Termination in Convex Sets of Distributions*

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Abstract

Convex algebras, also called (semi)convex sets, are at the heart of modelling probabilistic systems including probabilistic automata. Abstractly, they are the Eilenberg-Moore algebras of the finitely supported distribution monad. Concretely, they have been studied for decades within algebra and convex geometry.

In this paper we study the problem of extending a convex algebra by a single point. Such extensions enable the modelling of termination in probabilistic systems. We provide a full description of all possible extensions for a particular class of convex algebras: For a fixed convex subset D of a vector space satisfying additional technical condition, we consider the algebra of convex subsets of D. This class contains the convex algebras of convex subsets of distributions, modelling (nondeterministic) probabilistic automata. We also provide a full description of all possible extensions for the class of free convex algebras, modelling fully probabilistic systems. Finally, we show that there is a unique functorial extension, the so-called black-hole extension.

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1 Introduction

In this paper we study the question of how to extend a convex algebra by a single element. Convex algebras have been studied for many decades in the context of algebra, vector spaces, and convex geometry, see e.g. [32, 11, 13] and from a categorical viewpoint, see e.g. [12, 33, 24, 22, 3]. Recently they have attracted more attention in computer science as well, see e.g. [9, 15, 17]. One reason is that probability distributions, the main ingredient for modelling various probabilistic systems, see e.g. [34, 1, 30], have a natural convex algebra structure. Even more than that, the set of finitely supported distributions over a set S carries the free convex algebra over S. As a consequence, on the concrete side, convexity has notably appeared in the semantics of probabilistic systems, in particular probabilistic automata [27, 26, 19, 14]. One particularly interesting development in the last decade in the theory of probabilistic systems is to consider probabilistic automata as transformers of *belief states*, i.e., probability distributions over states, resulting in semantics on distributions, see [14, 4, 5, 7, 8, 6, 21] to name a few. Convexity is inherent to this modelling and the resulting semantics that we call *distribution bisimilarity.*

* Full version `http://www.asc.tuwien.ac.at/preprint/2017/asc08x2017.pdf`.

Additionally, on the abstract side, coalgebras over (categories of) algebras have attracted significant attention [29, 16]. They make explicit the algebraic structure that is present in (the states of) transition systems and allow for its utilisation in the notion of semantics. For coalgebras over convex algebras, the most important observation is that convex algebras are the Eilenberg-Moore algebras of the finitely supported distribution monad [33, 9, 10, 15]. The first author, with coauthors, has recently studied the abstract coalgebraic foundation of probabilistic automata as coalgebras over convex algebras in [2], by providing suitable functors and monads on the category $\mathcal{EM}(\mathcal{D}_f)$ of Eilenberg-Moore algebras that model probabilistic automata. As a result, one gets a neat generic treatment and understanding of distribution bisimilarity.

One contribution of [2] is identifying a convex-powerset monad $\mathcal{P}_c$ on $\mathcal{EM}(\mathcal{D}_f)$ that together with a constant-exponent functor can be used to model probabilistic automata as coalgebras over $\mathcal{EM}(\mathcal{D}_f)$. However, the convex-powerset monad allows only for *nonempty* convex subsets, and hence there is no notion of termination. As a consequence, one can only model *input enabled* probabilistic automata. Hence, the question arises of how to add termination. One obvious way is to add termination that rules over any other behaviour: Consider a probabilistic automaton with two states s and t; A distribution $ps+\bar{p}t$ over states s and t terminates if and only if one of the states terminates. We refer to this approach as *black-hole* termination. Several distribution-bisimilarity semantics in the literature disagree on the treatment of termination, see e.g. the discussion in [14] as well as [7, 8, 6, 4]. Understanding termination in probabilistic automata as transformers of belief states is the motivation for this work. On the level of convex algebras, termination amounts to the question of extending a convex algebra by a single element.

Stated algebraically, the questions we address in this paper are:

1. Given a convex algebra $\mathbb{X}$, is it possible to extend it by a single point?
2. If yes, what are all possible extensions?
3. Which extensions are functorial, i.e., provide a functor on $\mathcal{EM}(\mathcal{D}_f)$ that could then be used for modelling probabilistic automata as coalgebras over $\mathcal{EM}(\mathcal{D}_f)$?

Observe that extensions by a single point are different from the coproduct $\mathbb{X}+1$ in $\mathcal{EM}(\mathcal{D}_f)$; the coproduct was concretely described in [17, Lemma 4], and it has a much larger carrier than the set $X+1$.

Despite a large body of work on convex algebras, to the best of our knowledge, the problem of extending a convex algebra by a single element has not been studied, except for the black-hole extension mentioned above, see [12].

We answer the stated questions, and in particular our answers and main results are:

1. Yes, it is possible and there are many possible extensions in general. One of them is the mentioned black-hole extension.
2. We describe all possible extensions for the free convex algebra $\mathbb{D}_S$ of finitely supported probability distributions over a set S, see Theorem 16 in Section 5. Furthermore, we describe all possible extensions of an algebra $\mathcal{P}_c\mathbb{D}$ for $\mathbb{D}$ being a convex subset of a vector space, satisfying a boundedness condition, see Theorem 29 in Section 6. As $\mathbb{D}_S$ is a particular subset of a vector space, we get a description of all possible extensions of $\mathcal{P}_c\mathbb{D}_S$ which is exactly what is needed to understand termination in convex sets of distributions.
3. We prove that only the black-hole extension is functorial, see Theorem 18 in Section 5.

In addition, we provide many smaller results, observations, and examples that add to the vast knowledge on convex algebras.

We mention that reading our results and proofs in detail does not require any prior knowledge beyond basics of algebra, with two notable exceptions: (1) We do use some topological and geometric arguments in the appendix in order to prove claims for the construction of some of our examples; (2) We add small remarks about coalgebras and categories as we already did in this introduction, assuming that readers are familiar with these basic notions (or will otherwise ignore the remarks made).

2 Convex Algebras

By $\mathcal{C}$ we denote the signature of convex algebras

$$\mathcal{C} = \{(p_i)_{i=0}^n \mid n \in \mathbb{N}, p_i \in [0,1], \sum_{i=0}^n p_i = 1\}.$$

Intuitively, the $(n+1)$-ary operation symbol $(p_i)_{i=0}^n$ will be interpreted by a convex combination with coefficients p_i for $i = 0, \ldots, n$. For a real number $p \in [0,1]$ we set $\bar{p} = 1 - p$.

▶ **Definition 1.** A *convex algebra* $\mathbb{X}$ is an algebra with signature $\mathcal{C}$, i.e., a set X together with an operation $\sum_{i=0}^n p_i(-)_i$ for each operational symbol $(p_i)_{i=0}^n \in \mathcal{C}$, such that the following two axioms hold:

- Projection: $\sum_{i=0}^n p_i x_i = x_j$ if $p_j = 1$.
- Barycenter: $\sum_{i=0}^n p_i \left(\sum_{j=0}^m q_{i,j} x_j\right) = \sum_{j=0}^m \left(\sum_{i=0}^n p_i q_{i,j}\right) x_j$.

Convex algebras are the Eilenberg-Moore algebras of the finitely-supported distribution monad $\mathcal{D}_f$ on **Sets**, cf. [33, 4.1.3] and [28], see also [9, 10] or [15, Theorem 4] where a concrete and simple proof is given. A convex algebra homomorphism is a morphism in the Eilenberg-Moore category $\mathcal{EM}(\mathcal{D}_f)$. Concretely, a convex algebra homomorphism h from $\mathbb{X}$ to $\mathbb{Y}$ is a *convex* (synonymously, *affine*) map, i.e., $h\colon X \to Y$ with the property $h\left(\sum_{i=0}^n p_i x_i\right) = \sum_{i=0}^n p_i h(x_i)$.

▶ Remark 2. Let $\mathbb{X}$ be a convex algebra. Then (for $p_n \neq 1$)

$$\sum_{i=0}^n p_i x_i = \overline{p_n} \left(\sum_{j=0}^{n-1} \frac{p_j}{\overline{p_n}} x_j \right) + p_n x_n. \tag{1}$$

Hence, an $(n+1)$-ary convex combination can be written as a binary convex combination using an n-ary convex combination. As a consequence, if X is a set that carries two convex algebras $\mathbb{X}_1$ and $\mathbb{X}_2$ with operations $\sum_{i=0}^n p_i(-)_i$ and $\bigoplus_{i=0}^n p_i(-)_i$, respectively (and binary versions $+$ and $\oplus$, respectively) such that $px + \bar{p}y = px \oplus \bar{p}y$ for all p, x, y, then $\mathbb{X}_1 = \mathbb{X}_2$.

One can also see Equation (1) as a definition – the classical definition of Stone [32, Definition 1]. We make the connection explicit with the next proposition.

▶ **Proposition 3.** *Let X be a set with binary operations $px + \bar{p}y$ for $x, y \in X$ and $p \in (0,1)$. Assume*

- *Idempotence: $px + \bar{p}x = x$ for all $x \in X, p \in (0,1)$.*
- *Parametric commutativity: $px + \bar{p}y = \bar{p}y + px$ for all $x, y \in X, p \in (0,1)$.*
- *Parametric associativity: $p(qx + \bar{q}y) + \bar{p}z = pqx + \overline{pq}\left(\frac{p\bar{q}}{\overline{pq}}y + \frac{\bar{p}}{\overline{pq}}z\right)$ for all $x, y, z \in X, p, q, \in (0,1)$.*

Define n-ary convex operations inductively by the projection axiom and the formula (1). Then X becomes a convex algebra.

Proof. By induction, cf. [32, Lemma 1–Lemma 4][1]. ◀

This allows us to focus on binary convex combinations whenever more convenient.

▶ **Definition 4.** Let $\mathbb{X}$ be a convex algebra, and $C \subseteq X$. C is *convex* if it is the carrier of a subalgebra of $\mathbb{X}$, i.e., if $px + \bar{p}y \in C$ for all $x, y \in C$ and $p \in (0,1)$.

▶ **Definition 5.** Let $\mathbb{X}$ be a convex algebra. An element $z \in X$ is $\mathbb{X}$-*cancellable* if

$$\forall x, y \in X.\ \forall p \in (0,1).\ px + \bar{p}z = py + \bar{p}z \Rightarrow x = y.$$

The convex algebra $\mathbb{X}$ is *cancellative* if every element of X is $\mathbb{X}$-cancellable.

▶ **Definition 6.** Let $\mathbb{X}$ be a convex algebra. An element $x \in X$ *adheres to* an element $y \in X$, notation $x \mathrel{\circ\!\!-\!\!\bullet} y$, if $px + \bar{p}y = y$ for all $p \in (0,1)$.

Observe that for a cancellative algebra the adherence relation equals the identity relation. The following simple properties of adherence will be needed on multiple occasions.

▶ **Lemma 7.** *Let $\mathbb{X}$ be a convex algebra. The following properties hold.*
1. *For all $x, y \in X$, $x \mathrel{\circ\!\!-\!\!\bullet} y$ if and only if $px + \bar{p}y = y$ for some $p \in (0,1)$.*
2. *The adherence relation is reflexive and convex.*
3. *For all $x, y \in X$, if $x \mathrel{\circ\!\!-\!\!\bullet} y$ then $pz + \bar{p}x \mathrel{\circ\!\!-\!\!\bullet} pz + \bar{p}y$ for all $z \in X$ and $p \in (0,1)$.*
4. *If z is $\mathbb{X}$-cancellable, then for all $x, y \in X$ and $p \in (0,1)$*

$$pz + \bar{p}x \mathrel{\circ\!\!-\!\!\bullet} pz + \bar{p}y \quad \Rightarrow \quad x \mathrel{\circ\!\!-\!\!\bullet} y.$$

Proof.
1. Let $x, y \in X$. Consider the map $\varphi\colon [0,1] \to \mathbb{X}$ defined by $\varphi(p) = px + \bar{p}y$. Easy calculations show that

$$(qp + \bar{q}r)x + \overline{(qp + \bar{q}r)}y = q(px + \bar{p}y) + \bar{q}(rx + \bar{r}y), \tag{2}$$

showing that φ is convex. The implication $\Rightarrow$ trivially holds. For the implication $\Leftarrow$ assume that $rx + \bar{r}y = y$ for some $r \in (0,1)$. Then $\varphi(0) = y = \varphi(r)$ showing that the kernel of φ is a congruence of $[0,1]$ which is not the diagonal. By [11, Lemma 3.2], φ is constant on $(0,1)$. This shows that for all $p \in (0,1)$, $px + \bar{p}y = y$ and hence $x \mathrel{\circ\!\!-\!\!\bullet} y$.
2. Reflexivity is a direct consequence of idempotence. Let $x, y, u, v \in X$ and assume $x \mathrel{\circ\!\!-\!\!\bullet} y$ and $u \mathrel{\circ\!\!-\!\!\bullet} v$. Then

$$q(px + \bar{p}u) + \bar{q}(py + \bar{p}v) = p(qx + \bar{q}y) + \bar{p}(qu + \bar{q}v) = py + \bar{p}v.$$

3. Direct consequence of reflexivity and convexity of adherence.
4. Assume $pz + \bar{p}x \mathrel{\circ\!\!-\!\!\bullet} pz + \bar{p}y$ and z is $\mathbb{X}$-cancellable. Let $q \in (0,1)$. Then

$$pz + \bar{p}y = q(pz + \bar{p}x) + \bar{q}(pz + \bar{p}y) = pz + \bar{p}(qx + \bar{q}y)$$

implies $qx + \bar{q}y = y$, after cancelling z. Hence $x \mathrel{\circ\!\!-\!\!\bullet} y$. ◀

[1] Stone's cancellation Postulate V is not used in his Lemma 1–Lemma 4.

▶ **Example 8.** Here are some examples of convex algebras of interest in this paper:

1. Let $\mathbb{V}$ be a vector space over $\mathbb{R}$ and $X \subseteq V$ a convex subset. Then X with the operations inherited from $\mathbb{V}$ is a cancellative convex algebra $\mathbb{X}$. Conversely, every cancellative convex algebra is isomorphic to a convex subset of a vector space, cf. [32, Theorem 2][2].
2. In particular, we consider the vector space $\ell^1(S)$ for a set S. Recall, $\ell^1(S) = \{(r_s)_{s\in S} \mid r_s \in \mathbb{R}, \sum_{s\in S} |r_s| < \infty\}$ with the norm $\|(r_s)_{s\in S}\|_1 = \sum_{s\in S} |r_s|$. The set $\mathcal{D}_f S$ of finitely supported probability distributions over S forms a convex subset of $\ell^1(S)$ and hence a cancellative convex algebra $\mathbb{D}_S$. It is in fact a well-known convex algebra, the free convex algebra generated by S, cf. [20, Lemma 1].
3. Given a convex algebra $\mathbb{X}$, $\mathcal{P}_c X$ is the set of nonempty[3] convex subsets of X, i.e., carriers of subalgebras of $\mathbb{X}$. $\mathcal{P}_c X$ forms a convex algebra with the pointwise operations: $pA + \bar{p}B = \{pa + \bar{p}b \mid a \in A, b \in B\}$. We write $\mathcal{P}_c\mathbb{X}$ for this algebra. We note that $\mathcal{P}_c$ is a monad on $\mathcal{EM}(\mathcal{D}_f)$ as shown in [2]. On morphisms, $\mathcal{P}_c$ acts as the powerset monad. The original algebra $\mathbb{X}$ embeds in $\mathcal{P}_c\mathbb{X}$ via the unit of the monad $\eta\colon x \mapsto \{x\}$.
   For convex subsets of a finite dimensional vector space, the pointwise operations are known as Minkowski addition and are a basic construction in convex geometry, cf. [25]. The algebra $\mathcal{P}_c\mathbb{X}$ is in general not cancellative and has a nontrivial adherence relation, cf. [12, Example 6.3]. However it contains cancellative elements: It is easy to check that for each $\mathbb{X}$-cancellable element x the element $\{x\}$ is $\mathcal{P}_c\mathbb{X}$-cancellable.
4. The motivating example for this work is the convex algebra $\mathcal{P}_c\mathbb{D}_S$ of convex subsets of distributions over a set S.
5. Let X be the carrier of a meet-semilattice and define $px + \bar{p}y = x \wedge y$ for $x, y \in X$ and $p \in (0,1)$. Then X becomes a convex algebra $\mathbb{X}$ with these operations, cf. [20, §4.5]. This algebra is not cancellative, in fact $\multimap\!\bullet = \{(x,y) \mid x \geq y\}$. For the categorically minded, we remark that behind this construction is the support monad map from $\mathcal{D}_f$ to $\mathcal{P}_f$, the finite powerset monad, and semilattices are the Eilenberg-Moore algebras of $\mathcal{P}_f$.

We now present a construction that provides a beautiful way of constructing convex algebras out of existing ones.

▶ **Example 9.** The *semilattice construction*, cf. [12, p.22f]: Let S be the carrier of a meet-semilattice and let $(\mathbb{X}_s)_{s\in S}$ be an S-indexed family of convex algebras. Moreover, let $(f_s^t)_{\substack{s,t\in S\\ s\leq t}}$ be a family of convex algebra homomorphisms $f_s^t\colon \mathbb{X}_t \to \mathbb{X}_s$ that satisfy $f_s^t \circ f_t^u = f_s^u$ for all $s \leq t \leq u$, and $f_s^s = \mathrm{id}_{X_s}$ for all $s \in S$. Let X be the disjoint union of all X_s for $s \in S$. Then X becomes a convex algebra $\mathbb{X}$ with operations defined by $px + \bar{p}y = pf_{s\wedge t}^s(x) + \bar{p}f_{s\wedge t}^t(y)$ for $x \in X_s$, $y \in X_t$, and $p \in (0,1)$. The algebra $\mathbb{X}$ obtained in this way is the direct limit of the diagram formed by the algebras $\mathbb{X}_s$ and the maps f_s^t.

▶ **Definition 10.** Let $\mathbb{X}$ be a convex algebra, and $P, Q \subseteq X$.

- P is an *ideal* if $\forall x \in P.\ \forall y \in X.\ \forall p \in (0,1).\ px + \bar{p}y \in P$.
- P is a *prime ideal* if it is an ideal and its complement $X \setminus P$ is convex.
- Q is an *extremal set*[4] if $px + \bar{p}y \in Q \Rightarrow x, y \in Q$ for all $x, y \in X, p \in (0,1)$.
- $z \in X$ is an *extremal point* if $\{z\}$ is an extremal set. Explicitly: z is an extremal point if whenever $px + \bar{p}y = z$ for $x, y \in X, p \in (0,1)$, it follows that $x = y = z$. The set of all extremal points of $\mathbb{X}$ is denoted as $\operatorname{Ext}\mathbb{X}$.[5]

[2] Proved independently in [18, Satz 3].
[3] Non-emptiness is necessary for the projection axiom to hold.
[4] In [15, Definition 7] extremal sets are called *filters*.
[5] The construction Ext is not functorial.

▶ **Lemma 11.** *Let $\mathbb{X}$ be a convex algebra, and $P \subseteq X$. Then P is an ideal if and only if $X \setminus P$ is an extremal set.*

Proof. Assume P is an ideal. Let $x, y \in X, p \in (0,1)$ such that $px + \bar{p}y \in X \setminus P$. If $x \in P$ or $y \in P$, then since P is an ideal also $px + \bar{p}y \in P$, a contradiction. Hence $x, y \in X \setminus P$.

For the converse, assume $X \setminus P$ is extremal and let $x \in P, y \in X, p \in (0,1)$. If $px + \bar{p}y \notin P$, then $px + \bar{p}y \in X \setminus P$ which implies $x, y \in X \setminus P$, a contradiction. Hence, $px + \bar{p}y \in P$. ◀

3 The Problem and Some Example Solutions

Let $\mathbb{X}$ be a convex algebra. Can one extend it for one element to a convex algebra $\mathbb{X}_*$ with carrier $X \cup \{*\}$ where $* \notin X$? If yes, what are all possible extensions?

We will show that an arbitrary convex algebra $\mathbb{X}$ can be extended in many ways, and describe all possible ways of extending $\mathbb{X} = \mathbb{D}_S$ and $\mathbb{X} = \mathcal{P}_c\mathbb{D}_S$.

First, we provide four examples of extensions, two of which are instances of the semilattice construction from Example 9.

▶ **Example 12.** Let $\mathbb{X}$ be a convex algebra and $* \notin X$. We denote the operations of $\mathbb{X}$ as before by $p(-) + \bar{p}(-)$. In each of the examples we construct a convex algebra $\mathbb{X}_*$ with operations denoted by $p(-) \oplus \bar{p}(-)$ satisfying $p(-) \oplus \bar{p}(-) = p(-) + \bar{p}(-)$, $x, y \in X$, $p \in [0,1]$.

1. Black-hole behaviour, cf. [12, Example 6.1]: In this example, $*$ behaves like a black hole and swallows everything in the sense that $x \circ\!\!-\!\!\bullet *$ for all $x \in X$. To be precise, consider the semilattice $S = \{0,1\}$ with $0 \leq 1$. Let $\mathbb{X}_0$ be the trivial convex algebra with $X_0 = \{*\}$ and $\mathbb{X}_1 = \mathbb{X}$. Let $f_0^1 : \mathbb{X}_1 \to \mathbb{X}_0$ be the unique homomorphism (mapping everything to $*$). Then the semilattice construction gives us a convex algebra $\mathbb{X}_*$ with the property
$$px \oplus \bar{p}y = \begin{cases} px + \bar{p}y\,, & x, y \in X, \\ * \qquad\quad\;\, , & x = * \text{ or } y = *. \end{cases} \tag{3}$$
2. Imitating behaviour: In this example, $*$ imitates the behaviour of a given element $w \in X$. Consider again the semilattice $S = \{0,1\}$ with $0 \leq 1$. Let $\mathbb{X}_0 = \mathbb{X}$ and $\mathbb{X}_1$ be the trivial convex algebra with $X_1 = \{*\}$. Let $f_0^1 : \mathbb{X}_1 \to \mathbb{X}_0$ be the homomorphism mapping $*$ to w. Then the semilattice construction gives us a convex algebra $\mathbb{X}_*$ with the property
$$px \oplus \bar{p}y = \begin{cases} px + \bar{p}y\;, & x, y \in X, \\ px + \bar{p}w\,, & x \in X, y = *, \\ pw + \bar{p}y\,, & x = *, y \in X, \\ * \qquad\quad\;\;\, , & x = y = *. \end{cases} \tag{4}$$
3. Imitating an outer element: Assume we are given a convex algebra $\mathbb{Y}$ which contains $\mathbb{X}$ as a subalgebra. Let $w \in Y \setminus X$ be such that $X \cup \{w\}$ is convex. Then we obtain an extension $\mathbb{X}_*$ by identifying $X \cup \{*\}$ with $X \cup \{w\}$ via $x \mapsto x$ for $x \in X$ and $* \mapsto w$. We say that $*$ imitates the outer element w.
This way of defining extensions is of course trivial, but it is useful in presence of a natural larger algebra. For example, we will apply it when D is a convex subset of a vector space $\mathbb{V}$, $\mathbb{X} = \mathcal{P}_c\mathbb{D}$, and $\mathbb{Y} = \mathcal{P}_c\mathbb{V}$.

4. Mixed behaviour: Let w be an extremal point of $\mathbb{X}$. In this example, $*$ imitates $w \in X$ on $X \setminus \{w\}$ and swallows w. That is, setting

$$px \oplus \bar{p}y = \begin{cases} px + \bar{p}y \ , & x, y \in X, \\ px + \bar{p}w \ , & x \in X \setminus \{w\}, y = *, \\ pw + \bar{p}y \ , & x = *, y \in X \setminus \{w\}, \\ * \qquad\quad , & \text{otherwise.} \end{cases} \tag{5}$$

provides an extension $\mathbb{X}_*$. This example is not an instance of the semilattice construction and requires a proof. It will be proven in Section 5 (p.8).

4 Extensions of Convex Algebras - The Prime Ideal

The following two notions provide a crucial characteristic of an extension $\mathbb{X}_*$ for a convex algebra $\mathbb{X}$.

▶ **Definition 13.** Let $\mathbb{X}$ be a convex algebra, and let $\mathbb{X}_*$ be an extension. Then its *set of adherence* $\mathrm{Ad}(\mathbb{X}_*)$ is $\mathrm{Ad}(\mathbb{X}_*) = \{x \in X \mid x \circ\!\!-\!\!\bullet *\}$ and its *prime ideal* is $\mathrm{P}(\mathbb{X}_*) = X \setminus \mathrm{Ad}(\mathbb{X}_*)$.

▶ **Lemma 14.** *Let $\mathbb{X}$ be a convex algebra, and let $\mathbb{X}_*$ be an extension of $\mathbb{X}$. The set $\mathrm{P}(\mathbb{X}_*)$ is indeed a prime ideal of $\mathbb{X}$.*

Proof. Let $x \in \mathrm{P}(\mathbb{X}_*)$, $y \in X$, $p \in (0,1)$. Then

$$q(px + \bar{p}y) + \bar{q}* = \overline{q\bar{p}}\left(\frac{qp}{\overline{q\bar{p}}}x + \frac{\bar{q}}{\overline{q\bar{p}}}*\right) + q\bar{p}y \in X$$

since $y \in X$ and $\frac{qp}{\overline{q\bar{p}}}x + \frac{\bar{q}}{\overline{q\bar{p}}}* \in X$ due to $x \in \mathrm{P}(\mathbb{X}_*)$ and hence $x \notin \mathrm{Ad}(\mathbb{X}_*)$. Therefore, $px + \bar{p}y \in \mathrm{P}(\mathbb{X}_*)$ proving that $\mathrm{P}(\mathbb{X}_*)$ is an ideal in $\mathbb{X}$. By Lemma 7.2 $\mathrm{Ad}(\mathbb{X}_*)$ is convex and hence $\mathrm{P}(\mathbb{X}_*)$ is prime. ◀

The next lemma gives a way to conclude that $*$ imitates an element.

▶ **Lemma 15.** *Let $\mathbb{Y}$ be a convex algebra, $\mathbb{X} \leq \mathbb{Y}$ a subalgebra, and let $\mathbb{X}_*$ be an extension of $\mathbb{X}$. Further, let $z \in \mathrm{P}(\mathbb{X}_*)$ and assume that z is $\mathbb{Y}$-cancellable. If there exist $w \in \mathbb{Y}$ and $q \in (0,1)$ with $qz+\bar{q}* = qz+\bar{q}w$, then $*$ imitates w on $\mathrm{P}(\mathbb{X}_*)$ and $\mathrm{Ad}(\mathbb{X}_*) \subseteq \{x \in X \mid x \circ\!\!-\!\!\bullet w\}$.*

Proof. Let $x \in \mathrm{P}(\mathbb{X}_*)$, $p \in (0,1)$, and set $s = \frac{\bar{p}}{\overline{pq}}$. Then $s \in (0,1)$ and $\overline{sq} \cdot p = \bar{s}$, $\overline{sq} \cdot \bar{p} = s\bar{q}$, and we have

$$sqz + \overline{sq}(\underbrace{px + \bar{p}*}_{\in \mathrm{P}(\mathbb{X}_*) \subseteq Y}) = s(qz + \bar{q}*) + \bar{s}x = s(qz + \bar{q}w) + \bar{s}x = sqz + \overline{sq}(\underbrace{px + \bar{p}w}_{\in Y}).$$

Cancelling z yields $px + \bar{p}* = px + \bar{p}w$. We conclude that indeed $*$ imitates w on all of $\mathrm{P}(\mathbb{X}_*)$. Assume now that $x \in \mathrm{Ad}(\mathbb{X}_*)$. Then by Lemma 7.3.

$$pz + \bar{p}x \circ\!\!-\!\!\bullet pz + \bar{p}* = pz + \bar{p}w, \qquad \text{for } p \in (0,1).$$

Again using cancellability of z, it follows that $x \circ\!\!-\!\!\bullet w$ by Lemma 7.4. ◀

5 Extensions of Free Algebras and Functoriality

Let S be a nonempty set and consider the free convex algebra over S. As noted in Example 8.2, this is the algebra $\mathbb{D}_S$ of finitely supported distributions on S. In the next theorem we determine all possible one-point extensions of $\mathbb{D}_S$.

▶ **Theorem 16.** *Let S be a nonempty set and consider the free convex algebra $\mathbb{D}_S$. Extensions $(\mathbb{D}_S)_*$ can be constructed as follows:*
1. *The black-hole behaviour, where $\mathrm{Ad}((\mathbb{D}_S)_*) = \mathcal{D}_f S$.*
2. *Let $w \in \mathcal{D}_f S$, and let $*$ imitate w on all of $\mathcal{D}_f S$.*
3. *Let w be an extremal point of $\mathbb{D}_S$, and let $*$ imitate w on $\mathcal{D}_f S \setminus \{w\}$ and adhere w.*

Every extension $(\mathbb{D}_S)_*$ can be obtained in this way, and each two of these extensions are different.

Note that $w \in \mathrm{Ext}\,\mathbb{D}_S$ if and only if w is a corner point, in other words, a Dirac measure concentrated at one of the points of S.

The fact that the constructions (1) and (2) give extensions is Example 12.1/2. The construction in (3) is Example 12.4, for which we will now provide evidence. First, we prove a more general statement that we call the gluing lemma, which will be needed later as well. It gives a way to produce extensions with a prescribed set of adherence.

▶ **Lemma 17** (Gluing Lemma). *Let $\mathbb{X}$ be a convex algebra, and $P \subseteq X$ a prime ideal. Assume we have convex operations $p(-) \boxplus \bar{p}(-)$ on $\mathbb{P}_*$ that extend $\mathbb{P}$ (whose operations are inherited from $\mathbb{X}$). Assume further that $\mathrm{Ad}(\mathbb{P}_*) = \emptyset$ and that*

$$px + \bar{p}y \multimapdotboth px \boxplus \bar{p}*, \qquad \text{for } x \in P,\ y \in X \setminus P,\ p \in (0,1). \tag{6}$$

Then the operations $p(-) \oplus \bar{p}(-)$, $p \in (0,1)$, defined as follows extend $\mathbb{X}$ to a convex algebra $\mathbb{X}_*$ with $\mathrm{Ad}(\mathbb{X}_*) = X \setminus P$:

$$px \oplus \bar{p}y = \begin{cases} px + \bar{p}y, & x, y \in X, \\ px \boxplus py, & x = *, y \in P \text{ or } x \in P, y = *, \\ *, & \text{otherwise.} \end{cases}$$

Proof of Example 12.4. Assume we are in the situation of Example 12.4, i.e., $\mathbb{X}$ is a convex algebra and w is an extremal point of $\mathbb{X}$. Set $P = X \setminus \{w\}$, then P is a prime ideal. Further, let $\mathbb{P}_*$ be obtained as in Example 12.3 with $\mathbb{P} \leq \mathbb{X}$ by letting $*$ imitate w. Condition (6) is satisfied with equality, and hence the Gluing Lemma provides $\mathbb{X}_*$. The operations $p(-) \oplus \bar{p}(-)$ obtained in this way coincide with those written in Example 12.4. ◀

Proof of Theorem 16. The uniqueness part is easy to see. First, the action of $*$ determines which case of (1)–(3) occurs since $\mathrm{Ad}((\mathbb{D}_S)_*)$ is $\mathcal{D}_f S$ in case (1), $\emptyset$ in case (2), and $\{w\}$ in case (3). Now uniqueness of the point w in (2) and (3) follows since $\mathbb{D}_S$ is cancellative.

We have to show that every extension occurs in one of the described ways. Hence, let an extension $(\mathbb{D}_S)_*$ be given. If $\mathrm{P}((\mathbb{D}_S)_*) = \emptyset$, case (1) takes place. Assume that $\mathrm{P}((\mathbb{D}_S)_*) \neq \emptyset$ and choose $z \in \mathrm{P}((\mathbb{D}_S)_*)$ and $q \in (0,1)$. Set

$$w = \frac{1}{\bar{q}}\big([qz + \bar{q}*] - qz\big) \in \ell^1(S),$$

then $qz + \bar{q}* = qz + \bar{q}w$ by definition. We apply Lemma 15 with $\mathbb{D}_S \leq \ell^1(S)$ and z, w, q. This yields

$$px + \bar{p}* = px + \bar{p}w, \quad x \in \mathrm{P}((\mathbb{D}_S)_*), p \in (0,1), \tag{7}$$

and $\mathrm{Ad}((\mathbb{D}_S)_*) \subseteq \{x \in \mathcal{D}_f S \mid x \circ\!\!-\!\!\bullet w\} \subseteq \{w\}$.

As a linear combination of two elements of $\mathcal{D}_f S$, the element w is finitely supported. Further, by (7),

$$1 = \frac{1}{\bar{p}}\big(\|pz + \bar{p} * \|_1 - p\|z\|_1\big) \leq \|w\|_1 \leq \frac{1}{\bar{p}}\big(\|pz + \bar{p} * \|_1 + p\|z\|_1\big) = \frac{1+p}{1-p}$$

for all $p \in (0,1)$, and we see that $\|w\|_1 = 1$. Together, $w \in \mathcal{D}_f S$.

If $\mathrm{P}((\mathbb{D}_S)_*) = \mathcal{D}_f S$, we are in case (2) of the theorem. Otherwise, $\mathrm{P}((\mathbb{D}_S)_*) = (\mathcal{D}_f S)\backslash\{w\}$. This implies that w is an extremal point of $\mathbb{D}_S$, and we are in case (3). ◀

Next we investigate functoriality of one-point extensions. We say that a functor $F\colon \mathcal{EM}(\mathcal{D}_f) \to \mathcal{EM}(\mathcal{D}_f)$ naturally provides a one-point extension, if $\mathbb{X} \leq F\mathbb{X}$ and $F\mathbb{X}$ has carrier $X \cup \{*\}$ for $* \notin X$ for every algebra $\mathbb{X}$, and $(Ff)|_X = f$ for every convex map $f\colon \mathbb{X} \to \mathbb{Y}$. The latter property is (literally) a natural property: it says that the family of inclusion maps $\iota_X\colon \mathbb{X} \to F\mathbb{X}$ is a natural transformation of the identity functor to F.

An example of a functor possessing these properties is obtained by the black-hole construction: for an algebra $\mathbb{X}$ let $F\mathbb{X}$ be its black-hole extension, and for a convex map $f : \mathbb{X} \to \mathbb{Y}$ let Ff be the extension of f mapping $*$ (of $F\mathbb{X}$) to $*$ (of $F\mathbb{Y}$).

▶ **Theorem 18.** *Let $F : \mathcal{EM}(\mathcal{D}_f) \to \mathcal{EM}(\mathcal{D}_f)$ be a functor such that for all objects $\mathbb{X}$ and for all morphisms $f : \mathbb{X} \to \mathbb{Y}$*

$$\mathbb{X} \leq F\mathbb{X},\ \textit{the carrier of } F\mathbb{X} \textit{ is } X \cup \{*\} \textit{ with } * \notin X, \qquad \begin{array}{ccc} F\mathbb{X} & \xrightarrow{Ff} & F\mathbb{Y} \\ {\scriptstyle \iota_X}\uparrow & & \uparrow{\scriptstyle \iota_Y} \\ \mathbb{X} & \xrightarrow[f]{} & \mathbb{Y} \end{array} \tag{8}$$

Then, for all $\mathbb{X}$, $F\mathbb{X}$ is the black-hole extension, and for all $f\colon \mathbb{X} \to \mathbb{Y}$, Ff is the extension of f mapping $*$ (of $F\mathbb{X}$) to $*$ (of $F\mathbb{Y}$).

We present the proof using two lemmas.

▶ **Lemma 19.** *Assume that $F\colon \mathcal{EM}(\mathcal{D}_f) \to \mathcal{EM}(\mathcal{D}_f)$ satisfies (8), and let $f\colon \mathbb{X} \to \mathbb{Y}$ be a convex map. Then $(Ff)(*) = *$ and $f(\mathrm{P}(F\mathbb{X})) \subseteq \mathrm{P}(F\mathbb{Y})$, $f(\mathrm{Ad}(F\mathbb{X})) \subseteq \mathrm{Ad}(F\mathbb{Y})$.*

Proof. For the proof of $(Ff)(*) = *$, note that $(Ff)^{-1}(\{*\}) \subseteq \{*\}$ since $(Ff)|_X = f$. If f has a right inverse, say $g : \mathbb{Y} \to \mathbb{X}$ with $f \circ g = \mathrm{id}_\mathbb{Y}$, then $(Ff)((Fg)(*)) = *$, and hence $(Fg)(*) = *$. In turn also $(Ff)(*) = *$. Now let f be arbitrary. Let $\mathbb{Z}$ be an algebra which has only one element, a final object of $\mathcal{EM}(\mathcal{D}_f)$, and let $h : \mathbb{Y} \to \mathbb{Z}$ be the unique convex map. The map $h \circ f$ has a right inverse, and therefore $(Fh)((Ff)(*)) = (F(h \circ f))(*) = *$. Again, we obtain $(Ff)(*) = *$.

It remains to prove that f maps the respective prime ideals (sets of adherence) into each other. Let $x \in X$ and $p \in (0,1)$. Then

$$pf(x) + \bar{p}* = p(Ff)(x) + \bar{p}(Ff)(*) = (Ff)(px + \bar{p}*) = \begin{cases} f(px + \bar{p}*) \in Y\,, & x \in \mathrm{P}(F\mathbb{X}), \\ (Ff)(*) = * & , \ x \in \mathrm{Ad}(F\mathbb{X}). \end{cases} \tag{9}$$

Thus, indeed, $f(x) \in \mathrm{P}(F\mathbb{Y})$ if $x \in \mathrm{P}(F\mathbb{X})$, and $f(x) \in \mathrm{Ad}(F\mathbb{Y})$ if $x \in \mathrm{Ad}(F\mathbb{X})$. ◀

▶ **Lemma 20.** *Assume that $F\colon \mathcal{EM}(\mathcal{D}_f) \to \mathcal{EM}(\mathcal{D}_f)$ satisfies (8), and let S be an infinite set. Then $F\mathbb{D}_S$ is the black-hole extension of $\mathbb{D}_S$.*

Proof. Assume towards a contradiction that $\mathrm{P}(F\mathbb{D}_S) \neq \emptyset$. By Theorem 16 we find $w \in \mathcal{D}_f S$ such that $px + \bar{p}* = px + \bar{p}w$, $x \in \mathrm{P}(F\mathbb{D}_S)$, $p \in (0,1)$. Fix $x \in \mathrm{P}(F\mathbb{D}_S)$ and $p \in (0,1)$, and let $f\colon \mathbb{D}_S \to \mathbb{D}_S$ be an automorphism. Then $f(x) \in \mathrm{P}(F\mathbb{D}_S)$ by Lemma 19, and we can compute

$$pf(x) + \bar{p}w = pf(x) + \bar{p}* \overset{(9)}{=} f(px + \bar{p}*) = f(px + \bar{p}w) = pf(x) + \bar{p}f(w).$$

Cancelling $f(x)$ gives $w = f(w)$. Hence w is a fixpoint of every automorphism.

Since S is infinite, we can choose a point $s_1 \in S$ which lies outside of the support of w. Further, let $s_2 \in S$ be in the support of w, and let $\sigma\colon S \to S$ be the permutation of S which exchanges s_1 and s_2 and leaves all other points fixed. Since $\mathbb{D}_S$ is free with basis S, this permutation extends to an automorphism f of $\mathbb{D}_S$. But now $f(w) \neq w$, a contradiction. ◀

Proof of Theorem 18. The fact that Ff is the extension of f mapping $*$ to $*$ was shown in Lemma 19. It remains to show that, for every algebra $\mathbb{X}$, $F\mathbb{X}$ is the black-hole extension. Given $\mathbb{X}$, choose an infinite set S and a surjective convex map $f : \mathbb{D}_S \to \mathbb{X}$. This is possible since every convex algebra is the image of a free convex algebra, and if $S \supseteq S'$ then there is a surjective homomorphism from $\mathbb{D}_S$ to $\mathbb{D}_{S'}$. Then, by Lemma 19 and Lemma 20, $\mathrm{Ad}(F\mathbb{X}) \supseteq f(\mathrm{Ad}(F\mathbb{D}_S)) = f(\mathbb{D}_S) = \mathbb{X}$. ◀

6 Extensions of $\mathcal{P}_c\mathbb{D}$

In this section we formulate and prove Theorem 29 where we describe the set of all extensions $(\mathcal{P}_c\mathbb{D})_*$ for convex algebras $\mathbb{D}$ which are convex subsets of a vector space (equivalently, cancellative) and satisfy a certain linear boundedness condition. Theorem 29 applies in particular to the algebra $\mathbb{D} = \mathbb{D}_S$ of finitely supported distributions over S.

We start with some algebraic preliminaries. First, we recall the notion of linear boundedness, see e.g. [3, Definition 1.1].

▶ **Definition 21.** A convex algebra $\mathbb{X}$ is *linearly bounded*, if every homomorphism of the convex algebra $(0, \infty)$ into $\mathbb{X}$ is constant.

Intuitively, a convex algebra is linearly bounded if it does not contain an infinite ray. A large class of examples of linearly bounded algebras is given by topologically bounded subsets of a topological vector space.

▶ **Example 22.** Let $\mathbb{V}$ be a topological vector space. A subset $D \subseteq V$ is *bounded*, if for every neighbourhood U of 0 there exists $r_0 > 0$ such that $D \subseteq rU$, $r > r_0$. In particular, if $\mathbb{V}$ is a normed space (with a norm denoted by $\|.\|$), then a subset D is bounded in this sense if and only if $\sup_{x\in D} \|x\| < \infty$.

Let $\mathbb{V}$ be a topological vector space. Then for every bounded convex subset D of $\mathbb{V}$, the convex algebra $\mathbb{D}$ is linearly bounded. We could not find an explicit reference for this (intuitive) fact, and hence provide a proof in [31, Appendix B].

▶ Remark 23. Let $\mathbb{V}$ be a vector space over $\mathbb{R}$. Then, for each fixed $w \in \mathbb{V}$ and $t \in \mathbb{R}\setminus\{0\}$, we have the translation map $x \mapsto x + w$ and the scaling map $x \mapsto tx$. They are bijective convex maps on $\mathbb{V}$. Applying $\mathcal{P}_c$ on these maps gives bijective convex maps on $\mathcal{P}_c\mathbb{V}$. Moreover, a subset $A \in \mathcal{P}_c\mathbb{V}$ is linearly bounded if and only if $t(A + w)$ is linearly bounded.

The following observation holds for all cancellative convex algebras $\mathbb{D}$.

▶ **Lemma 24.** *Let $\mathbb{D}$ be a convex algebra and consider $\mathbb{X} = \mathcal{P}_c\mathbb{D}$. If $\mathbb{D}$ is cancellative, then $A \circ\!\!-\!\!\bullet \{x\} \Rightarrow A = \{x\}$ for all $A \in X$, $x \in D$.*

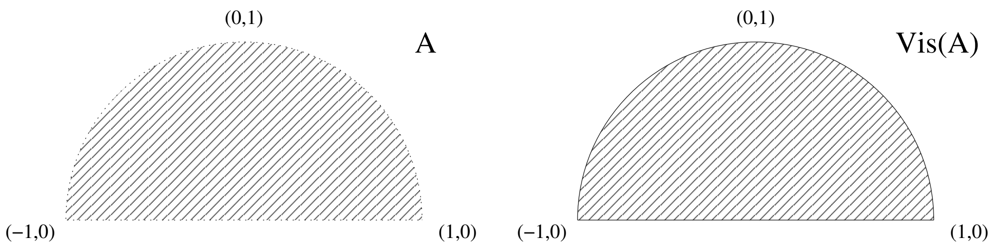

Figure 1 A and $\mathrm{Vis}(A)$.

Proof. Let $a \in A$. Then $pa + \bar{p}x = x = px + \bar{p}x$ which after cancelling with x yields $a = x$. Since A is nonempty, as it belongs to $\mathcal{P}_c\mathbb{D}$, we get $A = \{x\}$. ◀

Under a linear boundedness condition, the roles of A and $\{x\}$ can be exchanged.

▶ **Lemma 25.** *Let $\mathbb{V}$ be a vector space over $\mathbb{R}$, let $A \in \mathcal{P}_c\mathbb{V}$, and assume that $A - A$ is linearly bounded. Then*

$$\bigcap_{p\in[0,1)} \big(p\{x\} + \bar{p}A\big) \subseteq \{x\}, \qquad \text{for } x \in V.$$

In particular, $\{x\} \multimap\!\!\bullet A \Rightarrow A = \{x\}$ for all $x \in V$.

Proof. Note first that $A - A$ is convex. Let y belong to the intersection. Then $y \in A$ and for each $p \in (0,1)$ we find $a_p \in A$ with $y = px + \bar{p}a_p$. This implies

$$\frac{p}{\bar{p}}(x-y) = y - a_p \in A - A, \quad \text{for } p \in (0,1).$$

Any positive real number t can be represented as $\frac{p}{\bar{p}}$, namely with $p = \frac{t}{1+t} \in (0,1)$. It is easy to check then that $\varphi\colon t \mapsto t(x-y)$ is a convex homomorphism from $(0,\infty)$ to $A - A$. Since $A - A$ is linearly bounded, φ is constant, which further implies $x = y$. ◀

In order to construct extensions where $*$ imitates an outer element, we need the following notion of visibility closure.

▶ **Definition 26.** Let $\mathbb{X}$ be a convex algebra and $A \in \mathcal{P}_c\mathbb{X}$. The *visibility hull* of A is

$$\mathrm{Vis}(A) = \big\{x \in X \mid \forall a \in A.\ \forall p \in (0,1).\ px + \bar{p}a \in A\big\}.$$

The set A is *visibility closed* if $A = \mathrm{Vis}(A)$.

▶ **Example 27.** Let $A \subseteq \mathbb{R}^2$ be the open half-disk $A = \{(t_1,t_2) \in \mathbb{R}^2 \mid t_1^2 + t_2^2 < 1, t_2 > 0\}$. Then $\mathrm{Vis}(A)$ is the closed half disk, shown in Figure 1.

Now consider $B = A \cup \{(0,0)\}$. Then the part of the boundary of B located on the t_1-axis does not belong to $\mathrm{Vis}(B)$, see Figure 2.

Let $\mathbb{V}$ be a vector space over $\mathbb{R}$. The *affine hull* of a subset $A \subseteq V$ is

$$\mathrm{aff}(A) = \{\sum_{i=1}^n t_i x_i \mid n \geq 1, x_i \in A, t_i \in \mathbb{R}, \sum_{i=1}^n t_i = 1\}.$$

The affine hull of A is the smallest affine subspace of $\mathbb{V}$ containing A, see e.g. [23, p.6].

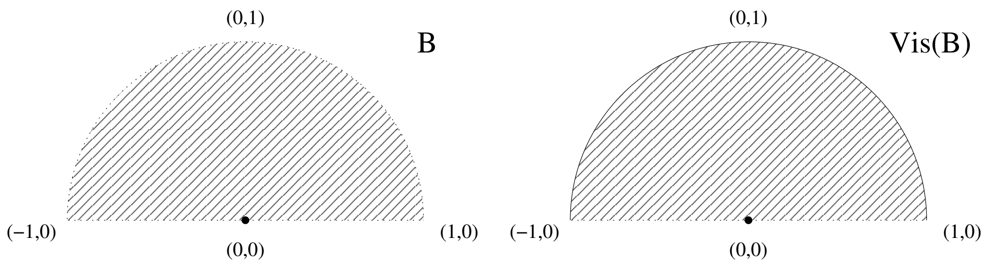

Figure 2 B and $\mathrm{Vis}(B)$.

▶ **Lemma 28.** *Let $\mathbb{V}$ be a vector space over $\mathbb{R}$, and $A \in \mathcal{P}_c\mathbb{V}$. Then*

1. $\mathrm{Vis}(A) = \displaystyle\bigcap_{\substack{a\in A\\ p\in(0,1)}} \frac{1}{p}(A - \bar{p}a) \subseteq \mathrm{aff}(A)$.
2. $\mathrm{Vis}(A)$ *is convex.*
3. $A \subseteq \mathrm{Vis}(A)$ *and* $\mathrm{Vis}(\mathrm{Vis}(A)) = \mathrm{Vis}(A)$.
4. $\mathrm{Vis}(\{z\}) = \{z\}$ *for all* $z \in V$.
5. *If $\mathbb{V}$ is a topological vector space, then* $\mathrm{Vis}(A) \subseteq \overline{A}$. [6]

Proof.

1. We have

$$x \in \mathrm{Vis}(A) \ \Leftrightarrow\ \forall a \in A.\ \forall p \in (0,1).\ px + \bar{p}a \in A \ \Leftrightarrow\ \forall a \in A.\ \forall p \in (0,1).\ x \in \frac{1}{p}(A - \bar{p}a)$$

2. By 1., the set $\mathrm{Vis}(A)$ is the intersection of convex sets.
3. Let $x \in A$. Then $px + \bar{p}a \in A$, $a \in A$, $p \in (0,1)$, since A is convex. Thus $A \subseteq \mathrm{Vis}(A)$. Assume that $x \in \mathrm{Vis}(\mathrm{Vis}(A))$, and let $a \in A$, $p, q \in (0,1)$. Then $px + \bar{p}a \in \mathrm{Vis}(A)$, since $a \in A \subseteq \mathrm{Vis}(A)$, and hence $qpx + \overline{qp}a = q(px + \bar{p}a) + \bar{q}a \in A$. Every number $r \in (0,1)$ can be represented as $r = pq$ with some $p, q \in (0,1)$, and we conclude that $x \in \mathrm{Vis}(A)$.
4. We have $\frac{1}{p}(\{z\} - \bar{p}z) = \{z\}$, $p \in (0,1)$. By 1., $\mathrm{Vis}(\{z\}) = \{z\}$.
5. Let $x \in \mathrm{Vis}(A)$ and $a \in A$. Then $x = \lim_{p\to 1}(px + \bar{p}a) \in \overline{A}$. ◀

The operator $\mathrm{Vis}\colon \mathcal{P}_c\mathbb{V} \to \mathcal{P}_c\mathbb{V}$ is not monotone, as demonstrated in Example 27. Hence, it is not the restriction of a topological closure operator to $\mathcal{P}_c\mathbb{V}$. Still, it is related with topological closures: Let $\mathbb{V}$ be a topological vector space and $A \in \mathcal{P}_c\mathbb{V}$ *relatively closed*, i.e., closed in $\mathrm{aff}(A)$ w.r.t. the subspace topology. Then A is visibility closed (remember footnote 6). This observation shows for example that $\mathrm{Vis}(\mathcal{D}_f S) = \mathcal{D}_f S$. The converse does not hold, as demonstrated by the set $\mathrm{Vis}(B)$ from Example 27.

We can now formulate our description of extensions of $\mathcal{P}_c\mathbb{D}$.

▶ **Theorem 29.** *Let $\mathbb{V}$ be a vector space over $\mathbb{R}$, let D be a convex subset of V with more than one element, and consider the convex algebra $\mathbb{X} = \mathcal{P}_c\mathbb{D}$. Extensions $\mathbb{X}_*$ can be constructed as follows:*

1. *The black-hole behaviour, where* $\mathrm{Ad}(\mathbb{X}_*) = X$.
2. *Let* $C \in \mathcal{P}_c(\mathrm{Vis}(D))$, *and let $*$ imitate C on all of X.*
3. *Let w be an extremal point of $\mathbb{D}$, and let $*$ imitate $\{w\}$ on $X \setminus \{\{w\}\}$ and adhere $\{w\}$.*

[6] In view of 1., $\mathrm{Vis}(A)$ is contained in the *relative closure* of A — the closure of A in $\mathrm{aff}(A)$ w.r.t. the subspace topology.

4. *Let* $C \in \mathcal{P}_c(\mathrm{Vis}(D))$ *with at least two elements, assume* $\mathrm{conv}\{A \in X \mid A \mathrel{\circ\!\!\not\!-\!\!\bullet} C\} \neq X$*, and let* $I = \mathrm{conv}\{A \in X \mid A \mathrel{\circ\!\!\not\!-\!\!\bullet} C\}$*. Let* $P \neq X$ *be a prime ideal in* $\mathbb{X}$ *with* $I \subseteq P$*, and let* $*$ *imitate* C *on* P *and adhere* $X \setminus P$*.*

Assume in addition that $D - D$ *is linearly bounded. Then every extension* $\mathbb{X}_*$ *can be obtained in this way. Each two of these extensions are different: the point* w *in case* (3)*, the set* C *in cases* (2), (4)*, and the prime ideal* P *in case* (4)*, are uniquely determined by* $\mathbb{X}_*$*.*

We are familiar with the constructions (1)–(3) from Example 12 and Theorem 16. That (4) gives extensions follows from the Gluing Lemma, Lemma 17. We give an explicit proof in [31, Appendix C] and illustrative examples in [31, Appendix D].

Assume that $D - D$ is linearly bounded. Our task is to show that every given extension $\mathbb{X}_*$ can be realised as described in (1)–(4) of the theorem, and show uniqueness. The proof relies on the following lemma.

▶ **Lemma 30.** *Assume* $\mathbb{X}_*$ *is an extension with* $\mathrm{P}(\mathbb{X}_*) \neq \emptyset$*. Then* $\mathrm{Ad}(\mathbb{X}_*)$ *contains at most one singleton set.*

Proof. Assume that $\{x\}, \{y\} \in \mathrm{Ad}(\mathbb{X}_*)$, and choose $A \in \mathrm{P}(\mathbb{X}_*)$. Then, for each $p \in (0,1)$, by Lemma 7.3

$$pA + \bar p\{x\} \mathrel{\circ\!\!-\!\!\bullet} pA + \bar p *, \quad pA + \bar p\{y\} \mathrel{\circ\!\!-\!\!\bullet} pA + \bar p * .$$

Set $C = pA + \bar p *$. Then $C \in \mathrm{P}(\mathbb{X}_*)$ and for each $q \in (0,1)$

$$q(pA + \bar p\{x\}) + \bar q C = C = q(pA + \bar p\{y\}) + \bar q C.$$

Thus, for each $a \in A, c \in C$ we find $a_1 \in A, c_1 \in C$ with

$$q\bar p x + \overline{q\bar p}\Big(\frac{qp}{\overline{q\bar p}} a + \frac{\bar q}{\overline{q\bar p}} c\Big) = q\bar p y + \overline{q\bar p}\Big(\frac{qp}{\overline{q\bar p}} a_1 + \frac{\bar q}{\overline{q\bar p}} c_1\Big),$$

and hence

$$\frac{q\bar p}{\overline{q\bar p}}(x - y) \in D - D.$$

Any positive real number t can be represented as $\frac{q\bar p}{\overline{q\bar p}}$ with some $p, q \in (0,1)$, for example use $p = \frac{1}{2t+1}, q = \frac{2t+1}{2t+2}$. Thus $\varphi\colon t \mapsto t(x-y)$ is a homomorphism of $(0,\infty)$ to $D - D$. Since $D - D$ is linearly bounded, φ is constant, and hence $x = y$. ◀

Proof (all $\mathbb{X}_*$ are obtained). Let an extension $\mathbb{X}_*$ of $\mathbb{X}$ be given. If $\mathrm{P}(\mathbb{X}_*) = \emptyset$ then case (1) of the theorem holds. Assume in the following that $\mathrm{P}(\mathbb{X}_*) \neq \emptyset$.

By Lemma 30, $\mathrm{Ad}(\mathbb{X}_*)$ contains at most one singleton set. Since D has more than one element, we find $z \in D$ with $\{z\} \in \mathrm{P}(\mathbb{X}_*)$. Choose $q \in (0,1)$. Then $q\{z\} + \bar q * \in \mathrm{P}(\mathbb{X}_*) \subseteq \mathcal{P}_c\mathbb{V}$. We will show that $*$ imitates the convex set

$$C = \frac{1}{\bar q}\big([q\{z\} + \bar q *] - qz\big) \in \mathcal{P}_c\mathbb{V}.$$

By definition, C satisfies $q\{z\} + \bar q * = q\{z\} + \bar q C$. Since singletons are $\mathcal{P}_c\mathbb{V}$-cancellable, cf. Example 8.3, the hypothesis of Lemma 15 are fulfilled. We conclude that $*$ imitates C on $\mathrm{P}(\mathbb{X}_*)$ and that $\mathrm{Ad}(\mathbb{X}_*) \subseteq \big\{A \in X \mid A \mathrel{\circ\!\!-\!\!\bullet} C\big\}$.

Consider the case that $\mathrm{Ad}(\mathbb{X}_*)$ contains a singleton, say $\{w\} \in \mathrm{Ad}(\mathbb{X}_*)$. Since $C \subseteq \frac{1}{\bar q}(D - qz)$, the set $C - C$ is linearly bounded, cf. Remark 23. Lemma 25 implies that

$C = \{w\}$ and Lemma 24 that $\{A \in X \mid A \circ\!\!-\!\!\bullet C\} = \{\{w\}\}$. We see that $\mathrm{P}(\mathbb{X}_*) = X \setminus \{\{w\}\}$ and that $*$ imitates $\{w\}$ on $\mathrm{P}(\mathbb{X}_*)$. Since $X \setminus \{\{w\}\}$ is an ideal in $\mathbb{X}$, also $D \setminus \{w\}$ is an ideal in $\mathbb{D}$, i.e., w is an extremal point of $\mathbb{D}$. Thus $\mathbb{X}_*$ has the form described in case (3).

Consider the case that $\mathrm{Ad}(\mathbb{X}_*)$ contains no singleton. Hence all singletons are in $\mathrm{P}(\mathbb{X}_*)$. Then

$$p\{y\} + \bar{p}C = p\{y\} + \bar{p}* \subseteq X, \qquad \text{for } y \in D,\ p \in (0,1).$$

Thus $C \subseteq \bigcap_{\substack{y \in D \\ p \in (0,1)}} \frac{1}{\bar{p}}(D - py) = \mathrm{Vis}(D)$, by Lemma 28.1. If $\mathrm{Ad}(\mathbb{X}_*) = \emptyset$, case (2) of the theorem holds. Assume that $\mathrm{Ad}(\mathbb{X}_*) \neq \emptyset$. If C contains only one element, say $C = \{w\}$, we would have

$$\emptyset \neq \mathrm{Ad}(\mathbb{X}_*) \subseteq \{A \in X \mid A \circ\!\!-\!\!\bullet \{w\}\} = \{\{w\}\}.$$

From this $\mathrm{Ad}(\mathbb{X}_*) = \{\{w\}\}$, a contradiction. Thus C has at least two elements. Since

$$X \neq \mathrm{P}(\mathbb{X}_*) = X \setminus \mathrm{Ad}(\mathbb{X}_*) \supseteq \{A \in X \mid A \circ\!\!\not\!-\!\!\bullet C\},$$

the convex hull of $\{A \in X \mid A \circ\!\!\not\!-\!\!\bullet C\}$ is not X and case (4) of the theorem holds. ◀

Proof (uniqueness). The uniqueness assertion of the theorem follows since $\mathrm{P}(\mathbb{X}_*)$ always contains singletons by Lemma 30, and singletons are cancellable in $\mathcal{P}_c\mathbb{V}$. ◀

The following example shows that the linear boundedness condition in Theorem 29 cannot be dropped without admitting other types of constructions. We give the proof in [31, Appendix C].

▶ **Example 31.** Let $\mathbb{V} = \mathbb{R}^2$ and $D = \{(t_1, t_2) \in \mathbb{R}^2 \mid t_2 > 0\} \cup \{(0,0)\}$. Set $P = \{A \in \mathcal{P}_c\mathbb{D} \mid (0,0) \notin A\}$, then P is a prime ideal of $\mathcal{P}_c\mathbb{D}$ (see proof in [31, Appendix C]).

Set $C = D \cup \{(t_1, t_2) \in \mathbb{R}^2 \mid t_2 = 0, t_1 > 0\}$. Then $C \subseteq \mathrm{Vis}(D \setminus \{(0,0)\})$, and we can define an extension $\mathbb{P}_*$ by letting $*$ imitate C on all of P. The assumptions of the Gluing Lemma are satisfied (see proof in [31, Appendix C]), and we obtain an extension $(\mathcal{P}_c\mathbb{D})_*$. This extension is not among the ones listed in Theorem 29, since $C \nsubseteq \mathrm{Vis}(D)$.

Unboundedness of D enters in this example in the way that it enables us to let $*$ imitate a cone. In fact, dropping linear boundedness, one can still show that an extension which is not of type (1)–(4) of the theorem, must be such that $*$ imitates some cone whose apex lies in D. However, we have no description which cones occur that way.

7 Conclusions

We have studied the possibility of extending a convex algebra by a single element. We have proven that many different extensions are possible of which only one gives rise to a functor on $\mathcal{EM}(\mathcal{D}_f)$. We have described all extensions of $\mathbb{D}_S$, the free convex algebra of probability distributions over a set S, and of $\mathcal{P}_c\mathbb{D}$, the convex algebra of convex subsets of a particular kind of convex subset of a vector space. As a consequence of the latter result, we have described all extensions of $\mathcal{P}_c\mathbb{D}_S$ used for modelling probabilistic automata.

We expect that the methods developed here can be useful in the study of Eilenberg-Moore algebras for the Giry monad. Detailed investigation is left for future work.

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