Bounds on order of indeterminate moment sequences

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Abstract: We investigate the order ρ of the four entire functions in the Nevanlinna matrix of an indeterminate Hamburger moment sequence. In operator theoretic language, this is the asymptotic behaviour of the eigenvalues of the associated Jacobi operator. We give upper and lower estimates for ρ which are explicit in terms of the parameters of the canonical system associated with the moment sequence via its three-term recurrance. Under a weak regularity assumption these estimates coincide, and hence ρ becomes computable. Dropping regularity leads to examples where the bounds do not coincide and do not coincide with the order. In particular this provides examples that in an estimate for order due to M.S.Livšic in 1939 equality does not always hold. Our proofs proceed via the theory of canonical systems.

AMS MSC 2010: 44A60, 30D15, 37J99, 47B36, 34L20
Keywords: Indeterminate moment problem, canonical system, order of entire function, asymptotic of eigenvalues

1 Introduction

Let $(s_n)_{n=0}^{\infty}$ be a of sequence real numbers, and assume that the Hamburger power moment problem for this sequence is solvable and indeterminate. Then the totality of all positive Borel measures on $\mathbb{R}$ with power moments $(s_n)_{n=0}^{\infty}$ is parameterised via their Cauchy-transforms with help of an entire 2×2-matrix function called the *Nevanlinna matrix* of the moment sequence. A classical result of M.Riesz is that the entries of this matrix are entire functions of minimal exponential type, cf. [Rie23]. Further, all the entries of the Nevanlinna matrix have the same order, cf. [BW06; BP94]. We denote this common number by $\rho((s_n)_{n=0}^{\infty})$ and call it the *order of the moment sequence.*

The order can be computed for several moment sequences for which the moment problem is explicitly solvable. As for theorems rather than examples, the only known estimate for $\rho((s_n)_{n=0}^{\infty})$ in terms of the sequence $(s_n)_{n=0}^{\infty}$ itself (and – probably – the first result in this context dealing with growth properties other than the exponential type) is due to M.S.Livšic back in 1939, cf. [Liv39]. It asserts that

$$\rho((s_n)_{n=0}^{\infty}) \geq \limsup_{n\to\infty} \frac{2n\ln n}{\ln s_{2n}}, \tag{1.1}$$

the right hand side being the order of the entire function $\sum_{n=0}^{\infty} \frac{z^{2n}}{s_{2n}}$. The question whether there exist moment problems for which the order is different from its Livšic estimate appears to have remained open since then. In particular, it is mentioned as such in [BS14]. The difficulty can be explained as follows. Let $P_n(z) = \sum_{k=0}^{n} b_{k,n} z^k$, $n \in \mathbb{N}_0$, be the orthonormal polynomials of the first kind associated with the moment sequence $(s_n)_{n=0}^{\infty}$. Then the order $\rho((s_n)_{n=0}^{\infty})$ is

[‡]This work was supported by a joint project of the Austrian Science Fund (FWF, I 1536–N25) and the Russian Foundation for Basic Research (RFBR, 13-01-91002-ANF).

expressed in terms of the coefficients $b_{k,n}$ as

$$\rho((s_n)_{n=0}^\infty) = \limsup_{k\to\infty} \frac{-2k\ln k}{\ln \sum\limits_{n=k}^{\infty} b_{k,n}^2}, \tag{1.2}$$

cf. [BS14, Theorem 3.1]. Livšic' estimate (1.1) is obtained when dropping all summands but $b_{n,n}^2$. While the term $b_{n,n}$, being the leading coefficient of the orthonormal polynomial P_n, is easily expressed via the Jacobi parameters of the sequence $(s_n)_{n=0}^\infty$, see (1.4) below, and can in turn be estimated by s_{2n}, the other terms $b_{k,n}$, $k<n$, are nearly impossible to control.

Our first objective in the present paper is to establish upper and lower bounds for the order of the functions in the monodromy matrix of a canonical system whose Hamiltonian H has a very particular form. Namely, H has determinant zero, is piecewise constant, and constancy intervals accumulate only to its right endpoint, cf. Definition 1.1. It is well-known that such canonical systems mimick the three-term recurrance of a Hamburger moment sequence, see, e.g., [Kac99]. We employ the recent work [Rom] about the order of the monodromy matrix of a canonical system to establish the upper estimate Theorem 2.7. The lower estimate Proposition 2.15 is easy to see and follows, e.g., from [BS14]. Under a weak regularity assumption these bounds coincide, and hence yield a formula for order, cf. Theorem 2.22.

The second theme in the paper is to construct examples which show that there exist Hamburger moment sequences whose order is strictly larger than the right hand side of (1.1). We shall provide explicit examples that the gap between the actual order and the Livšic estimate can be arbitrarily close to 1, cf. Corollary 3.6. Actually, such examples can be found already in the class of symmetric moment sequences (meaning that $s_n = 0$ for odd n). The proof again depends on [Rom].

Structuring of the paper is as follows. In the remaining part of this introduction we recall the connections between moment sequence and Jacobi matrices on the one hand, and canonical systems on the other. It is vital to have this connection on hand, since our proofs proceed via the theory of canonical systems. Section 2 is in some sense the core of the paper. We establish the upper and lower bounds for order and discuss regularily distributed sequences. The subject of Section 3 is the construction of examples showing that equality in Livšic' estimate (1.1) may fail. In fact we show – slightly stronger – that the bound obtained from (1.2) by dropping all summands but $b_{n,n}^2$ can differ from the order by any pregiven number (only taking into account that the order is always between 0 and 1), cf. Corollary 3.6.

Moment sequences, Jacobi matrices, and canonical systems

We establish our results on order taking the viewpoint of canonical systems. To translate to moment sequences and/or Jacobi parameters, it is necessary to have these connections on hand explicitly.

The relation between moment sequences and Jacobi matrices is most classical and commonly exploited. A standard reference is [Akh61]. Given a moment sequence $(s_n)_{n=0}^\infty$, the orthogonal polynomials P_n, $n\in\mathbb{N}_0$, satisfy a three-term recurrence relation

$$zP_n(z) = \rho_n P_{n+1}(z) + q_n P_n(z) + \rho_{n-1} P_{n-1}(z), \quad n = 0, 1, 2, \dots. \tag{1.3}$$

The coefficients ρ_n and q_n in this recurrency are called the *Jacobi parameters* associated with the sequence $(s_n)_{n=0}^{\infty}$. They can be computed from the sequence $(s_n)_{n=0}^{\infty}$ via determinantal formulae. However, these expressions are hardly accessible to practical computation. One connection we need frequently is that

$$b_{n,n} = \left(\prod_{k=1}^{n-1} \rho_k \right)^{-1}. \tag{1.4}$$

Let us now recall the – maybe less commonly used – relation with canoncial systems. The basic idea is that the three-term recurrence is nothing but a canonical system with a piecewise constant Hamiltonian. This idea can be made precise: Hamburger moment problems correspond to a certain type of canonical systems, and, under a suitable normalisation, this correspondence is one-to-one. An explicit presentation of these matters can be found in [Kac99].

Let $L \in (0, \infty]$ and $H\colon [0, L) \to \mathbb{R}^{2\times 2}$ be a measurable function such that for almost every $x \in [0, L)$ the matrix $H(x)$ is positive semidefinite with $\operatorname{tr} H(x) = 1$. Then the equation

$$\frac{\partial}{\partial x} y(x, z) = -zJH(x)y(x, z), \quad x \in [0, L),$$

where $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and z is a complex parameter, is called the *canonical system* with *Hamiltonian* H. If $L < \infty$, we say that *limit circle case* takes place for H, whereas for $L = \infty$ one speaks of *limit point case.*

1.1 Definition. Let $\vec{l} = (l_n)_{n=1}^{\infty}$ be a sequence of positive numbers and let $\vec{\phi} = (\phi_n)_{n=1}^{\infty}$ be a sequence of real numbers with $\phi_{n+1} \not\equiv \phi_n \mod \pi$, $n \in \mathbb{N}$. Set

$$x_0 := 0, \qquad x_n := \sum_{k=1}^{n} l_k, \ n \in \mathbb{N}, \qquad x_\infty := \sum_{k=1}^{\infty} l_k \in (0, \infty]. \tag{1.5}$$

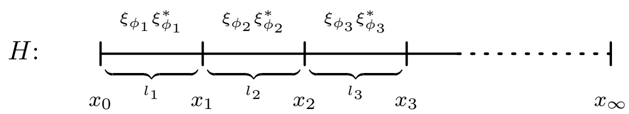
Denote

$$\xi_\phi := \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}, \quad \phi \in \mathbb{R}.$$

Then we call the function $H : [0, x_\infty) \to \mathbb{R}^{2\times 2}$ defined by

$$H(x) := \xi_{\phi_n} \xi_{\phi_n}^*, \quad x \in [x_{n-1}, x_n), \ n \in \mathbb{N},$$

the *Hamburger Hamiltonian* with *lengths* $\vec{l}$ and *angles* $\vec{\phi}$. Sometimes we refer to the points x_n as the *nodes* of H.



$\Diamond$

The correspondence between moment sequences, Jacobi parameters, and Hamburger Hamiltonians is given via the formulae (here Q_n, $n \in \mathbb{N}$, denote the orthogonal polynomials of the second kind)

$$l_n = P_n(0)^2 + Q_n(0)^2, \quad n \in \mathbb{N}, \tag{1.6}$$

$$\frac{1}{\rho_n} = |\sin(\phi_{n+1} - \phi_n)|\sqrt{l_n l_{n+1}}, \quad n \in \mathbb{N}, \tag{1.7}$$

$$q_n = -\frac{1}{l_n}\big[\cot(\phi_{n+1} - \phi_n) + \cot(\phi_n - \phi_{n-1})\big], \quad n \in \mathbb{N}. \tag{1.8}$$

Of course, this relation is again somewhat implicit since the above formulae contain the off-diagonal Jacobi parameters and, by their structure, cannot easily be inverted. However, the moment sequence $(s_n)_{n=0}^{\infty}$ is indeterminate if and only if the sequence $\vec{l} = (l_n)_{n=1}^{\infty}$ is summable. And if $(s_n)_{n=0}^{\infty}$ is indeterminate, then its Nevanlinna matrix coincides with the monodromy matrix of the canonical system with the corresponding Hamburger Hamiltonian, i.e., the matrix $W(L, z)$ where $W(x, z)$ is the unique solution of the inital value problem

$$\begin{cases} J\frac{\partial}{\partial x}W(x,z) = zH(x)W(x,z), \quad x \in [0, L], \\ W(0,z) = I. \end{cases}$$

2 Estimates for the order

We use a pointwise and an averaged measure for the decay of a sequence of positive numbers.

2.1 Definition. Let $\vec{y} = (y_n)_{n=1}^{\infty}$ be a bounded sequence of positive real numbers and let $\alpha \geq 0$. Then we set

$$\Delta^*(\vec{y}) := \sup\big\{\tau \geq 0 : y_n = \mathrm{O}(n^{-\tau})\big\},$$

$$\Delta(\vec{y}) := \sup\Big\{\tau \geq 0 : \frac{1}{n}\sum_{k=n}^{2n-1} y_k = \mathrm{O}(n^{-\tau})\Big\},$$

$$\delta(\vec{y}, \alpha) := \liminf_{n\to\infty} G(n; \vec{y}, \alpha) \quad \text{where} \quad G(n; \vec{y}, \alpha) := \frac{-1}{n \ln n} \ln\Big(y_n^{\alpha}\prod_{k=1}^{n-1} y_k\Big).$$

◊

The numbers $\Delta^*(\vec{y})$ and $\Delta(\vec{y})$ are understood as elements of $[0, \infty]$. Note here that the sets appearing in their definition are nonempty, since $\vec{y}$ is bounded. The number $\delta(\vec{y}, \alpha)$ is, a priori, an element of $[-\infty, \infty]$.

2.2 Lemma. *Let $\vec{y} = (y_n)_{n=1}^{\infty}$ be a bounded sequence of positive real numbers and let $\alpha \geq 0$. Then*

$$\Delta^*(\vec{y}) \leq \Delta(\vec{y}) \leq \delta(\vec{y}, \alpha). \tag{2.1}$$

Proof. The inequality $\Delta^*(\vec{y}) \leq \Delta(\vec{y})$ is clear. To show that $\Delta(\vec{y}) \leq \delta(\vec{y}, \alpha)$, consider $\tau \geq 0$ and $c \geq 1$ with $\frac{1}{n}\sum_{k=n}^{2n-1} y_k \leq cn^{-\tau}$, $n \in \mathbb{N}$.

For $m \in \mathbb{N}$, $m \geq 2$, let $p(m) \in \mathbb{N}_0$ and $r(m) \in \{0, \dots, 2^{p(m)} - 1\}$ be the unique numbers with $m - 1 = 2^{p(m)} + r(m)$. The arithmetic-geometric mean inequality gives

$$\begin{aligned} \prod_{k=1}^{m-1} y_k &= \prod_{j=1}^{p(m)} \prod_{k=2^{j-1}}^{2^j-1} y_k \cdot \prod_{k=2^{p(m)}}^{m-1} y_k \\ &\leq \prod_{j=1}^{p(m)} \underbrace{\Big(\frac{1}{2^{j-1}} \sum_{k=2^{j-1}}^{2^j-1} y_k \Big)}_{\leq c(2^{j-1})^{-\tau}}{}^{2^{j-1}} \cdot \underbrace{\Big(\frac{1}{r(m)+1} \sum_{k=2^{p(m)}}^{m-1} y_k \Big)}_{\leq \frac{2^{p(m)}}{r(m)+1} c(2^{p(m)})^{-\tau}}{}^{r(m)+1}, \end{aligned}$$

and we obtain ($\log_2$ denotes the logarithm to the basis 2)

$$\begin{aligned} \frac{-1}{m \log_2 m} \log_2 \Big(y_m^\alpha \prod_{k=1}^{m-1} y_k \Big) &\geq \frac{1}{m \log_2 m} \Big[-\alpha \underbrace{\log_2 y_m}_{\leq \log_2 \|\vec{y}\|_\infty} - \log_2 c \underbrace{\sum_{j=1}^{p(m)} 2^{j-1}}_{=2^{p(m)}-1} \\ &\quad + \tau \underbrace{\sum_{j=1}^{p(m)} (j-1) 2^{j-1}}_{=p(m)2^{p(m)} - 2^{p(m)+1} + 2} - (r(m)+1) \log_2 \frac{2^{p(m)} c}{r(m)+1} + (r(m)+1) \tau p(m) \Big] \\ &\geq \tau \frac{p(m)}{\log_2 m} - \frac{r(m)+1}{m} \frac{1}{\log_2 m} \log_2 \frac{2^{p(m)} c}{r(m)+1} + \mathrm{o}(1). \end{aligned}$$

The second summand is nonnegative and, since $\log_2 x \leq x \ln 2$, it is $\mathrm{o}(1)$. Moreover, $p(m) = \lfloor \log_2(m-1) \rfloor$, and hence $\lim_{m\to\infty} \frac{p(m)}{\ln m} = 1$. Therefore $\delta(\vec{y}, \alpha) \geq \tau$. ❑

2.1 An upper bound for $\rho(H)$

For a Hamiltonian H in the limit circle case we denote by $\rho(H)$ the order of the entries of its monodromy matrix. The following fact is, up to using the connection (1.6)–(1.8), nothing but [BS14, Theorem 1.2].

2.3 Proposition. *Let H be a limit circle Hamburger Hamiltonian with lengths $\vec{l}$ and angles $\vec{\phi}$. Then the order of H does not exceed the convergence exponent of $\vec{l}$, i.e.,*

$$\rho(H) \leq \inf \big\{ p > 0 : \vec{l} \in \ell^p \big\}.$$

In our first main result, Theorem 2.7 below, we give an upper bound for $\rho(H)$ which takes the asymptotic behaviour of the sequence of angles into consideration.

To quantify the behaviour of length- and angle sequences of a Hamburger Hamiltonian, we use the power scale and a pointwise measure for the decay of lengths, an averaged measure for the decay of angle-differences, and a measure for the speed of possible convergence of angles weighted with lengths, i.e., taking into account peaks of lengths.

2.4 Definition. Let H be a Hamburger Hamiltonian with lengths $\vec{l}$ and angles $\vec{\phi}$. Set

$$\begin{aligned} \Delta_l(H) &:= \Delta^*(\vec{l}), \qquad \Delta_l^+(H) := \max\big\{1, \Delta_l(H)\big\}, \\ \Delta_\phi(H) &:= \Delta\big((|\sin(\phi_{n+1}-\phi_n)|)_{n=1}^\infty\big). \end{aligned}$$

Provided that $\Delta_l^+(H) < \infty$, set

$$\Lambda(H) := \sup_{\phi\in[0,\pi)} \sup\Big\{\tau \geq 0 : \sum_{j=n}^\infty l_j|\sin(\phi_j-\phi)| = \mathrm{O}\big(n^{1-\Delta_l^+-\tau}\big)\Big\} \in [0,\infty].$$

When no confusion is possible, we drop explicit notation of H. ♢

2.5 Remark. A pointwise estimate for the speed of convergence of angles has an implication on $\Lambda(H)$. Denoting

$$\Lambda^*(H) := \begin{cases} \Delta^*\big((|\sin(\phi_n-\phi)|)_{n=1}^\infty\big), & \phi \mod \pi = \lim_{n\to\infty}\phi_n \mod \pi \text{ exists}, \\ 0, & \vec{\phi} \text{ not convergent modulo } \pi, \end{cases} \tag{2.2}$$

it holds that $\Lambda^*(H) \geq \Lambda(H)$. ♢

2.6 Lemma. *The quantities Δ_ϕ and Λ are related by*

$$\Delta_\phi - 1 \leq \Lambda. \tag{2.3}$$

In particular, if $\Delta_\phi < \infty$, then $\Delta_l^+ - \Delta_\phi + \Lambda > 0$ unless $\Delta_l^+ = 1$ and $\Delta_\phi = \Lambda + 1$.

In this, and subsequent proofs, we use the following notation: Let X, Y be functions taking nonnegative numbers as values. Then

$$\begin{aligned} X \lesssim Y &\iff \exists c > 0;\ X \leq cY \\ X \asymp Y &\iff X \lesssim Y \text{ and } Y \lesssim X \end{aligned}$$

Proof. The inequality is trivial if $\Delta_\phi \leq 1$. Hence, assume that $\Delta_\phi > 1$. For arbitrary $\tau \in (1, \Delta_\phi)$ we have

$$\sum_{j=n}^{2n} |\sin(\phi_{j+1}-\phi_j)| = \mathrm{O}\big(n^{1-\tau}\big),$$

and in turn

$$\sum_{j=n}^\infty |\sin(\phi_{j+1}-\phi_j)| \leq \sum_{l=0}^\infty \sum_{j=2^l n}^{2^{l+1}n} |\sin(\phi_{j+1}-\phi_j)| \lesssim \sum_{l=0}^\infty (2^l n)^{1-\tau} = \mathrm{O}\big(n^{1-\tau}\big).$$

By adding a proper multiple of π to each ϕ_n, we can assume without loss of generality that $|\phi_{n+1}-\phi_n| \leq \frac{\pi}{2}$. Then

$$\sum_{j=n}^\infty |\phi_{j+1}-\phi_j| \lesssim \sum_{j=n}^\infty |\sin(\phi_{j+1}-\phi_j)| = \mathrm{O}\big(n^{1-\tau}\big),$$

hence $\vec{\phi}$ has the limit $\phi := \phi_1 + \sum_{n=1}^{\infty}(\phi_{n+1} - \phi_n)$. If $\Delta_l < 1$, and hence $\Delta_l^+ = 1$, we get

$$\sum_{j=n}^{\infty} l_j |\sin(\phi_j - \phi)| \lesssim n^{1-\tau} \sum_{j=n}^{\infty} l_j = \mathrm{O}\big(n^{-(\tau-1)}\big).$$

If $\Delta_l \geq 1$, and hence $\Delta_l^+ = \Delta_l$, we have for arbitrary $\epsilon > 0$

$$\sum_{j=n}^{\infty} l_j |\sin(\phi_j - \phi)| \lesssim \sum_{j=n}^{\infty} j^{-\Delta_l^+ + \epsilon} \cdot j^{1-\tau} = \mathrm{O}\big(n^{1-\Delta_l^+ - (\tau-1)+\epsilon}\big).$$

In both cases, this shows that $\tau - 1 \leq \Lambda$. ❑

Our first theorem can now be formulated.

2.7 Theorem. *Let H be a limit circle Hamburger Hamiltonian with lengths $\vec{l}$ and angles $\vec{\phi}$. Assume that $(\Delta_l^+, \Delta_\phi, \Lambda) \neq (1, 1, 0)$.*

(*i*) *Generic region: If $\Delta_l^+ + \Delta_\phi \geq 2$, then*

$$\rho(H) \leq \frac{1}{\Delta_l^+ + \Delta_\phi}.$$

(*ii*) *Critical triangle: If $\Delta_l^+ + \Delta_\phi < 2$, then*

$$\rho(H) \leq \max\Big\{\frac{1}{\Delta_l^+ + \Delta_\phi}, \frac{1 - \Delta_\phi + \frac{1}{2}\Lambda}{\Delta_l^+ - \Delta_\phi + \Lambda}\Big\}.$$

The proof of this theorem will be carried out in §2.2. Before that we discuss several aspects.

2.8 Remark (Sharpness). We will see that for a large class of Hamburger Hamiltonians in the generic region the stated upper bound is equal to their order, cf. Theorem 2.22. Contrasting this, in the critical triangle (and $\Delta_\phi > 0$), we do not know whether the given bound is sharp. In fact, we have no example which lies inside the critical triangle where $\rho(H)$ can be computed. In particular, it is unknown whether the speed of possible convergence of angles measured by Λ influences the order. We believe the answer is affirmative.

In the case excluded in the assumption, namely if $(\Delta_l^+, \Delta_\phi, \Lambda) = (1, 1, 0)$, we have only the trivial bound "$\rho(H) \leq 1$", and again do not know if this bound is attained by some Hamburger Hamiltonian. ♢

Note that, in some vague sense, the division into generic region and critical triangle corresponds to the cases of "order $\leq \frac{1}{2}$" or "order $\in (\frac{1}{2}, 1]$", respectively. That may be an explanation for the occurrence of this case distinction. As experience tells, it would not be a surprise to witness a fundamentally different behaviour in these cases.

Next let us have a closer look at the two expressions whose maximum establishes the upper bound in the critical triangle. Set

$$D := \big\{(x, y, z) \in \mathbb{R}^3 : x \geq 1,\, y \geq 0,\, z \geq 0,\, y \leq z + 1,\, x - y + z > 0\big\},$$
$$g(x, y, z) := \frac{1 - y + \frac{1}{2}z}{x - y + z}, \quad (x, y, z) \in D.$$

Then

$$\frac{1}{x+y} \le g(x,y,z) \;\Leftrightarrow\; 0 \le (2-x-y)(y-\frac{1}{2}z) \tag{2.4}$$

$$g(x,y,z) = \frac{1}{2} \text{ if } x+y=2, \qquad g(x,y,z)=1 \Leftrightarrow (x,z)=(1,0), \tag{2.5}$$

the function $g(x,y,\cdot)$ is decreasing if $x+y<2$, increasing if $x+y>2$, and the function $g(x,\cdot,z)$ is monotone nonincreasing.

The relation (2.4) shows that the given bound in the critical triangle equals $\frac{1}{\Delta_l^+ + \Delta_\phi}$ if and only if $\Delta_\phi \le \frac{1}{2}\Lambda$.

As we have already observed in Lemma 2.2 and Remark 2.5, pointwise estimates lead to estimates for Δ_ϕ and Λ. From this we obtain the following corollary where the – easier to handle – quantities

$$\Delta_\phi^* := \Delta\big((|\sin(\phi_{n+1}-\phi_n)|)_{n=1}^\infty\big),$$

and Λ^* appear instead of Δ_ϕ and Λ. Of course, this statement is weaker than Theorem 2.7.

2.9 Corollary. *Let H be a Hamburger Hamiltonian with lengths $\vec{l}$ and angles $\vec{\phi}$. Assume that $(\Delta_l^+, \Delta_\phi^*, \Lambda^*) \ne (1,1,0)$. Then*

$$\rho(H) \le \begin{cases} \frac{1}{\Delta_l^+ + \Delta_\phi^*} , & \Delta_l^+ + \Delta_\phi^* \ge 2, \\ \frac{1-\Delta_\phi^* + \frac{1}{2}\Lambda^*}{\Delta_l^+ - \Delta_\phi^* + \Lambda^*} , & \Delta_l^+ + \Delta_\phi^* < 2. \end{cases}$$

Deduction of Corollary 2.9 *from Theorem* 2.7*.* We distinguish three cases.

— *Case $\Delta_l^+ + \Delta_\phi^* \ge 2$:* We have $\Delta_l^+ + \Delta_\phi \ge \Delta_l^+ + \Delta_\phi^* \ge 2$. Towards a contradiction assume that $(\Delta_l^+, \Delta_\phi, \Lambda) = (1,1,0)$. Then $\Delta_\phi^* \le 1$ and $\Delta_l^+ + \Delta_\phi^* \ge 2$ implies $\Delta_\phi^* = 1$. Moreover, $\Lambda^* \le \Lambda$, and hence $\Lambda^* = 0$. This case, however, is excluded by assumption. Thus Theorem 2.7 applies and yields

$$\rho(H) \le \frac{1}{\Delta_l^+ + \Delta_\phi} \le \frac{1}{\Delta_l^+ + \Delta_\phi^*}.$$

— *Case $\Delta_l^+ + \Delta_\phi^* < 2, \Delta_l^+ + \Delta_\phi \ge 2$:* If $(\Delta_l^+, \Lambda) = (1,0)$, then also $\Lambda^* = 0$, and hence $g(\Delta_l^+, \Delta_\phi^*, \Lambda^*) = 1$, cf. (2.5). Assume that $(\Delta_l^+, \Lambda) \ne (1,0)$. Then Theorem 2.7 and the obvious fact that $\Lambda^* \le \Delta_\phi^*$ yields

$$\rho(H) \le \frac{1}{\Delta_l^+ + \Delta_\phi} \le \frac{1}{\Delta_l^+ + \Delta_\phi^*} \le g(\Delta_l^+, \Delta_\phi^*, \Lambda^*).$$

— *Case $\Delta_l^+ + \Delta_\phi^* < 2, \Delta_l^+ + \Delta_\phi < 2$:* We have

$$\begin{aligned} \rho(H) \le \max\Big\{\frac{1}{\Delta_l^+ + \Delta_\phi}, g(\Delta_l^+, \Delta_\phi, \Lambda)\Big\} \\ \le \max\Big\{\frac{1}{\Delta_l^+ + \Delta_\phi^*}, g(\Delta_l^+, \Delta_\phi^*, \Lambda^*)\Big\} = g(\Delta_l^+, \Delta_\phi^*, \Lambda^*). \end{aligned}$$

$\square$

Comparing the nature of the quantities Δ_ϕ, Λ and Δ_ϕ^*, Λ^*, and having in mind Proposition 2.3, suggests that there might be room for improvement by introducing an averaged measure for the decay of lengths. However, as examples show, there seems to be an intrinsic obstacle.

2.2 Proof of Theorem 2.7

To start with we settle two simple cases.

— If $(\Delta_l^+, \Lambda) = (1, 0)$ and hence $\Delta_\phi \leq 1$, or if $(\Delta_l^+, \Delta_\phi) = (1, 0)$, the assertion reduces to the trivial bound "$\rho(H) \leq 1$".

— The convergence exponent of $(l_n)_{n=0}^\infty$ is not larger than $\frac{1}{\Delta_l^+}$. Therefore Proposition 2.3 entails

$$\rho(H) \leq \frac{1}{\Delta_l^+}. \tag{2.6}$$

In particular, the assertion of the theorem holds if $\Delta_l^+ = \infty$.

These observations justify that throughout the following we may assume

$$(\Delta_l^+, \Lambda) \neq (1, 0),\ (\Delta_l^+, \Delta_\phi) \neq (1, 0),\ \Delta_l < \infty.$$

Note that these assumptions imply that the right hand side in asserted bound is strictly less than 1.

We are going to employ [Rom, Theorem 1] which provides an upper bound for the order of a (arbitrary) Hamiltonian. This theorem is based on finding appropriate approximations of a given Hamiltonian by simple ones.

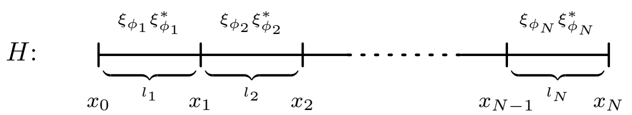
2.10 Definition. Let $N \in \mathbb{N}$, let $(l_n)_{n=1}^N$ be a finite sequence of positive numbers, and let $(\phi_n)_{n=1}^N$ be a finite sequence of real numbers with $\phi_{n+1} \not\equiv \phi_n \mod \pi$, $n = 1, \dots, N-1$. Set

$$x_0 := 0, \qquad x_n := \sum_{k=1}^n l_n,\ n = 1, \dots, N.$$

Then we speak of the function $H : [0, x_N) \to \mathbb{R}^{2\times 2}$ which is defined by

$$H(x) := \xi_{\phi_n} \xi_{\phi_n}^*, \quad x \in [x_{n-1}, x_n),\ n = 1, \dots, N,$$

as the *finite rank Hamiltonian with parameters* $\langle N, (l_n)_{n=1}^N, (\phi_n)_{n=1}^N \rangle$.



$\Diamond$

2.11 Theorem ([Rom]). *Let $L \in (0,\infty)$, and let $H : [0,L) \to \mathbb{R}^{2\times 2}$ be a Hamiltonian with $\operatorname{tr} H = 1$ a.e. Let $d \in (0,1]$, and assume that there exists a family of finite rank Hamiltonians*

$$H^\star(R),\ R>1 \quad \Big(\textit{parameters } \big\langle N^\star(R), (l_n^\star(R))_{n=1}^{N^\star(R)}, (\phi_n^\star(R))_{n=1}^{N^\star(R)} \big\rangle\Big)$$

and a family of sequences of weights

$$\big(a_n(R)\big)_{n=1}^{N^\star(R)},\ R>1 \quad \textit{with} \quad a_n(R) \in (0,1],$$

*such that (*O*-notation is understood for $R \to \infty$ and $\|.\|$ denotes any matrix norm)*

$$(i)\ \ \sum_{k=1}^{N^\star(R)} \frac{1}{a_k(R)^2} \int\limits_{x_{k-1}^\star(R)}^{x_k^\star(R)} \big\|H(x) - [H^\star(R)](x)\big\|\,dx = \mathrm{O}\big(R^{d-1}\big),$$

$$(ii)\ \ \sum_{k=1}^{N^\star(R)} a_k(R)^2 l_k^\star(R) = \mathrm{O}\big(R^{d-1}\big),$$

$$(iii)\ \ \sum_{k=1}^{N^\star(R)-1} \ln\left(1 + \frac{\big|\sin\big(\phi_{k+1}^\star(R) - \phi_k^\star(R)\big)\big|}{a_{k+1}(R)a_k(R)}\right) = \mathrm{O}\big(R^d\big),$$

$$(iv)\ \ \big|\ln a_1(R)\big| + \big|\ln a_{N^\star(R)}(R)\big| + \sum_{k=1}^{N^\star(R)-1} \left|\ln \frac{a_{k+1}(R)}{a_k(R)}\right| = \mathrm{O}\big(R^d\big).$$

Then the order of the entries of the monodromy matrix of H does not exceed d.

2.12. Notation: It turns out useful to agree on the following abbreviations.

(*i*) If $X, Y : [1,\infty) \to [0,\infty)$, then we define

$$X(R) \preceq Y(R) \quad \Longleftrightarrow \quad \forall\,\epsilon>0\ \exists\,C>0\ \forall\,R\geq 1:\ X(R) \leq CR^\epsilon Y(R)$$

(*ii*) If $x,y \in \mathbb{R}$ with $x<y$, and $X_k \in \mathbb{C}$ for $k \in [x,y]\cap\mathbb{Z}$, then we write

$$\sum_{k\geq x}^{y} X_k := \sum_{k\in[x,y]\cap\mathbb{Z}} X_k.$$

$\Diamond$

We set $\Delta'_\phi := \Delta_\phi$ if $\Delta_\phi < \infty$, and let Δ'_ϕ be an arbitrary number larger than 1 if $\Delta_\phi = \infty$. Next, for $\phi \in [0,\pi)$, set

$$\Lambda(\phi) := \sup\Big\{\tau \geq 0 : \sum_{j=n}^{\infty} l_j|\sin(\phi_j - \phi)| = \mathrm{O}\big(n^{1-\Delta_l^+ - \tau}\big)\Big\} \in [0,\infty].$$

Again, set $\Lambda(\phi)' := \Lambda(\phi)$ if $\Lambda(\phi) < \infty$, and let $\Lambda(\phi)'$ be an arbitrary number larger than 1 if $\Lambda(\phi) = \infty$.

Consider $\phi \in [0,\pi)$ such that $(\Delta_l^+, \Lambda(\phi)') \neq (1,0)$, and let $d(\phi)$ be any number with

$$1 > d(\phi) > \begin{cases} \frac{1}{\Delta_l^+ + \Delta_\phi'} & , \quad \Delta_l^+ + \Delta_\phi' \geq 2, \\ \max\left\{ \frac{1}{\Delta_l^+ + \Delta_\phi'}, \frac{1-\Delta_\phi' + \frac{1}{2}\Lambda(\phi)'}{\Delta_l^+ - \Delta_\phi' + \Lambda(\phi)'} \right\}, & \quad \Delta_l^+ + \Delta_\phi' < 2. \end{cases} \tag{2.7}$$

Given $R > 1$ we define an approximating Hamiltonian as a cut-off of H prolonged by one interval with angle ϕ. The cutting point will be the node $x_{N(R)}$ where

$$N(R) = \left\lfloor R^{\frac{1-d(\phi)}{\Delta_l^+ - 1 + \Lambda(\phi)'/2}} \right\rfloor.$$

Note that the value $(1-d(\phi))(\Delta_l^+ - 1 + \Lambda(\phi)'/2)^{-1}$ appearing in the exponent is positive. Now we define

$$N^\star(R) := N(R) + 1,$$

$$l_n^\star(R) := \begin{cases} l_n & , \quad n = 1, \dots, N(R) \\ x_\infty - x_{N(R)}, & \quad n = N^\star(R) \end{cases}$$

$$\phi_n^\star(R) := \begin{cases} \phi_n, & \quad n = 1, \dots, N(R) \\ \phi & , \quad n = N^\star(R) \end{cases}$$

and let $H^\star(R)$ be the finite rank Hamiltonian given by this data.

The required weights $a_n(R)$ are defined by (here we set $\sigma := (\Delta_l^+ + \Delta_\phi')^{-1}$)

$$a_n(R)^2 := \begin{cases} \frac{1}{R} n^{\Delta_l} & , \quad 1 \leq n \leq R^\sigma, \\ \frac{1}{\sqrt{R}} n^{\frac{1}{2}(\Delta_l^+ - \Delta_\phi')}, & \quad R^\sigma < n \leq N(R), \\ n^{-\frac{1}{2}\Lambda(\phi)'} & , \quad n = N^\star(R). \end{cases}$$

We need to check that $a_n(R) \leq 1$. This is clear in all cases except when $n \in (R^\sigma, N(R)]$ and $\Delta_l^+ > \Delta_\phi'$. Then it amounts to showing

$$\frac{1-d(\phi)}{\Delta_l^+ - 1 + \Lambda(\phi)'/2} (\Delta_l^+ - \Delta_\phi') \leq 1, \tag{2.8}$$

or, equivalently,

$$\frac{1 - \Delta_\phi' - \frac{1}{2}\Lambda(\phi)'}{\Delta_l^+ - \Delta_\phi'} \leq d(\phi).$$

We distinguish the cases that $\Delta_l^+ + \Delta_\phi' \geq 2$ or $\Delta_l^+ + \Delta_\phi' < 2$.

— *Case* $\Delta_l^+ + \Delta_\phi' \geq 2$*:* We have

$$\begin{aligned} \frac{2}{\Delta_l^+ + \Delta_\phi'} \leq 1 \ &\Rightarrow \ \frac{2\Delta_\phi'}{\Delta_l^+ + \Delta_\phi'} \leq \Delta_\phi' + \frac{1}{2}\Lambda(\phi)' \\ &\Rightarrow \ 1 - \Delta_\phi' - \frac{1}{2}\Lambda(\phi)' \leq 1 - \frac{2\Delta_\phi'}{\Delta_l^+ + \Delta_\phi'} = \frac{\Delta_l^+ - \Delta_\phi'}{\Delta_l^+ + \Delta_\phi'}, \end{aligned}$$

and hence

$$\frac{1 - \Delta_\phi' - \frac{1}{2}\Lambda(\phi)'}{\Delta_l^+ - \Delta_\phi'} \leq \frac{1}{\Delta_l^+ + \Delta_\phi'} < d(\phi).$$

— *Case* $\Delta_l^+ + \Delta'_\phi < 2$*:* We have

$$\Delta_l^+ \geq 1, \Lambda(\phi)' \geq 0 \;\Rightarrow\; \frac{1}{\Delta_l^+ - \Delta'_\phi + \Lambda(\phi)'} \leq \frac{1}{1 - \Delta'_\phi + \frac{1}{2}\Lambda(\phi)'}$$

$$\Rightarrow \underbrace{1 - \frac{\Lambda(\phi)'}{1 - \Delta'_\phi + \frac{1}{2}\Lambda(\phi)'}}_{\textstyle \frac{1 - \Delta'_\phi - \frac{1}{2}\Lambda(\phi)'}{1 - \Delta'_\phi + \frac{1}{2}\Lambda(\phi)'}} \leq \underbrace{1 - \frac{\Lambda(\phi)'}{\Delta_l^+ - \Delta'_\phi + \Lambda(\phi)'}}_{\textstyle \frac{\Delta_l^+ - \Delta'_\phi}{\Delta_l^+ - \Delta'_\phi + \Lambda(\phi)'}},$$

and hence, since $\Delta'_\phi = \Delta_\phi < 1$,

$$\frac{1 - \Delta'_\phi - \frac{1}{2}\Lambda(\phi)'}{\Delta_l^+ - \Delta'_\phi} \leq \frac{1 - \Delta'_\phi + \frac{1}{2}\Lambda(\phi)'}{\Delta_l^+ - \Delta'_\phi + \Lambda(\phi)'} < d(\phi).$$

We show that with the above approximation and $d := d(\phi) + \epsilon$, where $\epsilon > 0$ is arbitrary, the hypotheses of Theorem 2.11 are satisfied. To shorten notation, we drop explicit notation of the argument R.

Item (i):

$$\sum_{k=1}^{N+1} \frac{1}{a_k^2} \int_{x_{k-1}^\star}^{x_k^\star} \|H(x) - H^\star(x)\|\, dx = \frac{1}{a_{N+1}^2} \int_{x_N}^{x_\infty} \|H(x) - \xi_\phi \xi_\phi^T\|\, dx$$
$$\lesssim N^{\Lambda(\phi)'/2} \sum_{j=N+1}^{\infty} l_j |\sin(\phi_j - \phi)| \preceq N^{\Lambda(\phi)'/2} N^{1-\Delta_l^+ - \Lambda(\phi)'} \leq R^{d(\phi)-1}.$$

Item (ii):

$$\sum_{k=1}^{N+1} a_k^2 l_k^\star = \frac{1}{R} \sum_{k\geq 1}^{R^\sigma} k^{\Delta_l} l_k + \frac{1}{\sqrt{R}} \sum_{k\geq R^\sigma}^{N} k^{(\Delta_l^+ - \Delta'_\phi)/2} l_k + N^{-\Lambda(\phi)'/2}(x_\infty - x_N). \quad (2.9)$$

Since $l_k \preceq k^{-\Delta_l}$, the first term in the right hand side satisfies

$$\frac{1}{R} \sum_{k\geq 1}^{R^\sigma} k^{\Delta_l} l_k \preceq R^{\sigma-1} \leq R^{d(\phi)-1}.$$

Since $x_\infty - x_N = \sum_{k=N+1}^{\infty} l_k \preceq N^{1-\Delta_l^+}$, we have

$$N^{-\Lambda(\phi)'/2}(x_\infty - x_N) \preceq N^{-(\Delta_l^+ - 1 + \Lambda(\phi)'/2)} \lesssim R^{d(\phi)-1}.$$

In order to estimate the second term on the right side of (2.9), we distinguish the cases that $\Delta_l^+ > 1$ and $\Delta_l^+ = 1$.

— *Case* $\Delta_l^+ > 1$: Then $\Delta_l = \Delta_l^+$, and we obtain

$$\frac{1}{\sqrt{R}} \sum_{k \geq R^\sigma}^{N} k^{(\Delta_l^+ - \Delta_\phi')/2} l_k \preceq \frac{1}{\sqrt{R}} \sum_{k \geq R^\sigma}^{N} k^{-(\Delta_l^+ + \Delta_\phi')/2}$$
$$\lesssim \frac{1}{\sqrt{R}} \begin{cases} (R^\sigma)^{1-(\Delta_l^+ + \Delta_\phi')/2}, & \Delta_l^+ + \Delta_\phi' > 2 \\ \ln N, & \Delta_l^+ + \Delta_\phi' = 2 \\ N^{1-(\Delta_l^+ + \Delta_\phi')/2}, & \Delta_l^+ + \Delta_\phi' < 2 \end{cases}$$

In the first case, since $\sigma \leq \frac{1}{2}$ and $(\Delta_l^+ + \Delta_\phi)/2 \geq 1$, it holds that $(R^\sigma)^{1-(\Delta_l^+ + \Delta_\phi')/2} = R^{\sigma - \frac{1}{2}} \leq R^{d(\phi) - \frac{1}{2}}$. In the second case $d(\phi) > \frac{1}{2}$ and hence $\ln N \lesssim R^{d(\phi) - \frac{1}{2}}$. To settle the third case, we have to show that

$$\frac{1 - d(\phi)}{\Delta_l^+ - 1 + \Lambda(\phi)'/2} \Big(1 - \frac{\Delta_l^+ + \Delta_\phi'}{2} \Big) \leq d(\phi) - \frac{1}{2}. \tag{2.10}$$

To achieve this, observe

$$\begin{aligned} (2.10) \iff & \frac{1 - \frac{\Delta_l^+ + \Delta_\phi'}{2}}{\Delta_l^+ - 1 + \Lambda(\phi)'/2} + \frac{1}{2} \leq d(\phi) \Big[1 + \frac{1 - \frac{\Delta_l^+ + \Delta_\phi'}{2}}{\Delta_l^+ - 1 + \Lambda(\phi)'/2} \Big] \\ \iff & (2 - \Delta_l^+ - \Delta_\phi') + (\Delta_l^+ - 1 + \Lambda(\phi)'/2) \\ & \qquad \leq d(\phi) \big[2(\Delta_l^+ - 1 + \Lambda(\phi)'/2) + (2 - \Delta_l^+ - \Delta_\phi') \big] \\ \iff & 1 - \Delta_\phi' + \frac{\Lambda(\phi)'}{2} \leq d(\phi) \big[\Delta_l^+ - \Delta_\phi' + \Lambda(\phi)' \big]. \end{aligned}$$

The last relation holds by the choice of $d(\phi)$. The inequality (2.10) now gives $N^{1-(\Delta_l^+ + \Delta_\phi')/2} \lesssim R^{d(\phi) - \frac{1}{2}}$.

— *Case* $\Delta_l^+ = 1$:

$$\frac{1}{\sqrt{R}} \sum_{k \geq R^\sigma}^{N} k^{(\Delta_l^+ - \Delta_\phi')/2} l_k \leq \frac{1}{\sqrt{R}} \Big[\sum_{k \geq R^\sigma}^{N} l_k \Big] \max_{R^\sigma \leq k \leq N} k^{(1 - \Delta_\phi')/2}$$
$$\leq \frac{1}{\sqrt{R}} \begin{cases} (R^\sigma)^{(1-\Delta_\phi')/2}, & \Delta_\phi' \geq 1 \\ N^{(1-\Delta_\phi')/2}, & \Delta_\phi' < 1 \end{cases}$$

In the case $\Delta_\phi' \geq 1$ observe that

$$\sigma \frac{1 - \Delta_\phi'}{2} \leq d(\phi) - \frac{1}{2} \;\Leftrightarrow\; 1 - \Delta_\phi' \leq 2 \frac{d(\phi)}{\sigma} - (1 + \Delta_\phi') \;\Leftrightarrow\; 1 \leq \frac{d(\phi)}{\sigma},$$

and in the case $\Delta_\phi' < 1$ that

$$\frac{1 - d(\phi)}{\Lambda(\phi)'/2} \frac{1 - \Delta_\phi'}{2} \leq d(\phi) - \frac{1}{2} \;\Leftrightarrow\; 1 - \Delta_\phi' + \frac{\Lambda(\phi)'}{2} \leq d(\phi) \big[1 - \Delta_\phi' + \Lambda(\phi)' \big].$$

Thus, the right hand side in (2.9) is $\mathrm{O}(R^{d(\phi)-1})$ in all cases.

Item (iii): The weights a_n are, independently of n, bounded from below by an appropriate power of R. Hence, we have $\ln\Big(1+\frac{|\sin(\phi^\star_{k+1}-\phi^\star_k)|}{a_{k+1}a_k}\Big)\lesssim \ln R$, and obtain

$$\sum_{k\geq 1}^{R^\sigma+1}\ln\left(1+\frac{|\sin(\phi^\star_{k+1}-\phi^\star_k)|}{a_{k+1}a_k}\right)\lesssim R^\sigma\ln R\lesssim R^{d(\phi)},$$
$$\ln\left(1+\frac{|\sin(\phi^\star_{N+1}-\phi^\star_N)|}{a_{N+1}a_N}\right)\lesssim \ln R\lesssim R^{d(\phi)}.$$

In the remaining part of the sum the second line of the definition of weights applies to the effect that

$$\begin{aligned}
\sum_{k\geq R^\sigma+1}^{N-1}\ln\left(1+\frac{|\sin(\phi^\star_{k+1}-\phi^\star_k)|}{a_{k+1}a_k}\right) &\leq \sum_{k\geq R^\sigma+1}^{N-1}\frac{1}{a_{k+1}a_k}\big|\sin(\phi_{k+1}-\phi_k)\big|\\
&= \sum_{k\geq R^\sigma+1}^{N-1}\sqrt{R}\cdot[(k+1)k]^{-\frac{1}{4}(\Delta_l^+-\Delta'_\phi)}\cdot\big|\sin(\phi_{k+1}-\phi_k)\big|\\
&\lesssim \sqrt{R}\cdot\sum_{j\geq 1}^{\log_2(N/R^\sigma)+1}\Big(\sum_{k\geq 2^{j-1}R^\sigma}^{2^jR^\sigma-1}\big|\sin(\phi_{k+1}-\phi_k)\big|\Big)\max_{k\in[2^{j-1}R^\sigma,2^jR^\sigma]}k^{-\frac{1}{2}(\Delta_l^+-\Delta'_\phi)}\\
&\lesssim \sqrt{R}\cdot\sum_{j\geq 1}^{\log_2(N/R^\sigma)+1}\Big(\sum_{k\geq 2^{j-1}R^\sigma}^{2^jR^\sigma-1}\big|\sin(\phi_{k+1}-\phi_k)\big|\Big)\big(2^jR^\sigma\big)^{-\frac{1}{2}(\Delta_l^+-\Delta'_\phi)}\\
&\preceq \sqrt{R}\cdot(R^\sigma)^{-\frac{1}{2}(\Delta_l^+-\Delta'_\phi)}\sum_{j\geq 1}^{\log_2(N/R^\sigma)+1}\big(2^{j-1}R^\sigma\big)^{1-\Delta'_\phi}\cdot 2^{-j\frac{1}{2}(\Delta_l^+-\Delta'_\phi)}\\
&\asymp \sqrt{R}\cdot(R^\sigma)^{1-(\Delta_l^++\Delta'_\phi)/2}\sum_{j\geq 1}^{\log_2(N/R^\sigma)+1}2^{j[1-(\Delta_l^++\Delta'_\phi)/2]}\\
&\lesssim \sqrt{R}\cdot\begin{cases}(R^\sigma)^{1-(\Delta_l^++\Delta'_\phi)/2}, & \Delta_l^++\Delta'_\phi>2\\ \ln R, & \Delta_l^++\Delta'_\phi=2\\ N^{1-(\Delta_l^++\Delta'_\phi)/2}, & \Delta_l^++\Delta'_\phi<2\end{cases}
\end{aligned}$$

By what we showed in the proof of "Item (ii)", the last expression is in all cases $\lesssim R^{d(\phi)}$.

Item (iv): As we already observed, the weights a_n are bounded below by some power of R. Hence, each summand appearing in "Item (iv)" is $\lesssim \ln R$. Moreover, we have

$$\Big(\frac{a_{k+1}}{a_k}\Big)^2=\begin{cases}\big(\frac{k+1}{k}\big)^{\Delta_l}, & 1\leq k\leq R^\sigma-1,\\ \big(\frac{k+1}{k}\big)^{\frac{1}{2}(\Delta_l^+-\Delta'_\phi)}, & R^\sigma<k\leq N-1,\end{cases}$$

and together it follows that

$$\big|\ln a_1\big|+\big|\ln a_{N+1}\big|+\sum_{k=1}^{N}\Big|\ln\frac{a_{k+1}}{a_k}\Big|\lesssim \ln R+\sum_{k=1}^{N}\ln\big(1+\frac{1}{k}\big)\lesssim \ln R\lesssim R^{d(\phi)}.$$

We see that Theorem 2.11 is indeed applicable, and yields that $\rho(H) \leq d(\phi)+\epsilon$.

The proof of Theorem 2.7 is completed by letting $\epsilon \searrow 0$, passing to the infimum over all $d(\phi)$ subject to (2.7), passing to the supremum over all ϕ subject to $(\Delta_l^+, \Lambda(\phi)') \neq (1,0)$, and – if necessary – letting $\Delta'_\phi \nearrow \infty$ and $\Lambda(\phi)' \nearrow \infty$.

2.13 Remark. Choosing the approximating Hamiltonian as a cut-off of H is of course natural. The choice of the weights a_j in the proof is based on term-by-term optimization in conditions *(ii)* and *(iii)* assuming that the replacement of $a_j a_{j+1}$ in the denominator in *(iii)* by a_j^2 does not spoil the estimate too much.
$\Diamond$

2.3 A lower bound for $\rho(H)$

The following limes inferior appears naturally in estimating order from below.

2.14 Definition. Let H be a Hamburger Hamiltonian with lengths $\vec{l}$ and angles $\vec{\phi}$. Then we set

$$\delta_{l,\phi}(H) := \liminf_{m\to\infty} \Big[G(m;\vec{l},\tfrac{1}{2}) + G\big(m; (|\sin(\phi_{n+1}-\phi_n)|)_{n=1}^\infty, 0\big)\Big].$$

$\Diamond$

The meaning of $\delta_{l,\phi}$ is easily explained. Recall the notation $b_{n,n}$ for the leading coefficient of the orthogonal polynomial of the first kind of degree n. Plugging (1.4) into (1.7) we have

$$b_{m,m} = \Big(\prod_{k=1}^{m-1} \rho_k\Big)^{-1} = \prod_{k=1}^{m-1} |\sin(\phi_{k+1}-\phi_k)|\sqrt{l_k l_{k+1}},$$

from whence

$$\delta_{l,\phi} = \liminf_{m\to\infty} \frac{-\ln b_{m,m}}{m \ln m} \tag{2.11}$$

The following fact is now nothing but [BS14, Proposition 7.1(iii)].

2.15 Proposition. *Let H be a limit circle Hamburger Hamiltonian with lengths $\vec{l}$ and angles $\vec{\phi}$. Then*

$$\rho(H) \geq \frac{1}{\delta_{l,\phi}(H)}. \tag{2.12}$$

Proof. According to [BS14, Proposition 7.1(iii)], $\rho(H)$ is greater or equal to the order of the entire function $\sum_{n=0}^\infty b_{n,n} z^n$. The assertion follows from the standard formula for the order of an entire function in terms of its Taylor coefficients, see, e.g., [Boa54, Theorem 2.2.2]. ❑

Let us provide an alternative, direct, proof of Proposition 2.15.

Alternative proof of Proposition 2.15. Without loss of generality we assume that $\phi_1 \neq \frac{\pi}{2}$. This is possible since cutting of one interval of the Hamiltonian neither changes its order nor the value of $\delta_{l,\phi}$.

On one indivisible interval of type ϕ the matrix function of the canonical system is given by

$$W_\phi(t,z) = e^{J\phi}\begin{pmatrix} 1 & tz \\ 0 & 1 \end{pmatrix} e^{-J\phi} = I + tz e^{J\phi}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} e^{-J\phi}.$$

Notice that we have $e^{J\phi}\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)e^{-J\phi} = -\xi_\phi\xi_\phi^T J$. The fundamental solution W of H at a point x_n is therefore given as

$$W(x_n,z) = W_{\phi_1}(l_1,z)W_{\phi_2}(l_2,z)\dots W_{\phi_{n-1}}(l_{n-1},z).$$

In particular, we are interested in the leading coefficient of the polynomial $z \mapsto W(x_n,z)$, i.e.

$$(-1)^n l_1 l_2 \cdot \ldots \cdot l_{n-1}\xi_{\phi_1}\xi_{\phi_1}^T J\xi_{\phi_2}\xi_{\phi_2}^T J\dots\xi_{\phi_{n-1}}\xi_{\phi_{n-1}}^T J.$$

Due to $\xi_\phi^T J\xi_\psi = \sin(\phi-\psi)$, it can be written as a product,

$$\begin{aligned}(-1)^n l_1 l_2 \cdot\ldots\cdot l_{n-1}\sin(\phi_2-\phi_1)\sin(\phi_3-\phi_2)\cdot\ldots\cdot\sin(\phi_{n-1}-\phi_{n-2})\xi_{\phi_1}\xi_{\phi_{n-1}}^T J =\\ = (-1)^n\Big(\prod_{i=1}^{n-1} l_i\Big)\Big(\prod_{i=1}^{n-2}\sin(\phi_{i+1}-\phi_i)\Big)\xi_{\phi_1}\xi_{\phi_{n-1}}^T J.\end{aligned} \tag{2.13}$$

We denote the first line of the fundamental solution W by Θ, $\Theta := W^T(1,0)^T = (\Theta_+,\Theta_-)^T$. Let $\rho(H)\in[0,1]$ be the order of $W(L,z)$. For all $\epsilon>0$ there are constants $R,C,c>0$ such that for all z with $|z|\geq R$, we have

$$Ce^{c|z|^{\rho+\epsilon}} \geq \operatorname{Im}\Big(\Theta_+(L,z)\overline{\Theta_-(L,z)}\Big) = \operatorname{Im} z\int_0^L \big(H(t)\Theta(t,z),\Theta(t,z)\big)_{\mathbb{C}^2}\,dt.$$

On each indivisible interval (x_n,x_{n+1}) we integrate a function which is constant in t,

$$\begin{aligned}(H(x_n)\Theta(x_n,z))^T\Theta(x_n,z) = (1,0)W(x_n,z)H(x_n)W(x_n,z)^T(1,0)^T =\\ = |(1,0)W(x_n,z)\xi_{\phi_n}|^2.\end{aligned}$$

Since H consists only of indivisible intervals, we get

$$Ce^{c|z|^{\rho+\epsilon}} \geq \operatorname{Im} z\sum_{n=1}^\infty l_n|(1,0)W(x_n,z)\xi_{\phi_n}|^2.$$

The polynomial $z\mapsto(1,0)W(x_n,z)\xi_{\phi_n}$ of degree $n-1$ has real zeros only. Therefore, its absolute value for $z=i\tau$, $\tau>0$, can be estimated from below by the absolute value of its leading coefficient times τ^{n-1}. With the calculation done in (2.13), we get

$$\begin{aligned}Ce^{c\tau^{\rho+\epsilon}} \geq \tau\sum_{n=1}^\infty l_n\prod_{i=1}^{n-1} l_i^2\Big(\prod_{i=1}^{n-2}\sin^2(\phi_{i+1}-\phi_i)\Big)|(1,0)\xi_{\phi_1}\xi_{\phi_{n-1}}^T J\xi_{\phi_n}|^2\tau^{2n-2} =\\ = \sum_{n=1}^\infty l_n\prod_{i=1}^{n-1} l_i^2\prod_{i=1}^{n-1}\sin^2(\phi_{i+1}-\phi_i)\cos^2(\phi_1)\tau^{2n-1}.\end{aligned}$$

The classical result [Boa54, Theorem 2.2.2] states that the order of a power series $\sum_{n=0}^\infty a_n z^n$ can be obtained from its coefficients by the following limes superior

$$\limsup_{n\to\infty}\frac{n\ln n}{\ln(1/|a_n|)}.$$

We end up with the lower bound for the order

$$\begin{aligned}\rho(H) &\geq \limsup_{n\to\infty} \frac{(2n-1)\ln(2n-1)}{\ln\left(1/|a_{2n-1}|\right)} \\ &= \limsup_{n\to\infty} \frac{(2n-1)\ln(2n-1)}{-\ln\left(l_n \prod_{i=1}^{n-1} l_i^2 \prod_{i=1}^{n-1} \sin^2(\phi_{i+1}-\phi_i)\cos^2(\phi_1)\right)}.\end{aligned}$$

This expression can be simplified to

$$\limsup_{n\to\infty} \frac{n\ln n}{-\ln\left(\sqrt{l_n}\prod_{i=1}^{n-1} l_i\,|\sin(\phi_{i+1}-\phi_i)|\right)} = \frac{1}{\delta_{l,\phi}}.$$

❑

Sometimes it is useful to look at the lengths and angles of a Hamburger Hamiltonian separately. In fact, this viewpoint is vital when comparing the lower bound (2.12) for $\rho(H)$ with the upper bound established in Theorem 2.7. Also, it helps when considering concrete examples.

2.16 Definition. Let H be a Hamburger Hamiltonian with lengths $\vec{l}$ and angles $\vec{\phi}$. Then we set

$$\begin{aligned}\delta_l(H) &:= \liminf_{m\to\infty} G(m;\vec{l},\tfrac{1}{2}),\\ \delta_\phi(H) &:= \liminf_{m\to\infty} G\big(m;(|\sin(\phi_{n+1}-\phi_n)|)_{n=1}^\infty,0\big).\end{aligned}$$

◊

The next statement is an immediate corollary of Proposition 2.15.

2.17 Corollary. *Let H be a Hamburger Hamiltonian with lengths $\vec{l}$ and angles $\vec{\phi}$. Assume that at least one of δ_l and δ_ϕ exists as a limit. Then*

$$\rho(H) \geq \frac{1}{\delta_l+\delta_\phi}.$$

Let us provide an example that the additional assumption in the corollary cannot be dropped.

2.18 Proposition. *Let $\alpha>\beta>1$ and $\gamma>0$ with $\beta+\frac{\gamma}{2}<\alpha<\beta+2\gamma$ be given. Consider the Hamburger Hamiltonian H with lengths and angles given by*

$$l_n := \begin{cases} n^{-\alpha}, & 2^{2j}\leq n<2^{2j+1}, j\in\mathbb{N}_0,\\ n^{-\beta}, & 2^{2j-1}\leq n<2^{2j}, j\in\mathbb{N},\end{cases}$$

$$\phi_1 := 0, \qquad \phi_{n+1}-\phi_n := \begin{cases} \frac{\pi}{2}, & 2^{2j}\leq n<2^{2j+1}, j\in\mathbb{N}_0,\\ n^{-\gamma}, & 2^{2j-1}\leq n<2^{2j}, j\in\mathbb{N}.\end{cases}$$

Then $\rho(H)<\frac{1}{\delta_l+\delta_\phi}$.

Proof. By the choice of α being greater than β,

$$\delta_l(H) = \lim_{m\to\infty} G(2^{2m}; \vec{l}, \tfrac{1}{2}) = \frac{2\beta+\alpha}{3}.$$

Then

$$\delta_\phi(H) = \lim_{m\to\infty} G\big(2^{2m+1}; (|\sin(\phi_{n+1}-\phi_n)|)_{n=1}^\infty, 0\big) = \frac{\gamma}{3}.$$

On the other hand, if $M_n(z)$ is the monodromy matrix corresponding to n-th interval of the Hamiltonian, then for any $\epsilon > 0$

$$\log \Big\| \prod_{n=2^{2j}}^{2^{2j+1}-1} M_n(z) \Big\| \leq \log \prod_{n=2^{2j}}^{2^{2j+1}} \Big(1 + \frac{|z|}{n^\alpha}\Big) \leq C_\epsilon |z|^{1/\alpha+\epsilon} \sum_{n=2^{2j}}^{2^{2j+1}} \frac{1}{n^{1+\epsilon}},$$

hence the product of the norms of M_n over $n \in \bigcup_{j\in\mathbb{N}_0}[2^{2j}, 2^{2j+1})$ is estimated above by $e^{C|z|^{1/\alpha+\epsilon}}$. Then,

$$\log \Big\| \prod_{n=2^{2j-1}}^{2^{2j}-1} M_n(z) \Big\| \leq C_\epsilon |z|^{1/(\beta+\gamma)+\epsilon} \sum_{n=2^{2j-1}}^{2^{2j}} \frac{1}{n^{1+\epsilon}}.$$

The proof of this fact can actually be taken verbatim from the proof of the relevant part of Theorem 1 in [Rom] by choosing the a_n^2 as in the proof of Theorem 2.7 above (in the case $\Delta_l > 1$, $\Delta_l + \Delta_\phi > 2$). Adding up the obtained estimates in j we find that the product

$$\prod_{j=1}^{\infty} \Big\| \prod_{n=2^{2j-1}}^{2^{2j}-1} M_n(z) \Big\|$$

is estimated above by $e^{C|z|^{1/(\beta+\gamma)+\epsilon}}$. By the chain rule for the monodromy matrix it follows that $\rho(H) \leq \max\{\alpha^{-1}, (\beta+\gamma)^{-1}\}$. Now the condition on parameters α, β and γ in the assumption ensures that

$$\frac{1}{\alpha} < \delta_l + \delta_\phi \quad \text{and} \quad \frac{1}{\beta+\gamma} < \delta_l + \delta_\phi.$$

❑

2.4 Regularly distributed data

The formula for $\rho(H)$ given in Theorem 2.22 below is obtained by comparing the upper and lower bounds of Theorem 2.7 and Proposition 2.15. The decisive property which enables to show that these bounds coincide is a certain regularity of the distribution of the sequences of lengths and angle-differences.

2.19 Definition. We call a sequence $\vec{y} = (y_n)_{n=1}^\infty$ of positive real numbers *regularly distributed*, if

$$\frac{y_n}{\Big(\prod_{k=1}^{n} y_k\Big)^{\frac{1}{n}}} = \mathrm{O}(1).$$

◊

This notion of regularity rules out heavy oscillations but also sparse peaks where very large or very small elements occur.

2.20 Remark. Many examples of regularly distributed sequences are provided by the following observations.

(i) Each monotonically decreasing sequence is regularly distributed.

(ii) If $\vec{y}$ is regularly distributed and $u_n \asymp y_n$, then $\vec{u}$ is regularly distributed.

$\Diamond$

2.21 Lemma. *Let $\vec{y} = (y_n)_{n=1}^\infty$ be a bounded and regularly distributed sequence of positive real numbers and let $\alpha \geq 0$. Then*

$$\Delta^*(\vec{y}) = \Delta(\vec{y}) = \delta(\vec{y}, \alpha).$$

Proof. By Lemma 2.2 it remains to show that $\delta(\vec{y}, \alpha) \leq \Delta^*(\vec{y})$. If $\delta(\vec{y}, \alpha) = 0$, this inequality holds trivially. Assume that $\delta(\vec{y}, \alpha)$ is positive, and consider $\tau \in [0, \delta(\vec{y}, \alpha))$. For all sufficiently large m we have

$$\tau \leq G(m; \vec{y}, \alpha)) = \frac{-1}{m \ln m} \ln\Big(y_m^\alpha \prod_{k=1}^{m-1} y_k\Big).$$

This implies that

$$m^{-\tau} \geq \Big(y_m^\alpha \prod_{k=1}^{m-1} y_k\Big)^{\frac{1}{m}} = y_m^{\frac{\alpha-1}{m}} \Big(\prod_{k=1}^{m} y_k\Big)^{\frac{1}{m}} \gtrsim b^{\frac{\alpha-1}{m}} y_m,$$

and we conclude that $y_n \lesssim n^{-\tau}$. ❑

In the formulation of the next result the quantity Λ from Definition 2.4 is used.

2.22 Theorem. *Let H be a limit circle Hamburger Hamiltonian with lengths $\vec{l}$ and angles $\vec{\phi}$. Assume that $\vec{l}$ is regularly distributed, that at least one of δ_l and δ_ϕ exists as a limit, and that either*

(A) *The sequence $(|\sin(\phi_{n+1} - \phi_n)|)_{n=1}^\infty$ is regularly distributed, $\delta_l + \delta_\phi \geq 2$, and $(\delta_l, \delta_\phi, \Lambda) \neq (1, 1, 0)$,*

or

(B) $\delta_\phi = 0$.

Then

$$\rho(H) = \frac{1}{\delta_l + \delta_\phi}.$$

Proof. The sequence $\vec{l}$ is summable, and hence $\Delta(\vec{l}) \geq 1$. Lemma 2.21 gives

$$\delta_l = \delta(\vec{l}, \frac{1}{2}) = \Delta(\vec{l}) = \Delta_l^+.$$

Moreover, the overall assumptions of the theorem ensure that $\delta_{l,\phi} = \delta_l + \delta_\phi$. Assume that (A) holds. Then

$$\delta_\phi = \delta\big((|\sin(\phi_{n+1} - \phi_n)|)_{n=1}^\infty, 0\big) = \Delta((|\sin(\phi_{n+1} - \phi_n)|)_{n=1}^\infty) = \Delta_\phi,$$

and the further assumption in (A) just say that Theorem 2.7 is applicable and that we are in the generic region. Thus

$$\frac{1}{\Delta_l^+ + \Delta_\phi} = \frac{1}{\delta_l + \delta_\phi} = \frac{1}{\delta_{l,\phi}} \leq \rho(H) \leq \frac{1}{\Delta_l^+ + \Delta_\phi}.$$

If (B) holds, it is enough to remember Proposition 2.3 and Proposition 2.15 which yield

$$\frac{1}{\Delta_l^+} = \frac{1}{\delta_l} = \frac{1}{\delta_{l,\phi}} \leq \rho(H) \leq \frac{1}{\Delta_l^+}.$$

❑

Let us now discuss two basic examples.

2.23 Example (Power-like behaviour). Let H be a Hamburger Hamiltonian with lengths $\vec{l}$ and angles $\vec{\phi}$, and assume that

$$l_n \asymp n^{-\alpha}, \quad |\sin(\phi_{n+1} - \phi_n)| \asymp n^{-\beta},$$

with $\alpha > 1$ and $\beta \geq 0$. Then both sequences $\vec{l}$ and $(|\sin(\phi_{n+1} - \phi_n)|)_{n=1}^\infty$ are regularly distributed, cf. Remark 2.20, δ_l and δ_ϕ exist as limits, and

$$\delta_l = \Delta_l^+ = \alpha, \quad \delta_\phi = \Delta_\phi = \Delta_\phi^* = \beta.$$

If $\alpha + \beta \geq 2$, we obtain

$$\rho(H) = \frac{1}{\alpha + \beta},$$

if $\alpha + \beta < 2$, we have the bounds

$$\frac{1}{\alpha + \beta} \leq \rho(H) \leq \frac{1 - \beta}{\alpha - \beta}.$$

◊

2.24 Example (jumping angles). Consider a limit circle Hamburger Hamiltonian whose angles $\vec{\phi}$ satisfy $|\sin(\phi_{n+1} - \phi_n)| \asymp 1$. Then $\delta_\phi = \Delta_\phi = \Delta_\phi^* = 0$, and δ_ϕ exists as a limit. Proposition 2.3 and Proposition 2.15 imply

$$\frac{1}{\delta_l} \leq \rho(H) \leq \frac{1}{\Delta_l^+},$$

remember here (2.6). Using the more involved Theorem 2.7 inside the critical triangle does not improve this bound. Regardless of the value of Λ, the maximum in Theorem 2.7, (ii), equals $\frac{1}{\Delta_l^+}$. ◊

3 Diagonal Hamiltonians with irregularly distributed lengths and the Livšic estimate

If the lengths and angle-differences of a Hamburger Hamiltonian are not regularly distributed, the upper and lower bounds from Proposition 2.3, Theorem 2.7 and Proposition 2.15 need not coincide. This section is devoted to the construction of – simple and explicit – examples which show that neither of these bounds necessarily coincides with the order.

Such examles are already found in the class of *diagonal* Hamburger Hamiltonians, i.e., Hamburger Hamiltonian with angles ϕ_n all being integer multiples of $\frac{\pi}{2}$. To make the connection with moment problems, by (1.8), diagonal Hamburger Hamiltonians correspond to moment sequences with $s_n = 0$ for all odd n. In turn, these are the Hamburger moment problems which arise from symmetrising a Stieltjes moment problem.

Remember Example 2.24 which shows in particular that for a diagonal Hamburger Hamiltonian H always $\delta_\phi(H) = \Delta_\phi(H) = \Delta_\phi^*(H) = 0$ and so

$$\frac{1}{\delta_l(H)} \leq \rho(H) \leq \inf\{p > 0 \colon \vec{l} \in \ell^p\} \leq \frac{1}{\Delta_l^+(H)}. \tag{3.1}$$

3.1 Theorem. *Let $\alpha \in [1, \infty)$ and $\beta \in (\alpha, \infty)$ be arbitrarily prescribed numbers. Let $q \in \mathbb{N}$, $q \geq 2$, and consider the Hamburger Hamiltonian H_q with angles $\phi_n := n\frac{\pi}{2}$, $n \in \mathbb{N}$, and lengths*

$$l_n := \begin{cases} 1 \qquad\qquad, & n = 1, \\ (n \ln^2 n)^{-\alpha}, & n \not\equiv 0 \mod q,\ n \geq 2, \\ (n \ln^2 n)^{-\beta}, & n \equiv 0 \mod q. \end{cases}$$

Then

$$\inf\{p > 0 \colon \vec{l} \in \ell^p\} = \Delta_l^+(H_q) = \frac{1}{\alpha}, \quad \delta_l(H_q) = \frac{q-1}{q}\alpha + \frac{1}{q}\beta, \quad \Lambda(H_q) = \beta - \alpha \tag{3.2}$$

and

$$\rho(H_q) = \begin{cases} \frac{1}{\delta_l(H_q)}, & q = 2, \\ \frac{1}{\alpha} \qquad, & q \geq 3. \end{cases}$$

On first sight it may seem peculiar that $\rho(H_q)$ is different for $q = 2$ and $q = 3$, but constant for $q \geq 3$. One intuitive explanation might be that the dominating subsequence of lengths ($n \not\equiv 0 \mod q$) sees the jumps of angles if $q \geq 3$ whereas for $q = 2$ it does not.

Proof of Theorem 3.1*; the equalities* (3.2)*.* The first equality in (3.2) is obvious. In order to compute $\delta_l(H_q)$, consider first the sequence $\vec{\lambda}$ defined by

$$\lambda_n := \begin{cases} n^{-\alpha}, & n \not\equiv 0 \mod q, \\ n^{-\beta}, & n \equiv 0 \mod q. \end{cases}$$

Then

$$\prod_{k=1}^n \lambda_k = \prod_{k=1}^n k^{-\alpha} \cdot \Big(\prod_{j=1}^{\lfloor \frac{n}{q} \rfloor} (qj)^\alpha \Big) \cdot \Big(\prod_{j=1}^{\lfloor \frac{n}{q} \rfloor} (qj)^{-\beta} \Big)$$
$$= (n!)^{-\alpha} \cdot q^{\alpha \lfloor \frac{n}{q} \rfloor} \big(\lfloor \tfrac{n}{q} \rfloor ! \big)^\alpha \cdot q^{-\beta \lfloor \frac{n}{q} \rfloor} \big(\lfloor \tfrac{n}{q} \rfloor ! \big)^{-\beta}.$$

From Stirling's formula $\ln(n!) = n \ln n + \mathrm{o}(n \ln n)$. Moreover,

$$\frac{\lfloor \frac{n}{q} \rfloor \ln \lfloor \frac{n}{q} \rfloor}{n \ln n} = \frac{1}{q} \cdot \underbrace{\frac{q \lfloor \frac{n}{q} \rfloor}{n}}_{\to 1} \cdot \underbrace{\frac{\ln \lfloor \frac{n}{q} \rfloor}{\ln n}}_{\to 1},$$

and alltogether
$$\lim_{n\to\infty} G(n;\vec{\lambda},\tfrac{1}{2}) = \frac{q-1}{q}\alpha + \frac{1}{q}\beta.$$
We have $\frac{l_n}{\lambda_n} = \ln^{-2\alpha} n$, and hence
$$G(n;(\frac{l_k}{\lambda_k})_{k=1}^\infty,\frac{1}{2}) = \frac{-1}{n\ln n}\Big[-\alpha\ln\ln n - 2\alpha\sum_{k=1}^{n-1}\ln\ln k\Big] \leq \frac{1}{n\ln n}\cdot n\ln\ln n = \mathrm{o}(1).$$
Hence,
$$\delta_l(H_q) = \lim_{n\to\infty} G(n;\vec{l},\frac{1}{2}) = \lim_{n\to\infty} G(n;\vec{\lambda},\frac{1}{2}) = \frac{q-1}{q}\alpha + \frac{1}{q}\beta.$$
It remains to calculate $\Lambda(H_q)$. For $\phi \not\equiv \frac{\pi}{2} \mod \pi$ we have $|\sin(\phi_j-\phi)| \gtrsim 1$, j odd, and hence
$$\sum_{j=n}^\infty l_j|\sin(\phi_j-\phi)| \gtrsim \sum_{\substack{j=n\\ j\text{ odd}}}^\infty l_j = \sum_{\substack{j=n\\ j\text{ odd}}}^\infty (j\ln^2 j)^{-\alpha}.$$
Given $\tau \geq 0$, the right hand side is $\lesssim n^{1-\alpha-\tau}$ if and only if $\tau = 0$. For $\phi = \frac{\pi}{2}$ we have
$$\sum_{j=n}^\infty l_j|\sin(\phi_j-\phi)| = \sum_{\substack{j=n\\ j\text{ even}}}^\infty l_j = \sum_{\substack{j=n\\ j\text{ even}}}^\infty (j\ln^2 j)^{-\beta},$$
and the right hand side is $\lesssim n^{1-\alpha-\tau}$ if and only if $\tau \leq \beta-\alpha$. We conclude that $\Lambda(H_q) = \beta - \alpha$. ❑

Computing the order of H_q for $q \geq 3$ relies on [Rom, Theorem 2] which tells us how to compute the order of a diagonal Hamiltonian. Let us recall this theorem. To formulate it, one more notation is needed.

3.2 Definition. Consider a nonempty interval $[a,b)$. We denote by $\mathrm{Cov}[a,b)$ the set of all coverings Ω of $[a,b)$ by finitely many pairwise disjoint left-closed and right-open intervals contained in $[a,b)$. ◊

Moreover, we denote by λ the Lebesgue-measure on $\mathbb{R}$ and by $\#F$ the number of elements of a finite set F.

3.3 Theorem ([Rom]). *Let $L \in (0,\infty)$, and let $H : [0,L) \to \mathbb{R}^{2\times 2}$ be a Hamiltonian with $\operatorname{tr} H = 1$ a.e. and*
$$\det H(x) = 0, H(x) \text{ diagonal}, \quad x \in [0,L) \text{ a.e.} \tag{3.3}$$
Set
$$\begin{aligned} M_1 &:= \Big\{x \in [0,L) : H(x) = \begin{pmatrix} 1 & 0\\ 0 & 0\end{pmatrix}\Big\},\\ M_2 &:= \Big\{x \in [0,L) : H(x) = \begin{pmatrix} 0 & 0\\ 0 & 1\end{pmatrix}\Big\}.\end{aligned} \tag{3.4}$$
Then $\rho(H)$ is equal to the infimum of all numbers $d \in (0,1]$ for which there exists a family $(\Omega(R))_{R>1}$ of coverings $\Omega(R) \in \mathrm{Cov}[0,L)$ such that
$$\#\Omega(R) = \mathrm{O}\big(R^d\big), \tag{3.5}$$
$$\sum_{\omega\in\Omega(R)} \sqrt{\lambda(\omega\cap M_1)\cdot\lambda(\omega\cap M_2)} = \mathrm{O}\big(R^{d-1}\big). \tag{3.6}$$

Observe that, since the real and symmetric 2×2-matrix $H(x)$ satisfies $\operatorname{tr} H(x) = 1$, it holds that $\det H(x) = 0$ and $H(x)$ is diagonal if and only if

$$H(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad H(x) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

It is an important observation that in Theorem 3.3 it suffices to consider coverings by intervals with endpoints at nodes x_n.

3.4 Definition. Let H be a diagonal Hamburger Hamiltonian, and let x_n, $n = 0, 1, \dots, \infty$ be as in (1.5). We write $\Omega \in \operatorname{Cov}(H)$, if

(*i*) $\Omega \in \operatorname{Cov}[0, x_\infty)$,

(*ii*) $\forall \omega \in \Omega \, \exists n_-, n_+ \in \mathbb{N}_0 \cup \{\infty\} : \ \omega = [x_{n_-}, x_{n_+})$.

$\Diamond$

3.5 Lemma. *Let H be a diagonal limit circle Hamburger Hamiltonian. Then $\rho(H)$ is equal to the infimum of all numbers $d \in (0,1]$ for which there exists a family $(\Omega(R))_{R>1}$ of coverings $\Omega(R) \in \operatorname{Cov}(H)$ such that* (3.5) *and* (3.6) *hold.*

Proof. It is enough to show that for each number $d \in (0,1]$ and family $(\Omega(R))_{R>1}$, $\Omega(R) \in \operatorname{Cov}[x_0, x_\infty)$, with (3.5) and (3.6), there exists a family $(\tilde{\Omega}(R))_{R>1}$, $\tilde{\Omega}(R) \in \operatorname{Cov}(H)$, such that (3.5) and (3.6) still hold.

The coverings $\tilde{\Omega}(R)$ are constructed by modifying $\Omega(R)$ in the obvious way. Let $\omega \in \Omega(R)$.

— *Case 1:* Assume that there exists an $n \in \mathbb{N}_0$ such that $\omega \subseteq [x_n, x_{n+1})$. Then we include the interval $[x_n, x_{n+1})$ into $\tilde{\Omega}(R)$.

— *Case 2:* Assume that Case 1 does not take place. Then there exists an $n \in \mathbb{N}$ such that x_n lies in the interior of ω. Set

$$\begin{aligned} n_- &:= \min \big\{ n \in \mathbb{N} : x_n \text{ inner point of } \omega \big\}, \\ n_+ &:= \max \big\{ n \in \mathbb{N} : x_n \text{ inner point of } \omega \big\}, \end{aligned}$$

and include the intervals (the middle interval appears only if $n_- < n_+$)

$$[x_{n_- - 1}, x_{n_-}), \quad [x_{n_-}, x_{n_+}), \quad [x_{n_+}, x_{n_+ + 1})$$

into $\tilde{\Omega}(R)$.

Then $\tilde{\Omega}(R) \in \operatorname{Cov}(H)$ and $\#\tilde{\Omega}(R) \leq 4 \cdot \#\Omega(R)$. In particular, (3.5) holds for $(\tilde{\Omega}(R))_{R>1}$.

Consider the sum in (3.6) for the covering $\tilde{\Omega}(R)$. Then only intervals of the form $[x_{n_-}, x_{n_+})$ constructed from some $\omega \in \Omega(R)$ contribute a possibly nonzero summand. However, $[x_{n_-}, x_{n_+}) \subseteq \omega$ and hence

$$\lambda\big([x_{n_-}, x_{n_+}) \cap M_i\big) \leq \lambda(\omega \cap M_i), \quad i = 1, 2.$$

We see that

$$\sum_{\tilde{\omega} \in \tilde{\Omega}(R)} \sqrt{\lambda(\tilde{\omega} \cap M_1) \cdot \lambda(\tilde{\omega} \cap M_2)} \leq \sum_{\omega \in \Omega(R)} \sqrt{\lambda(\omega \cap M_1) \cdot \lambda(\omega \cap M_2)},$$

and conclude that (3.6) holds. ❑

Proof of Theorem 3.1*; computing* $\rho(H_q)$. The estimate (3.1) and (3.2) give

$$\Big[\frac{q-1}{q}\alpha+\frac{1}{q}\beta\Big]^{-1}\leq\rho(H_q)\leq\frac{1}{\alpha}. \tag{3.7}$$

First, we consider the case that $q=2$. We are going to employ [BS15, Theorem 1.2]. Since H is diagonal, the orthogonal polynomials P_n are even for even n and odd for odd n, and the Q_n are odd for even n and even for odd n. Hence,

$$P_{2n}(0)^2=l_{2n},\ Q_{2n-1}(0)^2=l_{2n-1},\quad n\in\mathbb{N},$$

and hence $(P_{2n}(0)^2)_{n=1}^\infty\in\ell^{1/\beta}$ and $(Q_{2n-1}(0)^2)_{n=1}^\infty\in\ell^{1/\alpha}$. Moreover, both sequences are monotonically decreasing and

$$\frac{P_{2n}(0)^{2/\alpha}}{Q_{2n-1}(0)^{2/\beta}}=\frac{(2n-1)\ln^2(2n-1)}{2n\ln^2(2n)}\to 1.$$

Hence [BS15, Theorem 1.2] is indeed applicable, and yields $\rho(H_2)\leq[\frac{1}{2}(\alpha+\beta)]^{-1}$. Together with (3.7), thus $\rho(H_2)=\frac{2}{\alpha+\beta}=\delta_l(H_2)^{-1}$.

Now assume that $q\geq 3$. In view of (3.7) we have to show that $\rho(H_q)\geq\frac{1}{\alpha}$. To this end, consider the auxiliary diagonal Hamburger Hamiltonian $\tilde{H}$ with lengths

$$h_n:=\begin{cases}1 & ,\quad n=1,\\ \big(n\ln^2 n\big)^{-\alpha}, & n\in\mathbb{N},\ n\geq 2,\end{cases}$$

and the same angles as H. By monotonicity, $\vec{h}$ is regularly distributed and (3.1) gives $\rho(\tilde{H})=\frac{1}{\alpha}$.

Let $d>\rho(H_q)$ and choose, by virtue of Theorem 3.3, a family of coverings $(\Omega(R))_{R>1}\in\mathrm{Cov}(H)$ such that (3.5) and (3.6) hold for d. We are going to modify this family so as to obtain a family of coverings for $\tilde{H}$. First we refine the given coverings to construct $(\Omega'(R))_{R>1}\in\mathrm{Cov}(H)$ such that (3.5) and (3.6) hold for $(\Omega'(R))_{R>1}$ and d, and such that:

If $j\in\mathbb{N}$ and $\omega\in\Omega'(R)$ contains $[x_{qj-1},x_{qj})$, then either $\omega=[x_{qj-1},x_{qj})$ or $\omega\supseteq[x_{qj-3},x_{qj-2})$. (3.8)

Indeed, since $q\geq 3$, this property can be achieved by splitting the intervals $\omega\in\Omega(R)$ in at most three smaller ones, namely by cutting off the first or the first two intervals of H which lie in ω if necessary, and adding them to $\Omega'(R)$. We have $\#\Omega'(R)\leq 3\cdot\#\Omega(R)$ and the sum in (3.6) for $\Omega'(R)$ does not exceed the one for $\Omega(R)$. Hence (3.5) and (3.6) hold for $(\Omega'(R))_{R>1}$ and d.

The property (3.8) and monotonicity of $(h_j)_{j=1}^\infty$ implies that for each interval ω which is not equal to a single interval of H,

$$\sum_{\substack{n\equiv 0 \bmod q\\ [x_{n-1},x_n)\subseteq\omega\cap M_i}} h_n \ \leq \sum_{\substack{n\not\equiv 0 \bmod q\\ [x_{n-1},x_n)\subseteq\omega\cap M_i}} h_n\ \leq\ \lambda(\omega\cap M_i),\quad \omega\in\Omega'(R),\ i=1,2,$$

and hence

$$\sum_{\substack{n\in\mathbb{N}\\ [x_{n-1},x_n)\subseteq\omega\cap M_i}} h_n\ \leq\ 2\lambda(\omega\cap M_i),\quad \omega\in\Omega'(R),\ i=1,2. \tag{3.9}$$

Denote by $\tilde{x}_n$ the nodes of $\tilde{H}$, and define $\tilde{\Omega}(R) \in \mathrm{Cov}(\tilde{H})$ to be the covering

$$\tilde{\Omega}(R) := \big\{ [\tilde{x}_n, \tilde{x}_m) : \ n, m \in \mathbb{N}, [x_n, x_m) \in \Omega'(R) \big\}.$$

Then we have $\#\tilde{\Omega}(R) = \#\Omega(R)$ and, by (3.9),

$$\lambda([\tilde{x}_n, \tilde{x}_m) \cap \tilde{M}_i) \leq 2\lambda([x_n, x_m) \cap M_i), \quad [\tilde{x}_n, \tilde{x}_m) \in \tilde{\Omega}(R),\ i = 1, 2.$$

We see that $\tilde{\Omega}(R)$ satisfies (3.5) and (3.6). Referring again to Theorem 3.3, this time for $\tilde{H}$, gives $d \geq \rho(\tilde{H}) = \frac{1}{\alpha}$, and the result follows. ❑

Finally, let us discuss the Livšic estimate (1.1). To translate Theorem 3.1 into this language, we recall the modern proof of (1.1) based on Proposition 2.15.

Deduction of (1.1) *from Proposition* 2.15. Given a sequence $(s_n)_{n=0}^{\infty}$ of power moments of a measure μ on the real axis, we have for the corresponding orthogonal polynomials P_n

$$1 = (P_n, P_n)_{L^2(\mu)} = b_{n,n}(z^n, P_n)_{L^2(\mu)} \leq b_{n,n} \underbrace{\|z^n\|_{L^2(\mu)}}_{=\sqrt{s_{2n}}} \underbrace{\|P_n\|_{L^2(\mu)}}_{=1},$$

from whence

$$b_{n,n} \geq \frac{1}{\sqrt{s_{2n}}}. \tag{3.10}$$

Using Proposition 2.15, (2.11), and plugging (3.10), yields

$$\limsup_{n\to\infty} \frac{2n \ln n}{\ln s_{2n}} \leq \limsup_{n\to\infty} \frac{n \ln n}{\ln b_{n,n}^{-1}} \leq \rho\big((s_n)_{n=0}^{\infty}\big). \tag{3.11}$$

❑

From Theorem 3.1 we now obtain examples for which the second inequality in (3.11) is strict.

3.6 Corollary. *For any $\rho \in (0, 1]$ and $r \in (0, \rho)$ there exists an indeterminate moment sequence $(s_n)_{n=1}^{\infty}$ such that*

$$\rho\big((s_n)_{n=0}^{\infty}\big) = \rho \quad \textit{and} \quad \limsup_{n\to\infty} \frac{n \ln n}{\ln b_{n,n}^{-1}} = r.$$

The sequence can be chosen so that $s_n = 0$ for all odd n.

Proof. Let $q \in \mathbb{N}$, $q \geq 3$, and set $\alpha := \frac{1}{\rho}$ and $\beta := \frac{q}{r} - (q-1)\alpha$. Then the moment sequence corresponding to H_q has all the required properties. ❑

We close the paper with formulating an open question. We have just established that the second inequality in (3.11) may be strict. Are there moment problems for which

$$\limsup_{n\to\infty} \frac{2n \ln n}{\ln s_{2n}} < \limsup_{n\to\infty} \frac{n \ln n}{\ln b_{n,n}^{-1}} \quad ?$$

The answer is expected to be affirmative.

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