_1 \in (0, \infty]$. However, since $\frac{l_{n+1}}{l_{n-1}}$ tends to 1,

$$\frac{1}{t_0} = \lim_{k\to\infty} \frac{l_{2k-1}}{l_{2k}} = \lim_{k\to\infty} \frac{l_{2k+1}}{l_{2k}} = t_1,$$

in particular, $t_0, t_1 < \infty$. Now we pass to the sequence

$$l_n' := \begin{cases} t_0 l_n\,, & n \text{ odd} \\ l_n \quad, & n \text{ even} \end{cases}$$

Then the quotient sequences $\big(\frac{l_{2k}'}{l_{2k-1}'}\big)_{k=1}^\infty$ and $\big(\frac{l_{2k+1}'}{l_{2k}'}\big)_{k=1}^\infty$ are still nondecreasing and both tend to 1. Thus $\vec{l'}$ is nonincreasing. Monotonicty implies that the convergence exponents of $({l_n'}^{-1})_{n=1}^\infty$ and $([l_n' l_{n+1}']^{-\frac{1}{2}})_{n=1}^\infty$ coincide. Since $l_n' \asymp l_n$, these are the same as the convergence exponents of $(l_n^{-1})_{n=1}^\infty$ and of $(\rho_n)_{n=1}^\infty$, respectively. Moreover, being comparable with a monotone sequence, $\vec{l}$ is regularly distributed in the sense of [PRW16, Definition 2.19].

Now we can compute order from [PRW16, Theorem 2.22(B)]. Since angles alternate, we have $\delta_\phi = 0$ and hence the order equals δ_l^{-1} which in turn equals the convergence exponent of $(l_n^{-1})_{n=1}^\infty$ and hence the convergence exponent of $(\rho_n)_{n=1}^\infty$.

If the convergence exponent of $(\rho_n)_{n=1}^\infty$ is less than $1/2$, we also can compute order from Theorem 4.4. To this end we pass to the Jacobi matrix $\frac{1}{\sqrt{t_0}}J$ and add an offset $-\frac{\pi}{4}$ to the sequence of angles. This leads to the Hamburger Hamiltonian with lengths $(l'_n)_{n=1}^\infty$ and angles alternating between $\pm\frac{\pi}{4}$. Thus the order equals the convergence exponent of $(\frac{\sqrt{2}}{l'_n})_{n=1}^\infty$ which is equal to the convergence exponent of $(\rho_n)_{n=1}^\infty$.

Having seen that Theorem 4.7 (for orders $< 1/2$) can be deduced from Theorem 4.4, we shall now show that Theorem 4.4 actually goes far beyond the Berezanskii case.

4.8 Example. We revisit the Hamiltonians constructed in Proposition 4.5 (so to make sure that order cannot be computed already from [PRW16]), and consider the associated Jacobi matrices. Let $\vec{h}$ be a decreasing sequence with (4.3) which has the property that $\frac{h_n}{h_{n+1}} \asymp 1$. For instance use $h_n = \frac{n^{-\frac{1}{\alpha}}}{(\ln n)^3}$ where $\alpha \in (0, 1/2]$. Let $\vec{s}, \vec{l}, \vec{\phi}$ be the sequences constructed in the proof of Proposition 4.5. Then we know that

$$\lim_{n\to\infty} s_n = \infty, \quad |\sin(\phi_{n+1} - \phi_n)| \asymp \frac{1}{s_n^2}, \quad l_n \asymp h_n \cdot s_n^2,$$

and hence

$$\frac{q_n}{\rho_n} = -\underbrace{\frac{\sqrt{l_n l_{n+1}}}{l_n}}_{\to 1} \cdot \operatorname{sgn}(\phi_{n+1} - \phi_n) \cdot$$
$$\cdot \left(\underbrace{\cos(\phi_{n+1} - \phi_n)}_{\to 1} + \underbrace{\cos(\phi_n - \phi_{n-1})}_{\to 1} \frac{\sin(\phi_{n+1} - \phi_n)}{\sin(\phi_n - \phi_{n-1})} \right)$$

Since s_n is unbounded but $|s_{n+1} - s_n| = 1$, we find a subsequence $(\phi_{n_k})_{k=1}^\infty$ with $\phi_{n_k - 1} > \phi_{n_k} > \phi_{n_k + 1}$. Along this subsequence

$$\inf_{k\in\mathbb{N}} \frac{\sin(\phi_{n_k+1} - \phi_{n_k})}{\sin(\phi_{n_k} - \phi_{n_k-1})} > 0,$$

and we conclude that $\limsup_{n\to\infty} \left|\frac{q_n}{\rho_n}\right| > 1$. This shows that the Jacobi matrix associated with $H_{\vec{l},\vec{\phi}}$ is far from being a small perturbation of the corresponding zero-diagonal matrix in the sense of Theorem 4.7. ♢

References

[Ber56] Yu.M. Berezanskiĭ. "Expansion according to eigenfunction of a partial difference equation of order two". In: *Trudy Moskov. Mat. Obšč.* 5 (1956), pp. 203–268.

[BP07] C. Berg and H.L. Pedersen. "Logarithmic order and type of indeterminate moment problems". In: *Difference equations, special functions and orthogonal polynomials.* With an appendix by Walter Hayman. World Sci. Publ., Hackensack, NJ, 2007, pp. 51–79.

[BP94] C. Berg and H.L. Pedersen. "On the order and type of the entire functions associated with an indeterminate Hamburger moment problem". In: *Ark. Mat.* 32.1 (1994), pp. 1–11.

[Bra61] L. de Branges. "Some Hilbert spaces of entire functions. II". In: *Trans. Amer. Math. Soc.* 99 (1961), pp. 118–152.

[Bra68] L. de Branges. *Hilbert spaces of entire functions.* Englewood Cliffs, N.J.: Prentice-Hall Inc., 1968, pp. ix+326.

[BS14] C. Berg and R. Szwarc. "On the order of indeterminate moment problems". In: *Adv. Math.* 250 (2014), pp. 105–143.

[Fle96] A. Fleige. *Spectral theory of indefinite Krein-Feller differential operators.* Vol. 98. Mathematical Research. Akademie Verlag, Berlin, 1996, p. 133.

[Fre05] U. Freiberg. "Spectral asymptotics of generalized measure geometric Laplacians on Cantor like sets". In: *Forum Math.* 17.1 (2005), pp. 87–104.

[GK69] I.C. Gohberg and M.G. Krein. *Introduction to the theory of linear nonselfadjoint operators.* Translated from the Russian by A. Feinstein. Translations of Mathematical Monographs, Vol. 18. American Mathematical Society, Providence, R.I., 1969, pp. xv+378.

[HSW00] S. Hassi, H. de Snoo, and H. Winkler. "Boundary-value problems for two-dimensional canonical systems". In: *Integral Equations Operator Theory* 36.4 (2000), pp. 445–479.

[Kac85] I.S. Kac. "Linear relations, generated by a canonical differential equation on an interval with a regular endpoint, and expansibility in eigenfunctions". Russian. In: *VINITI Deponirovannye Nauchnye Raboty* 195.1 (1985). Deposited in Ukr NIINTI, No. 1453, 1984, 50 pp., b.o. 720.

[Kac86a] I.S. Kac. "Expansibility in eigenfunctions of a canonical differential equation on an interval with singular endpoints and associated linear relations". Russian. In: *VINITI Deponirovannye Nauchnye Raboty* 282.12 (1986). Deposited in Ukr NIINTI, No. 2111, 1986, 64 pp., b.o. 1536.

[Kac86b] I.S. Kac. "Integral estimates for the distribution of the spectrum of a string". In: *Sibirsk. Mat. Zh.* 27.2 (1986), pp. 62–74, 221.

[Kac90] I.S. Kac. "Thightness of the spectrum of a singular string". In: *Izv. Vyssh. Uchebn. Zaved. Mat.* 3 (1990), pp. 23–30.

[Kac99] I.S. Kac. "Inclusion of the Hamburger power moment problem in the spectral theory of canonical systems". Russian. In: *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* 262.Issled. po Linein. Oper. i Teor. Funkts. 27 (1999). English translation: J. Math. Sci. (New York) 110 (2002), no. 5, 2991–3004, pp. 147–171, 234.

[KK68] I.S. Kac and M.G. Krein. “R-functions – Analytic functions mapping the upper half plane into itself”. Russian. In: Addition I in F.V.Atkinson, Дискретные и непрерывные граничные задачи (Russian translation). English translation: Amer. Math. Soc. Transl. (2) 103 (1974), 1–19. Izdat. “Mir”, Moscow, 1968, pp. 629–647.

[KL79] M.G. Krein and H. Langer. “On some extension problems which are closely connected with the theory of Hermitian operators in a space Π_κ. III. Indefinite analogues of the Hamburger and Stieltjes moment problems. Part I”. In: *Beiträge Anal.* 14 (1979), 25–40 (loose errata).

[KL80] M.G. Krein and H. Langer. “On some extension problems which are closely connected with the theory of Hermitian operators in a space Π_κ. III. Indefinite analogues of the Hamburger and Stieltjes moment problems. Part II”. In: *Beiträge Anal.* 15 (1980), 27–45 (1981).

[KL85] M.G. Krein and H. Langer. “On some continuation problems which are closely related to the theory of operators in spaces Π_κ. IV. Continuous analogues of orthogonal polynomials on the unit circle with respect to an indefinite weight and related continuation problems for some classes of functions”. In: *J. Operator Theory* 13.2 (1985), pp. 299–417.

[Kre51] M.G. Krein. “On the theory of entire matrix functions of exponential type”. In: *Ukrain. Mat. Žurnal* 3 (1951), pp. 164–173.

[KW06] M. Kaltenbäck and H. Woracek. “Pontryagin spaces of entire functions. IV”. In: *Acta Sci. Math. (Szeged)* 72.3-4 (2006), pp. 709–835.

[KW10] M. Kaltenbäck and H. Woracek. “Pontryagin spaces of entire functions. VI”. In: *Acta Sci. Math. (Szeged)* 76.3-4 (2010), pp. 511–560.

[KW11] M. Kaltenbäck and H. Woracek. “Pontryagin spaces of entire functions. V”. In: *Acta Sci. Math. (Szeged)* 77.1-2 (2011), pp. 223–336.

[KWW06] M. Kaltenbäck, H. Winkler, and H. Woracek. “Singularities of generalized strings”. In: *Operator theory and indefinite inner product spaces*. Vol. 163. Oper. Theory Adv. Appl. Basel: Birkhäuser, 2006, pp. 191–248.

[KWW07] M. Kaltenbäck, H. Winkler, and H. Woracek. “Strings, dual strings, and related canonical systems”. In: *Math. Nachr.* 280.13-14 (2007), pp. 1518–1536.

[Lev80] B.Ja. Levin. *Distribution of zeros of entire functions*. Revised. Vol. 5. Translations of Mathematical Monographs. Translated from the Russian by R. P. Boas, J. M. Danskin, F. M. Goodspeed, J. Korevaar, A. L. Shields and H. P. Thielman. Providence, R.I.: American Mathematical Society, 1980, pp. xii+523.

[LW98] H. Langer and H. Winkler. “Direct and inverse spectral problems for generalized strings”. In: *Integral Equations Operator Theory* 30.4 (1998). Dedicated to the memory of Mark Grigorievich Krein (1907–1989), pp. 409–431.

[PRW16] R. Pruckner, R. Romanov, and H. Woracek. “Bounds on order of indeterminate moment sequences”. In: *Constr. Approx.* (2016), p. 27.

[Win95] H. Winkler. “The inverse spectral problem for canonical systems”. In: *Integral Equations Operator Theory* 22.3 (1995), pp. 360–374.

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