Directing functionals and de Branges space completions in almost Pontryagin spaces

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Dedicated to Daniel Alpay on the occasion of his 60th birthday

Abstract: The following theorem holds: Let $\mathcal{L}$ be a – not necessarily nondegenerated or complete – positive semidefinite inner product space carrying an anti-isometric involution, and let S be a symmetric operator in $\mathcal{L}$. If S possesses a universal directing functional $\Phi : \mathcal{L} \times \mathbb{C} \to \mathbb{C}$ which is real w.r.t. the given involution, and the closure of S in the completion of $\mathcal{L}$ has defect index $(1, 1)$, then there exists a de Branges (Hilbert-) space $\mathcal{B}$ such that $x \mapsto \Phi(x, \cdot)$ maps $\mathcal{L}$ isometrically onto a dense subspace of $\mathcal{B}$ and the multiplication operator in $\mathcal{B}$ is the closure of the image of S under this map.

In this paper we consider a version of universal directing functionals defined on an open set $\Omega \subseteq \mathbb{C}$ instead of the whole plane, and inner product spaces $\mathcal{L}$ having finite negative index. We seek for representations of S in a class of reproducing kernel almost Pontryagin spaces of functions on Ω having de Branges-type properties. Our main result is a version of the above stated theorem, which gives conditions making sure that Φ establishes such a representation. This result is accompanied by a converse statement and some supplements.

As a corollary, we obtain that if a de Branges-type inner product space of analytic functions on Ω has a reproducing kernel almost Pontryagin space completion, then this completion is a de Branges-type almost Pontryagin space. This is an important fact in applications. The corresponding result in the case that $\Omega = \mathbb{C}$ and $\mathcal{L}$ is positive semidefinite is well-known, often used, and goes back (at least) to work of M.Riesz in the 1920's.

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1 Introduction

The method of directing functionals was orginally developed by M.G.Krein in the 1940's. Nowadays it comes in various flavours and generalities: scalar- or operator valued, defined locally around the real line or on the whole complex plane, for positive semidefinite inner product spaces or indefinite ones leading to Pontryagin or Krein spaces, for symmetric or isometric operators, etc.

One feature of directing functionals, which is the mainly exploited one, is that they can be used to establish existence of spectral functions μ for a symmetric operator S which is given on a *non-complete* inner product space. They provide *explicit* representations of S as a part of the multiplication operator in $L^2(\mu)$. Another feature of the method of directing functionals is that it can be

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combined with the theory of entire operators, and then leads to *explicit* representations of S as the multiplication operator in a *de Branges space* of entire functions.

In many concrete situations directing functionals are naturally present and lead in a most elegant way to interesting and deep results. Surprisingly, despite its power and ease in application, the method of directing functionals seems to be not widely known.

In the present paper we develop a generalisation of the second mentioned feature to the setting of almost Pontryagin spaces and spaces of analytic functions on arbitrary open base sets $\Omega \subseteq \mathbb{C}$. Our main result is a theorem which constructs in presence of an "Ω-directing functional" a representation of S as the multiplication operator in an "Ω-space".

We continue with a rather detailed discussion of the known theory and available literature. Only after that, in §1.2, we explain the contribution of the present paper in detail, since then it will be much clearer how it relates to the existing theory and what our major achivements are[1].

1.1 Review of the existing theory

Krein considered in his pioneering work [Kre48] finite families of directing functionals, in other words directing mappings into a finite dimensional space $\mathbb{C}^p$. For the reason of simplicity, we shall restrict ourselves to the scalar valued case. Moreover, at least in the subsequent informal discussion, we always think of densely defined linear operators (rather than linear relations).

Directing functionals in positive semidefinite inner product spaces

The power of the method of directing functionals can – to the taste of the author – be best understood in the context of models for symmetric operators. Hence, let us start with a short digression about representations of symmetric operators as the operator of multiplication by the independent variable in certain classes of spaces.

Generally speaking, it is an important problem in spectral theory to find such representations. The first result in this direction which comes to ones mind is of course the spectral theorem providing a normal form for a selfadjoint operator. A few concrete situations where such results play a role are: The spectral theory of differential operators where one speaks of a Fourier-transform or – maybe slightly old-fashioned – of eigenfunction expansions; power moment problems where representations of a certain operator yield measures having a given sequence of numbers as moments; the theory of continuous positive definite functions on the real line or an interval where a proof of Bochner's theorem or certain continuation results can be obtained using representations of an associated selfadjoint or symmetric operator.

We will now state three theorems which provide representations. To do that some notation is required:

1.1 Definition. Let $\mathcal{H}$ be a Hilbert space and T a closed symmetric operator in $\mathcal{H}$.

[1]We compiled the following §1.1 also with the idea to give a very brief survey of the presently available theory of directing functionals.

(*i*) We call $\beta_+ := \dim[\mathcal{H}/\operatorname{ran}(T-i)]$ and $\beta_- := \dim[\mathcal{H}/\operatorname{ran}(T+i)]$ the *upper- and lower deficiency indices* of T, and the pair (β_+, β_-) the *defect index* of T.

(*ii*) T is *completely non-selfadjoint*, if $\bigcap_{\zeta\in\mathbb{C}\setminus\mathbb{R}} \operatorname{ran}(T-\zeta) = \{0\}$.

(*iii*) T is *regular*, if its set of points of regular type equals $\mathbb{C}$.

Moreover:

(*iv*) If $\mathcal{K}$ is a Hilbert space of functions, we denote by $S(\mathcal{K})$ the *operator of multiplication by the independent variable* in $\mathcal{K}$ defined on its natural maximal domain.

$\Diamond$

Remember in this context that every regular operator is completely non-selfadjoint.

The first theorem is a simple consequence of the spectral theorem applied to selfadjoint extensions of T.

1.2 Theorem. *Let T be a closed symmetric operator in a Hilbert space $\mathcal{H}$. Assume that T is completely non-selfadjoint and that the deficiency indices of T do not exceed* 1. *Then there exists a positive Borel measure μ on $\mathbb{R}$ and an isometric operator $\mathcal{F} : \mathcal{H} \to L^2(\mu)$, such that $\mathcal{F} \circ T \subseteq S(L^2(\mu)) \circ \mathcal{F}$.*

The other two theorems deal with the defect $(1, 1)$ case. These results go back at least to work of M.S.Livšic [Liv60b; Liv60a] on the characteristic function (when transferred via Caley transforms), or to work of M.G.Krein [Kre44; Kre46b] on the Q-function of a symmetric operator (when combined with reproducing kernel models for Q-functions). There is a vast literature on the topic from many different points of view, for instance let us also mention the more function-theoretic viewpoint taken up in [Nik02].

A *Herglotz space* $\mathcal{B}$ is a reproducing kernel Hilbert space of functions analytic on $\mathbb{C}\setminus\mathbb{R}$ with certain additional properties. For details see [AMR13, p.1042], from where we also borrowed the terminology, and [Bra68, Theorems 5,6].

1.3 Theorem. *Let T be a closed symmetric operator in a Hilbert space $\mathcal{H}$. Assume that T has defect index $(1, 1)$ and is completely non-selfadjoint. Then there exists a Herglotz space $\mathcal{B}$ and a unitary operator $\mathcal{F} : \mathcal{H} \to \mathcal{B}$, such that $\mathcal{F} \circ T = S(\mathcal{B}) \circ \mathcal{F}$.*

A *de Branges (Hilbert-) space* $\mathcal{B}$ is a reproducing kernel Hilbert space of entire functions with certain additional properties. For details see [Bra68, §19,Problem 50,Theorem 23].

1.4 Theorem. *Let T be a closed symmetric operator in a Hilbert space $\mathcal{H}$. Assume that T has defect index $(1, 1)$ and is regular. Then there exists a de Branges space $\mathcal{B}$ and a unitary operator $\mathcal{F} : \mathcal{H} \to \mathcal{B}$, such that $\mathcal{F} \circ T = S(\mathcal{B}) \circ \mathcal{F}$.*

References which contain explicit proofs are [AMR13, Theorem 6.3] for Theorem 1.3 and [Mar11, Theorem 5.0.7] for Theorem 1.4. In these papers reproducing kernel space theory is systematically exploited, and proofs are based on Livšic' characteristic functions.

Usually in concrete situations one is given a linear space $\mathcal{L}$ with a positive semidefinite inner product and a symmetric operator S in this space. However, this explicitly accessible space $\mathcal{L}$ is (usually) not complete. For instance, when dealing with differential operators one has a linear space of compactly supported C^∞-functions, or when dealing with power moments one has the linear space of polynomials. In order to apply general results as the ones mentioned above, one first has to pass to the Hilbert space completion $\mathcal{H}$ of $\mathcal{L}$ and the closure T of S in this completion. Whereas $\mathcal{L}$ and S are given directly by the data of the problem under consideration, the space $\mathcal{H}$ and the operator T are often not anymore easily and explicitly accessible, simply by the inconstructive nature of the completion process. And this is the point where the method of directing functionals comes into play: It can be used to produce *explicit* representations of S and to draw conclusions *about the closure* T of S.

In the sequel we explain two aspects related to representations as in Theorem 1.2 and Theorem 1.4 in more detail.

Aspect 1. Representations in spaces $L^2(\mu)$:

At this point recall the definition of a directing functional, cf. [Kre48, Definition 1] (see also [GG97, Definition II.8.1]).

1.5 Definition. Let $\mathcal{L}$ be a positive semidefinite inner product space (not necessarily nondegenerated or complete) and let S be a symmetric operator (not necessarily everywhere defined) in $\mathcal{L}$. A function $\Phi : \mathcal{L} \times \mathbb{R} \to \mathbb{C}$ is called a *directing functional* for S, if:

- **(df1)** For each $x \in \mathcal{L}$ the map $\zeta \mapsto \Phi(x, \zeta)$, $\zeta \in \mathbb{R}$, has a continuation to an analytic function defined on some open set Ω_x containing $\mathbb{R}$.
- **(df2)** For each $\zeta \in \mathbb{R}$ the map $x \mapsto \Phi(x, \zeta)$, $x \in \mathcal{L}$, is linear.
- **(df3)** For each $x \in \mathcal{L}$ and $\zeta \in \mathbb{R}$ we have $x \in \operatorname{ran}(S - \zeta)$ if and only if $\Phi(x, \zeta) = 0$.
- **(df4)** There exists an element $u \in \mathcal{L}$ such that the function $\Phi(u, \cdot)$ does not vanish identically.

$\diamondsuit$

Regardless of the properties of the closure T of S in the Hilbert space completion $\mathcal{H}$ of $\mathcal{L}$, presence of a directing functional for S yields an explicit representation of S in a space $L^2(\mu)$.

1.6 Theorem. *Let $\mathcal{L}$ be a positive semidefinite inner product space, let S be a symmetric operator in $\mathcal{L}$, and let Φ be a directing functional for S. Then there exists a positive Borel measure μ on $\mathbb{R}$, such that the map $\mathcal{F} : x \mapsto [\Phi(x, \cdot)]_\mu$ (here $[\cdot]_\mu$ denotes the equivalence class μ-almost everywhere) is an isometry of $\mathcal{L}$ into $L^2(\mu)$ and satisfies $\mathcal{F} \circ S \subseteq S(L^2(\mu)) \circ \mathcal{F}$.*

A proof can be found in the original reference [Kre48, Theorem 1] or in [GG97, Theorem II.8.1][2].

[2]When referring to the treatment of directing functionals in [GG97] one has to be careful. Theorem II.8.1 in this reference is correctly proved, but Theorems II.8.4 and II.8.5 are not.

As far as we understand history and literature, validity of this result was considered the main feature of directing functionals. However, we believe that the features explained below as “Aspect 2” are equally valuable.

Aspect 2. Representations in reproducing kernel spaces:

Recall the definition of a universal directing functional, cf. [GG97, Definition II.8.2] (we do not know where this notion appeared for the first time).

1.7 Definition. Let $\mathcal{L}$ be a positive semidefinite inner product space (not necessarily nondegenerated or complete) and let S be a symmetric operator (not necessarily everywhere defined) in $\mathcal{L}$. A function $\Phi : \mathcal{L} \times \mathbb{C} \to \mathbb{C}$ is called a *universal directing functional* for S, if:

(udf1) For each $x \in \mathcal{L}$ the map $\zeta \mapsto \Phi(x, \zeta)$, $\zeta \in \mathbb{C}$, is analytic.

(udf2) For each $\zeta \in \mathbb{C}$ the map $x \mapsto \Phi(x, \zeta)$ is linear.

(udf3) For each $x \in \mathcal{L}$ and $\zeta \in \mathbb{C}$ we have $x \in \operatorname{ran}(S - \zeta)$ if and only if $\Phi(x, \zeta) = 0$.

(udf4) There exists an element $u \in \mathcal{L}$ such that the function $\Phi(u, \cdot)$ does not vanish identically.

$\Diamond$

Note that, if S has a universal directing functional, the deficiency indices of the closure T of S cannot exceed 1.

Presence of a universal directing functional plus assuming existence of an entire gauge (using the terminology of [GG97]) plus assuming that T (the closure of S in the Hilbert space completion of $\mathcal{L}$) has defect $(1, 1)$ invokes Krein's theory of entire operators, and hence leads to an explicit representation in a de Branges space (this is probably a classical fact, an explicit mentioning can be found in [ST10]).

1.8 Theorem. *Let $\mathcal{L}$ be a positive semidefinite inner product space, let S be a symmetric operator in $\mathcal{L}$, and let Φ be a universal directing functional for S. Assume that T has defect index $(1, 1)$, and that there exists an element $u \in \mathcal{L}$ with $\Phi(u, \cdot) = 1$. Then there exists a de Branges space $\mathcal{B}$, such that the map $\mathcal{F} : x \mapsto \Phi(x, \cdot)$ is an isometry of $\mathcal{L}$ onto a dense subspace of $\mathcal{B}$ and satisfies $\mathcal{F} \circ S \subseteq S(\mathcal{B}) \circ \mathcal{F}$ where (thinking of operators in terms of their graphs) $S(\mathcal{B}) = \operatorname{Clos}_{\mathcal{B}\times\mathcal{B}}[(\mathcal{F}\times\mathcal{F})(S)]$. In particular, T is regular.*

It is worth to compare the notions of directing and universal directing functionals in more detail. Apparently, the conditions (udf1) and (udf3) in Definition 1.7 are strengthenings of Definition 1.5, (df1) and (df3). The fact in (udf3) that the description of $\operatorname{ran}(S-\zeta)$ holds for nonreal points, together with analyticity along

For Theorem II.8.4 the mistake is obvious: what if $\Phi(v, \cdot)$ vanishes identically (compare with the correct statement [GL74, Behauptung 3.5]). The mistake in the proof of Theorem II.8.5 is more difficile: In course of the proof Theorem II.3.3 is employed. This result however requires that T is completely non-selfadjoint, which is not known at that point. Adding the additional assumption that T is completely non-selfadjoint implies correctness of the proof of Theorem II.8.5. Despite this mistake in the proof, we do not know whether the actual assertion of Theorem II.8.5 is true or false. However, compare Proposition 1.9.

$\mathbb{R}$ in (udf1) already implies existence of a representation in a de Branges space (combine the below Proposition 1.9 with Theorem 1.4). Having entire functions in (udf1) in conjunction with existence of an entire gauge (the element u) is responsible for one representation in a de Branges space to be actually given by $x \mapsto \Phi(x, \cdot)$. For completeness, let us note that (df2) together with analyticity on all of $\mathbb{C}$ readily implies (udf2).

1.9 Proposition. *Let $\mathcal{L}$ be a positive semidefinite inner product space and let S be a symmetric operator in $\mathcal{L}$ whose domain is dense in $\mathcal{L}$ w.r.t. the seminorm induced by the inner product. Let $\Omega \subseteq \mathbb{C}$ be an open neighbourhood of $\mathbb{R}$, and let $\Phi : \mathcal{L} \times \mathbb{R} \to \mathbb{C}$ be a function which satisfies*

(df1$^+$) *For each $x \in \mathcal{L}$ the map $\zeta \mapsto \Phi(x, \zeta)$, $\zeta \in \mathbb{R}$, has a continuation to an analytic function on Ω (which we shall again denote by $\Phi(x, \cdot)$).*

(df2$^+$) *For each $\zeta \in \Omega$ the map $x \mapsto \Phi(x, \zeta)$, $x \in \mathcal{L}$, is linear.*

(df3$^+$) *For each $x \in \mathcal{L}$ and $\zeta \in \Omega$ we have $x \in \operatorname{ran}(S - \zeta)$ if and only if $\Phi(x, \zeta) = 0$.*

(df4$^+$) *There exists an element $u \in \mathcal{L}$ such that the function $\Phi(u, \cdot)$ does not vanish identically on $\mathbb{R}$.*

Assume that T has defect index $(1, 1)$. Then T is regular.

The proof of this fact will be given in §4.

It is noteworthy that Theorem 1.8 has a partial converse; one half of (udf3) should be replaced by an approximative variant of the condition.

1.10 Proposition. *Let $\mathcal{L} \neq \{0\}$ be a positive semidefinite inner product space and let S be a symmetric operator in $\mathcal{L}$. Assume that there exists a de Branges space $\mathcal{B}$ and an isometric map $\mathcal{F} : \mathcal{L} \to \mathcal{B}$ such that $\operatorname{ran} \mathcal{F}$ is dense in $\mathcal{B}$ and $S(\mathcal{B}) = \operatorname{Clos}_{\mathcal{B} \times \mathcal{B}}[(\mathcal{F} \times \mathcal{F})(S)]$. Then the map*

$$\Phi(x, \zeta) := [\mathcal{F}(x)](\zeta), \quad x \in \mathcal{L}, \zeta \in \mathbb{C},$$

satisfies (udf1), (udf2), (udf4), *and*

(udf3') *If $\zeta \in \mathbb{C}$ and $x \in \operatorname{ran}(S - \zeta)$, then $\Phi(x, \zeta) = 0$.*

(udf3") *If $x \in \mathcal{L}$, $\zeta \in \mathbb{C} \setminus \mathbb{R}$, and $\Phi(x, \zeta) = 0$, then there exists a sequence $(x_n)_{n \in \mathbb{N}}$ with $x_n \in \operatorname{ran}(S - \zeta)$, $n \in \mathbb{N}$, such that*

$$\lim_{n \to \infty} (x_n - x, x_n - x)_{\mathcal{L}} = 0, \quad \lim_{n \to \infty} \Phi(x_n - x, \eta) = 0, \eta \in \mathbb{C}.$$

The proof is straightforward (using verbatim the same argument as in the proof of Proposition 4.6, (v)).

Directing functionals in indefinite inner product spaces

Let us turn to the presently available versions for indefinite inner product spaces. These are [GL74, §3] treating the Pontryagin space case and [Tex06] for the Krein space case.

The two aspects of directing functionals discussed above now become indistinct. The reason for this is that selfadjoint operators in a Pontryagin or Krein space may have nonreal spectrum. When seeking representations in spaces of the type $L^2(\mu)$, μ should be substituted by a suitable distribution ϕ and $L^2(\mu)$ by the model space $\Pi(\phi)$ as introduced in [JLT92] for the Pontryagin space case and in [Kle10] for the Krein spaces case. Analyticity of the functions $\Phi(x, \cdot)$ is needed on the spectrum of selfadjoint extensions of T. Thus, contrasting the Hilbert space case, analyticity on the whole plane must be required even if only interested in "Aspect 1".

Maybe it is for the reason just explained that the definitions of a directing functional in [GL74] and [Tex06] (when written out for the scalar valued case) read exactly as Definition 1.7. The results establishing the analogues of Theorem 1.6 are [GL74, Behauptung 3.1] and [Tex06, Theorem 5.1]. These statements have a different form, but can be reformulated in a way that matches Theorem 1.6. For the analogue of Theorem 1.8 we do not know an explicit reference, but the results of [GL74, §3] certainly go in that direction. In fact, it can be obtained by combining the "defect $(1, 1)$" case in [GL74, §3] with the indefinite version of entire operators developed a bit later in [KL78, §6].

About the literature

Our research for literature about theory and applications of directing functionals produced the following list of references which seems to be exhaustive at this point in time. If the reader knows about further references, we would be grateful to learn about them.

`(1) Positive definite theory:` First of all, of course, the pioneering work [Kre48] of M.G.Krein (announced in [Kre46a]). There he considered directing mappings into a finite dimensional space $\mathbb{C}^p$. The basic theorem is a vector-valued version of Theorem 1.6, cf. [Kre48, Theorem 1]. The paper contains two applications, namely a matrix-valued version of Bochner's theorem and a Fourier-transform for a first-order differential operator. An overview (including more details and also other topics) can be found in [Ber00].

Soon after that the theory was supplemented by A.Ya.Povzner in [Pov50] who proved inversion formulas, and a bit later F.S.Rofe-Beketov implicitly used directing mappings into infinite-dimensional spaces in his consideration [Rof60] of differential operators with an operator valued potential. A systematic treatment of the theory of directing mappings into infinite-dimensional spaces was given by H.Langer in [Lan70]. A generalisation to linear relations, including applications to differential expressions and an operator-valued moment problem, was given by H.Langer and B.Textorius in [LT78; LT84; LT85]. These results were somewhat refined and transferred to isometric operators by V.A.Yavryan and A.V.Yavryan in [YY95], where also some more applications are found.

A further generalisation can be found in [BK88, Chapter V,§5]. A version of directing functionals for isometric operators (instead of symmetric ones) was investigated by M.B.Bekker in [Bek89]. In [ST10] universal directing functionals and the theory of entire operators is applied to obtain representations in de Branges spaces.

Finally we mention the book [AG81] where a treatment (somewhat incomplete and bound to a differential operator) can be found in Appendix II, §7, and the book [GG97] where it is attempted to elaborate the scalar-valued case (remember footnote 2).

`(2) Indefinite theory:` A treatment of directing functionals in the Pontryagin space setting was given by M.Großmann and H.Langer in [GL74], where the theory was applied to a continuation problem for hermitian indefinite functions on an interval. It is noteworthy that in this paper the method of directing functionals was also combined with ideas from the theory of entire operators.

A version of the basic Theorem 1.6 in the setting of Krein spaces was given by B.Textorius in [Tex06].

`(3) Applications:` The following is a list of references where typical applications of the method of directing functionals are given, i.e., where existence of spectral functions for various problems is established by applying an appropriate variant of Theorem 1.6. In chronological order: [GG71], [Ovc80], [Kac85; Kac86] (announced in [Kac83]), [Bek98] (where moreover the method is combined with the theory of entire operators), [LM00], [Bek01], [FLL12] (this paper is remarkable since it deals with an equation having two singular endpoints). Most recently, an application to the transfer function of a Sturm-Liouville equation was elaborated by H.Langer in [Lan16, §5.2,5.3].

1.2 The contribution of the present paper

In this paper we investigate "Aspect 2" and establish a variant of Theorem 1.8. Our results generalise this theorem in different directions:

(1) *Geometrically:* We allow $\mathcal{L}$ to be indefinite with finite index of negativity, and ask for an explicit representation in reproducing kernel almost Pontryagin spaces $\mathcal{B}$ with certain additional properties. At this point it should be said that such representations may exists even if there are no in reproducing kernel Pontryagin spaces.

(2) *Function theoretically:* We neither require the elements of $\mathcal{B}$ to be analytic on $\mathbb{C} \setminus \mathbb{R}$ nor to be entire. We ask for analyticity on some pregiven subset Ω of $\mathbb{C}$. In particular, $\Omega \cap \mathbb{R}$ may be a proper – but nonempty – subset of the real line.

(3) *Operator theoretically:* We allow S to be a linear relation. Of course, this is only a minor point.

We improve upon the conditions given in Theorem 1.8:

(4) Using an approximate version as in Proposition 1.10, an "if and only if" statement is obtained.

(5) The condition that T has defect index $(1,1)$ is replaced by a pair of conditions which are more directly formulated in terms of $\mathcal{L}$ and Φ. Admittedly, these conditions are still not easy to check (but we believe that this is in the nature of the problem).

The version of directing functionals used in this paper – we call them Ω-directing functionals – is introduced in Definition 4.1. The type of spaces $\mathcal{B}$ in which we seek for representations of S – we call them Ω-spaces – is introduced in

Definition 2.19. The main result is Theorem 4.3, where we show that presence of an Ω-directing functional together with some further conditions yields an explicit representation in an Ω-space. This theorem is accompanied by the converse result Proposition 4.6, and a supplement taking care of an additional algebraic structure, cf. Proposition 4.8. Matching the presently used conditions with the ones more familiar from the literature is done in the second part of §4.

Besides what we already mentioned above, a striking difference between Definition 4.1 and Definition 1.7 is that Definition 4.1 does not pose any assumptions on $\operatorname{ran}(S-\eta)$ when η is real. However, this does not mean that such assumptions are not necessary; they just take a different form, namely, they appear as the separate assumption *(ii)* in Theorem 4.3.

1.3 Organisation of the manuscript

Section 2 is of preliminary nature. However, we do not only set up notation and recall some basics, we also introduce and study some essential notions: *sets of* Φ*-containement*, cf. Definitions 2.7 and 2.8, and Ω*-spaces*, cf. Definition 2.19. Moreover, we formulate an appropriate version of a theorem of Krein about preservation of analyticity, cf. Theorem 2.15.

Section 3 contains the core arguments needed for the proof of our main result Theorem 4.3. These are arranged in the form of eight lemmata; an outline is given in the beginning of Section 3. Concerning proofs and technique, in this section the major part of the work is done.

In Section 4 we define Ω*-directing functionals*, cf. Definition 4.1, and establish our main theorem. Its proof is obtained by applying the lemmata from the previous section. Moreover, we establish the afore mentioned converse and supplement.

In Section 5 we introduce a non-complete variant of Ω-spaces, cf. Definition 5.1, and show that these geometric properties are inherited when passing to a reproducing kernel completion, cf. Proposition 5.2. This includes the case of *de Branges space completions*, cf. Corollary 5.8, a fact which is of importance in many concrete situations (and which is well-known for the positive definite case, at least going back to work of M.Riesz from the 1920's). One such application, namely a treatment of (an indefinite version of) the Hamburger power moment problem and the index of determinacy of a measure, will be presented in forthcoming work.

The paper closes with an appendix where we elaborate the proof of two results on preservation of analyticity for which we cannot appoint an explicit reference. The intention is to make our work more easily accessible also to the reader who is not fully into the subject.

Finally, some standard references: for the geometry of Pontryagin spaces [IKL82], for indefinite inner product spaces in general [Bog74; AI89], for almost Pontryagin spaces [KWW05], [SW12], and [Wor14, Appendix], for complex analysis, e.g., [BG91; Con78; Rem98], and for linear relation in indefinite inner product spaces [DS87a].

2 Preliminaries

2.1 Some general notation

In the following items we introduce some notation which is used throughout the paper.

1° For a subset Ω of $\mathbb{C}$, set $\Omega^{\#} := \{\eta \in \mathbb{C} : \overline{\eta} \in \Omega\}$, and for a complex-valued function f on Ω, set

$$f^{\#} : \begin{cases} \Omega^{\#} & \to & \mathbb{C} \\ \eta & \mapsto & \overline{f(\overline{\eta})} \end{cases} \tag{2.1}$$

2° Let Ω be an open and nonempty subset of $\mathbb{C}$. Then we denote by $\mathbb{H}(\Omega)$ the set of all complex-valued analytic functions on Ω.

We denote by $\mathbb{C}_\infty$ the *Riemann sphere* considered as a Riemann surface in the usual way, and by $\mathbb{H}(\Omega, \mathbb{C}_\infty)$ the set of all analytic functions of Ω into $\mathbb{C}_\infty$. In other words, $\mathbb{H}(\Omega, \mathbb{C}_\infty)$ is the set of all meromorphic functions on Ω.

The spaces $\mathbb{H}(\Omega)$ and $\mathbb{H}(\Omega, \mathbb{C}_\infty)$ are always endowed with the compact-open topology, that is, the topology of locally uniform convergence.

3° For a function $f \in \mathbb{H}(\Omega, \mathbb{C}_\infty)$, we denote by $\mathfrak{d}_f : \Omega \to \mathbb{Z} \cup \{\pm\infty\}$ its *divisor*. This is the function defined as $\mathfrak{d}_f(\eta) := -\infty$ if $f = \infty$ on the connected component of Ω containing η, as $\mathfrak{d}_f(\eta) := +\infty$ if $f = 0$ on the component of Ω containing η, and as $\mathfrak{d}_f(\eta)$ being the power of the first nonvanishing term in the Laurent-expansion of f at η if non of the previous cases occurs.

For a nonempty family $\mathcal{F} \subseteq \mathbb{H}(\Omega, \mathbb{C}_\infty)$, we set

$$\mathfrak{d}_{\mathcal{F}}(\eta) := \inf\{\mathfrak{d}_f(\eta) : f \in \mathcal{F}\} \in \mathbb{Z} \cup \{\pm\infty\}, \quad \eta \in \Omega.$$

4° For two nonempty sets Ω and X we denote by $\chi_\eta : X^\Omega \to X$ the *point evaluation functional* at the point $\eta \in \Omega$, i.e.,

$$\chi_\eta : \begin{cases} X^\Omega & \to & X \\ f & \mapsto & f(\eta) \end{cases}, \quad \eta \in \Omega.$$

If Ω is an open and nonempty subset of $\mathbb{C}$, we denote

$$\chi_\eta^{(l)} : \begin{cases} \mathbb{H}(\Omega) & \to & \mathbb{C} \\ f & \mapsto & f^{(l)}(\eta) \end{cases}, \quad \eta \in \Omega, l \in \mathbb{N}_0. \tag{2.2}$$

5°: Let $\mathcal{X}, \mathcal{Y}$ be linear spaces. A linear subspace of V of $\mathcal{X} \times \mathcal{Y}$ is called a *linear relation.* We set

$$\begin{aligned} \operatorname{dom} V &:= \big\{x \in \mathcal{X} : \exists\, y \in \mathcal{Y} \text{ with } (x; y) \in V\big\}, \\ \operatorname{ran} V &:= \big\{y \in \mathcal{Y} : \exists\, x \in \mathcal{X} \text{ with } (x; y) \in V\big\}, \\ \ker V &:= \big\{x \in \mathcal{X} : (x; 0) \in V\big\}, \\ \operatorname{mul} V &:= \big\{y \in \mathcal{Y} : (0; y) \in V\big\}, \end{aligned}$$

and speak of the *domain*, *range*, *kernel*, and *multivalued part* of V. Moreover, set

$$V^{-1} := \{(y; x) : (x; y) \in V\}.$$

If $v : \mathcal{L} \subseteq \mathcal{X} \to \mathcal{Y}$ is a linear map defined on some linear subspace $\mathcal{L}$ of $\mathcal{X}$, then its *graph*

$$\operatorname{graph} v := \big\{(x; v(x)) : x \in \mathcal{L}\big\} \subseteq \mathcal{X} \times \mathcal{Y}$$

is a linear relation. A linear relation is the graph of some linear map if and only if its multivalued part equals $\{0\}$.

6°: Let $\mathcal{X}$ be a linear space, and let $V \subseteq \mathcal{X} \times \mathcal{X}$ be a linear relation. For $\eta \in \mathbb{C}$ set

$$V - \eta := \big\{(x; y - \eta x) : (x; y) \in V\big\}$$

and

$$\begin{aligned}\sigma_p(V) :=& \big\{\eta \in \mathbb{C} : \ker(V - \eta) \neq \{0\}\big\}\\ =& \big\{\eta \in \mathbb{C} : \ \exists x \in \mathcal{X} \setminus \{0\} \text{ with } (x; \eta x) \in V\big\}.\end{aligned}$$

7°: Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces, and let $V \subseteq \mathcal{X} \times \mathcal{Y}$ be a linear relation. Then we say that V is *closed*, if it is closed as a subset of $\mathcal{X} \times \mathcal{Y}$ w.r.t. the product topology.

Let $\mathcal{X}$ be a normed space, and let $V \subseteq \mathcal{X} \times \mathcal{X}$ be a linear relation. We say a point $\eta \in \mathbb{C}$ is a *point of regular type* of V, if $(V - \eta)^{-1}$ is (the graph of) a bounded linear operator (with domain and range not necessarily equal to all of $\mathcal{X}$). The set of all points of regular type is denoted by $r(V)$. If $\mathcal{X}$ is a Banach space and V is closed, then $\eta \in r(V)$ if and only if $\operatorname{ran}(V - \eta)$ is closed and $\eta \notin \sigma_p(V)$.

8°: Let $\mathcal{X}$ be a linear space. A map $[\cdot, \cdot]_{\mathcal{X}}$ is called an *inner product*, if it is linear in the first argument and $[x, y]_{\mathcal{X}} = \overline{[y, x]_{\mathcal{X}}}$, $x, y \in \mathcal{X}$. We call $\langle \mathcal{X}, [\cdot, \cdot]_{\mathcal{X}}\rangle$ an *inner product space*. Note that we do not assume any definiteness properties.

Let $\langle \mathcal{X}, [\cdot, \cdot]_{\mathcal{X}}\rangle$ be an inner product space. Then we denote

$$\begin{aligned}\operatorname{ind}_+ \mathcal{X} &:= \sup \big\{ \dim \mathcal{L} : \mathcal{L} \text{ positive definite subspace of } \mathcal{X}\big\},\\ \operatorname{ind}_- \mathcal{X} &:= \sup \big\{ \dim \mathcal{L} : \mathcal{L} \text{ negative definite subspace of } \mathcal{X}\big\}.\end{aligned}$$

We write $x \perp y$ if $[x, y]_{\mathcal{X}} = 0$, set $M^\perp := \{x \in \mathcal{X} : x \perp y, y \in M\}$ for $M \subseteq \mathcal{X}$, and set $\mathcal{X}^\circ := \mathcal{X}^\perp$ and $\operatorname{ind}_0 \mathcal{X} := \dim \mathcal{X}^\circ$.

Let us point out that we do not distinguish different cardinalities of infinity. All dimensions, as well as $\operatorname{ind}_\pm$ and ind_0, are understood as elements of $\mathbb{N}_0 \cup \{\infty\}$. In particular, if we write expressions like "$d < \dim \mathcal{L}$", then this implies that d is finite.

2.2 Almost Pontryagin spaces

Almost Pontryagin spaces are a type of complete topological inner product spaces which generalise Pontryagin spaces. We recall only the definition; precise references to the literature for geometric and topological properties will be given during the presentation.

2.1 Definition. We call a triple $\langle \mathcal{A}, [\cdot, \cdot]_{\mathcal{A}}, \mathcal{O}\rangle$ an *almost Pontryagin space*, if $\mathcal{A}$ is a linear space, $[\cdot, \cdot]_{\mathcal{A}}$ is an inner product on $\mathcal{A}$, and $\mathcal{O}$ is a topology on $\mathcal{A}$, such that the following axioms hold.

(aPs1) The topology $\mathcal{O}$ is a Hilbert space topology on $\mathcal{A}$, i.e., it is induced by some inner product which turns $\mathcal{A}$ into a Hilbert space.

(aPs2) The inner product $[\cdot,\cdot]_{\mathcal{A}}$ is $\mathcal{O}$-continuous, i.e., it is continuous as a map of $\mathcal{A}\times\mathcal{A}$ into $\mathbb{C}$ where $\mathcal{A}\times\mathcal{A}$ carries the product topology $\mathcal{O}\times\mathcal{O}$ and $\mathbb{C}$ the euclidean topology.

(aPs3) There exists an $\mathcal{O}$-closed linear subspace $\mathcal{M}$ of $\mathcal{A}$ with finite codimension in $\mathcal{A}$, such that $\langle\mathcal{M},[\cdot,\cdot]_{\mathcal{A}}|_{\mathcal{M}\times\mathcal{M}}\rangle$ is a Hilbert space.

If $\langle\mathcal{A},[\cdot,\cdot]_{\mathcal{A}},\mathcal{O}\rangle$ and $\langle\mathcal{B},[\cdot,\cdot]_{\mathcal{B}},\mathcal{T}\rangle$ are two almost Pontryagin spaces, then we call a map $\psi:\mathcal{A}\to\mathcal{B}$ an *isomorphism* if ψ is a linear and isometric homeomorphism of $\mathcal{A}$ onto $\mathcal{B}$. Here isometry is understood w.r.t. the inner products $[\cdot,\cdot]_{\mathcal{A}}$ and $[\cdot,\cdot]_{\mathcal{B}}$, and bicontinuity w.r.t. $\mathcal{O}$ and $\mathcal{T}$. $\Diamond$

When no confusion is possible, we drop explicit notation of inner product and topology, and shortly speak of an almost Pontryagin space $\mathcal{A}$.

An almost Pontryagin space $\mathcal{A}$ is a Pontryagin space if and only if $\operatorname{ind}_0\mathcal{A}=0$. If $\operatorname{ind}_0\mathcal{A}=0$, the topology $\mathcal{O}$ is uniquely determined by the linear space $\mathcal{A}$ and the inner product $[\cdot,\cdot]_{\mathcal{A}}$ on $\mathcal{A}$. If $\operatorname{ind}_0\mathcal{A}>0$, this is not the case, see, e.g., [KWW05, Lemma 2.8].

Our focus in the present paper lies on a special kind of almost Pontryagin spaces, namely such whose elements are complex-valued functions and which have the property that point evaluation functionals are continuous.

2.2 Definition. Let Ω be a nonempty set. We call an almost Pontryagin space $\langle\mathcal{A},[\cdot,\cdot]_{\mathcal{A}},\mathcal{O}\rangle$ a *reproducing kernel almost Pontryagin space on* Ω, if the following axioms hold.

(RKS1) The elements of $\mathcal{A}$ are complex-valued functions on Ω, and the linear operations of $\mathcal{A}$ are given by pointwise addition and scalar multiplication.

(RKS2) For each $\eta\in\Omega$ the point evaluation functional $\chi_\eta|_{\mathcal{A}}:\mathcal{A}\to\mathbb{C}$ is continuous w.r.t. the topology $\mathcal{O}$ on $\mathcal{A}$.

$\Diamond$

A systematic treatment of reproducing kernel almost Pontryagin spaces is given in [Wor14]. Among other things, it is shown in this paper that for a reproducing kernel almost Pontryagin space an analogue of the reproducing kernel of a reproducing kernel Pontryagin space exists (which also justifies the choice of terminology).

If $\langle\mathcal{A},[\cdot,\cdot]_{\mathcal{A}}\rangle$ is an inner product space subject to (RKS1), then there exists at most one topology on $\mathcal{A}$ such that $\langle\mathcal{A},[\cdot,\cdot]_{\mathcal{A}},\mathcal{O}\rangle$ is a reproducing kernel almost Pontryagin space. This is due to the presence of the point separating family $\{\chi_\eta|_{\mathcal{A}}:\eta\in\Omega\}$ of continuous functionals, cf. [KWW05, Proposition 2.9].

2.3 Almost Pontryagin space completions

Given an inner product space $\mathcal{L}$, the question appears naturally whether $\mathcal{L}$ can be embedded into an almost Pontryagin space as a dense subspace. This leads to the notion of completions which was formally introduced in [KWW05], with some ideas going back to [JLT92]. A more systematic study was undertaken in [SW12] and supplemented in [Wor14].

Again we recall only some definitions; precise references to the literature are provided in course of the presentation.

2.3 Definition. Let $\mathcal{L}$ be an inner product space. We call a pair $\langle \iota, \mathcal{A}\rangle$ an *almost Pontryagin space completion of* $\mathcal{L}$, if $\mathcal{A}$ is an almost Pontryagin space, and ι is a linear and isometric map of $\mathcal{L}$ onto a dense subspace of $\mathcal{A}$.

Let $\langle \iota_i, \mathcal{A}_i\rangle$, $i = 1, 2$, be two almost Pontryagin space completions of $\mathcal{L}$. We say that $\langle \iota_1, \mathcal{A}_1\rangle$ and $\langle \iota_2, \mathcal{A}_2\rangle$ are *isomorphic*, if there exists an isomorphism φ of $\mathcal{A}_1$ onto $\mathcal{A}_2$ with $\varphi \circ \iota_1 = \iota_2$. $\Diamond$

It is easy to see that an inner product space $\mathcal{L}$ has an almost Pontryagin space completion if and only if $\operatorname{ind}_- \mathcal{L} < \infty$. If $\operatorname{ind}_- \mathcal{L} < \infty$, the totality of all completions of $\mathcal{L}$ corresponds to certain spaces of linear functionals on $\mathcal{L}$. The relevant notion in this context is given in Definition 2.4 below. We state this definition in the intrinsic form given in [Wor14, Lemma A.14]. The description of isomorphy classes of completions was given in [SW12, Theorem 6.8].

2.4 Definition. Let $\mathcal{L}$ be an inner product space with $\operatorname{ind}_- \mathcal{L} < \infty$. We denote by $\mathcal{L}'$ the set of all linear functionals φ on $\mathcal{L}$ with

$$\forall\, (x_n)_{n\in\mathbb{N}}, x_n \in \mathcal{L} :$$

$$\left.\begin{array}{l} \lim_{n\to\infty}[x_n, x_n]_{\mathcal{L}} = 0 \\ \lim_{n\to\infty}[x_n, y]_{\mathcal{L}} = 0, y \in \mathcal{L} \end{array}\right\} \quad\Longrightarrow\quad \lim_{n\to\infty} \varphi(x_n) = 0$$

$\Diamond$

The choice of notation $\mathcal{L}'$, resembling the standard notation for the topological dual of a topological vector space, is motivated by the fact that indeed $\mathcal{L}'$ is the topological dual of $\mathcal{L}$ w.r.t. a certain intrinsically contructed seminorm. However, for a quick approach, it is simpler to just write out the explicit conditions as done above.

It is an important fact that $\mathcal{L}'$ can be topologised in a canonical way (different than with a weak topology); for a proof of the following see [Wor14, Lemma A.13, Lemma A.17].

2.5. The topology $\mathcal{T}(\mathcal{L}')$ on $\mathcal{L}'$: Let $\mathcal{L}$ be an inner product space with $\operatorname{ind}_- \mathcal{L} < \infty$ and let $\langle \iota, \mathcal{A}\rangle$ be an almost Pontryagin space completion of $\mathcal{L}$. Then the restriction $\iota^*|_{\mathcal{A}'}$ of the algebraic dual map ι^* of ι to the topological dual $\mathcal{A}'$ of $\mathcal{A}$ maps the closed subspace

$$\mathcal{A}^{\wr} := \{[\cdot, y]_{\mathcal{A}} : y \in \mathcal{A}\}$$

of $\mathcal{A}'$ onto $\mathcal{L}'$. We denote by $\mathcal{T}(\mathcal{L}')$ the final topology on $\mathcal{L}'$ w.r.t. the map $\iota^*|_{\mathcal{A}^{\wr}}$ and the norm topology on $\mathcal{A}^{\wr}$ inherited from $\mathcal{A}'$. The topology $\mathcal{T}(\mathcal{L}')$ does not depend on the choice of $\langle \iota, \mathcal{A}\rangle$. It can be constructed using only the Pontryagin space completion of $\mathcal{L}$. $\Diamond$

Let us return to our focus: reproducing kernel almost Pontryagin spaces on a set Ω. In this context the following question occurs:

> *Given an inner product space $\mathcal{L}$ whose elements are complex-valued functions on a set Ω, does there exist a reproducing kernel almost Pontryagin space $\mathcal{A}$ which contains $\mathcal{L}$ isometrically as a dense subspace ?*

2.6 Definition. Let Ω be a nonempty set, $\mathcal{L} \subseteq \mathbb{C}^\Omega$ an inner product space, and $\mathcal{A}$ a reproducing kernel almost Pontryagin space. We say that $\mathcal{A}$ is a *reproducing kernel space completion* of $\mathcal{L}$, if $\mathcal{A}$ contains $\mathcal{L}$ isometrically as a dense subspace. $\Diamond$

Observe that, if $\mathcal{A}$ is a reproducing kernel space completion of $\mathcal{L}$, then $\langle \subseteq, \mathcal{A} \rangle$ is an almost Pontryagin space completion of $\mathcal{L}$.

The obvious necessary condition on $\mathcal{L}$ that a reproducing kernel space completion exists, namely that $\operatorname{ind}_- \mathcal{L} < \infty$, is far from sufficient. In [Wor14, Theorem 4.1] a characterisation of existence was given. Moreover, it is shown that a reproducing kernel space completion is unique (provided there exists one).

2.4 Sets of Φ-containement

The following sets play a central role in the present paper.

2.7 Definition. Let $\mathcal{L}$ be a linear space, let $V \subseteq \mathcal{L} \times \mathcal{L}$ be a linear relation, let $\Omega \subseteq \mathbb{C}$, and $\Phi : \mathcal{L} \times \Omega \to \mathbb{C}$. Then we denote

$$\begin{aligned} r_{\subseteq}(V, \Phi) &:= \big\{\eta \in \Omega : \operatorname{ran}(V - \eta) \subseteq \ker \Phi(\cdot, \eta)\big\} \\ r_{\supseteq}(V, \Phi) &:= \big\{\eta \in \Omega : \operatorname{ran}(V - \eta) \supseteq \ker \Phi(\cdot, \eta)\big\} \\ r_{=}(V, \Phi) &:= r_{\subseteq}(V, \Phi) \cap r_{\supseteq}(V, \Phi) \end{aligned}$$

$\Diamond$

Moreover, we introduce an approximative variant of $r_{\supseteq}(\cdot, \cdot)$.

2.8 Definition. Let $\langle \mathcal{L}, [\cdot, \cdot]_{\mathcal{L}} \rangle$ be an inner product space, let $V \subseteq \mathcal{L} \times \mathcal{L}$ be a linear relation, let $\Omega \subseteq \mathbb{C}$, let $\Phi : \mathcal{L} \times \Omega \to \mathbb{C}$, and $M \subseteq \Omega$. Then we denote

$$\begin{aligned} r_{\supseteq}^{\mathrm{app}}(V, \Phi; M) := \Big\{ &\eta \in \Omega : \ \forall x \in \ker \Phi(\cdot, \eta)\ \exists (x_n)_{n \in \mathbb{N}} \text{ with } x_n \in \operatorname{ran}(V - \eta), \\ &\lim_{n \to \infty} [x_n, x_n]_{\mathcal{L}} = [x, x]_{\mathcal{L}}, \quad \lim_{n \to \infty} [x_n, y]_{\mathcal{L}} = [x, y]_{\mathcal{L}},\ y \in \mathcal{L}, \\ &\lim_{n \to \infty} \Phi(x_n, \zeta) = \Phi(x, \zeta),\ \zeta \in M \Big\}. \end{aligned}$$

$\Diamond$

Clearly, the inclusion $r_{\supseteq}(V, \Phi) \subseteq r_{\supseteq}^{\mathrm{app}}(V, \Phi; M)$ holds; for given $x \in \ker \Phi(\cdot, \eta)$ simply use the constant sequence $x_n := x$.

For later reference let us explicitly state the following fact.

2.9 Remark. Assume that $\mathcal{A}$ is an almost Pontryagin space, $V \subseteq \mathcal{A} \times \mathcal{A}$ is a linear relation, $\Omega \subseteq \mathbb{C}$, and $\Phi : \mathcal{A} \times \Omega \to \mathbb{C}$. Let $M \subseteq \Omega$ be a subset with the properties

(*i*) $\Phi(\cdot, \eta) \in \mathcal{A}'$, $\eta \in M$;

(*ii*) the family $\{\Phi(\cdot, \eta) : \eta \in M\}$ is *point separating on* $\mathcal{A}^\circ$ which means that

$$\mathcal{A}^\circ \cap \bigcap_{\eta \in M} \ker \Phi(\cdot, \eta) = \{0\}.$$

Then

$$r_{\supseteq}(V,\Phi) \supseteq r_{\supseteq}^{\mathrm{app}}(V,\Phi;M) \cap \{\eta \in \mathbb{C} : \operatorname{ran}(V-\eta) \text{ closed}\}.$$

To see this, choose a sequence $(x_n)_{n\in\mathbb{N}}$ according to the definition of $r_{\supseteq}^{\mathrm{app}}(V,\Phi;M)$. By [Wor14, Proposition A.5] it holds that $\lim_{n\to\infty} x_n = x$ in the norm of $\mathcal{A}$, whence $x \in \operatorname{ran}(V-\eta)$. $\Diamond$

2.5 Symmetric relations in almost Pontryagin spaces

2.10 Definition. Let $\langle \mathcal{L}, [\cdot,\cdot]_{\mathcal{L}}\rangle$ be an inner product space, and let $S \subseteq \mathcal{L}\times\mathcal{L}$ be a linear relation in $\mathcal{L}$. We say that S is *symmetric*, if

$$[y_1, x_2]_{\mathcal{L}} = [x_1, y_2]_{\mathcal{L}}, \quad (x_1; y_1), (x_2; y_2) \in S.$$

$\Diamond$

For a linear map (considered as a linear relation via its graph) this condition reduces to the usual symmetry condition $[Sx, y]_{\mathcal{L}} = [x, Sy]_{\mathcal{L}}$, $x, y \in \operatorname{dom} S$.

The following basic result on symmetric linear relations in a Pontryagin space is shown in [DS87a].

2.11 Theorem. *Let $\mathcal{P}$ be a Pontryagin space and let S be a closed symmetric relation in $\mathcal{P}$. Then the following statements hold.*

(*i*) $\operatorname{ran}(S-\eta)$ *is closed for* $\eta \in \mathbb{C}\setminus\mathbb{R}$.

(*ii*) *The codimension of* $\operatorname{ran}(S-\eta)$ *is constant on connected components of* $r(S)$.

(*iii*) *There exist numbers* $\alpha, \beta_+, \beta_- \in \mathbb{N}_0$ *with* $\alpha \leq \operatorname{ind}_- \mathcal{P}$, *and an exceptional set* $\Upsilon \subseteq \mathbb{C}\setminus\mathbb{R}$, *such that*

$$\max\big\{|\Upsilon \cap \mathbb{C}^+|, |\Upsilon \cap \mathbb{C}^-|\big\} \leq \operatorname{ind}_- \mathcal{P} - \alpha, \tag{2.3}$$

$$\operatorname{codim}_{\mathcal{P}} \operatorname{ran}(S-\eta) = \beta_\pm,\ \dim\ker(S-\eta) = \alpha, \quad \eta \in \mathbb{C}^\pm \setminus \Upsilon, \tag{2.4}$$

$$\operatorname{codim}_{\mathcal{P}} \operatorname{ran}(S-\eta) - \beta_\pm = \dim\ker(S-\eta) - \alpha > 0, \quad \eta \in \mathbb{C}^\pm \cap \Upsilon, \tag{2.5}$$

$$\alpha \leq \dim \operatorname{mul} S. \tag{2.6}$$

(*iv*) *The numbers* $\beta_\pm$ *can be evaluated as*

$$\beta_\pm = \min\big\{\operatorname{codim}_{\mathcal{A}} \operatorname{ran}(S-\eta) : \eta \in L\big\}, \qquad L \subseteq \mathbb{C}^\pm,\ |L| > \operatorname{ind}_- \mathcal{P}. \tag{2.7}$$

Item (*i*) is [DS87a, Proposition 4.3, (*ii*)], item (*ii*) is [DS87a, Corollary to Theorem 2.4], item (*iii*) is [DS87a, Proposition 4.4 and Theorem 2.4], and item (*iv*) follows immediately by combining (2.4) and (2.5).

Using [KWW05, Proposition 3.2], Theorem 2.11 transfers immediately to the almost Pontryagin space situation.

2.12 Corollary. *Let $\mathcal{A}$ be an almost Pontryagin space and let S be a closed symmetric relation in $\mathcal{A}$. Then the assertions of Theorem* 2.11 *remain true when* $\operatorname{ind}_- \mathcal{P}$ *is everywhere replaced by* $\operatorname{ind}_- \mathcal{A} + \operatorname{ind}_0 \mathcal{A}$.

Proof. Due to [KWW05, Proposition 3.2], we can choose a Pontryagin space $\mathcal{P}$ which contains $\mathcal{A}$ as a closed subspace with codimension $\operatorname{ind}_0 \mathcal{A}$. It holds that $\operatorname{ind}_- \mathcal{P} = \operatorname{ind}_- \mathcal{A} + \operatorname{ind}_0 \mathcal{A}$. ❑

We refer to β_+ and β_- as the *upper-* and *lower defect indices* of S in $\mathcal{A}$, and to the pair (β_+, β_-) as the *defect index* of S in $\mathcal{A}$.

The concept of a completely non-selfadjoint symmetric relation in an almost Pontryagin space is defined in the usual way (only taking care of possible unsymmetrically located spectral points).

2.13 Definition. Let $\mathcal{A}$ be an almost Pontryagin space and let S be a closed symmetric relation in $\mathcal{A}$ with $r(S) \neq \emptyset$. We say that S is *completely non-selfadjoint*, if

$$\bigcap_{\eta \in r(S) \cap r(S)^{\#}} \operatorname{ran}(S - \eta) = \{0\}.$$

◊

The usual consequence of analyticity holds true (an explicit proof can be found in [SW]):

2.14 Lemma. *Let $\mathcal{A}$ be an almost Pontryagin space and let S be a closed symmetric relation in $\mathcal{A}$ with $r(S) \neq \emptyset$. Let Ω be an open and nonempty subset of $r(S) \cap r(S)^{\#} \cap \mathbb{C}^+$, and set*

$$U := \begin{cases} r(S) \cap r(S)^{\#} & , \quad r(S) \cap \mathbb{R} \neq \emptyset \\ r(S) \cap r(S)^{\#} \cap \mathbb{C}^+ , & r(S) \cap \mathbb{R} = \emptyset \end{cases}$$

Then

$$\bigcap_{\eta \in \Omega} \operatorname{ran}(S - \eta) = \bigcap_{\eta \in U} \operatorname{ran}(S - \eta).$$

The same holds with $\mathbb{C}^+$ replaced by $\mathbb{C}^-$.

2.6 Krein's Theorem on preservation of analyticity

A central result in the theory of entire operators and directing functionals is a theorem on preservation of analyticity which goes back to M.G.Krein. The below Theorem 2.15 is a variant of this result formulated in a sufficiently general way to serve our present needs. The crux of the proof can be extracted from [Kre49], see also the argument leading to [GG97, Ch.2,Theorem 3.2] and the argument indicated in [KL78] leading to Items (*i*)–(*iii*) on p.430 of this reference. The reader who is not familiar with this type of arguments can find a fully elaborated proof in Appendix A of this paper.

2.15 Theorem ([Kre49], [KL78]). *Let $\mathcal{K}$ be a Krein space and $A \subseteq \mathcal{K} \times \mathcal{K}$ a definitisable selfadjoint linear relation. Let $a_0 \in \mathcal{K}$, $\zeta_0 \in \rho(A)$, and set*

$$a(\zeta) := \big(I + (\zeta - \zeta_0)(A - \zeta)^{-1}\big) a_0, \quad \zeta \in \rho(A).$$

Moreover, assume that $y \in \mathcal{K}$ with the property that $[a(\cdot), y]_{\mathcal{K}}$ does not vanish identically on any component of $\rho(A)$. Then the following statements hold.

(*i*) *Let $x \in \mathcal{K}$. Consider the set of all open subsets $O \subseteq \mathbb{C}$ with the property that the function*

$$\begin{cases} \{\zeta \in \rho(A) : [a(\zeta), y]_{\mathcal{K}} \neq 0\} & \to \quad \mathbb{C} \\ \qquad\qquad\qquad\qquad\qquad \zeta & \mapsto \quad \frac{[a(\zeta),x]_{\mathcal{K}}}{[a(\zeta),y]_{\mathcal{K}}} \end{cases}, \tag{2.8}$$

has a meromorphic continuation to O. This set has a largest element.

We denote this largest element by $\Omega_{x,y}$, and the meromorphic continuation of the function (2.8) *to $\Omega_{x,y}$ by $\Theta_{x,y}$.*

(*ii*) *Let $\Omega \subseteq \mathbb{C}$ be open and let $\mathfrak{d} : \Omega \to \mathbb{Z}$ be a function with discrete support. Choose a function $W \in \mathbb{H}(\Omega, \mathbb{C}_\infty)$ with $\mathfrak{d}_W = -\mathfrak{d}$. Then the family ($\|\cdot\|_{\mathcal{K}}$ denotes some norm induced by a fundamental decomposition of $\mathcal{K}$)*

$$\mathcal{F} := \big\{W\Theta_{x,y} : O_{x,y} \supseteq \Omega, \mathfrak{d}_{\Theta_{x,y}} \geq \mathfrak{d}, \|x\|_{\mathcal{K}} \leq 1\big\}$$

is normal in $\mathbb{H}(\Omega)$.

(*iii*) *Let $\Omega \subseteq \mathbb{C}$ be open and let $\mathfrak{d} : \Omega \to \mathbb{Z}$ be a function with discrete support. Then the linear subspace*

$$\mathcal{M} := \big\{x \in \mathcal{K} : O_{x,y} \supseteq \Omega, \mathfrak{d}_{\Theta_{x,y}}|_\Omega \geq \mathfrak{d}\big\} \subseteq \mathcal{K}$$

is closed.

We will also make use of the following lemma which depends on analyticity. This fact is folklore, however, we cannot appoint an explicit reference. For completeness, a proof is provided in Appendix A.

2.16 Lemma. *Let $\mathcal{X}$ be a complete metrisable topological vector space, let $\Omega \subseteq \mathbb{C}$ be open, nonempty, and connected, and let $\Phi : \mathcal{X} \times \Omega \to \mathbb{C}$. Assume that*

(*i*) *for each $x \in \mathcal{X}$, the map $\Phi(x, \cdot)$ is analytic;*

(*ii*) *the set $L := \{\eta \in \Omega : \Phi(\cdot, \eta) \in \mathcal{X}'\}$ has an accumulation point in Ω.*

Then $\Phi(\cdot, \eta)$ is linear for all $\eta \in \Omega$, and the map

$$\Phi_{\mathcal{X}} : \begin{cases} \mathcal{X} & \to & \mathbb{H}(\Omega) \\ x & \mapsto & \Phi(x, \cdot) \end{cases}$$

is continuous. In particular, we have $\frac{\partial^l}{\partial \eta^l}\Phi(\cdot, \eta) \in \mathcal{X}'$, $\eta \in \Omega$. For every bounded subset M of $\mathcal{X}$, the image $\{\Phi(x, \cdot) : x \in M\}$ of M under $\Phi_{\mathcal{X}}$ is a normal family.

2.7 The multiplication operator

In this subsection we introduce the type of spaces of analytic functions which we are going to deal with, and study the operator of multiplication by the independent variable in such spaces.

2.17 Definition. Let $\Omega \subseteq \mathbb{C}$ be open and nonempty, and let $\mathcal{B}$ be a reproducing kernel almost Pontryagin space of analytic functions on Ω. Then we denote by $S(\mathcal{B})$ the *multiplication operator in $\mathcal{B}$*. Written as a linear relation this is

$$S(\mathcal{B}) := \big\{(f(\zeta); \zeta f(\zeta)) : f(\zeta), \zeta f(\zeta) \in \mathcal{B}\big\}.$$

$\Diamond$

The operator $S(\mathcal{B})$ is closed and (remember the notation (2.2))

$$\sigma_p(S(\mathcal{B})) = \emptyset, \quad \operatorname{mul} S(\mathcal{B}) = \{0\}, \tag{2.9}$$

$$\operatorname{ran}(S(\mathcal{B}) - \eta) \subseteq \ker \chi_\eta^{(\mathfrak{d}_{\mathcal{B}}(\eta))}, \ \eta \in \Omega, \mathfrak{d}_{\mathcal{B}}(\eta) < \infty. \tag{2.10}$$

Moreover, for each subset $L \subseteq \Omega$ which has accumulation points in each connected component of Ω, it holds that $\bigcap_{\eta \in L} \operatorname{ran}(S(\mathcal{B}) - \eta) = \{0\}$. The above properties give rise to the following fact.

2.18 Remark. Assume that $S(\mathcal{B})$ is symmetric. Then $\mathbb{C} \setminus \mathbb{R} \subseteq r(S(\mathcal{B}))$ and $S(\mathcal{B})$ is completely non-selfadjoint. $\Diamond$

The following class of spaces is an analogue of de Branges spaces for arbitrary open subsets Ω of $\mathbb{C}$.

2.19 Definition. Let Ω be an open and nonempty subset of $\mathbb{C}$, and let $\mathcal{B}$ be a reproducing kernel almost Pontryagin space on Ω. We call $\mathcal{B}$ an Ω-*space*, if it satisfies the following axioms.

(Ω1) The elements of $\mathcal{B}$ are analytic functions on Ω.

(Ω2) $\operatorname{supp} \mathfrak{d}_{\mathcal{B}}$ is a discrete subset of Ω (in particular, thus, $\mathcal{B} \neq \{0\}$ and $\mathfrak{d}_{\mathcal{B}}(\eta) < \infty$, $\eta \in \Omega$).

(Ω3) $\forall \xi \in \Omega \ \forall f \in \mathcal{B}, f^{(\mathfrak{d}_{\mathcal{B}}(\xi))}(\xi) = 0 : \quad \dfrac{f(\zeta)}{\zeta - \xi} \in \mathcal{B}.$

(Ω4) $\forall \xi \in \Omega \ \forall f, g \in \mathcal{B}, f^{(\mathfrak{d}_{\mathcal{B}}(\xi))}(\xi) = g^{(\mathfrak{d}_{\mathcal{B}}(\xi))}(\xi) = 0 :$

$$\Big[\frac{\zeta - \overline{\xi}}{\zeta - \xi} f(\zeta), \frac{\zeta - \overline{\xi}}{\zeta - \xi} g(\zeta)\Big]_{\mathcal{B}} = \big[f(\zeta), g(\zeta)\big]_{\mathcal{B}}.$$

$\Diamond$

An Ω-space has several operator theoretic properties which follow immediately from the definition.

2.20 Remark. Let $\mathcal{B}$ be an Ω-space. Then $S(\mathcal{B})$ is symmetric, completely non-selfadjoint, and

$$\operatorname{ran}(S(\mathcal{B}) - \eta) = \ker \chi_\eta^{(\mathfrak{d}_{\mathcal{B}}(\eta))}, \ \eta \in \Omega, \qquad \Omega \cup (\mathbb{C} \setminus \mathbb{R}) \subseteq r(S(\mathcal{B})). \tag{2.11}$$

If $\Omega \cap \mathbb{C}^+ \neq \emptyset$, then the upper defect index β_+ of $S(\mathcal{B})$ equals 1. The same holds with $\mathbb{C}^+$ and β_+ replaced by $\mathbb{C}_-$ and β_-.

Using Lemma 2.14 and analyticity, we obtain the following fact: For each $f \in \mathcal{B}$, the set $\operatorname{supp} \mathfrak{d}_f \cap \mathbb{C}^+$ is either a discrete subset of Ω or all of $\Omega \cap \mathbb{C}^+$. The same holds with $\mathbb{C}^+$ replaced by $\mathbb{C}^-$. As a consequence, if $\Omega \cap \mathbb{R} \neq \emptyset$, then $\operatorname{supp} \mathfrak{d}_f$ is a discrete subset of Ω whenever $f \in \mathcal{B} \setminus \{0\}$. $\Diamond$

These operator theoretic properties are characteristic for Ω-spaces.

2.21 Proposition. *Let $\Omega \subseteq \mathbb{C}$ be open and nonempty, and let $\mathcal{B}$ be a reproducing kernel almost Pontryagin space on Ω. Assume that $\mathcal{B}$ satisfies* (Ω1), (Ω2), *and*

(*i*) *$S(\mathcal{B})$ is symmetric;*

(*ii*) *if* $\Omega \cap \mathbb{C}^+ \neq \emptyset$ *then* $\beta_+ = 1$, *and if* $\Omega \cap \mathbb{C}^- \neq \emptyset$ *then* $\beta_- = 1$.

Then $\mathcal{B}$ *is an* Ω*-space.*

Proof. We need to show that the condition in (Ω3), i.e., invariance under division, holds for all $\xi \in \Omega$. Since $\operatorname{supp} \mathfrak{d}_{\mathcal{B}}$ is discrete, for no $\eta \in \Omega$ the functional $\chi_\eta^{(\mathfrak{d}_{\mathcal{B}}(\eta))}$ vanishes identically. If $\xi \in \Omega \cap r(S(\mathcal{B}))$, then by (*ii*) we have $\operatorname{codim}_{\mathcal{B}} \operatorname{ran}(S(\mathcal{B}) - \eta) = 1$. Hence, in the inclusion (2.10) equality must hold. This means that the condition in (Ω3) holds for ξ. Since $\mathbb{C} \setminus \mathbb{R} \subseteq S(\mathcal{B})$, the condition in ($\Omega$3) holds for all $\xi \in \Omega \setminus \mathbb{R}$.

Assume that $\Omega \cap \mathbb{R} \neq \emptyset$. Then Ω certainly intersects both half-planes $\mathbb{C}^+$ and $\mathbb{C}^-$, whence by (*ii*) the defect index of $S(\mathcal{B})$ is $(1, 1)$. We are going to show that $\Omega \cap \mathbb{R} \subseteq r(S(\mathcal{B}))$. This is achieved by reducing to the Hilbert space case.

Fix $\xi \in \Omega \cap \mathbb{R}$, and choose a nonempty interval $(a, b) \subseteq \Omega$ with $\xi \in (a, b)$. Let $f \in \mathcal{B}$, and assume that f vanishes on (a, b). Then it vanishes identically on the component Ω_0 of Ω which contains (a, b). Using Lemma 2.14, it follows that

$$f \in \bigcap_{\eta \in (\Omega_0 \cap \Omega_0^{\#}) \setminus \mathbb{R}} \operatorname{ran}\big(S(\mathcal{B}) - \eta\big) = \bigcap_{\eta \in r(S(\mathcal{B})) \cap r(S(\mathcal{B}))} \operatorname{ran}\big(S(\mathcal{B}) - \eta\big) = \{0\}.$$

In other words, we may say that the family $\{\chi_\eta : \eta \in (a, b)\}$ is a point separating subset of $\mathcal{B}'$. From [Wor14, Proposition A.9] we obtain $\gamma \in \mathbb{R}$ and $\eta_1, \dots, \eta_n \in (a, b)$, such that the inner product

$$(f, g)_{\mathcal{B}} := [f, g]_{\mathcal{B}} + \gamma \sum_{l=1}^{n} f(\eta_l)\overline{g(\eta_l)}, \quad f, g \in \mathcal{B},$$

turns $\mathcal{B}$ into a Hilbert space and induces the topology of $\mathcal{B}$.

Since the points $\eta_1, \dots, \eta_n$ are real, it is straightforward to check that $S(\mathcal{B})$ is also symmetric w.r.t. $(\cdot, \cdot)_{\mathcal{B}}$. All algebraic and topological properties of $S(\mathcal{B})$ are independent of the inner product, and hence remain valid. In particular, $S(\mathcal{B})$ is completely non-selfadjoint and has defect index $(1, 1)$ when considered as an operator in the Hilbert space $\langle \mathcal{B}, (\cdot, \cdot)_{\mathcal{B}} \rangle$. Moreover, $r(S(\mathcal{B}))$ is the same whether considered in the almost Pontryagin space $\mathcal{B}$ or in the Hilbert space $\langle \mathcal{B}, (\cdot, \cdot)_{\mathcal{B}} \rangle$.

Choose $u \in \mathcal{B} \setminus \{0\}$. Then $M := \{\eta \in \Omega_0 : u(\eta) = 0\}$ is a discrete subset of Ω_0. We have

$$f - \frac{f(\eta)}{u(\eta)} u \in \ker \chi_\eta = \operatorname{ran}(S(\mathcal{B}) - \eta), \quad f \in \mathcal{B}, \eta \in \Omega_0 \setminus (\mathbb{R} \cup M).$$

Hence, for each $\eta \in \Omega_0 \setminus (\mathbb{R} \cup M)$, the projection of f onto $\operatorname{span}\{u\}$ according to the decomposition $\mathcal{B} = \operatorname{ran}(S(\mathcal{B}) - \eta) \dot{+} \operatorname{span}\{u\}$ is equal to $\frac{f(\eta)}{u(\eta)} u$. Since M is discrete it holds that $\mathfrak{d}_u(\eta) < \infty$, $\eta \in \Omega_0$, and we see that the function $p_f := \frac{f(\eta)}{u(\eta)}$ has an extension $P_f \in \mathbb{H}(\Omega, \mathbb{C}_\infty)$ with $\mathfrak{d}_{P_f} \geq -\mathfrak{d}_u$. Applying [GG97, Ch.2,Theorem 3.3] yields $\xi \in r(S(\mathcal{B}))$.

Now it follows that the condition in (Ω3) holds also for all $\xi \in \Omega \cap \mathbb{R}$. Finally, the condition in (Ω4) is void if $\xi \in \Omega \cap \mathbb{R}$, and for $\xi \in \Omega \setminus \mathbb{R}$ it is just the isometry property of the Caley-transform of $S(\mathcal{B})$. ❑

3 The core arguments

In this section the core arguments needed to establish Theorem 4.3 are proved. We present the matters in the form of eight lemmata. In order to point out "what is needed where", we tried to give minimal assumptions in each of them, rather than proving "just" what is needed somewhere afterwards.

We start with explaining the setting which will mainly be present.

3.1. Frequently considered setup: Data $\mathcal{L}, S, \langle\iota, \mathcal{A}\rangle, \Omega, \Phi, M$ has the meaning

(1) $\mathcal{L}$ is an inner product space and S is a symmetric linear relation in $\mathcal{L}$;

(2) $\langle\iota, \mathcal{A}\rangle$ is an almost Pontryagin space completion of $\mathcal{L}$;

(3) Ω is a subset of $\mathbb{C}$ and Φ is a map $\Phi : \mathcal{L} \times \Omega \to \mathbb{C}$ with the property that for each $\eta \in \Omega$ the map $\Phi(\cdot, \eta) : \mathcal{L} \to \mathbb{C}$ is linear;

(4) M is a subset of Ω with the property that

$$\iota^*(\mathcal{A}') = \mathcal{L}' + \operatorname{span}\big\{\Phi(\cdot, \eta) : \eta \in M\big\}. \tag{3.1}$$

$\Diamond$

Whenever we are given data $\mathcal{L}, S, \langle\iota, \mathcal{A}\rangle$ according to $3.1_{(1,2)}$, we set

$$T := \operatorname{Clos}_{\mathcal{A}}\big[(\iota \times \iota)(S)\big]. \tag{3.2}$$

Since ι is isometric, T is a closed symmetric relation in $\mathcal{A}$.

Whenever we are given data according to $3.1_{(1-3)}$, we set

$$\tilde{\Phi}_\eta := (\iota^*|_{\mathcal{A}'})^{-1}\Phi(\cdot, \eta) \text{ whenever } \Phi(\cdot, \eta) \in \iota^*(\mathcal{A}').$$

The above described situation arises in particular when we already start with an almost Pontryagin space.

3.2. Inclusion of the complete case: Assume we are given data $\mathcal{A}, T, \Omega, \Psi, M$ such that

($1'$) $\mathcal{A}$ is an almost Pontryagin space, T is a closed symmetric relation in $\mathcal{A}$;

($2'$) Ω is a subset of $\mathbb{C}$ and $\Psi : \mathcal{A} \times \Omega \to \mathbb{C}$ is a map with the property that for each $\eta \in \Omega$ the map $\Psi(\cdot, \eta) : \mathcal{A} \to \mathbb{C}$ is linear;

($3'$) M is a subset of Ω, such that $\Psi(\cdot, \eta) \in \mathcal{A}'$, $\eta \in M$, and $\{\Psi(\cdot, \eta) : \eta \in M\}$ is point separating on $\mathcal{A}^\circ$.

Set

$$\mathcal{L} := \mathcal{A}, \quad S := T, \quad \iota := \mathrm{id}_{\mathcal{A}}, \quad \Phi := \Psi.$$

Then $\mathcal{L}, S, \langle\iota, \mathcal{A}\rangle, \Omega, \Phi, M$ satisfies $3.1_{(1-4)}$. Moreover, the relation (3.2) equals the given relation T, and $\tilde{\Phi}_\eta = \Psi(\cdot, \eta)$ whenever $\Psi(\cdot, \eta) \in \mathcal{A}'$.

If we are given $\mathcal{A}, T, \Omega, \Psi$ with ($1'$), ($2'$), then the above defined data $\mathcal{L}, S, \langle\iota, \mathcal{A}\rangle, \Omega, \Phi$ satisfies $3.1_{(1-3)}$. $\Diamond$

Now let us give a brief outline of what happens in the lemmata of this section.

`(1) Lemma` 3.3: Some basic relations among $r_\subseteq(S, \Phi)$, $r_\supseteq^{\mathrm{app}}(S, \Phi; M)$, $\operatorname{ran}(T-\eta)$, and $\ker \tilde{\Phi}_\eta$.

`(2) Lemma` 3.4`:` Data as in 3.1. Conclusion about the defect indices of T, in particular a way to conclude that T has defect indices equal to 1; properties of points in $r_{\supseteq}^{\mathrm{app}}(S,\Phi;M)$.

`(3) Lemma` 3.5`:` Data as in 3.1. Conclusion towards complete non-selfadjointness of T; conclusion about the size of $r(T)$.

`(4) Lemma` 3.6`:` Data as in 3.2. Construction of a reproducing kernel almost Pontryagin space $\mathcal{B}$; comparison of T and $S(\mathcal{B})$; properties of $r_{\subseteq}(S,\Phi)$.

`(5) Lemma` 3.7 `(cont. of Lemma 3.6):` Data as in 3.2. More properties of $S(\mathcal{B})$; identification with T; properties of $r_{\supseteq}(S,\Phi)$.

`(6) Lemma` 3.8`:` Data as in 3.1. Construction of a lifting of Φ to the completion $\mathcal{A}$. The central assumption is that T has defect index $(1,1)$. This is the point where the connection with Krein's representation by analytic functions is made and where preservation of analyticity is applied.

`(7) Lemma` 3.9 `(cont. of Lemma 3.8):` Data as in 3.1. Conclusion towards complete non-selfadjointness of T; properties of $r_{\subseteq}(T,\Psi)$, $r_{=}(T,\Psi)$.

`(8) Lemma` 3.10 `(cont. of Lemmata 3.8 and 3.9):` Data as in 3.1. Showing that $\mathcal{B}$ is an Ω-space.

✰ ✰ ✰ ✰ ✰ ✰ ✰ ✰ ✰

3.3 Lemma. *Let data $\mathcal{L},S,\langle\iota,\mathcal{A}\rangle,\Omega,\Phi$ be given according to* $3.1_{(1-3)}$. *Then it holds that (remember the definitions of Φ-containement sets from Definition* 2.7 *and Definition* 2.8*)*

$$\eta\in r_{\subseteq}(S,\Phi),\Phi(\cdot,\eta)\in\iota^*(\mathcal{A}')\quad\Longrightarrow\quad\operatorname{ran}(T-\eta)\subseteq\ker\tilde{\Phi}_\eta \tag{3.3}$$

If in addition a set M is given according to $3.1_{(4)}$, *then it holds that*

$$\eta\in r_{\supseteq}^{\mathrm{app}}(S,\Phi;M),\operatorname{ran}(T-\eta)\ \textit{closed}\quad\Longrightarrow\quad\operatorname{codim}_{\mathcal{A}}\operatorname{ran}(T-\eta)\leq 1 \tag{3.4}$$

$$\begin{aligned}&\eta\in r_{\supseteq}^{\mathrm{app}}(S,\Phi;M),\operatorname{ran}(T-\eta)\ \textit{closed},\operatorname{codim}_{\mathcal{A}}\operatorname{ran}(T-\eta)=1\\&\qquad\Longrightarrow\quad\eta\in r_{\subseteq}(S,\Phi),\Phi(\cdot,\eta)\in\iota^*(\mathcal{A}')\setminus\{0\},\operatorname{ran}(T-\eta)=\ker\tilde{\Phi}_\eta\end{aligned} \tag{3.5}$$

Proof.
1° *The implication* (3.3)*:* We have

$$\operatorname{ran}(T-\eta)\subseteq\operatorname{Clos}_{\mathcal{A}}\big[\iota\big(\operatorname{ran}(S-\eta)\big)\big]\subseteq\operatorname{Clos}_{\mathcal{A}}\big[\iota\big(\ker\Phi(\cdot,\eta)\big)\big]\subseteq\ker\tilde{\Phi}_\eta.$$

Here the last but one inclusion holds since $\eta\in r_{\subseteq}(S,\Phi)$, and the last inclusion because $\ker\tilde{\Phi}_\eta$ is closed and $\tilde{\Phi}_\eta\circ\iota=\Phi(\cdot,\eta)$.

2° *The implication* (3.4)*:* Choose $y_0\in\mathcal{L}$ with $\ker\Phi(\cdot,\eta)+\operatorname{span}\{y_0\}=\mathcal{L}$. Since $\iota(\mathcal{L})$ is dense in $\mathcal{A}$, it follows that

$$\operatorname{Clos}_{\mathcal{A}}\big[\iota\big(\ker\Phi(\cdot,\eta)\big)\big]+\operatorname{span}\{\iota y_0\}=\mathcal{A},$$

and hence that $\operatorname{codim}_{\mathcal{A}} \operatorname{Clos}_{\mathcal{A}}[\iota(\ker \Phi(\cdot,\eta))] \leq 1$.

Let $x \in \ker \Phi(\cdot,\eta)$ and choose a sequence $(x_n)_{n\in\mathbb{N}}$ according to the definition of $r_{\supseteq}^{\mathrm{app}}(S,\Phi;M)$. By (3.1) the family of functionals $\mathcal{F} := \{\tilde{\Phi}_\eta : \eta \in M\}$ is point separating on $\mathcal{A}^\circ$, cf. [Wor14, Remark A.4, Lemma A.13]. We apply [Wor14, Proposition A.5] with the dense set $\iota(\mathcal{L})$ and the family $\mathcal{F}$. This yields that $\lim_{n\to\infty} \iota x_n = \iota x$ in the norm of $\mathcal{A}$. Since $\iota x_n \in \operatorname{ran}(T-\eta)$ and $\operatorname{ran}(T-\eta)$ is closed, it follows that $\iota x \in \operatorname{ran}(T-\eta)$. We conclude that $\operatorname{Clos}_{\mathcal{A}}[\iota(\ker \Phi(\cdot,\eta))] \subseteq \operatorname{ran}(T-\eta)$, and hence that $\operatorname{codim}_{\mathcal{A}} \operatorname{ran}(T-\eta) \leq 1$.

3° *The implication* (3.5)*:* Since $\operatorname{codim}_{\mathcal{A}} \operatorname{ran}(T-\eta) = 1$, we can choose $y \in \mathcal{A}$ with

$$\mathcal{A} = \operatorname{ran}(T-\eta) \dot{+} \operatorname{span}\{y\}.$$

Denote by $\varphi : \mathcal{A} \to \mathbb{C}$ the linear functional defined by

$$x - \varphi(x)y \in \operatorname{ran}(T-\eta), \quad x \in \mathcal{A}.$$

Clearly, φ does not vanish identically. Since $\operatorname{ran}(T-\eta)$ is closed, φ is continuous. If $x \in \operatorname{ran}(S-\eta)$, then $\iota x \in \operatorname{ran}(T-\eta)$, and hence $\varphi(\iota x) = 0$. This shows that

$$\iota\big[\operatorname{ran}(S-\eta)\big] \subseteq \ker \varphi. \tag{3.6}$$

Let $x \in \ker \Phi(\cdot,\eta)$ and choose $(x_n)_{n\in\mathbb{N}}$ according to the definition of $r_{\supseteq}^{\mathrm{app}}(S,\Phi;M)$. Again, we have $\lim_{n\to\infty} \iota x_n = \iota x$ in $\mathcal{A}$. Moreover, $\iota x_n \in \ker \varphi$ by (3.6). Since $\ker \varphi$ is closed, it follows that $\iota x \in \ker \varphi$. We conclude that

$$\ker \Phi(\cdot,\eta) \subseteq \ker(\iota^* \varphi). \tag{3.7}$$

Since $\iota^*|_{\mathcal{A}'}$ is injective, we have $\iota^*\varphi \neq 0$. This implies that in (3.7) equality holds, and hence that $\Phi(\cdot,\eta)$ is a nonzero scalar multiple of $\iota^*\varphi$. In particular, $\Phi(\cdot,\eta) \in \iota^*(\mathcal{A}') \setminus \{0\}$. Now (3.6) says that $\eta \in r_{\subseteq}(S,\Phi)$, and the already proved implication (3.3) yields $\operatorname{ran}(T-\eta) \subseteq \ker \tilde{\Phi}_\eta$. Since $\operatorname{codim}_{\mathcal{A}} \operatorname{ran}(T-\eta) = 1$ and $\tilde{\Phi}_\eta \neq 0$, equality must hold. ❑

3.4 Lemma. *Let data $\mathcal{L}, S, \langle \iota, \mathcal{A}\rangle, \Omega, \Phi, M$ be given according to $3.1_{(1-4)}$. Assume that*

$<1>$ *the set $M_0 := M \cap r_{\subseteq}(S,\Phi) \cap \{\eta \in \Omega : \Phi(\cdot,\eta) \neq 0\}$ satisfies*

$$\dim\Big(\big[\mathcal{L}' + \operatorname{span}\{\Phi(\cdot,\eta) : \eta \in M\}\big]\Big/\mathcal{L}'\Big) + \operatorname{ind}_- \mathcal{L} < |M_0 \cap \mathbb{C}^+|.$$

Then the upper defect index β_+ of T is larger or equal to 1. *We have*

$$\begin{aligned} r_{\supseteq}^{\mathrm{app}}(S,\Phi;M) \cap \mathbb{C}^+ \subseteq r_{\subseteq}(S,\Phi) \cap \big\{\eta \in \mathbb{C} : \Phi(\cdot,\eta) \neq 0\big\} \cap \\ \cap \big\{\eta \in \mathbb{C} : \Phi(\cdot,\eta) \in \iota^*(\mathcal{A}'), \operatorname{ran}(T-\eta) = \ker \tilde{\Phi}_\eta\big\}. \end{aligned} \tag{3.8}$$

In particular $\Upsilon \cap (r_{\supseteq}^{\mathrm{app}}(S,\Phi;M) \cap \mathbb{C}^+) = \emptyset$ where Υ is the exceptional set for T as in Theorem 2.11*/Corollary* 2.12*, and $r_{\supseteq}^{\mathrm{app}}(S,\Phi;M) \cap \mathbb{C}^+ \neq \emptyset$ implies $\beta_+ = 1$.*

The same statement holds with $\mathbb{C}^+, \beta_+$ replaced by $\mathbb{C}^-, \beta_-$.

Proof. Let $\alpha, \beta_\pm, \Upsilon$ be the data obtained by applying Corollary 2.12 with T in $\mathcal{A}$. If $\eta \in M_0$, then $\Phi(\cdot,\eta) \in \iota^*(\mathcal{A}')$, $\Phi(\cdot,\eta) \neq 0$, and $\eta \in r_\subseteq(S,\Phi)$. The implication (3.3) yields $\operatorname{codim}_{\mathcal{A}} \operatorname{ran}(T-\eta) \geq 1$. Remembering that

$$\operatorname{ind}_0 \mathcal{A} = \dim \big(\iota^*(\mathcal{A}')/\mathcal{L}'\big),$$

our assumption <1> ensures that M_0 contains sufficiently many points to evaluate β_+ by means of (2.7). Doing so yields $\beta_+ \geq 1$.

Let $\eta \in r_\supseteq^{\mathrm{app}}(S,\Phi;M) \cap \mathbb{C}^+$. Then $\operatorname{ran}(T-\eta)$ is closed and (3.4) implies that $\operatorname{codim}_{\mathcal{A}} \operatorname{ran}(T-\eta) \leq 1$. Since $\beta_+ \geq 1$, this codimension (and hence also β_+ itself) must be equal to 1 and the point η cannot belong to the exceptional set Υ. Applying (3.5) yields $\eta \in r_\subseteq(S,\Phi)$, $\Phi(\cdot,\eta) \in \iota^*(\mathcal{A}') \setminus \{0\}$, $\operatorname{ran}(T-\eta) = \ker \tilde{\Phi}_\eta$. ❑

3.5 Lemma. *Let data $\mathcal{L}, S, \langle\iota,\mathcal{A}\rangle, \Omega, \Phi, M$ be given according to* 3.1$_{(1-4)}$. *If*

<2> *a set $M' \subseteq r_\subseteq(S,\Phi) \cap \{\eta \in \Omega : \Phi(\cdot,\eta) \in \iota^*(\mathcal{A}')\}$ is given such that $\mathcal{L}' \cap \operatorname{span}\{\Phi(\cdot,\eta) : \eta \in M'\}$ is dense in $\mathcal{L}'$ (w.r.t. $\mathcal{T}(\mathcal{L}')$, cf.* 2.5*),*

then

$$\bigcap_{\eta \in M'} \operatorname{ran}(T-\eta) \subseteq \bigcap_{\eta \in M'} \ker \tilde{\Phi}_\eta \subseteq \mathcal{A}^\circ.$$

If, in addition to <2>*,*

<3> *a set $M'' \subseteq r_\subseteq(S,\Phi) \cap \{\eta \in \Omega : \Phi(\cdot,\eta) \in \iota^*(\mathcal{A}')\}$ is given such that $\mathcal{L}' + \operatorname{span}\{\Phi(\cdot,\eta) : \eta \in M''\} = \iota^*(\mathcal{A}')$,*

then

$$\bigcap_{\eta \in M' \cup M''} \operatorname{ran}(T-\eta) = \bigcap_{\eta \in M' \cup M''} \ker \tilde{\Phi}_\eta = \{0\}, \quad \operatorname{mul} T = \{0\}.$$

If, in addition to <2> *and* <3>*, the hypothesis* <1> *of Lemma* 3.4 *is fullfilled, then*

$$r_\supseteq^{\mathrm{app}}(S,\Phi;M) \cap \mathbb{C}^+ \subseteq r(T). \tag{3.9}$$

The same holds with $\mathbb{C}^+$ replaced by $\mathbb{C}^-$ in <1> *and* (3.9).

Proof. Assume that a set M' with the properties stated in <2> is given. We know that $(\iota^*|_{\mathcal{A}'})^{-1}$ maps $\mathcal{L}'$ homeomorphically onto the closed subspace $\mathcal{A}^\wr$ of $\mathcal{A}'$ whose annihilator is equal to $\mathcal{A}^\circ$, cf. [Wor14, Lemma A.17, Proposition A.1]. Using (3.3) it follows that (here $\perp$ denotes the annihilator w.r.t. the usual duality between $\mathcal{A}$ and $\mathcal{A}'$)

$$\begin{aligned}
&\bigcap_{\eta \in M'} \operatorname{ran}(T-\eta) \subseteq \bigcap_{\eta \in M'} \ker \tilde{\Phi}_\eta \subseteq \\
&\qquad \subseteq \big[(\iota^*|_{\mathcal{A}'})^{-1}\big(\mathcal{L}' \cap \operatorname{span}\{\Phi(\cdot,\eta) : \eta \in M'\}\big)\big]^\perp = \big[(\iota^*|_{\mathcal{A}'})^{-1}(\mathcal{L}')\big]^\perp = \mathcal{A}^\circ.
\end{aligned}$$

Assume that in addition a set M'' with the properties stated in <3> is given. Then $\{\tilde{\Phi}_\eta : \eta \in M''\}$ is point separating on $\mathcal{A}^\circ$, cf. [Wor14, Remark A.4, Lemma A.13]. Hence,

$$\bigcap_{\eta \in M' \cup M''} \operatorname{ran}(T-\eta) \subseteq \bigcap_{\eta \in M' \cup M''} \ker \tilde{\Phi}_\eta \subseteq \mathcal{A}^\circ \cap \bigcap_{\eta \in M''} \ker \tilde{\Phi}_\eta = \{0\}.$$

In particular, $\operatorname{mul} T = \{0\}$.

Since $\operatorname{mul} T = \{0\}$, the constant α from Theorem 2.11/Corollary 2.12 equals 0. This implies that $r(T) \setminus \mathbb{R} = \mathbb{C} \setminus (\mathbb{R} \cup \Upsilon)$.

Assume that in addition also $<1>$ holds. If $\eta \in r^{\mathrm{app}}_{\supseteq}(S, \Phi; M) \cap \mathbb{C}^+$, then we already saw in Lemma 3.4 that $\eta \notin \Upsilon$. ❏

3.6 Lemma. *Let data $\mathcal{A}, T, \Omega, \Psi$ be given according to $3.2_{(1',2')}$. Assume that*

$<4>$ $\displaystyle\bigcap_{\eta \in \Omega} \ker \Psi(\cdot, \eta) \subseteq \mathcal{A}^\circ$;

$<5>$ *the set Ω is open and for each $x \in \mathcal{A}$ the function $\Psi(x, \cdot)$ is analytic;*

$<6>$ *both sets $\{\eta \in \Omega : \Psi(\cdot, \eta) \in \mathcal{A}'\}$ and $r_{\subseteq}(T, \Psi)$ have accumulation points in each connected component of Ω;*

Set $\Psi_{\mathcal{A}} : x \mapsto \Psi(x, \cdot)$ and

$$\mathcal{B} := \operatorname{ran} \Psi_{\mathcal{A}} = \big\{\Psi(x, \cdot) : x \in \mathcal{A}\big\}.$$

Then $\mathcal{B}$ can be endowed with a unique almost Pontryagin space structure such that $\Psi_{\mathcal{A}} : \mathcal{A} \to \mathcal{B}$ becomes isometric, continuous and open. In this way $\mathcal{B}$ becomes a reproducing kernel almost Pontryagin space of analytic functions on Ω. We have (here $S(\mathcal{B})$ denotes the multiplication operator in $\mathcal{B}$)

$$\Psi(\cdot, \eta) \in \iota^*(\mathcal{A}'), \eta \in \Omega, \quad (\Psi_{\mathcal{A}} \times \Psi_{\mathcal{A}})(T) \subseteq S(\mathcal{B}), \quad r_{\subseteq}(T, \Psi) = \Omega. \tag{3.10}$$

Proof. First, existence of an almost Pontrygain space structure on $\mathcal{B}$ such that $\Psi_{\mathcal{A}}$ becomes isometric, continuous and open follows from [Wor14, Proposition A.7]. The hypothesis necessary to apply this result is satisfied by our assumption $<4>$. Uniqueness is clear.

The topology of $\mathcal{B}$ is the final topology w.r.t. the map $\Psi_{\mathcal{A}}$. Applying Lemma 2.16 with each component of Ω separately (which is possible by the first half of $<6>$), it follows that it is finer than the topology of locally uniform convergence. In particular, all point evaluation functionals are continuous, i.e., $\mathcal{B}$ is a reproducing kernel almost Pontryagin space. Also Lemma 2.16 already gives $\Psi(\cdot, \eta) \in \mathcal{A}'$, $\eta \in \Omega$.

To show that $(\Psi_{\mathcal{A}} \times \Psi_{\mathcal{A}})(T) \subseteq S(\mathcal{B})$, let $(x; y) \in T$. Then

$$y - \eta x \in \operatorname{ran}(T - \eta) \subseteq \ker \Psi(\cdot, \eta), \quad \eta \in r_{\subseteq}(T, \Psi),$$

and hence

$$\Psi(y, \eta) - \eta \Psi(x, \eta) = 0, \quad \eta \in r_{\subseteq}(T, \Psi).$$

By the second half of $<6>$ and analyticity, this identity holds for all $\eta \in \Omega$. Hence,

$$(\Psi_{\mathcal{A}}(x); \Psi_{\mathcal{A}}(y)) = (\Psi(x, \cdot); \Psi(y, \cdot)) \in S(\mathcal{B}).$$

It remains to show that $r_{\subseteq}(T, \Psi) = \Omega$. However, the just proved inclusion in (3.10) yields

$$\Psi_{\mathcal{A}}\big(\operatorname{ran}(T - \eta)\big) = \operatorname{ran}\big((\Psi_{\mathcal{A}} \times \Psi_{\mathcal{A}})(T) - \eta\big) \subseteq \operatorname{ran}\big(S(\mathcal{B}) - \eta\big) \subseteq \ker(\chi_\eta|_{\mathcal{B}}).$$

Hence, for each $x \in \operatorname{ran}(T - \eta)$,

$$\Psi(x, \eta) = \big[\Psi_{\mathcal{A}}(x)\big](\eta) = \chi_\eta\big(\Psi_{\mathcal{A}}(x)\big) = 0.$$

❏

3.7 Lemma. *Let data $\mathcal{A}, T, \Omega, \Psi$ be given according to* $3.2_{(1',2')}$, *and assume that the hypothesis $<4>$, $<5>$, $<6>$ are fullfilled. Assume in addition that*

$<7>$ $r_{\supseteq}(T,\Psi) \neq \emptyset$.

Then $S(\mathcal{B})$ is symmetric,

$$(\Psi_{\mathcal{A}} \times \Psi_{\mathcal{A}})(T) = S(\mathcal{B}), \quad r_{\supseteq}(T,\Psi) \cup (\mathbb{C} \setminus \mathbb{R}) \subseteq r(S(\mathcal{B})),$$

and

$$\Big(\forall \eta \in r_{\supseteq}(T,\Psi) \cap \mathbb{C}^+ :\ \Psi(\cdot,\eta) \neq 0\Big) \text{ or } \Big(\forall \eta \in r_{\supseteq}(T,\Psi) \cap \mathbb{C}^+ :\ \Psi(\cdot,\eta) = 0\Big). \tag{3.11}$$

Assume in addition to $<4>$–$<7>$ that $r_{\supseteq}(T,\Psi) \cap \mathbb{C}^+ \neq \emptyset$. If the first case in (3.11) *takes place, then the upper defect index β_+ of $S(\mathcal{B})$ is equal to* 1. *In the second case $\beta_+ = 0$ and $\Psi|_{\mathcal{A}\times(\Omega\cap\mathbb{C}^+)} = 0$. The same statement holds true when $\mathbb{C}^+, \beta_+$ are replaced by $\mathbb{C}^-, \beta_-$.*

If $r_{\supseteq}(T,\Psi) \cap \mathbb{R} \neq \emptyset$, then either Ψ vanishes identically or the defect index of $S(\mathcal{B})$ is $(1,1)$.

Proof. Using the inclusion in (3.10), the fact that $\Psi(\cdot,\eta) = \chi_\eta \circ \Psi_{\mathcal{A}}$, and that $\Psi_{\mathcal{A}}$ is surjective (onto $\mathcal{B}$), we obtain that for each $\eta \in r_{\supseteq}(T,\Psi)$

$$\begin{aligned} \ker(\chi_\eta|_{\mathcal{B}}) = \Psi_{\mathcal{A}}\big(\ker \Psi(\cdot,\eta)\big) \subseteq \Psi_{\mathcal{A}}\big(\operatorname{ran}(T-\eta)\big) = \\ = \operatorname{ran}\big((\Psi_{\mathcal{A}} \times \Psi_{\mathcal{A}})(T) - \eta\big) \subseteq \operatorname{ran}\big(S(\mathcal{B}) - \eta\big) \subseteq \ker(\chi_\eta|_{\mathcal{B}}), \end{aligned}$$

and hence

$$\operatorname{ran}\big((\Psi_{\mathcal{A}} \times \Psi_{\mathcal{A}})(T) - \eta\big) = \operatorname{ran}\big(S(\mathcal{B}) - \eta\big) = \ker(\chi_\eta|_{\mathcal{B}}), \quad \eta \in r_{\supseteq}(T,\Psi). \tag{3.12}$$

By $<7>$ the set $r_{\supseteq}(T,\Psi)$ is nonempty. Since $\sigma_p(S(\mathcal{B})) = \emptyset$, cf. (2.9), the known inclusion of relations and equality of ranges implies $(\Psi_{\mathcal{A}} \times \Psi_{\mathcal{A}})(T) = S(\mathcal{B})$. Now isometry of $\Psi_{\mathcal{A}}$ implies that $S(\mathcal{B})$ is symmetric. Moreover, we see from (3.12) that $\operatorname{ran}(S(\mathcal{B}) - \eta)$ is closed for all $\eta \in r_{\supseteq}(T,\Psi)$, and that (by surjectivity of $\Psi_{\mathcal{A}}$ we have $\chi_\eta|_{\mathcal{B}} = 0$ if and only if $\Psi(\cdot,\eta) = 0$)

$$\operatorname{codim}_{\mathcal{B}} \operatorname{ran}(S(\mathcal{B}) - \eta) = \begin{cases} 1\,, & \Psi(\cdot,\eta) \neq 0 \\ 0\,, & \Psi(\cdot,\eta) = 0 \end{cases}, \qquad \eta \in r_{\supseteq}(T,\Psi). \tag{3.13}$$

Remembering again that $\sigma_p(S(\mathcal{B})) = \emptyset$, we conclude that $r_{\supseteq}(T,\Psi) \subseteq r(S(\mathcal{B}))$. Moreover, the exceptional set Υ for $S(\mathcal{B})$ is empty. Hence, $\mathbb{C} \setminus \mathbb{R} \subseteq r(S(\mathcal{B}))$ and the codimension of $\operatorname{ran}(S(\mathcal{B}) - \eta)$ is constant on each component of $r(S(\mathcal{B}))$.

If $r_{\supseteq}(T,\Psi) \cap \mathbb{C}^+ = \emptyset$, (3.11) is void. Hence, assume that $r_{\supseteq}(T,\Psi) \cap \mathbb{C}^+ \neq \emptyset$. Then (3.13) shows that $\beta_+ \in \{0,1\}$ and, depending whether $\beta_+ = 1$ or $\beta_+ = 0$, the first or second alternative in (3.11) takes place. Let us further consider the case $\beta_+ = 0$. Then $\operatorname{ran}(S(\mathcal{B}) - \eta) = \mathcal{B}$, $\eta \in \mathbb{C}^+$. However, $\operatorname{ran}(S(\mathcal{B}) - \eta) \subseteq \ker(\chi_\eta|_{\mathcal{B}})$, $\eta \in \Omega$, and we conclude that each element of $\mathcal{B}$ (i.e. each function of the form $\Psi(x,\cdot)$) vanishes on $\Omega \cap \mathbb{C}^+$.

The case of the lower half-plane is treated in the same way. If $r_{\supseteq}(T,\Psi) \cap \mathbb{R} \neq \emptyset$, the set $r(S(\mathcal{B}))$ contains a real point, and hence is connected. Thus the defect numbers of $S(\mathcal{B})$ are equal, and they are either both equal to 1 or both equal to 0. In the latter case, it follows that each element of $\mathcal{B}$ vanishes on $\Omega \setminus \mathbb{R}$ and by continuity everywhere. ❑

3.8 Lemma. *Let data $\mathcal{L}, S, \langle\iota, \mathcal{A}\rangle, \Omega, \Phi$ be given according to $3.1_{(1-3)}$. Assume that*

<8> *the set Ω is open and for each $x \in \mathcal{L}$ the function $\Phi(x, \cdot)$ is analytic;*

<9> *the set $\{\eta \in \Omega : \Phi(\cdot, \eta) \in \iota^*(\mathcal{A}')\} \cap r_{\subseteq}(S, \Phi)$ has accumulation points in each connected component of $\Omega \setminus \mathbb{R}$;*

<10> *the relation $T := \mathrm{Clos}_{\mathcal{A}\times\mathcal{A}}\big[(\iota\times\iota)(S)\big]$ has defect index $(1,1)$ and $r(T) \neq \emptyset$.*

Then there exists a unique function $\Psi : \mathcal{A} \times \Omega \to \mathbb{C}$ with

$$\Psi(\cdot, \eta) \in \mathcal{A}',\ \eta \in \Omega, \qquad \Psi(x, \cdot) \in \mathbb{H}(\Omega),\ x \in \mathcal{A},$$

which lifts Φ:

$$\begin{tikzcd} \mathcal{A}\times\Omega \arrow[dr, dashed, "\Psi"] & \\ \mathcal{L}\times\Omega \arrow[u, "\iota\times\mathrm{id}_\Omega"] \arrow[r, "\Phi"'] & \mathbb{C} \end{tikzcd}$$

It holds that

$$\dim\Big(\mathcal{L}' + \mathrm{span}\{\Phi(\cdot,\eta) : \eta \in \Omega\}\Big/_{\mathcal{L}'}\Big) \leq \mathrm{ind}_0\,\mathcal{A}, \quad r_{\subseteq}(S,\Phi) \subseteq r_{\subseteq}(T,\Psi).$$

Proof. Choose a Pontryagin space $\mathcal{P}$ which contains $\mathcal{A}$ as a closed subspace with codimension $\mathrm{ind}_0\,\mathcal{A}$, and choose $\zeta_0 \in r(T)$. The relation T, considered in the Pontryagin space $\mathcal{P}$, has finite and equal defect numbers, namely equal to $1 + \mathrm{ind}_0\,\mathcal{A}$. Hence, we can choose a selfadjoint extension A_0 of T in $\mathcal{P}$ with $\zeta_0 \in \rho(A_0)$, see, e.g., [KW98, Lemma 2.1]. Since $\mathrm{ran}(T - \zeta_0) \subseteq \mathcal{A}$, there exists $y_0 \in \mathcal{P}$ with

$$\mathcal{P}[-]\,\mathrm{ran}(T - \zeta_0) = \mathcal{A}^\circ \dot{+} \mathrm{span}\{y_0\}. \tag{3.14}$$

We use y_0 to generate a family of defect elements of T: set

$$y(\zeta) := \big(I + (\zeta - \overline{\zeta_0})(A_0 - \zeta)^{-1}\big)y_0, \quad \zeta \in \rho(A_0).$$

Then $y(\zeta) \perp \mathrm{ran}(T - \overline{\zeta})$, $\zeta \in \rho(A_0)$. Since $y(\overline{\zeta_0}) = y_0 \notin \mathcal{A}^\circ$, the set $\{\eta \in \rho(A_0) : y(\eta) \in \mathcal{A}^\circ\}$ intersects the connected component of $\rho(A_0)$ which contains $\overline{\zeta_0}$ only in a discrete set.

In the subsequent items 1°–3° we carry out some reductions, in 4° we comment on uniqueness, and in 5° we provide an additional preliminary observation.

1°: We show that the choice of y_0 can be made such that $\{\eta \in \rho(A_0) : y(\eta) \in \mathcal{A}^\circ\}$ intersects each component of $\rho(A_0)$ only in a discrete set. To this end assume that $\rho(A_0)$ has two components and that $y(\zeta) \in \mathcal{A}^\circ$ for all ζ in the component which does not contain $\overline{\zeta_0}$. Fix a point ζ_1 in this component. The map $(I + (\zeta_1 - \overline{\zeta_0})(A_0 - \zeta_1)^{-1})$ is a bijection of $\mathcal{P}[-]\,\mathrm{ran}(T - \zeta_0) = \mathcal{A}^\circ \dot{+} \mathrm{span}\{y_0\}$ onto $\mathcal{P}[-]\,\mathrm{ran}(T - \overline{\zeta_1})$. Since $\rho(A_0) \subseteq r(T)$, the latter space contains $\mathcal{A}^\circ$ as a proper subset. Since $y(\zeta_1) \in \mathcal{A}^\circ$, there must exist $a_0 \in \mathcal{A}^\circ$ with $(I + (\zeta_1 - \overline{\zeta_0})(A_0 - \zeta_1)^{-1})a_0 \notin \mathcal{A}^\circ$. Set $\tilde{y}_0 := y_0 + a_0$, and let $\tilde{y}(\zeta)$, $\zeta \in \rho(A_0)$, be the correspondingly defined defect family. Note here that, clearly, (3.14) holds also for $\tilde{y}_0$ in place of y_0. Then $\tilde{y}(\overline{\zeta_0}) = \tilde{y}_0 \notin \mathcal{A}^\circ$ and also

$$\tilde{y}(\zeta_1) = y(\zeta_1) + (I + (\zeta_1 - \overline{\zeta_0})(A_0 - \zeta_1)^{-1})a_0 \notin \mathcal{A}^\circ.$$

Thus we may assume for the rest of the proof that $\{\eta \in \rho(A_0) : y(\eta) \in \mathcal{A}^\circ\}$ intersects each component of $\rho(A_0)$ only in a discrete set.

2°: In order to define a lifting $\Psi : \mathcal{A} \times \Omega \to \mathbb{C}$ we may define Ψ separately for each set $\mathcal{A} \times \Omega'$ where Ω' is a component of Ω. This holds because each component of $\Omega' \setminus \mathbb{R}$ is a component of $\Omega \setminus \mathbb{R}$, and hence our assumption $<9>$ remains valid when Ω is replaced by Ω'. Thus we may assume for the rest of the proof that Ω is connected.

3°: If Φ vanishes identically, the map $\Psi(x, \eta) := 0$, $x \in \mathcal{A}$, $\eta \in \Omega$, trivially satisfies all requirements. Thus we may assume for the rest of the proof that Φ does not vanish identically.

4°: Uniqueness of a lifting Ψ is clear since $\iota(\mathcal{L})$ is dense in $\mathcal{A}$ and $\Psi(\cdot, \eta)$ is required to be continuous.

5°: We show that $\{\eta \in \rho(A_0) : y(\eta) \in \mathcal{A}^\circ\}$ has no accumulation point in $\mathbb{C} \setminus \mathbb{R}$. Assume on the contrary that this set would accumulate in $\mathbb{C}^+$ (the case of $\mathbb{C}^-$ is treated in the same way). Nonreal spectral points of A_0 are poles of the resolvent. Hence, we find a polynomial p having $\sigma(A_0) \cap \mathbb{C}^+$ as its zero set, such that $p(\zeta)y(\zeta)$ has an analytic continuation to $\mathbb{C}^+$, say $z(\zeta)$. The set $\{\eta \in \mathbb{C}^+ : z(\eta) \in \mathcal{A}^\circ\}$ has an accumulation point in $\mathbb{C}^+$, and hence $z(\zeta) \in \mathcal{A}^\circ$ for all $\zeta \in \mathbb{C}^+$. Thus $y(\zeta) \in \mathcal{A}^\circ$ for all $\zeta \in \rho(A_0) \cap \mathbb{C}^+$, and we have reached a contradiction.

Now we come to the actual construction of Ψ. The facts that $y(\zeta) \perp \operatorname{ran}(T - \overline{\zeta})$, $\zeta \in \rho(A_0)$, and that T has defect index $1 + \operatorname{ind}_0 \mathcal{A}$ in $\mathcal{P}$, lead to

$$\mathcal{P}[-] \operatorname{ran}(T - \overline{\zeta}) = \mathcal{A}^\circ \dot{+} \operatorname{span}\{y(\zeta)\}, \quad \zeta \in \rho(A_0), y(\zeta) \notin \mathcal{A}^\circ.$$

We conclude that (note here that $\rho(A_0)$ is symmetric w.r.t. the real line)

$$x \in \operatorname{ran}(T - \zeta) \;\Leftrightarrow\; x \perp y(\overline{\zeta}), \qquad x \in \mathcal{A}, \zeta \in \rho(A_0), y(\overline{\zeta}) \notin \mathcal{A}^\circ. \tag{3.15}$$

Choose $b_0 \in \mathcal{L}$ with $\Phi(b_0, \cdot) \neq 0$, and consider the set

$$\begin{aligned} N := r_{\subseteq}(S, \Phi) \cap \big\{\eta \in \Omega : \Phi(\cdot, \eta) \in \iota^*(\mathcal{A}')\big\} \cap (\mathbb{C} \setminus \mathbb{R}) \cap \\ \cap \rho(A_0) \cap \big\{\eta \in \Omega : \Phi(b_0, \eta) \neq 0\big\} \cap \big\{\eta \in \rho(A_0) : y(\overline{\eta}) \notin \mathcal{A}^\circ\big\}. \end{aligned}$$

From our assumption $<9>$ and 5° it follows that this set has accumulation points in each component of $\Omega \setminus \mathbb{R}$.

If $\eta \in r_{\subseteq}(S, \Phi) \cap \{\eta \in \Omega : \Phi(\cdot, \eta) \in \iota^*(\mathcal{A}')\}$, then (3.3) gives $\operatorname{ran}(T - \eta) \subseteq \ker \tilde{\Phi}_\eta$. For $\eta \in N$ we have $\tilde{\Phi}_\eta(\iota b_0) = \Phi(b_0, \eta) \neq 0$ and hence $\iota b_0 \notin \operatorname{ran}(T - \eta)$. The equivalence (3.15) yields

$$[\iota b_0, y(\overline{\eta})]_{\mathcal{P}} \neq 0, \quad \eta \in N,$$

and

$$x - \frac{[x, y(\overline{\eta})]_{\mathcal{P}}}{[\iota b_0, y(\overline{\eta})]_{\mathcal{P}}} \iota b_0 \in \operatorname{ran}(T - \eta), \quad x \in \mathcal{A}, \eta \in N.$$

We conclude that

$$\tilde{\Phi}_\eta(x) = \frac{[x, y(\overline{\eta})]_{\mathcal{P}}}{[\iota b_0, y(\overline{\eta})]_{\mathcal{P}}} \cdot \Phi(b_0, \eta), \quad x \in \mathcal{A}, \eta \in N. \tag{3.16}$$

For $a \in \mathcal{L}$ and $x \in \mathcal{A}$ consider the quotients

$$p_a := \Big(\frac{\Phi(a,\cdot)}{\Phi(b_0,\cdot)}\Big)^{\#} \quad \text{and} \quad q_x := \frac{[y(\cdot),x]_{\mathcal{P}}}{[y(\cdot),\iota b_0]_{\mathcal{P}}}.$$

Then $p_a \in \mathbb{H}(\Omega^{\#},\mathbb{C}_\infty)$ and the divisor of p_a can be estimated from below by

$$\mathfrak{d}_{p_a} = \mathfrak{d}_{\Phi(a,\cdot)^{\#}} - \mathfrak{d}_{\Phi(b_0,\cdot)^{\#}} \geq -\mathfrak{d}_{\Phi(b_0,\cdot)^{\#}}.$$

Since the nonreal spectral points of A_0 are poles of the resolvent, we have $q_x \in \mathbb{H}(\mathbb{C}\setminus\mathbb{R},\mathbb{C}_\infty)$. Denoting

$$\mathfrak{e}_{A_0}(\eta) := \begin{cases} -\text{pole order of } (A_0-\eta)^{-1} \text{ at } \eta\,, & \eta \in \sigma(A_0)\setminus\mathbb{R} \\ 0 \qquad\qquad\qquad\qquad\qquad\qquad , & \eta \in \rho(A_0)\setminus\mathbb{R} \end{cases}$$

we can estimate the divisor of q_x from below by

$$\mathfrak{d}_{q_x} = \mathfrak{d}_{[y(\cdot),x]_{\mathcal{P}}} - \mathfrak{d}_{[y(\cdot),\iota b_0]_{\mathcal{P}}} \geq \mathfrak{e}_{A_0} - \mathfrak{d}_{[y(\cdot),\iota b_0]_{\mathcal{P}}}.$$

Let $a \in \mathcal{L}$. By (3.16) the functions p_a and $q_{\iota a}$ coincide on $N^{\#}$. Since $N^{\#}$ accumulates in each component of $\Omega^{\#}\cap(\mathbb{C}\setminus\mathbb{R})$, they coincide on all of $\Omega^{\#}\cap(\mathbb{C}\setminus\mathbb{R})$, i.e., are analytic extensions of each other. This shows that the quotient $q_{\iota a}$ has an extension $Q_{\iota a} \in \mathbb{H}(\Omega^{\#}\cup(\mathbb{C}\setminus\mathbb{R}),\mathbb{C}_\infty)$. The divisor of this extension can be estimated from below by[3]

$$\mathfrak{d}_{Q_{\iota a}} \geq \mathbb{1}_{(\mathbb{C}\setminus\mathbb{R})\setminus\Omega^{\#}} \cdot \big(\mathfrak{e}_{A_0} - \mathfrak{d}_{[y(\cdot),\iota b_0]_{\mathcal{P}}}\big) - \mathbb{1}_{\Omega^{\#}} \cdot \mathfrak{d}_{\Phi(b_0,\cdot)^{\#}}. \tag{3.17}$$

Notice that the right side is independent of $a \in \mathcal{L}$.

Now Theorem 2.15, (iii), applies and yields that for each $x \in \mathcal{A}$ the quotient q_x has an extension $Q_x \in \mathbb{H}(\Omega^{\#}\cup(\mathbb{C}\setminus\mathbb{R}),\mathbb{C}_\infty)$ whose divisor is bounded from below by the right side of (3.17). Set

$$\Psi(x,\cdot) := Q_x^{\#}\cdot\Phi(b_0,\cdot), \quad x \in \mathcal{A}.$$

Then $\Psi(x,\cdot) \in \mathbb{H}(\Omega,\mathbb{C})$, $x \in \mathcal{A}$, and

$$\Psi(\iota a,\eta) = \big[p_a^{\#}\cdot\Phi(b_0,\cdot)\big](\eta) = \Phi(a,\eta), \quad a \in \mathcal{L}, \eta \in \Omega.$$

Remember here that $\Phi(b_0,\cdot)$ vanishes only on a discrete subset of Ω. By (3.16) we have

$$\Psi(x,\eta) = \big[q_x^{\#}\cdot\Phi(b_0,\cdot)\big](\eta) = \tilde{\Phi}_\eta(x), \quad x \in \mathcal{A}, \eta \in N,$$

and hence $\Psi(\cdot,\eta) \in \mathcal{A}'$, $\eta \in N$. Since N accumulates in Ω, we may apply Lemma 2.16 to conclude that $\Psi(\cdot,\eta) \in \mathcal{A}'$, $\eta \in \Omega$. This implies that

$$\dim\Big(\mathcal{L}' + \operatorname{span}\{\Phi(\cdot,\eta) : \eta \in \Omega\}\Big/\mathcal{L}'\Big) \leq \dim\Big(\iota^*(\mathcal{A}')\Big/\mathcal{L}'\Big) = \operatorname{ind}_0\mathcal{A}.$$

The inclusion $r_\subseteq(S,\Phi) \subseteq r_\subseteq(T,\Psi)$ now follows from (3.3). ❑

3.9 Lemma. *Let data $\mathcal{L}, S, \langle\iota,\mathcal{A}\rangle, \Omega, \Phi$ be given according to* 3.1$_{(1-3)}$*, and assume that the hypothesis* <8>,<9>,<10>*, are fullfilled. Let Ψ be the lifting of Φ constructed in Lemma* 3.8*. If*

[3] Here $\mathbb{1}_E$ denotes the characteristic function of the set E.

$<11>$ *$\mathcal{L}' \cap \operatorname{span}\{\Phi(\cdot,\eta) : \eta \in \Omega\}$ is dense in $\mathcal{L}'$,*

then the following statements hold.

(i) For each subset $L \subseteq \Omega$ which has accumulation points in each connected component of Ω, we have

$$\bigcap_{\eta\in L} \ker \Psi(\cdot,\eta) \subseteq \mathcal{A}^\circ. \tag{3.18}$$

(ii)
$$r_{\subseteq}(T,\Psi) = \Omega, \quad r_{=}(T,\Psi) \supseteq r(T) \cap \big\{\eta \in \Omega : \Phi(\cdot,\eta) \neq 0\big\}. \tag{3.19}$$

(iii) For each subset $L' \subseteq \mathbb{C}$ which has accumulation points in each of the half-planes $\mathbb{C}^\pm$ intersecting Ω, we have

$$\bigcap_{\eta\in L'} \operatorname{ran}(T-\eta) \subseteq \mathcal{A}^\circ. \tag{3.20}$$

If, in addition to $<8>$–$<11>$,

$<12>$ *$\mathcal{L}' + \operatorname{span}\{\Phi(\cdot,\eta) : \eta \in \Omega\} = \iota^*(\mathcal{A}')$,*

then $\sigma_p(T) = \emptyset$, $\operatorname{mul} T = \{0\}$, and for sets L and L' as above

$$\bigcap_{\eta\in L} \ker \Psi(\cdot,\eta) = \{0\}, \quad \bigcap_{\eta\in L'} \operatorname{ran}(T-\eta) = \{0\}. \tag{3.21}$$

Proof. Assume that $<11>$ holds. Using analyticity and the same argument as in Lemma 3.5, it follows that

$$\begin{aligned}\bigcap_{\eta\in L} \ker \Psi(\cdot,\eta) = \bigcap_{\eta\in \Omega} \ker \Psi(\cdot,\eta) = \big[\operatorname{span}\{\Psi(\cdot,\eta) : \eta\in\Omega\}\big]^\perp \subseteq \\ \subseteq \big[(\iota^*|_{\mathcal{A}'})^{-1}\big(\mathcal{L}' \cap \operatorname{span}\{\Phi(\cdot,\eta) : \eta\in\Omega\}\big)\big]^\perp = \\ = \big[(\iota^*|_{\mathcal{A}'})^{-1}(\mathcal{L}')\big]^\perp = \mathcal{A}^\circ.\end{aligned}$$

By means of Lemma 3.8 and the just proved (3.18), all assumptions of Lemma 3.6 are fullfilled. Applying this lemma yields $r_{\subseteq}(T,\Psi) = \Omega$. If $\eta \in r(T)\cap\Omega$, the codimension of $\operatorname{ran}(T-\eta)$ is equal to 1. As we just showed, this range is contained in $\ker\Psi(\cdot,\eta)$. Hence, provided that $\Psi(\cdot,\eta)\neq 0$, we must have $\operatorname{ran}(T-\eta) = \ker\Psi(\cdot,\eta)$. This is (3.19).

It remains to prove (3.20). Assume that L' has an accumulation point in $\mathbb{C}^+$. Then (remember here that the resolvent of any selfadjoint extension is meromorphic in $\mathbb{C}^+$ and hence we may apply the identity theorem in $\mathbb{H}(\mathbb{C}^+,\mathbb{C}_\infty)$, see, e.g., [Rem91, p.319f])

$$\begin{aligned}\bigcap_{\eta\in L'\cap\mathbb{C}^+} \operatorname{ran}(T-\eta) \subseteq \bigcap_{L'\cap r(T)\cap\mathbb{C}^+} \operatorname{ran}(T-\eta) = \bigcap_{r(T)\cap\mathbb{C}^+} \operatorname{ran}(T-\eta) \subseteq \\ \subseteq \bigcap_{\Omega\cap r(T)\cap\mathbb{C}^+} \operatorname{ran}(T-\eta) \subseteq \bigcap_{\Omega\cap r(T)\cap\mathbb{C}^+} \ker\Psi(\cdot,\eta).\end{aligned} \tag{3.22}$$

In the same way, we obtain $\bigcap_{\eta\in L'\cap\mathbb{C}^-} \operatorname{ran}(T-\eta) \subseteq \bigcap_{\Omega\cap r(T)\cap\mathbb{C}^-} \ker\Psi(\cdot,\eta)$ provided that L' has an accumulation point in $\mathbb{C}^-$. Our assumption ensures that we may put together these inclusions and apply (3.18) with the set $L := \Omega\cap r(T)\cap(\mathbb{C}^+\cup\mathbb{C}^-)$.

Assume that in addition $<12>$ holds. Then $\{\Psi(\cdot,\eta) : \eta\in\Omega\}$ is point separating on $\mathcal{A}^\circ$. By analyticity, the family $\{\Psi(\cdot,\eta) : \eta\in L\}$ has the same property. Using the already proved inclusion (3.18), it follows that $\bigcap_{\eta\in L}\ker\Psi(\cdot,\eta) = \{0\}$. The argument which led to (3.22) thus also gives $\bigcap_{\eta\in L'}\operatorname{ran}(T-\eta) = \{0\}$. From this we get $\operatorname{mul} T = \{0\}$. Moreover, if $\eta\in\mathbb{C}$ and $x\in\ker(T-\eta)$, then $x\in\operatorname{ran}(T-\zeta)$, $\zeta\in\mathbb{C}\setminus\{\eta\}$, and it follows that $x=0$. ❑

3.10 Lemma. *Let data $\mathcal{L}, S, \langle\iota,\mathcal{A}\rangle, \Omega, \Phi$ be given according to 3.1$_{(1-3)}$, and assume that the hypothesis $<8>$–$<12>$ are fullfilled. Let Ψ be the lifting of Φ constructed in Lemma 3.8, and let $\mathcal{B} := \operatorname{ran}\Psi_{\mathcal{A}}$ (remember that, by (3.21), $\Psi_{\mathcal{A}}$ is injective). Assume, in addition to $<8>$–$<12>$ that*

$<13>$ *there is no nonempty open subset O of Ω, such that $\Phi|_{\mathcal{L}\times O} = 0$.*

Then $\mathcal{B}$ is an Ω-space. In particular, $\Omega\subseteq r(T)$ and $r_=(T,\Psi) = \{\eta\in\Omega : \Phi(\cdot,\eta)\neq 0\}$.

Proof. Our aim is to apply Proposition 2.21. We know from the proof of Lemma 3.9 that the assumptions $<4>$–$<6>$ of Lemma 3.6 are fullfilled. Thus, $\mathcal{B}$ is an reproducing kernel almost Pontryagin space of analytic functions on Ω. Our present assumption $<13>$ yields that $\operatorname{supp}\mathfrak{d}_{\mathcal{B}}$ is a discrete subset of Ω.

Since T has no eigenvalues, we have $\mathbb{C}\setminus\mathbb{R}\subseteq r(T)$. The relation (3.19) together with $<13>$ shows that the assumption $<7>$ of Lemma 3.7 is fullfilled. In fact, if $\Omega\cap\mathbb{C}^+\neq\emptyset$ then $r_=(T,\Psi)\cap\mathbb{C}^+\neq\emptyset$, and the same for the lower half-plane. Lemma 3.7 implies that $S(\mathcal{B})$ is symmetric and isomorphic to T via $\Psi_{\mathcal{A}}$. If $\Omega\cap\mathbb{C}^+\neq\emptyset$, the second alternative in (3.11) is ruled out by $<13>$, and it follows that $\beta_+ = 1$. The same holds for the lower half-plane.

Now Proposition 2.21 applies and yields that $\mathcal{B}$ is an Ω-space. Since $S(\mathcal{B})$ and T are isomorphic, the statements for T follow. ❑

4 Embeddings into spaces of analytic functions

The following version of directing functionals should be thought of as a generalisation of universal directing functionals. Remember the definition of $r_\subseteq(S,\Phi)$ and $r_\supseteq^{\mathrm{app}}(S,\Phi;\Omega)$ from Definitions 2.7 and 2.8.

4.1 Definition. Let $\mathcal{L}$ be an inner product space with $\operatorname{ind}_-\mathcal{L}<\infty$, let S be a symmetric linear relation in $\mathcal{L}$, let $\Omega\subseteq\mathbb{C}$, and let $\Phi:\mathcal{L}\times\Omega\to\mathbb{C}$. We call Φ an *Ω-directing functional* for S, if:

(Ωdf1) The set Ω is open and nonempty. For each $x\in\mathcal{L}$ the function $\Phi(x,\cdot):\Omega\to\mathbb{C}$ is analytic.

(Ωdf2) For each $\zeta\in\Omega$ the function $\Phi(\cdot,\zeta):\mathcal{L}\to\mathbb{C}$ is linear.

(Ωdf3') The set $r_\subseteq(S,\Phi)$ has accumulation points in each connected component of $\Omega\setminus\mathbb{R}$.

(Ωdf3") The set $r^{\text{app}}_{\supseteq}(S,\Phi;\Omega)$ has nonempty intersection with both half-planes $\mathbb{C}^+$ and $\mathbb{C}^-$.

(Ωdf4) There is no nonempty open subset O of Ω, such that $\Phi|_{\mathcal{L}\times O}=0$.

$\Diamond$

Note that (Ωdf3") implicitly contains that Ω intersects both half-planes $\mathbb{C}^{\pm}$.

4.2 Remark. Let us pause and revisit the discussions in §1.1.

- We seek for representations in almost Pontryagin spaces. Hence, we restrict from the start to spaces $\mathcal{L}$ with finite index of negativity.
- We want to have the representations of S to be established explicitly by Φ. Hence, $\Phi(x,\cdot)$ is defined and analytic on all of Ω.
- We aim towards an "if and only if" statement. Hence, we use the approximative version (Ωdf3").
- The conditions known from (udf3') and (udf3") are assumed only on sufficiently large sets. Weakening the first is possible by analyticity, weakening the second by constancy of defect on half-planes.

$\Diamond$

Our aim is to determine when presence of an Ω-directing functional for S gives rise to a representation of S in an Ω-space (remember Definition 2.19). The next theorem is the main result of this paper. We give conditions which ensure that such a representation exists and is established by Φ. It is not difficult to show that these conditions are also necessary. This is deferred to Proposition 4.6 below, where a slightly refined converse statement is given.

4.3 Theorem. *Let $\mathcal{L}$ be an inner product space with $\operatorname{ind}_-\mathcal{L}<\infty$, let S be a symmetric linear relation in $\mathcal{L}$, and let $\Omega\subseteq\mathbb{C}$. Assume that $\Phi:\mathcal{L}\times\Omega\to\mathbb{C}$ is an Ω-directing functional for S, and that there exists a subset $M\subseteq r_{\subseteq}(S,\Phi)$ which has accumulation points in each connected component of $\Omega\setminus\mathbb{R}$, such that (notation $\mathcal{L}'$ and $\mathcal{T}(\mathcal{L}')$ as in Definition 2.4 and 2.5)*

(i) $\dim\Big(\big[\mathcal{L}'+\operatorname{span}\{\Phi(\cdot,\eta):\eta\in M\}\big]\Big/\mathcal{L}'\Big)<\infty$,

(ii) $\mathcal{L}'\cap\operatorname{span}\big\{\Phi(\cdot,\zeta):\ \zeta\in r_{\subseteq}(S,\Phi),\Phi(\cdot,\zeta)\in\mathcal{L}'+\operatorname{span}\{\Phi(\cdot,\eta):\eta\in M\}\big\}$ *is dense in $\mathcal{L}'$ w.r.t. $\mathcal{T}(\mathcal{L}')$.*

Then there exists a unique reproducing kernel almost Pontryagin space $\mathcal{B}$, such that the assignment $\Phi_{\mathcal{L}}:x\mapsto\Phi(x,\cdot)$ maps $\mathcal{L}$ isometrically onto a dense subspace of $\mathcal{B}$. This space $\mathcal{B}$ is an Ω-space and $\operatorname{Clos}_{\mathcal{B}}\big[(\Phi_{\mathcal{L}}\times\Phi_{\mathcal{L}})(S)\big]=S(\mathcal{B})$. Moreover,

$$[\Phi_{\mathcal{L}}]^*(\mathcal{B}')=\mathcal{L}'+\operatorname{span}\big\{\Phi(\cdot,\eta):\eta\in M\big\}=\mathcal{L}'+\operatorname{span}\big\{\Phi(\cdot,\eta):\eta\in\Omega\big\},\quad(4.1)$$

$$\operatorname{ind}_0\mathcal{B}=\dim\Big(\big[\mathcal{L}'+\operatorname{span}\{\Phi(\cdot,\eta):\eta\in\Omega\}\big]\Big/\mathcal{L}'\Big).\quad(4.2)$$

4.4 Remark. In the proof we will not use the full strength of (Ωdf3"). It is only needed that $r^{\text{app}}_{\supseteq}(S,\Phi;M)$ has nonempty intersection with both half-planes $\mathbb{C}^+$ and $\mathbb{C}^-$. $\Diamond$

4.5 Remark. Notice the balance between the conditions (i) and (ii): Condition (i) gets stronger when M gets larger, whereas (ii) gets stronger when M gets smaller. Hence, it is important that (i) and (ii) hold with a common set M. ◊

Proof of Theorem 4.3. Choose a subset M according to the assumption of the theorem, and choose an almost Pontryagin space completion $\langle\iota,\mathcal{A}\rangle$ of $\mathcal{L}$ with

$$\iota^*(\mathcal{A}') = \mathcal{L}' + \operatorname{span}\big\{\Phi(\cdot,\eta) : \eta \in M\big\}. \tag{4.3}$$

This is possible by (i) and [Wor14, Theorem A.15]. Then the data $\mathcal{L}, S, \langle\iota,\mathcal{A}\rangle, \Omega, \Phi, M$ qualifies according to $3.1_{(1-4)}$.

1° *Applying Lemma* 3.4*:* We have $M \subseteq r_{\subseteq}(S,\Phi)$, and by ($\Omega$df4) and analyticity the set $\{\eta \in \Omega : \Phi(\cdot,\eta) = 0\}$ is a discrete subset of Ω. Therefore the set

$$M_0 := M \cap r_{\subseteq}(S,\Phi) \cap \big\{\eta \in \Omega : \Phi(\cdot,\eta) \neq 0\big\}$$

has accumulation points in each component of $\Omega \setminus \mathbb{R}$. Since Ω intersects both half-planes $\mathbb{C}^+$ and $\mathbb{C}^-$, the set M_0 in particular contains infinitely many points of each of these half-planes. Thus $<1>$ is satisfied for both half-planes and Lemma 3.4 is applicable. Since $r^{\mathrm{app}}_{\supseteq}(S,\Phi;M) \cap \mathbb{C}^{\pm} \neq \emptyset$, the relation $T := \operatorname{Clos}_{\mathcal{A}}[(\iota\times\iota)(S)]$ has defect index $(1,1)$. For later reference, note that Lemma 3.4 also gives

$$\operatorname{ran}(T-\eta) = \ker\tilde{\Phi}_\eta \neq \mathcal{A}, \quad \eta \in r^{\mathrm{app}}_{\supseteq}(S,\Phi;M) \setminus \mathbb{R}. \tag{4.4}$$

2° *Applying Lemma* 3.5*:* Set

$$M' := r_{\subseteq}(S,\Phi) \cap \big\{\eta \in \Omega : \Phi(\cdot,\eta) \in \iota^*(\mathcal{A}')\big\}, \qquad M'' := M.$$

Then $<2>$ holds by assumption (ii) of the theorem, and $<3>$ by the choice $\langle\iota,\mathcal{A}\rangle$, cf. (4.3). It follows that (note here that $M'' \subseteq M'$)

$$\bigcap_{\eta \in M'} \ker\tilde{\Phi}_\eta = \{0\}, \quad \emptyset \neq r^{\mathrm{app}}_{\supseteq}(S,\Phi;M) \cap \mathbb{C}^{\pm} \subseteq r(T). \tag{4.5}$$

3° *Applying Lemma* 3.8*:* The required hypothesis $<8>$ holds by (Ωdf1), $<9>$ since $M \subseteq M'$, and $<10>$ by what we showed above. Hence, we find a lifting $\Psi : \mathcal{A} \times \Omega \to \mathbb{C}$ of Φ with the properties stated in Lemma 3.8. We have $\Phi(\cdot,\eta) = \iota^*\Psi(\cdot,\eta)$, $\eta \in \Omega$, and hence

$$\mathcal{L}' + \operatorname{span}\big\{\Phi(\cdot,\eta) : \eta \in \Omega\big\} \subseteq \iota^*(\mathcal{A}') = \mathcal{L}' + \operatorname{span}\big\{\Phi(\cdot,\eta) : \eta \in M\big\}. \tag{4.6}$$

It follows that equality holds throughout.

4° *Applying Lemma* 3.6*:* Clearly the data $\mathcal{A}, T, \Omega, \Psi$ qualifies according to $3.2_{(1',2')}$. Let us check the hypothesis $<4>$, $<5>$, $<6>$. First, by (4.5),

$$\bigcap_{\eta\in\Omega} \ker\Psi(\cdot,\eta) \subseteq \bigcap_{\eta \in M'} \ker \underbrace{\Psi(\cdot,\eta)}_{\tilde{\Phi}_\eta} = \{0\},$$

Second, $\Psi(x,\cdot) \in \mathbb{H}(\Omega)$ by construction. Third, the first set in $<6>$ equals Ω and the second contains M.

It follows that the space

$$\mathcal{B} := \big\{ \Psi(x,\cdot) : x \in \mathcal{A} \big\}$$

becomes a reproducing kernel almost Pontryagin space and that $\Psi_{\mathcal{A}} : \mathcal{A} \to \mathcal{B}$ is an isomorphism; remember here that $\ker \Psi_{\mathcal{A}} = \bigcap_{\eta\in\Omega} \ker \Psi(\cdot,\eta) = \{0\}$.

We may view $\mathcal{B}$ as a completion of $\mathcal{L}$ via $\Phi_{\mathcal{L}}$. To be precise, $\langle \Psi_{\mathcal{A}} \circ \iota, \mathcal{B}\rangle$ is an almost Pontryagin space completion of $\mathcal{L}$ which is isomorphic to $\langle \iota, \mathcal{A}\rangle$. Now (4.1) follows from (4.6) since $\Psi_{\mathcal{A}} \circ \iota = \Phi_{\mathcal{L}}$, and (4.2) follows from (4.1) and [Wor14, Theorem A.15].

5° *Applying Lemma* 3.7*:* As we have seen in step 1°, cf. (4.4), it holds that

$$\emptyset \neq r_{\supseteq}^{\mathrm{app}}(S,\Phi;M) \setminus \mathbb{R} \subseteq r_{=}(T,\Psi) \cap \{\eta \in \Omega : \Psi(\cdot,\eta) \neq 0\}.$$

In particular, <7> is fullfilled. We conclude that

$$\begin{aligned} S(\mathcal{B}) = (\Psi_{\mathcal{A}} \times \Psi_{\mathcal{A}})(T) &= (\Psi_{\mathcal{A}} \times \Psi_{\mathcal{A}})\big(\operatorname{Clos}_{\mathcal{A}}[(\iota\times\iota)(S)]\big) \\ &= \operatorname{Clos}_{\mathcal{A}}\big[((\Psi_{\mathcal{A}}\circ\iota)\times(\Psi_{\mathcal{A}}\circ\iota))(S)\big] = \operatorname{Clos}_{\mathcal{A}}\big[(\Phi_{\mathcal{L}}\times\Phi_{\mathcal{L}})(S)\big]. \end{aligned}$$

Moreover, being isomorphic to T, the operator $S(\mathcal{B})$ has defect index $(1,1)$.

6° *Applying Lemma* 3.10*:* Hypothesis <11> holds by assumption (*ii*) of the theorem, <12> holds by (4.6), and <13> by (Ωdf4). It follows that $\mathcal{B}$ is an Ω-space.

7° *Uniqueness:* Assume that $\mathcal{B}_1$ and $\mathcal{B}_2$ are reproducing kernel almost Pontryagin spaces such that $\Phi_{\mathcal{L}}$ maps $\mathcal{L}$ isometrically onto dense subspaces of each of them. Then the inner products $[\cdot,\cdot]_{\mathcal{B}_1}$ and $[\cdot,\cdot]_{\mathcal{B}_2}$ coincide on the linear space

$$\mathcal{M} := \big\{ \Phi_{\mathcal{L}}(x) : x \in \mathcal{L} \big\}.$$

Hence, an inner product is well-defined on $\mathcal{M}$ by setting $[f,g]_{\mathcal{M}} := [f,g]_{\mathcal{B}_i}$, $f,g \in \mathcal{M}$, $i \in \{1,2\}$. The spaces $\mathcal{B}_1$ and $\mathcal{B}_2$ both contain $\langle \mathcal{M}, [\cdot,\cdot]_{\mathcal{M}}\rangle$ isometrically as a dense linear subspace and hence, by the uniqueness part of [Wor14, Theorem 4.1], are equal. ❑

Next we show the promised converse to Theorem 4.3.

4.6 Proposition. *Let $\mathcal{L}$ be an inner product space. Let $\Omega \subseteq \mathbb{C}$ be open and nonempty, $\mathcal{B}$ a reproducing kernel almost Pontryagin space of analytic functions on Ω, and $\iota : \mathcal{L} \to \mathcal{B}$ a linear and isometric map whose range is dense in $\mathcal{B}$. Define a map $\Phi : \mathcal{L} \times \Omega \to \mathbb{C}$ by (χ_η denotes point evaluation)*

$$\Phi(x,\eta) := \chi_\eta\big(\iota(x)\big), \quad x \in \mathcal{L}, \eta \in \Omega.$$

Then:

(*i*) Φ *satisfies* (Ωdf1) *and* (Ωdf2).

(*ii*) *Assume that $M \subseteq \Omega$ has accumulation points in each component of Ω. Then $\mathcal{L}' + \operatorname{span}\big\{\Phi(\cdot,\eta) : \eta \in M\big\} = \iota^*(\mathcal{B}')$, in particular,*

$$\dim\Big(\big[\mathcal{L}' + \operatorname{span}\{\Phi(\cdot,\eta) : \eta \in M\}\big]\Big/\mathcal{L}'\Big) = \operatorname{ind}_0 \mathcal{B} < \infty,$$

and the set $\mathcal{L}' \cap \operatorname{span}\{\Phi(\cdot,\eta) : \eta \in M\}$ is dense in $\mathcal{L}'$ w.r.t. $\mathcal{T}(\mathcal{L}')$.

Let, in addition, S be a linear relation in $\mathcal{L}$. Then:

(*iii*) *If* $(\Phi_{\mathcal{L}} \times \Phi_{\mathcal{L}})(S) \subseteq S(\mathcal{B})$, *then* $r_{\subseteq}(S, \Phi) = \Omega$, *in particular,* Φ *satisfies* (Ωdf3').

(*iv*) *If* $\operatorname{supp} \mathfrak{d}_{\mathcal{B}}$ *is discrete, then* Φ *satisfies* (Ωdf4).

(*v*) *If* $\mathcal{B}$ *is an* Ω*-space and* $\operatorname{Clos}_{\mathcal{B}} \big[(\Phi_{\mathcal{L}} \times \Phi_{\mathcal{L}})(S)\big] = S(\mathcal{B})$, *then*

$$r_{\supseteq}^{\mathrm{app}}(S, \Phi; \Omega) = \Omega \setminus \operatorname{supp} \mathfrak{d}_{\mathcal{B}}.$$

Provided that Ω *intersects both half-planes,* Φ *is an* Ω*-directing functional.*

Proof. Item (*i*) is obvious. For the proof of (*ii*), let $M \subseteq \Omega$ be given and assume that M accumulates in each component of Ω. By analyticity,

$$\bigcap_{\eta \in M} \ker \big(\chi_\eta|_{\mathcal{B}}\big) = \{0\}, \tag{4.7}$$

whence in particular $\{\chi_\eta|_{\mathcal{B}} : \eta \in M\}$ is point separating on $\mathcal{B}^\circ$. By its definition, $\Phi(\cdot, \eta) = \iota^*(\chi_\eta|_{\mathcal{B}})$, and we can invoke [Wor14, Proposition A.3, Lemma A.17] to conclude that

$$\iota^*(\mathcal{B}') = \mathcal{L}' + \operatorname{span}\big\{\Phi(\cdot, \eta) : \eta \in M\big\}.$$

To show the asserted density property we repeat an argument which already appeared in the proof of [Wor14, Proposition 4.3]. The relation (4.7) shows that $\operatorname{span}\{\chi_\eta|_{\mathcal{B}} : \eta \in M\}$ is w^*-dense in $\mathcal{B}'$. By reflexivity and convexity it is thus also dense w.r.t. the norm of $\mathcal{B}'$. Choose $\eta_1, \ldots, \eta_m$ such that $\{\chi_{\eta_i}|_{\mathcal{B}} : i = 1, \ldots, m\}$ is linearly independent and $\mathcal{B}' = \mathcal{B}^\wr \dot{+} \operatorname{span}\{\chi_{\eta_i}|_{\mathcal{B}} : i = 1, \ldots, m\}$, and denote by P the corresponding projection of $\mathcal{B}'$ onto $\mathcal{B}^\wr$. Then

$$\mathcal{B}^\wr \cap \operatorname{span}\big\{\chi_\eta|_{\mathcal{B}} : \eta \in M\big\} = P\Big(\operatorname{span}\big\{\chi_\eta|_{\mathcal{B}} : \eta \in M\big\}\Big),$$

and continuity of P implies that this space is norm-dense in $\mathcal{B}^\wr$. Applying the homeomorphism ι^* yields that $\mathcal{L}' \cap \operatorname{span}\{\Phi(\cdot, \eta) : \eta \in M\}$ is $\mathcal{T}(\mathcal{L}')$-dense in $\mathcal{L}'$.

Item (*iii*) is again obvious. For (*iv*) assume that $\operatorname{supp} \mathfrak{d}_{\mathcal{B}}$ is discrete, and let $O \subseteq \Omega$ be open and nonempty. Choose $\eta \in O \setminus \operatorname{supp} \mathfrak{d}_{\mathcal{B}}$, and $f \in \mathcal{B}$ with $f(\eta) \neq 0$. Since $\iota(\mathcal{L})$ is dense in $\mathcal{B}$, we find $x \in \mathcal{L}$ with $\Phi(x, \eta) = (\iota x)(\eta) \neq 0$.

We come to the proof of (*v*). Let $\eta \in \Omega \setminus \operatorname{supp} \mathfrak{d}_{\mathcal{B}}$ and $x \in \ker \Phi(\cdot, \eta)$. Then $(\iota x)(\eta) = 0$, and hence $\iota x \in \operatorname{ran}(S(\mathcal{B}) - \eta) = \operatorname{Clos}_{\mathcal{B}} \iota[\operatorname{ran}(S - \eta)]$. Choose $x_n \in \operatorname{ran}(S - \eta)$ with $\lim_{n\to\infty} \iota x_n = \iota x$ in the norm of $\mathcal{B}$, then

$$\begin{aligned}
&\lim_{n\to\infty} [x_n, x_n]_{\mathcal{L}} = \lim_{n\to\infty} [\iota x_n, \iota x_n]_{\mathcal{B}} = [\iota x, \iota x]_{\mathcal{B}} = [x, x]_{\mathcal{L}}, \\
&\lim_{n\to\infty} [x_n, y]_{\mathcal{L}} = \lim_{n\to\infty} [\iota x_n, \iota y]_{\mathcal{B}} = [\iota x, \iota y]_{\mathcal{B}} = [x, y]_{\mathcal{L}}, \quad y \in \mathcal{L}, \\
&\lim_{n\to\infty} \Phi(x_n, \eta) = \lim_{n\to\infty} (\iota x_n)(\eta) = (\iota x)(\eta) = \Phi(x, \eta), \quad \eta \in \Omega,
\end{aligned}$$

and we see that $\eta \in r_{\supseteq}^{\mathrm{app}}(S, \Phi; \Omega)$.

Assume that $\eta \in r_{\supseteq}^{\mathrm{app}}(S, \Phi; \Omega) \cap \operatorname{supp} \mathfrak{d}_{\mathcal{B}}$. Let $x \in \mathcal{L}$, then $\Phi(x, \eta) = (\iota x)(\eta) = 0$. Choose a sequence $(x_n)_{n\in\mathbb{N}}$ according to the definition of $r_{\supseteq}^{\mathrm{app}}(S, \Phi; \Omega)$. Then $\lim_{n\to\infty} \iota x_n = \iota x$ in the norm of $\mathcal{B}$, and we conclude that $\iota x \in \operatorname{ran}(S(\mathcal{B}) - \eta) = \ker \chi_\eta^{(\mathfrak{d}_{\mathcal{B}}(\eta))}$. This contradicts the fact that $\ker \chi_\eta^{(\mathfrak{d}_{\mathcal{B}}(\eta))}$ is not dense in $\mathcal{B}$. ❑

In many applications, the space $\mathcal{L}$ carries an additional algebraic structure.

4.7 Definition. Let $\mathcal{L}$ be an inner product space and let $.^{\#}$ be a conjugate-linear involution on $\mathcal{L}$ which is *anti-isometric*, i.e., satisfies

$$[x^{\#}, y^{\#}]_{\mathcal{L}} = [y, x]_{\mathcal{L}}, \quad x, y \in \mathcal{L}.$$

If $\Omega \subseteq \mathbb{C}$ and $\Phi : \mathcal{L} \times \Omega \to \mathbb{C}$, then we call Φ *real w.r.t.* $.^{\#}$, if[4]

$$\Omega = \Omega^{\#} \quad \text{and} \quad \Phi(x^{\#}, \cdot) = [\Phi(x, \cdot)]^{\#}, \quad x \in \mathcal{L}.$$

$\Diamond$

The following result is the supplement to Theorem 4.3 taking care of such involutions.

4.8 Proposition. *In the situation and under the hypothesis of Theorem* 4.3, *assume in addition that $\mathcal{L}$ carries a conjugate linear and anti-isometric involution $.^{\#}$, and that Φ is real w.r.t. this involution. Then the space $\mathcal{B}$ constructed in Theorem* 4.3 *is invariant under $.^{\#}$, and $.^{\#}|_{\mathcal{B}}$ is anti-isometric.*

Proof. The family $\mathcal{F} := \{\chi_\eta|_{\mathcal{B}} : \eta \in \Omega\}$ is a point-separating subfamily of $\mathcal{B}'$. Let $f \in \mathcal{B}$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements of $\mathcal{L}$ with $\lim_{n\to\infty} \Phi_{\mathcal{L}}(x_n) = f$ in the norm of $\mathcal{B}$. Then

$$\begin{aligned}
&\lim_{n,m\to\infty} \big[\Phi_{\mathcal{L}}(x_n) - \Phi_{\mathcal{L}}(x_m), \Phi_{\mathcal{L}}(y)\big]_{\mathcal{B}} = 0, \quad y \in \mathcal{L},\\
&\lim_{n,m\to\infty} \big[\Phi_{\mathcal{L}}(x_n) - \Phi_{\mathcal{L}}(x_m), \Phi_{\mathcal{L}}(x_n) - \Phi_{\mathcal{L}}(x_m)\big]_{\mathcal{B}} = 0,\\
&\lim_{n,m\to\infty} \chi_\eta\big(\Phi_{\mathcal{L}}(x_n) - \Phi_{\mathcal{L}}(x_m)\big) = 0, \quad \eta \in \Omega.
\end{aligned}$$

Using that $.^{\#}$ is anti-isometric, that $\mathcal{L}^{\#} = \mathcal{L}$, and that $\Omega^{\#} = \Omega$, we obtain

$$\begin{aligned}
&\lim_{n,m\to\infty} \big[\Phi_{\mathcal{L}}(x_n^{\#}) - \Phi_{\mathcal{L}}(x_m^{\#}), \Phi_{\mathcal{L}}(y)\big]_{\mathcal{B}} = 0, \quad y \in \mathcal{L},\\
&\lim_{n,m\to\infty} \big[\Phi_{\mathcal{L}}(x_n^{\#}) - \Phi_{\mathcal{L}}(x_m^{\#}), \Phi_{\mathcal{L}}(x_n^{\#}) - \Phi_{\mathcal{L}}(x_m^{\#})\big]_{\mathcal{B}} = 0,\\
&\lim_{n,m\to\infty} \chi_\eta\big(\Phi_{\mathcal{L}}(x_n^{\#}) - \Phi_{\mathcal{L}}(x_m^{\#})\big) = 0, \quad \eta \in \Omega.
\end{aligned}$$

By [Wor14, Proposition A.5], this implies that $(\Phi_{\mathcal{L}}(x_n^{\#}))_{n\in\mathbb{N}}$ is a Cauchy-sequence in the norm of $\mathcal{B}$. Thus it converges to some element of $\mathcal{B}$, say, $h := \lim_{n\to\infty} \Phi_{\mathcal{L}}(x_n^{\#})$. Continuity of point-evaluations implies that $h = f^{\#}$, and we see that $f^{\#} \in \mathcal{B}$.

Let $f, g \in \mathcal{B}$ be given, and choose approximating sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ as above. Then

$$[f, g]_{\mathcal{B}} = \lim_{n\to\infty} \big[\Phi_{\mathcal{L}}(x_n), \Phi_{\mathcal{L}}(y_n)\big]_{\mathcal{B}} = \lim_{n\to\infty} [x_n, y_n]_{\mathcal{L}} = \lim_{n\to\infty} [y_n^{\#}, x_n^{\#}]_{\mathcal{L}} = [g^{\#}, f^{\#}]_{\mathcal{B}}.$$

❑

[4] Observe the double meaning of the symbol $.^{\#}$. One, it is the given involution on the space $\mathcal{L}$, two, it is the natural involution present on $\mathbb{C}^{\Omega}$, cf. (2.1). This notice will apply throughout.

Matching with the commonly used conditions

The results in the literature providing representations in spaces of entire functions use two kinds of assumptions: one, that the closure T of S has defect index $(1,1)$ and, two, that the directing functional Φ is also defined on $\mathbb{R}$ and characterises the range of $(S-\eta)$ also for real points.

In the following we deduce a result based on similar assumptions, instead of (i) and (ii) of Theorem 4.3. This viewpoint also explains very clearly the different roles of assumptions on real and nonreal points η, cf. Remark 4.14. In order to point out the essentials and directly match the results from the literature, we use characterisation of ranges as in (udf3) instead of the pair of conditions (Ωdf3') and (Ωdf3").

4.9 Definition. Let $\mathcal{L}$ be an inner product space with $\operatorname{ind}_-\mathcal{L}<\infty$, let S be a symmetric linear relation in $\mathcal{L}$, let $\Omega\subseteq\mathbb{C}$, and let $\Phi:\mathcal{L}\times\Omega\to\mathbb{C}$. We call Φ a *strong Ω-directing functional* for S, if $\Omega\cap\mathbb{C}^\pm\neq\emptyset$, Φ satisfies (Ωdf1), (Ωdf2), (Ωdf4) and:

(Ωdf3) $\quad\Omega\setminus\mathbb{R}\subseteq r_=(S,\Phi)$.

$\Diamond$

4.10 Theorem. *Let $\mathcal{L}$ be an inner product space with $\operatorname{ind}_-\mathcal{L}<\infty$, let S be a symmetric linear relation in $\mathcal{L}$, let $\Omega\subseteq\mathbb{C}$, and let $\Phi:\mathcal{L}\times\Omega\to\mathbb{C}$ be a strong Ω-directing functional for S. Assume that there exists an almost Pontryagin space completion $\langle\iota,\mathcal{A}\rangle$ of $\mathcal{L}$, such that*

(i) $T:=\operatorname{Clos}_{\mathcal{A}\times\mathcal{A}}\big[(\iota\times\iota)(S)\big]$ has defect index $(1,1)$;

(ii) $\bigcap_{\eta\in\mathbb{C}\setminus\mathbb{R}}\operatorname{ran}(T-\eta)\subseteq\mathcal{A}^\circ$.

Then the function $\Phi_{\mathcal{L}}:x\mapsto\Phi(x,\cdot)$ establishes a representation of S in an Ω-space.

Clearly, the first condition is an assumption on nonreal points. Interestingly, the second condition corresponds to the assumption on real points in the classical case, see Remark 4.14 below.

In the proof we use two lemmata. In the first one we rewrite condition (i) of Theorem 4.10.

4.11 Lemma. *Let Φ be a strong Ω-directing functional for S, let $\langle\iota,\mathcal{A}\rangle$ be an almost Pontryagin space completion of $\mathcal{L}$, and set $T:=\operatorname{Clos}_{\mathcal{A}\times\mathcal{A}}\big[(\iota\times\iota)(S)\big]$. Then T has defect index $(1,1)$ if and only if $\Phi(\cdot,\eta)\in\iota^*(\mathcal{A}')$, $\eta\in\Omega\setminus\mathbb{R}$.*

Proof. The proof is based on a non-approximative variant of (3.4) and (3.5):

$$\eta\in r_\supseteq(S,\Phi),\operatorname{ran}(T-\eta)\text{ closed}\quad\Longrightarrow\quad\operatorname{codim}_{\mathcal{A}}\operatorname{ran}(T-\eta)\leq 1 \tag{4.8}$$

$$\begin{aligned}&\eta\in r_\supseteq(S,\Phi),\operatorname{ran}(T-\eta)\text{ closed},\operatorname{codim}_{\mathcal{A}}\operatorname{ran}(T-\eta)=1\\&\qquad\Longrightarrow\quad\eta\in r_\subseteq(S,\Phi),\Phi(\cdot,\eta)\in\iota^*(\mathcal{A}')\setminus\{0\},\operatorname{ran}(T-\eta)=\ker\tilde{\Phi}_\eta\end{aligned} \tag{4.9}$$

Their proof is similar as in Lemma 3.3, even simpler: For (4.8) note that the inclusion $\operatorname{ran}(S-\eta)\supseteq\ker\Phi(\cdot,\eta)$ implies $\operatorname{ran}(T-\eta)\supseteq\operatorname{Clos}_{\mathcal{A}}\iota[\ker\Phi(\cdot,\eta)]$ and the right side has codimension at most 1. For (4.9) consider again the projection φ introduced in the proof of (3.4), and observe that

$$\iota\big[\ker\Phi(\cdot,\eta)\big]\subseteq\operatorname{ran}(T-\eta)=\ker\varphi,$$

whence $\Phi(\cdot,\eta)=\iota^*\varphi$.

We come to the actual proof of the present lemma. Assume first that T has defect index $(1,1)$. By (4.8) we have $\operatorname{codim}_{\mathcal{A}}\operatorname{ran}(T-\eta)\leq 1$, $\eta\in\Omega\setminus\mathbb{R}$. Hence, the exceptional set from Theorem 2.11 does not intersect $\Omega\setminus\mathbb{R}$ and $\operatorname{codim}_{\mathcal{A}}\operatorname{ran}(T-\eta)=1$, $\eta\in\Omega\setminus\mathbb{R}$. Now (4.9) implies

$$\Phi(\cdot,\eta)\in\iota^*(\mathcal{A}')\setminus\{0\},\ \operatorname{ran}(T-\eta)=\ker\tilde{\Phi}_\eta,\quad \eta\in\Omega\setminus\mathbb{R}. \tag{4.10}$$

Conversely, assume that $\Phi(\cdot,\eta)\in\iota^*(\mathcal{A}')$, $\eta\in\Omega\setminus\mathbb{R}$. By (3.3) we have $\ker(T-\eta)\subseteq\ker\tilde{\Phi}_\eta$, and by ($\Omega$df4) the set $\{\eta\in\Omega\setminus\mathbb{R}:\tilde{\Phi}_\eta=0\}$ is discrete. We conclude that $\operatorname{codim}_{\mathcal{A}}\operatorname{ran}(T-\eta)\geq 1$ with possible ecxception of a discrete set. However, by (4.8), for such points in fact $\operatorname{codim}_{\mathcal{A}}\operatorname{ran}(T-\eta)=1$. Now we evaluate the defect indices of T with (2.7). ❑

From this lemma, we have an immediate corollary.

4.12 Corollary. *In the situation of Theorem* 4.10 *there exists* $\langle\iota,\mathcal{A}\rangle$ *such that the condition Theorem* 4.10*, (i), holds, if and only if*

$$\dim\Big(\big[\mathcal{L}'+\operatorname{span}\{\Phi(\cdot,\eta):\eta\in\Omega\setminus\mathbb{R}\}\big]\Big/_{\mathcal{L}'}\Big)<\infty. \tag{4.11}$$

If (4.11) *holds, the set of all completions satsfying Theorem* 4.10*, (i), has a smallest element (w.r.t. the partial order [SW12, Definition 6.2(ii)]), namely, the completion* $\langle\iota_0,\mathcal{A}_0\rangle$ *with*

$$\iota_0^*(\mathcal{A}_0')=\mathcal{L}'+\operatorname{span}\{\Phi(\cdot,\eta):\eta\in\Omega\setminus\mathbb{R}\}. \tag{4.12}$$

In the second lemma, we investigate condition (ii) of Theorem 4.10.

4.13 Lemma. *Assume that* (4.11) *holds. Then the following are equivalent.*

(i) *There exists a completion* $\langle\iota,\mathcal{A}\rangle$ *which satisfies Theorem* 4.10*, (i) and (ii).*

(ii) *Every completion that satisfies Theorem* 4.10*, (i), also satisfies (ii).*

(iii) *The smallest completion with Theorem* 4.10*, (i), satisfies*

$$\bigcap_{\eta\in\Omega\setminus\mathbb{R}}\operatorname{ran}(T-\eta)=\{0\}.$$

Proof. We need to show "$(i)\Rightarrow(ii),(iii)$". Hence, assume that a completion $\langle\iota,\mathcal{A}\rangle$ is given which satisfies Theorem 4.10, (i) and (ii). Set

$$\mathcal{D}:=\bigcap_{\eta\in\Omega\setminus\mathbb{R}}\operatorname{ran}(T-\eta),\quad B:=\mathcal{A}/_{\mathcal{D}},\quad \iota_{\mathcal{B}}:=\pi\circ\iota,$$

where $\pi:\mathcal{A}\to\mathcal{B}$ is the canonical projection. Consider $\mathcal{B}$ as an almost Pontryagin space being endowed with the inherited inner product and the factor topology.

The tuple $\langle \iota_{\mathcal{B}}, \mathcal{B} \rangle$ is a completion of $\mathcal{L}$. Due to the choice of $\mathcal{D}$ there exist linear functionals $\tilde{\Phi}'_\eta : \mathcal{B} \to \mathbb{C}$, $\eta \in \Omega \setminus \mathbb{R}$, with

$$\begin{tikzcd}
\mathcal{L} \arrow[r, "\iota"] \arrow[dr, "\iota_{\mathcal{B}}"'] & \mathcal{A} \arrow[r, "\tilde{\Phi}_\eta"] \arrow[d, "\pi"] & \mathbb{C} \\
& \mathcal{B} \arrow[ur, dashed, "\tilde{\Phi}'_\eta"'] &
\end{tikzcd}$$

Since $\mathcal{B}$ carries the final topology, we have $\tilde{\Phi}'_\eta \in \mathcal{B}'$. Moreover,

$$\iota_{\mathcal{B}}^*(\tilde{\Phi}'_\eta) = \tilde{\Phi}'_\eta \circ \iota_{\mathcal{B}} = \tilde{\Phi}'_\eta \circ \pi \circ \iota = \tilde{\Phi}_\eta \circ \iota = \Phi(\cdot, \eta),$$

whence $\Phi(\cdot, \eta) \in \iota_{\mathcal{B}}^*(\mathcal{B}')$. Thus $\langle \iota_{\mathcal{B}}, \mathcal{B} \rangle$ satisfies Theorem 4.10, (i). Remembering (4.10), we have

$$\pi^{-1}\Big(\bigcap_{\eta \in \Omega \setminus \mathbb{R}} \ker \tilde{\Phi}'_\eta \Big) = \bigcap_{\eta \in \Omega \setminus \mathbb{R}} \ker \tilde{\Phi}_\eta = \mathcal{D} = \ker \pi,$$

and hence

$$\bigcap_{\eta \in \Omega \setminus \mathbb{R}} \ker \tilde{\Phi}'_\eta = \{0\}.$$

Now let $\langle \iota_0, \mathcal{A}_0 \rangle$ be the completion with (4.12). The fact that $\langle \iota_0, \mathcal{A}_0 \rangle$ is the smallest w.r.t. the order [SW12, Definition 6.2], provides a surjective, continuous, and open map $\pi_0 : \mathcal{B} \to \mathcal{A}_0$ with $\iota_0 = \pi_0 \circ \iota_{\mathcal{B}}$. The right half of the diagram

$$\begin{tikzcd}
\mathcal{L} \arrow[r, "\iota_{\mathcal{B}}"] \arrow[dr, "\iota_0"'] & \mathcal{B} \arrow[r, "\tilde{\Phi}'_\eta"] \arrow[d, "\pi_0"] & \mathbb{C} \\
& \mathcal{B} \arrow[ur, "\tilde{\Phi}_{0,\eta}"'] &
\end{tikzcd}$$

commutes on $\operatorname{ran} \iota_{\mathcal{B}}$ since

$$\tilde{\Phi}'_\eta \circ \iota_{\mathcal{B}} = \Phi(\cdot, \eta) = \tilde{\Phi}_{0,\eta} \circ \iota_0 = (\tilde{\Phi}_{0,\eta} \circ \pi_0) \circ \iota_{\mathcal{B}}. \tag{4.13}$$

By continuity it thus commutes on all of $\mathcal{B}$. This yields

$$\ker \pi_0 \subseteq \bigcap_{\eta \in \Omega \setminus \mathbb{R}} \ker \tilde{\Phi}'_\eta = \{0\}$$

and we see that $\langle \iota_{\mathcal{B}}, \mathcal{B} \rangle$ and $\langle \iota_0, \mathcal{A}_0 \rangle$ are isomorphic. Thus, item (iii) of the present lemma holds.

To show item (ii), let $\langle \iota_{\mathcal{C}}, \mathcal{C} \rangle$ be an arbitrary completion with Theorem 4.10, (i), and let $\pi_{\mathcal{C}} : \mathcal{C} \to \mathcal{A}_0$ be surjective, continuous, and open with $\iota_{\mathcal{C}} = \iota_0 \circ \pi_{\mathcal{C}}$. Then, again using (4.13) and continuity, we obtain $\tilde{\Phi}_{\mathcal{C},\eta} = \tilde{\Phi}_{0,\eta} \circ \pi_{\mathcal{C}}$. It follows that

$$\bigcap_{\eta \in \Omega \setminus \mathbb{R}} \ker \tilde{\Phi}_{\mathcal{C},\eta} = \pi_{\mathcal{C}}^{-1}\Big(\bigcap_{\eta \in \Omega \setminus \mathbb{R}} \ker \tilde{\Phi}_{0,\eta} \Big) = \pi_{\mathcal{C}}^{-1}(\{0\}) \subseteq \mathcal{C}^\circ.$$

❑

It is now easy to deduce Theorem 4.10.

Proof of Theorem 4.10. Clearly, a strong Ω-directing functional is also an Ω-directing functional. We are going to apply Theorem 4.3 with $M := \Omega \setminus \mathbb{R}$. By means of Corollary 4.12, the present assumption (i) implies that Theorem 4.3, (i), is satisfied.

To show that Theorem 4.3, (ii), holds, we use the argument from the proof of Proposition 4.6, (ii). Consider the completion $\langle \iota_0, \mathcal{A}_0 \rangle$ with (4.12). Choose $\eta_1, \ldots, \eta_m \in \Omega \setminus \mathbb{R}$ such that $\{\Phi(\cdot, \eta_i) : i = 1, \ldots, m\}$ is linearly independent and

$$\mathcal{A}_0' = \mathcal{A}_0^{\wr} \dot{+} \operatorname{span}\{\tilde{\Phi}_{0,\eta_i} : i = 1, \ldots, m\},$$

and denote by P the corresponding projection of $\mathcal{A}_0'$ onto $\mathcal{A}_0^{\wr}$. Then

$$\mathcal{A}_0^{\wr} \cap \operatorname{span}\{\tilde{\Phi}_{0,\eta_i} : \eta \in \Omega \setminus \mathbb{R}\} = P\big(\operatorname{span}\{\tilde{\Phi}_{0,\eta_i} : \eta \in \Omega \setminus \mathbb{R}\}\big).$$

The fact that $\bigcap_{\eta \in \Omega \setminus \mathbb{R}} \ker \tilde{\Phi}_{0,\eta} = \{0\}$ means that $\operatorname{span}\{\tilde{\Phi}_{0,\eta_i} : \eta \in \Omega \setminus \mathbb{R}\}$ is dense in $\mathcal{A}_0'$, whence its image under P is dense in $\mathcal{A}_0^{\wr}$. Passing to $\mathcal{L}'$ with the isomorphism ι_0^* yields that indeed Theorem 4.3, (ii), is satisfied. ❑

4.14 Remark. Condition (i) in Theorem 4.10 is an assumption on nonreal points η. Contrasting this, (ii) actually corresponds to assumptions on real points (also if it does not look like this): instead of (ii) we could equally well use assumptions on real points in the definition of a directing functional similar as in Definition 1.5.

Proving this fact would require a sufficiently general version of Theorem 1.6, namely, for linear relations in almost Pontryagin spaces. Such a version is not yet available, and establishing it is beyond the scope of the present paper. However, for the positive definite case the necessary machinery would be available. ♢

The above remark leads to a proof of Proposition 1.9.

Proof of Proposition 1.9. Let $\langle \iota, \mathcal{A} \rangle$ be the Hilbert space completion of $\mathcal{L}$. By our present assumption the relation $T := \operatorname{Clos}_{\mathcal{A}\times\mathcal{A}}(\iota \times \iota)(S)$ has defect index $(1,1)$ and $\operatorname{mul} T = \{0\}$. By Lemma 4.11, therefore, $\Phi(\cdot, \eta) \in \iota^*(\mathcal{A}')$, $\eta \in \Omega \setminus \mathbb{R}$. Since $\mathcal{A}$ is positive definite we have $\mathbb{C}\setminus\mathbb{R} \subseteq r(T)$. The assumptions of Lemma 3.8 are fullfilled, and we find a lifting $\Psi : \mathcal{A} \times \Omega \to \mathbb{C}$ with the properties stated there. By (4.9) we have $r_=(T, \Psi) \supseteq \Omega \setminus \mathbb{R}$.

Now use Theorem 1.6: Let μ be a positive Borel measure on $\mathbb{R}$ such that

$$[x, y]_{\mathcal{L}} = \int_{\mathbb{R}} \Phi(x, t)\overline{\Phi(y, t)}\, d\mu(t), \quad x, y \in \mathcal{L}.$$

If $(x_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{L}$ such that $\lim_{n\to\infty} \iota x_n = x$ in $\mathcal{A}$, then $(\Phi(x_n, \cdot))_{n\in\mathbb{N}}$ is a Cauchy-sequence in $L^2(\mu)$. However, by continuity of $\Psi(\cdot, \eta)$,

$$\lim_{n\to\infty} \Phi(x_n, \eta) = \lim_{n\to\infty} \Psi(\iota x_n, \eta) = \Psi(x, \eta), \quad \eta \in \Omega.$$

Thus $\lim_{n\to\infty} \Phi(x_n, \cdot) = \Psi(x, \cdot)$ in $L^2(\mu)$. It follows that

$$[x, y]_{\mathcal{A}} = \int_{\mathbb{R}} \Psi(x, t)\overline{\Psi(y, t)}\, d\mu(t), \quad x, y \in \mathcal{A}. \tag{4.14}$$

Now consider $x \in \bigcap_{\eta \in \Omega \setminus \mathbb{R}} \operatorname{ran}(T - \eta)$. Then $\Psi(x, \eta) = 0$, $\eta \in \Omega \setminus \mathbb{R}$, and hence also for $\eta \in \mathbb{R}$. The relation (4.14) yields $x = 0$.

The proof is finished by referring to Theorem 4.10 and (2.11) (or to the correct version of [GG97, Theorem II.8.5]).

❑

5 Ω-space completions

The purely geometric version of an Ω-space reads as:

5.1 Definition. Let Ω be an open and nonempty subset of $\mathbb{C}$. We call an inner product space $\mathcal{L}$ an Ω-*inner product space*, if it satisfies the following axioms.

(Ω-IP1) The elements of $\mathcal{L}$ are analytic functions on Ω.

(Ω-IP2) $\operatorname{supp} \mathfrak{d}_{\mathcal{L}}$ is a discrete subset of Ω.

(Ω-IP3) There exists $\eta \in \mathbb{C} \setminus \mathbb{R}$ such that $\eta, \overline{\eta} \in \Omega$, $\mathfrak{d}_{\mathcal{L}}(\eta) = \mathfrak{d}_{\mathcal{L}}(\overline{\eta}) = 0$, and

$$\forall \xi \in \{\eta, \overline{\eta}\} \; \forall f \in \mathcal{L}, f(\xi) = 0 : \quad \frac{f(\zeta)}{\zeta - \xi} \in \mathcal{L}.$$

(Ω-IP4) There exists $\eta \in \mathbb{C} \setminus \mathbb{R}$ such that (Ω-IP3) holds and

$\forall f, g \in \mathcal{L}, f(\eta) = g(\eta) = 0 :$

$$\Big[\frac{\zeta - \overline{\eta}}{\zeta - \eta} f(\zeta), \frac{\zeta - \overline{\eta}}{\zeta - \eta} g(\zeta)\Big]_{\mathcal{L}} = \big[f(\zeta), g(\zeta)\big]_{\mathcal{L}}.$$

$\Diamond$

For an Ω-inner product space $\mathcal{L}$ the following question appears naturally:

> *Provided $\mathcal{L}$ has a reproducing kernel space completion, is this completion an Ω-space ?*

It is a consequence of Theorem 4.3 that the answer is affirmative.

5.2 Proposition. *Let $\Omega \subseteq \mathbb{C}$ be open and nonempty, and let $\mathcal{L}$ be an Ω-inner product space with* $\operatorname{ind}_- \mathcal{L} < \infty$*. Then $\mathcal{L}$ has a reproducing kernel space completion if and only if there exists a subset M of Ω which has accumulation points in each component of $\Omega \setminus \mathbb{R}$ and is such that (i), (ii) of Theorem* 4.3 *hold.*

If $\mathcal{L}$ has a reproducing kernel space completion, then this completion is an Ω-space.

Proof. By (Ω-IP3) and (Ω-IP4) the multiplication operator $S(\mathcal{L})$ is the inverse Caley-transform of an isometry, hence symmetric. We show that the map

$$\Phi(f, \eta) := f(\eta), \quad f \in \mathcal{L}, \eta \in \Omega, \tag{5.1}$$

is an Ω-directing functional for the multiplication operator $S(\mathcal{L})$. The properties (Ωdf1), (Ωdf2) are obvious, and (Ωdf3') holds since $r_{\subseteq}(S(\mathcal{L}), \Phi) = \Omega$. From ($\Omega$-IP3) we have $\eta, \overline{\eta} \in r_{\supseteq}(S(\mathcal{L}), \Phi)$, in particular ($\Omega$df3") holds. Finally, ($\Omega$df4) holds by ($\Omega$-IP2).

The map $\Phi_{\mathcal{L}}$ acts as the identity: $\Phi_{\mathcal{L}}(f) = \Phi(f, \cdot) = f$. The assertion now follows at once from Theorem 4.3 and Proposition 4.6. ❑

With each space which possesses a reproducing kernel space completion a structural constant is associated.

5.3 Definition. Let Ω be a nonempty set, $\mathcal{L} \subseteq \mathbb{C}^\Omega$ an inner product space, and assume that $\mathcal{L}$ has a reproducing kernel space completion (say $\mathcal{A}$). Then we denote

$$\Delta(\mathcal{L}) := \operatorname{ind}_0 \mathcal{A}.$$

$\Diamond$

In general it is a difficult task to compute $\Delta(\mathcal{L})$, and no effective algorithm is known. Recall [Wor14, Proposition 4.7]: *The number $\Delta(\mathcal{L})$ is the minimum of all numbers $n \in \mathbb{N}$ such that there exists an n-element subset $L \subseteq \Omega$ with*

$$\forall\, (f_n)_{n\in\mathbb{N}}, f_n \in \mathcal{L} :$$

$$\left.\begin{array}{l} \lim_{n\to\infty}[f_n, f_n]_{\mathcal{L}} = 0 \\ \lim_{n\to\infty}[f_n, g]_{\mathcal{L}} = 0, g \in \mathcal{L} \\ \lim_{n\to\infty} f_n(\eta) = 0, \eta \in L \end{array}\right\} \quad \Longrightarrow \quad \lim_{n\to\infty} f_n(\eta) = 0, \eta \in \Omega \tag{5.2}$$

In general it may happen that (5.2) holds for some subsets with $\Delta(\mathcal{L})$ elements, but does not hold for some others with the same cardinality. Ω-inner product spaces are in this respect better behaved.

5.4 Proposition. *Let $\Omega \subseteq \mathbb{C}$ be open and assume that $\Omega \cap \mathbb{R} \neq \emptyset$. Let $\mathcal{L}$ be an Ω-inner product space and assume that $\mathcal{L}$ has a reproducing kernel space completion. Then there exists a discrete subset $\mathcal{Z}$ of Ω which is symmetric w.r.t. the real line and contains at most $2 \operatorname{ind}_- \mathcal{L}$ nonreal points, such that* (5.2) *holds for all $L \subseteq \mathbb{C} \setminus \mathcal{Z}$, $|L| \geq \Delta(\mathcal{L})$.*

5.5 Remark. This result has some computational significance. Let $n \in \mathbb{N}$ be given. Then we can decide whether or not $\Delta(\mathcal{L}) \leq n$ by testing (5.2) for $(\operatorname{ind}_- \mathcal{L} + 1)$ many pairwise disjoint n-element subsets of $\mathbb{C}^+$. Of course, $\Delta(\mathcal{L})$ remains far from effectively computable, simply because (5.2) itself involves the universal quantifiers over the infinite sets of all sequences $(f_n)_{n\in\mathbb{N}}$ and all points $\eta \in \Omega$. $\Diamond$

In the proof of Proposition 5.4 we use the following fact.

5.6 Lemma. *Let $\Omega \subseteq \mathbb{C}$ be open and assume that $\Omega \cap \mathbb{R} \neq \emptyset$. Let $\mathcal{B}$ be an Ω-space and assume that $\mathcal{B}^\circ \neq \{0\}$. Then there exists a function $g \in \mathcal{B}$, such that*

$$\mathcal{B}^\circ = \big\{p(\zeta)g(\zeta) : p \in \mathbb{C}[\zeta], \deg p < \operatorname{ind}_0 \mathcal{B}\big\}, \quad |\operatorname{supp} \mathfrak{d}_g \cap \mathbb{C}^\pm| \leq \operatorname{ind}_- \mathcal{B}.$$

If $\Omega = \Omega^\#$, $\mathcal{B} = \mathcal{B}^\#$, and $.^\#$ is anti-isometric, then g can be chosen such that $g = g^\#$.

Proof. Our aim is to apply [KW99b, Proposition 1] with the multiplication operator $S(\mathcal{B})$. This requires to check the regularity assumptions [KW99b, (2.3),(2.4)]. The first is obvious. To show the second choose $f \in \mathcal{B}^\circ \setminus \{0\}$. By Remark 2.20, $\operatorname{supp} \mathfrak{d}_f$ is a discrete subset of Ω. Thus we find $\eta_\pm \in \Omega \cap \mathbb{C}^\pm$ with $f(\eta_\pm) \neq 0$. However, $\operatorname{ran}(S(\mathcal{B}) - \eta_\pm) = \ker \chi_{\eta_\pm}|_{\mathcal{B}}$, and it follows that

$$\operatorname{ran}(S(\mathcal{B}) - \eta_\pm) + \operatorname{span}\{f\} = \mathcal{B}.$$

Now [KW99b, Proposition 1] furnishes us with a basis $\{g_1, \dots, g_{\mathrm{ind}_0 \mathcal{B}}\}$ of $\mathcal{B}^\circ$ having the property that $(g_i; g_{i+1}) \in S(\mathcal{B})$, $i = 1, \dots, \mathrm{ind}_0 \mathcal{B} - 1$. This just means that $g_{i+1}(\zeta) = \zeta g_i(\zeta)$, whence

$$\mathcal{B}^\circ = \big\{p(\zeta) g_1(\zeta) : p \in \mathbb{C}[\zeta], \deg p < \mathrm{ind}_0 \mathcal{B}\big\}. \tag{5.3}$$

If η is a zero of g, then $\mathcal{B}^\circ \subseteq \mathrm{ran}(S(\mathcal{B})-\eta)$, and hence $\mathrm{ran}(S(\mathcal{B})/\mathcal{B}^\circ - \eta) \neq \mathcal{B}/\mathcal{B}^\circ$. Thus, each zero of g belongs to the spectrum of $S(\mathcal{B})/\mathcal{B}^\circ$. However, $\sigma(S(\mathcal{B})/\mathcal{B}^\circ)$ may contain at most $\mathrm{ind}_- \mathcal{B}/\mathcal{B}^\circ = \mathrm{ind}_- \mathcal{B}$ many points in the open upper or lower half-plane, cf. [DS87a, p.162, Corollary].

Assume now that $\mathcal{B}^\circ$ is invariant under the involution $.^\#$ and that this involution is anti-isometric. Then also $\mathcal{B}^\circ$ is invariant under $.^\#$, and we find a polynomial q with $g_1^\# = q g_1$. However, $\big|\frac{g_1^\#(\xi)}{g_1(\xi)}\big| = 1$, $\xi \in \mathbb{R}$, and it follows that q is identically equal to some unimodular constant. Passing from g_1 to a suitable (unimodular) scalar multiple, say h, we can thus achieve that $h^\# = h$. Clearly, (5.3) remains valid with h instead of g_1. ❑

Proof of Proposition 5.4. Let $\mathcal{B}$ be the reproducing kernel space completion of $\mathcal{L}$. Then $\mathcal{B}$ is an Ω-space and $\mathrm{ind}_- \mathcal{B} = \mathrm{ind}_- \mathcal{L}$. Observe that (5.2) is getting stronger when L gets smaller. Hence, it is enough to construct $\mathcal{Z}$ such that (5.2) holds for each $\Delta(\mathcal{L})$-element subset of $\Omega \setminus \mathcal{Z}$.

Choose $g \in \mathcal{B}^\circ$ according to Lemma 5.6, and set $\mathcal{Z} := \{\eta \in \Omega : g(\eta) = 0\}$. Let $\eta_1, \dots, \eta_{\Delta(\mathcal{L})}$ be pairwise different points of $\Omega \setminus \mathcal{Z}$. Due to the description of $\mathcal{B}^\circ$ in Lemma 5.6, the family $\{\chi_{\eta_i}|_{\mathcal{B}} : i = 1, \dots, \Delta(\mathcal{L})\}$ is point separating on $\mathcal{B}^\circ$. By [Wor14, Proposition A.5] the hypothesis in (5.2) implies that $\lim_{n\to\infty} f_n = 0$ in the norm of $\mathcal{B}$. Continuity of point evaluations now yields $\lim_{n\to\infty} f_n(\eta) = 0$, $\eta \in \Omega$. ❑

Finally, we turn to the case that $\Omega = \mathbb{C}$ which deserves particular attention. As in the classical case of universal directing functionals we are led to embeddings into de Branges spaces.

Recall the definition of a de Branges space, including the purely geometric variant:

5.7 Definition. We call an inner product space $\mathcal{L}$ a *de Branges inner product space*, if it is a $\mathbb{C}$-inner product space, is invariant under the involution $f \mapsto f^\#$, and

$$\big[f^\#, g^\#\big]_{\mathcal{L}} = [g, f]_{\mathcal{L}}, \quad f, g \in \mathcal{L}.$$

If $\mathcal{L}$ is actually a $\mathbb{C}$-space subject to the above, then we speak of a *de Branges space.* ◊

Combining Proposition 5.2 with Proposition 4.8 yields:

5.8 Corollary. *Let $\mathcal{L}$ be a de Branges inner product space. If $\mathcal{L}$ has a reproducing kernel space completion, then this completion is a de Branges space.*

Proof. The $\mathbb{C}$-directing functional (5.1) is obviously real w.r.t. to $.^\#$. ❑

For the positive definite case, i.e., when $\mathcal{L}$ and its completion $\mathcal{B}$ are required to be positive definite, this is well-known. In fact, it is a classical fact going back (at least) to work of M.Riesz on the power moment problem, cf. [Rie23]; a rather general approach is given by L.Pitt in [Pit76].

In [Wor14, Proposition 4.8] we gave a sufficient condition for an inner product space of functions to have a reproducing kernel space completion. For de Branges inner product spaces this condition is also necessary, in fact, one can say a bit more.

5.9 Proposition. *Let $\mathcal{L}$ be a de Branges inner product space which has a reproducing kernel space completion. Then there exists a positive Borel measure μ on $\mathbb{R}$ with discrete support, a nonnegative integer n, points $\eta_1, \ldots, \eta_n \in \mathbb{R}$ and $\gamma > 0$, such that*

(*i*) *each element $f \in \mathcal{L}$ is square integrable w.r.t. μ, and*

$$[f, g]_{\mathcal{L}} = \int_{\mathbb{R}} f(\zeta)\overline{g(\zeta)}\, d\mu - \gamma \sum_{i=1}^{n} f(\eta_i)\overline{g(\eta_i)}, \quad f, g \in \mathcal{L};$$

(*ii*) *for each $\eta \in \mathbb{C}$ the point evaluation functional $\chi_\eta|_{\mathcal{L}}$ is continuous w.r.t. the $L^2(\mu)$-norm on $\mathcal{L}$.*

Proof. Let $\mathcal{B}$ be the reproducing kernel space completion of $\mathcal{L}$. By Corollary 5.8, $\mathcal{B}$ is a de Branges space. The family $\{\chi_\eta : \eta \in \mathbb{R}\}$ is point separating, hence [Wor14, Proposition A.9] provides points $\eta_1, \ldots, \eta_n \in \mathbb{R}$ and $\gamma > 0$, such that the inner product

$$(f, g)_{\mathcal{L}} := [f, g]_{\mathcal{L}} + \gamma \sum_{i=1}^{n} f(\eta_i)\overline{g(\eta_i)}, \quad f, g \in \mathcal{B},$$

turns $\mathcal{B}$ into a Hilbert space and induces the topology of $\mathcal{B}$. Moreover, $\langle \mathcal{B}, (\cdot,\cdot)_{\mathcal{B}} \rangle$ is a de Branges–Hilbert space (which is seen by elementary computation or reference to [KW99a, Lemma 3.2]).

Due to [Bra68, Theorem 22], there exists a positive Borel measure μ on $\mathbb{R}$ with discrete support, such that

$$(f, g)_{\mathcal{B}} = \int_{\mathbb{R}} f(\lambda)\overline{g(\lambda)}\, d\mu, \quad f, g \in \mathcal{B}.$$

By the definition of $(\cdot,\cdot)_{\mathcal{B}}$ the formula required in (*i*) holds. Since $\mathcal{B}$ is a de Branges space if endowed with the $L^2(\mu)$-norm, point evaluations are continuous w.r.t. this norm. ❑

Appendix A. Preservation of analyticity

In this appendix we provide explicit proofs for Theorem 2.15 and Lemma 2.16.

Proof of Theorem 2.15(*i*). This is a general analyticity argument which works since the set $\Lambda := \{\zeta \in \rho(A) : [a(\zeta), y]_{\mathcal{K}} \neq 0\}$ is open and dense in $\mathbb{C}$. We give the general argument.

Let $\Lambda \subseteq \mathbb{C}$ be open and dense and let $F \in \mathbb{H}(\Lambda, \mathbb{C}_\infty)$. Consider the set

$$\mathcal{O} := \big\{ O \subseteq \mathbb{C} : \ O \text{ open}, O \supseteq \Lambda, \ \exists\, F_O \in \mathbb{H}(O, \mathbb{C}_\infty) : F_O|_\Lambda = F \big\},$$

and set $\Lambda^* := \bigcup \{O : O \in \mathcal{O}\}$. If $O_1, O_2 \in \mathcal{O}$ with corresponding extensions F_{O_1} and F_{O_2}, then Λ is dense in $O_1 \cap O_2$ and by continuity $F_{O_1}|_{O_1 \cap O_2} = F_{O_2}|_{O_1 \cap O_2}$.

Thus a function $F^*:\Lambda^*\to\mathbb{C}_\infty$ exists with $F^*|_O=F_O$, $O\in\mathcal{O}$. Since all $O\in\mathcal{O}$ are open and $F^*|_O\in\mathbb{H}(O,\mathbb{C}_\infty)$, it follows that $F^*\in\mathbb{H}(\Lambda^*,\mathbb{C}_\infty)$. Thus the set Λ^* belongs to $\mathcal{O}$. Clearly, it is the largest element of $\mathcal{O}$. ❑

Item (ii) is the crucial part of Theorem 2.15. It relies on H.Langer's spectral theorem for definitisable selfadjoint operators and estimates of subharmonic functions.

Proof of Theorem 2.15(ii).

Step 1: The essence is to estimate expressions $[(A-\zeta)^{-1}a,b]_{\mathcal{K}}$ where $\zeta\in\rho(A)$ and $a,b\in\mathcal{K}$. To this end we employ the functional calculus for A as developed in [Lan65, §I.7] (see [Lan82], and [DS87b] for an extention to linear relations). For practical reasons, we use the formulation given in [KP15]. Let p be a real definitising polynomial for A, $\zeta_0\in\rho(A)\cap\mathbb{C}^+$ and set

$$q(\zeta):=p(\zeta)(\zeta-\zeta_0)^{-\deg p}(\zeta-\overline{\zeta_0})^{-\deg p}.$$

Now let $\mathcal{V}$, and $T:\mathcal{V}\to\mathcal{K}$, and E, be the Hilbert space, the bounded operator, and the $\mathcal{V}$-spectral measure constructed in [KP15, Theorem 7.19]. The identity

$$\frac{1}{z-\zeta}=\frac{-1}{q(\zeta)}\frac{q(z)-q(\zeta)}{z-\zeta}+q(z)\cdot\frac{1}{q(\zeta)(z-\zeta)}$$

together with the identity in [Lan82, top of p.26] yields the representation of the resolvent of A as

$$(A-\zeta)^{-1}=\frac{1}{p(\zeta)}\sum_{j=0}^{2\deg p-1}D_j\zeta^j+\frac{1}{q(\zeta)}\int_{\mathbb{R}}\frac{1}{\xi-\zeta}dE_{T^+a,T^+b}(\xi),\quad \zeta\in\rho(A),\tag{A.1}$$

where D_j are appropriate bounded operators, T^+ denotes the Krein space adjoint of T, and E_{T^+a,T^+b} is the measure

$$E_{T^+a,T^+b}(\Delta):=\big(E(\Delta)T^+a,T^+b\big)_{\mathcal{V}},\quad \Delta\subseteq\mathbb{R}\text{ Borelset}.$$

The formula (A.1) allows to estimate

$$\Big|\log\big|[(A-\zeta)^{-1}a,b]_{\mathcal{K}}\big|\Big|\le\big|\log|p(\zeta)|\big|+$$
$$+\Big|\log\Big|\sum_{j=0}^{2\deg p-1}[D_ja,b]_{\mathcal{K}}\zeta^j+(\zeta-\zeta_0)^{\deg p}(\zeta-\overline{\zeta_0})^{\deg p}\int_{\mathbb{R}}\frac{1}{\xi-\zeta}dE_{T^+a,T^+b}(\xi)\Big|\Big|.$$

The integral on the right side is a function of bounded type in both half-planes $\mathbb{C}^+$ and $\mathbb{C}^-$, see, e.g., [GG97, Lemma I.4.4], and polynomials also have this property. Thus [GG97, Lemma I.4.3] yields (λ_2 denotes the 2-dimensional Lebesgue measure)

$$\int_{\mathbb{C}}\frac{\big|\log|[(A-\zeta)^{-1}a,b]_{\mathcal{K}}|\big|}{(1+|\zeta|)^4}\,d\lambda_2(\zeta)<\infty.\tag{A.2}$$

Note here that $\mathbb{C}\setminus\rho(A)$ is a Lebesgue zero set.

A more explicit estimate can be given for $\log^+ |[(A-\zeta)^{-1}a, b]_{\mathcal{K}}|$. We have

$$\log^+ \big|[(A-\zeta)^{-1}a, b]_{\mathcal{K}}\big| \leq \big|\log|p(\zeta)|\big| + \sum_{j=0}^{2\deg p-1} \Big(\log^+ \big|[D_j a, b]_{\mathcal{K}}\big| + j\log^+|\zeta|\Big)$$
$$+ \log^+ \big|(\zeta-\zeta_0)^{\deg p}(\zeta-\overline{\zeta_0})^{\deg p}\big| + \log^+ \Big|\int_{\mathbb{R}} \frac{1}{\xi-\zeta} dE_{T^+a,T^+b}(\xi)\Big|$$
$$+ (2\deg p + 1)\log 2.$$

We have $|[D_j a, b]_{\mathcal{K}}| \leq \|D_j\| \cdot \|a\|_{\mathcal{K}} \cdot \|b\|_{\mathcal{K}}$, where $\|\cdot\|$ is the operator norm corresponding to $\|\cdot\|_{\mathcal{K}}$. The total variation of E_{T^+a,T^+b} does not exceed $\|T^+\|^2 \cdot \|a\|_{\mathcal{K}} \cdot \|b\|_{\mathcal{K}}$. Using [GG97, Lemma I.4.5] to estimate the area integral of the term involving the integral, provides us with constants $c_1, c_2 > 0$ such that

$$\int_{\mathbb{C}} \frac{\log^+ |[(A-\zeta)^{-1}a, b]_{\mathcal{K}}|}{(1+|\zeta|)^4}\, d\lambda_2(\zeta) \leq c_1 \log^+ \big(\|a\|_{\mathcal{K}} \cdot \|b\|_{\mathcal{K}}\big) + c_2. \tag{A.3}$$

Step 2: We show that the family $\mathcal{F}$ is locally bounded in Ω. Let $\eta_0 \in \Omega$, and choose a closed disk $U_r(\eta_0) = \{\zeta \in \mathbb{C} : |\zeta - \eta_0| \leq r\}$ which is entirely contained in Ω. The function $W\Theta_{x,y}$ is analytic in Ω, and hence $\log^+ |W\Theta_{x,y}|$ is subharmonic. This implies that for $\eta \in D_{\frac{r}{2}}(\eta_0)$

$$\log^+ \big|(W\Theta_{x,y})(\eta)\big| \leq \frac{4}{\pi r^2} \int_{D_{\frac{r}{2}}(\eta)} \log^+ \big|(W\Theta_{x,y})(\zeta)\big|\, d\lambda_2(\zeta)$$
$$\leq \frac{4(1+r+|\eta_0|)^4}{\pi r^2} \int_{D_{\frac{r}{2}}(\eta)} \frac{\log^+ |(W\Theta_{x,y})(\zeta)|}{(1+|\zeta|)^4}\, d\lambda_2(\zeta)$$
$$\leq \frac{4(1+r+|\eta_0|)^4}{\pi r^2} \Bigg[\int_{D_{\frac{r}{2}}(\eta)} \frac{\log^+ |W(\zeta)|}{(1+|\zeta|)^4}\, d\lambda_2(\zeta)$$
$$+ \int_{\mathbb{C}} \frac{\big|\log |[a(\zeta), y]_{\mathcal{K}}|\big|}{(1+|\zeta|)^4}\, d\lambda_2(\zeta) + \int_{\mathbb{C}} \frac{\log^+ |[a(\zeta), x]_{\mathcal{K}}|}{(1+|\zeta|)^4}\, d\lambda_2(\zeta)\Bigg].$$

Note here that $\mathbb{C} \setminus \{\zeta \in \rho(A) : [a(\zeta), y]_{\mathcal{K}} \neq 0\}$ is a Lebesgue zero set. Plugging (A.2), (A.3), and further estimating yields constants $c_1', c_2' > 0$ such that

$$\log^+ \big|(W\Theta_{x,y})(\eta)\big| \leq c_1' \log^+ \|x\|_{\mathcal{K}} + c_2', \quad \eta \in D_{\frac{r}{2}}(\eta_0).$$

We see that the functions in $\mathcal{F}$ are uniformly bounded on the disk $D_{\frac{r}{2}}(\eta_0)$. ❑

Item *(iii)* is a simple consequence of *(ii)*.

Proof of Theorem 2.15*(iii)*. Choose W as in *(ii)*. Let $x_n \in \mathcal{M}$, $n \in \mathbb{N}$, $x \in \mathcal{K}$, and assume that $\lim_{n\to\infty} x_n = x$. Then

$$\lim_{n\to\infty} (W\Theta_{x_n,y})(\eta) = (W\Theta_{x,y})(\eta), \quad \eta \in \rho(A), [a(\eta), y]_{\mathcal{K}} \neq 0. \tag{A.4}$$

The sequence $(x_n)_{n\in\mathbb{N}}$ is bounded in the norm of $\mathcal{K}$, and by item (ii) the family $\{W\Theta_{x_n,y} : n \in \mathbb{N}\} \subseteq \mathbb{H}(\Omega)$ is normal. Vitali's theorem yields that $\lim_{n\to\infty} W\Theta_{x_n,y}$ exists in $\mathbb{H}(\Omega)$; let us denote this limit by G. By (A.4) we have

$$G(\eta) = (W\Theta_{x,y})(\eta), \quad \eta \in \rho(A), [a(\eta), y]_{\mathcal{K}} \neq 0,$$

and hence $\frac{G}{W}$ is a meromorphic continuation of $\Theta_{x,y}$ to Ω. Clearly, $\mathfrak{d}_{\frac{G}{W}} \geq -\mathfrak{d}_W = \mathfrak{d}$. Thus $x \in \mathcal{M}$. ❑

Finally, we give a proof of Lemma 2.16.

Proof of Lemma 2.16. Linearity of $\Phi(\cdot, \eta)$ is easy to see: For each $x, y \in \mathcal{X}$ and $\alpha, \beta \in \mathbb{C}$, the analytic functions

$$\Phi(\alpha x + \beta y, \cdot) \quad \text{and} \quad \alpha\Phi(x, \cdot) + \beta\Phi(y, \cdot)$$

coincide on L. Since L has an accumulation point in Ω and Ω is connected, they coincide everywhere.

We come to the proof of continuity of $\Phi_{\mathcal{X}}$ (which is the main assertion). Consider the map

$$\kappa : \begin{cases} \mathcal{X} & \to & \mathcal{X} \times \mathbb{H}(\Omega) \\ x & \mapsto & \big(x, \Phi(x, \cdot)\big) \end{cases}$$

As we just saw, this map is linear. The space $\mathbb{H}(\Omega)$, and hence also $\mathcal{X} \times \mathbb{H}(\Omega)$, is a complete metrisable topological vector space. We are going to apply the Closed Graph Theorem, cf. [Sch71, III.2.3 Theorem]. Let $x_n \in \mathcal{X}$, $x \in \mathcal{X}$, with $\lim_{n\to\infty} x_n = x$, and assume that $\lim_{n\to\infty} \Phi(x_n, \cdot) = f$ in $\mathbb{H}(\Omega)$. Then

$$f(\eta) = \lim_{n\to\infty} \Phi(x_n, \eta) = \Phi(x, \eta), \quad \eta \in L.$$

Since L accumulates, this implies that $f = \Phi(x, \cdot)$. Thus the graph of κ is closed, and we infer that κ is continuous. The map $\Phi_{\mathcal{X}}$ equals $\pi_2 \circ \kappa$, where $\pi_2 : \mathcal{X} \times \mathbb{H}(\Omega)$ is the projection onto the second component. Hence, $\Phi_{\mathcal{X}}$ is continuous. Since point evaluations of derivatives $\chi_\eta^{(l)}$ are continuous on $\mathbb{H}(\Omega)$ and $\frac{\partial^l}{\partial \eta^l}\Phi(\cdot, \eta) = \chi_\eta^{(l)} \circ \Phi_{\mathcal{X}}$, it follows that $\frac{\partial^l}{\partial \eta^l}\Phi(\cdot, \eta) \in \mathcal{X}'$.

Let K be a compact subset of Ω. The family $\{\Phi(\cdot, \eta) : \eta \in K\}$ is a pointwise bounded subfamily of $\mathcal{X}'$. By the Principle of Uniform Boundedness, cf. [Sch71, III.4.2 Theorem], it is equicontinuous. Thus we find $\epsilon > 0$ such that ($d_{\mathcal{X}}$ denotes a metric on $\mathcal{X}$ which establishes that $\mathcal{X}$ is a complete metrisable topological vector space)

$$|\Phi(x, \eta)| = |\Phi(x, \eta) - \Phi(0, \eta)| \leq 1, \quad d_{\mathcal{X}}(x, 0) \leq \epsilon, \quad \eta \in K.$$

This shows that $\sup_{x\in M} \sup_{\eta\in K} |\Phi(x, \eta)| \leq \frac{1}{\epsilon} \sup_{x\in M} d_{\mathcal{X}}(x, 0)$. ❑

References

[AG81] N.I. Akhiezer and I.M. Glazman. *Theory of linear operators in Hilbert space. Vol. II.* Vol. 10. Monographs and Studies in Mathematics. Translated from the third Russian edition by E. R. Dawson, Translation edited by W. N. Everitt. Pitman (Advanced Publishing Program), Boston, Mass.-London, 1981, pages.

[AI89] T.Ya. Azizov and I.S. Iohvidov. *Linear operators in spaces with an indefinite metric*. Pure and Applied Mathematics (New York). Translated from the Russian by E. R. Dawson, A Wiley-Interscience Publication. Chichester: John Wiley & Sons Ltd., 1989, pp. xii+304.

[AMR13] A. Aleman, R.T.W. Martin, and W.T. Ross. "On a theorem of Livsic". In: *J. Funct. Anal.* 264.4 (2013), pp. 999–1048.

[Bek01] M. Bekker. "On the extension problem for continuous positive definite generalized Toeplitz kernels". In: *Illinois J. Math.* 45.4 (2001), pp. 1315–1330.

[Bek89] M.B. Bekker. "Isometric operators with directing mappings". Russian. In: *Current analysis and its applications (Russian)*. "Naukova Dumka", Kiev, 1989, pp. 3–9, 218.

[Bek98] M.B. Bekker. "On integral representation of the continuous $O(n)$-positive definite functions". In: *Integral Equations Operator Theory* 31.2 (1998), pp. 133–149.

[Ber00] Yu.M. Berezansky. "The works of M.G.Krein on eigenfunction expansion for selfadjoint operators and their applications and development". In: *Differential operators and related topics, Vol. I (Odessa, 1997)*. Vol. 117. Oper. Theory Adv. Appl. Birkhäuser, Basel, 2000, pp. 21–43.

[BG91] C.A. Berenstein and R. Gay. *Complex variables*. Vol. 125. Graduate Texts in Mathematics. An introduction. New York: Springer-Verlag, 1991, pp. xii+650.

[BK88] Yu.M. Berezanskii and Yu.G. Kondratev. *Spektralnye metody v beskonechnomernom analize*. Russian. "Naukova Dumka", Kiev, 1988, p. 680.

[Bog74] J. Bognár. *Indefinite inner product spaces*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 78. New York: Springer-Verlag, 1974, pp. ix+224.

[Bra68] L. de Branges. *Hilbert spaces of entire functions*. Englewood Cliffs, N.J.: Prentice-Hall Inc., 1968, pp. ix+326.

[Con78] J.B. Conway. *Functions of one complex variable*. Second edition. Vol. 11. Graduate Texts in Mathematics. New York: Springer-Verlag, 1978, pp. xiii+317.

[DS87a] A. Dijksma and H. de Snoo. "Symmetric and selfadjoint relations in Kreĭn spaces. I". In: *Operators in indefinite metric spaces, scattering theory and other topics (Bucharest, 1985)*. Vol. 24. Oper. Theory Adv. Appl. Basel: Birkhäuser, 1987, pp. 145–166.

[DS87b] A. Dijksma and H. de Snoo. "Symmetric and selfadjoint relations in Kreĭn spaces. II". In: *Ann. Acad. Sci. Fenn. Ser. A I Math.* 12.2 (1987), pp. 199–216.

[FLL12] C. Fulton, H. Langer, and A. Luger. "Mark Krein's method of directing functionals and singular potentials". In: *Math. Nachr.* 285.14-15 (2012), pp. 1791–1798.

[GG71] V.I. Gorbachuk and M.L. Gorbachuk. “Gewisse Fragen der Spektraltheorie einer linearen Differentialgleichung zweiter Ordnung mit unbeschränkten Operatorkoeffizienten.” Russian. In: *Ukr. Mat. Zh.* 23 (1971), pp. 3–14.

[GG97] M.L. Gorbachuk and V.I. Gorbachuk. *M. G. Krein's lectures on entire operators.* Vol. 97. Operator Theory: Advances and Applications. Basel: Birkhäuser Verlag, 1997, pp. x+220.

[GL74] M. Großmann and H. Langer. “Über indexerhaltende Erweiterungen eines hermiteschen Operators im Pontrjaginraum”. German. In: *Math. Nachr.* 64 (1974), pp. 289–317.

[IKL82] I. S. Iohvidov, M.G. Krein, and H. Langer. *Introduction to the spectral theory of operators in spaces with an indefinite metric.* Vol. 9. Mathematical Research. Berlin: Akademie-Verlag, 1982, p. 120.

[JLT92] P. Jonas, H. Langer, and B. Textorius. “Models and unitary equivalence of cyclic selfadjoint operators in Pontrjagin spaces”. In: *Operator theory and complex analysis (Sapporo, 1991).* Vol. 59. Oper. Theory Adv. Appl. Basel: Birkhäuser, 1992, pp. 252–284.

[Kac83] I.S. Kac. “Linear relations generated by canonical differential equations”. In: *Funktsional. Anal. i Prilozhen.* 17.4 (1983), pp. 86–87.

[Kac85] I.S. Kac. “Linear relations, generated by a canonical differential equation on an interval with a regular endpoint, and expansibility in eigenfunctions”. Russian. In: *VINITI Deponirovannye Nauchnye Raboty* 195.1 (1985). Deposited in Ukr NIINTI, No. 1453, 1984, 50 pp., b.o. 720.

[Kac86] I.S. Kac. “Expansibility in eigenfunctions of a canonical differential equation on an interval with singular endpoints and associated linear relations”. Russian. In: *VINITI Deponirovannye Nauchnye Raboty* 282.12 (1986). Deposited in Ukr NIINTI, No. 2111, 1986, 64 pp., b.o. 1536.

[KL78] M.G. Krein and H. Langer. “Über einige Fortsetzungsprobleme, die eng mit der Theorie hermitescher Operatoren im Raume Π_κ zusammenhängen. II. Verallgemeinerte Resolventen, u-Resolventen und ganze Operatoren”. In: *J. Funct. Anal.* 30.3 (1978), pp. 390–447.

[Kle10] M. Kleinert. “Models for cyclic definitisable selfadjoint operators in Krein spaces”. PhD thesis. Vienna University of Technology, 2010.

[KP15] M. Kaltenbäck and R. Pruckner. “Functional Calculus for Definitizable Self-adjoint Linear Relations on Krein Spaces”. In: *Integral Equations Operator Theory* 83.4 (2015), pp. 451–482.

[Kre44] M.G. Krein. “On Hermitian operators whose deficiency indices are 1”. In: *C. R. (Doklady) Acad. Sci. URSS (N.S.)* 43 (1944), pp. 323–326.

[Kre46a] M. Krein. “On a general method of decomposing Hermite-positive nuclei into elementary products”. In: *C. R. (Doklady) Acad. Sci. URSS (N.S.)* 53 (1946), pp. 3–6.

[Kre46b] M.G. Krein. “Concerning the resolvents of an Hermitian operator with the deficiency-index (m, m)”. In: *C. R. (Doklady) Acad. Sci. URSS (N.S.)* 52 (1946), pp. 651–654.

[Kre48] M.G. Krein. “On Hermitian operators with directed functionals”. Russian. In: *Akad. Nauk Ukrain. RSR. Zbirnik Prac′ Inst. Mat.* 1948.10 (1948), pp. 83–106.

[Kre49] M.G. Krein. “The fundamental propositions of the theory of representations of Hermitian operators with deficiency index (m, m)”. In: *Ukrain. Mat. Žurnal* 1.2 (1949), pp. 3–66.

[KW98] M. Kaltenbäck and H. Woracek. “Generalized resolvent matrices and spaces of analytic functions”. In: *Integral Equations Operator Theory* 32.3 (1998), pp. 282–318.

[KW99a] M. Kaltenbäck and H. Woracek. “Pontryagin spaces of entire functions. I”. In: *Integral Equations Operator Theory* 33.1 (1999), pp. 34–97.

[KW99b] M. Kaltenbäck and H. Woracek. “The Krein formula for generalized resolvents in degenerated inner product spaces”. In: *Monatsh. Math.* 127.2 (1999), pp. 119–140.

[KWW05] M. Kaltenbäck, H. Winkler, and H. Woracek. “Almost Pontryagin spaces”. In: *Recent advances in operator theory and its applications*. Vol. 160. Oper. Theory Adv. Appl. Basel: Birkhäuser, 2005, pp. 253–271.

[Lan16] H. Langer. “Transfer functions and local spectral uniqueness for Sturm-Liouville operators, canonical systems and strings”. In: *Integral Equations Operator Theory* 85 (2016), pp. 1–23.

[Lan65] H. Langer. “Spektraltheorie linearer Operatoren in J-Räumen und einige Anwendungen auf die Schar $L(\lambda) = \lambda^2 I + \lambda B + C$”. German. Habilitation. Fakultät für Mathematik und Naturwissenschaften, Technische Universität Dresden, 1965.

[Lan70] H. Langer. “Über die Methode der richtenden Funktionale von M. G. Krein”. In: *Acta Math. Acad. Sci. Hungar.* 21 (1970), pp. 207–224.

[Lan82] H. Langer. “Spectral functions of definitizable operators in Kreĭn spaces”. In: *Functional analysis (Dubrovnik, 1981)*. Vol. 948. Lecture Notes in Math. Berlin: Springer, 1982, pp. 1–46.

[Liv60a] M. S. Livšic. “On a class of linear operators in Hilbert space”. In: *Amer. Math. Soc. Transl. (2)* 13 (1960), pp. 61–83.

[Liv60b] M.S. Livšic. “Isometric operators with equal deficiency indices, quasi-unitary operators”. In: *Amer. Math. Soc. Transl. (2)* 13 (1960), pp. 85–103.

[LM00] M. Lesch and M.M. Malamud. “The inverse spectral problem for first order systems on the half line”. In: *Differential operators and related topics, Vol. I (Odessa, 1997)*. Vol. 117. Oper. Theory Adv. Appl. Birkhäuser, Basel, 2000, pp. 199–238.

[LT78] H. Langer and B. Textorius. "A generalization of M. G. Kreĭn's method of directing functionals to linear relations". In: *Proc. Roy. Soc. Edinburgh Sect. A* 81.3-4 (1978), pp. 237–246.

[LT84] H. Langer and B. Textorius. "Spectral functions of a symmetric linear relation with a directing mapping. I". In: *Proc. Roy. Soc. Edinburgh Sect. A* 97 (1984), pp. 165–176.

[LT85] H. Langer and B. Textorius. "Spectral functions of a symmetric linear relation with a directing mapping. II". In: *Proc. Roy. Soc. Edinburgh Sect. A* 101.1-2 (1985), pp. 111–124.

[Mar11] R.T.W. Martin. "Representation of simple symmetric operators with deficiency indices $(1,1)$ in de Branges space". In: *Complex Anal. Oper. Theory* 5.2 (2011), pp. 545–577.

[Nik02] N.K. Nikolski. *Operators, functions, and systems: an easy reading. Vol. 2*. Vol. 93. Mathematical Surveys and Monographs. Model operators and systems, Translated from the French by Andreas Hartmann and revised by the author. Providence, RI: American Mathematical Society, 2002, pp. xiv+439.

[Ovc80] I.E. Ovčarenko. "The expansion of Hermitian forms and operators. The problem of obtaining special integral representations of positive definite functions. III". Russian. In: *Teor. Funktsiĭ Funktsional. Anal. i Prilozhen.* 34 (1980), pp. 81–89, iii.

[Pit76] L.D. Pitt. "Weighted L^p closure theorems for spaces of entire functions". In: *Israel J. Math.* 24.2 (1976), pp. 94–118.

[Pov50] A.Ya. Povzner. "On M. G. Kreĭn's method of directing functionals". Russian. In: *Učenye Zapiski Har′kov Gos. Univ. Zapiski Naučno-Issled. Inst. Mat. Meh. i Har′kov. Mat. Obšč. (4)* 28(20) (1950), pp. 43–52.

[Rem91] R. Remmert. *Theory of complex functions*. Vol. 122. Graduate Texts in Mathematics. Translated from the second German edition by Robert B. Burckel, Readings in Mathematics. Springer-Verlag, New York, 1991, pp. xx+453.

[Rem98] R. Remmert. *Classical topics in complex function theory*. Vol. 172. Graduate Texts in Mathematics. Translated from the German by Leslie Kay. New York: Springer-Verlag, 1998, pp. xx+349.

[Rie23] M. Riesz. "Sur le problème des moments. III." French. In: *Ark. Mat. Astron. Fys.* 17.16 (1923), p. 52.

[Rof60] F.S. Rofe-Beketov. "Expansion in eigenfunctions of infinite systems of differential equations in the non-self-adjoint and self-adjoint cases". Russian. In: *Mat. Sb. (N.S.)* 51 (93) (1960), pp. 293–342.

[Sch71] H.H. Schaefer. *Topological vector spaces*. Third printing corrected, Graduate Texts in Mathematics, Vol. 3. Springer-Verlag, New York-Berlin, 1971, pp. xi+294.

[ST10] L.O. Silva and J.H. Toloza. "On the spectral characterization of entire operators with deficiency indices $(1,1)$". In: *J. Math. Anal. Appl.* 367.2 (2010), pp. 360–373.

[SW] H. de Snoo and H. Woracek. “Compressed resolvents, Q-functions and h_0-resolvents in almost Pontryagin spaces”. In: *Oper. Theory Adv. Appl.* (). to appear, 59 pp.

[SW12] H. de Snoo and H. Woracek. “Sums, couplings, and completions of almost Pontryagin spaces”. In: *Linear Algebra Appl.* 437.2 (2012), pp. 559–580.

[Tex06] B. Textorius. “Directing mappings in Kreĭn spaces”. In: *Operator theory and indefinite inner product spaces.* Vol. 163. Oper. Theory Adv. Appl. Basel: Birkhäuser, 2006, pp. 351–363.

[Wor14] H. Woracek. “Reproducing kernel almost Pontryagin spaces”. In: *Linear Algebra Appl.* 461 (2014), pp. 271–317.

[YY95] V.A. Yavryan and A.V. Yavryan. “Spectral decomposition of single-pair complex integral operators”. In: *Izv. Nats. Akad. Nauk Armenii Mat.* 30.3 (1995), pp. 32–48.

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