Restriction and factorization for isometric and symmetric operators in almost Pontryagin spaces

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Abstract. We investigate symmetric linear relations in almost Pontryagin spaces. A notion of restriction and factorization is introduced. It applies to both spaces and relations.

The question under consideration is how symmetric extensions and inner products involving resolvents ("compressed resolvents") behave when a restriction-factorization process is applied. The main result, which holds under some natural conditions, is for a symmetric relation S and a restricted and factorized relation S_1 of S. Every compressed resolvent of S_1 can be realized as the compressed resolvent of a restriction-factorization of a symmetric extension of the original relation S. However, in general not every symmetric extension of S_1 coincides with the restriction-factorization of some symmetric extension of S. The difficulties one encounters, as well as the methods employed to overcome them, are mainly of geometric nature and are specific for the indefinite and degenerated situation.

The present results form the core needed to understand minimality notions for symmetric and selfadjoint linear relations in almost Pontryagin spaces.

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1. Introduction

The theory of extensions of symmetric operators in Hilbert spaces frequently appears in problems of classical analysis. For instance it is applied in the

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spectral theory of differential operators, interpolation problems, or moment problems, under the assumption that these problems are definite. Usually in the application of the extension theory one needs minimality of the involved symmetric operator or its selfadjoint extensions. The reduction of operators or extensions to minimal operators or minimal extensions is straightforward in the case of Hilbert spaces. When dealing with indefinite analogs of such classical problems the extension theory takes place in spaces with an indefinite inner product, more specifically, in Pontryagin spaces or in almost Pontryagin spaces. The same minimality requirements as in the definite case appear; however, the process to pass from operators or extensions to minimal ones is not straightforward anymore. The reason behind this is of a geometric nature as certain basic subspaces need not be orthocomplemented. The process towards minimality is not a pure restriction procedure, but it also involves factorization of (all or a part of) the isotropic parts of the subspaces. Before turning to the indefinite case in detail, we briefly explain the classical case.

When dealing with the spectral theory of symmetric operators S in Hilbert spaces $\mathcal{H}$, the notion of minimality appears in at least two variants. One, it appears for the basic symmetric operators and, two, for the selfadjoint extensions of such operators. First recall that a symmetric operator S is called *minimal* or, equivalently, *completely nonselfadjoint*, if

$$\bigcap_{w\in\gamma(S)} \operatorname{ran}(S-w)=\{0\},$$

where $\gamma(S)$ denotes the set of points of regular type of S. The property of being minimal has striking consequences. For simplicity we explain this for the case of a symmetric operator with deficiency index $(1,1)$; see [GG97], our standard reference concerning this topic.

- With S there is associated a family of analytic functions, the so-called Q-functions of S. The family of Q-functions contains all spectral information about S. For instance, the set $\gamma(S)$ coincides with the union of the domains of analyticity of Q-functions of S.
- The symmetry S is isomorphic to the operator of multiplication by the independent variable in a Hilbert space of analytic or meromorphic functions.
- Spectral measures for S can be constructed from integral representations of Q-functions and are described via Krein's resolvent formula.

Secondly, recall that a selfadjoint extension A of a symmetric operator S which acts in a possibly larger Hilbert space $\widetilde{\mathcal{H}}\supseteq\mathcal{H}$ is called *minimal*, if

$$\widetilde{\mathcal{H}}=\operatorname{cls}\Big(\mathcal{H}\cup\bigcup_{w\in\rho(A)}(A-w)^{-1}\mathcal{H}\Big),$$

where "cls" stands for "closed linear space". Again, the property of being minimal has strong consequences.

- The extension A is uniquely determined up to isomorphisms by its compressed resolvent, i.e., the operator family

$$R_A(w) := P(A-w)^{-1}|_{\mathcal{H}}, \quad w \in \rho(A),$$

where P denotes the orthogonal projection of $\widetilde{\mathcal{H}}$ onto $\mathcal{H}$.
- The resolvent set $\rho(A)$ is the maximal domain of analyticity of R_A.
- The totality of minimal extensions is parametrized via its compressed resolvents by Krein's resolvent formula.

Furthermore, recall that there exists a more refined notion of minimality of an extension, which is often useful. Namely, for a subset L of $\mathcal{H}$, possibly containing only one element, the extension A is said to be L-minimal, if

$$\widetilde{\mathcal{H}} = \operatorname{cls}\Big(L \cup \bigcup_{w\in\rho(A)} (A-w)^{-1}L\Big).$$

This property plays a role especially in applications of extension theory to concrete problems where one is only interested in the functions

$$\big[(A-w)^{-1}u, u\big], \quad w \in \rho(A), u \in L, \tag{1.1}$$

rather than in the whole compressed resolvent of A. For instance, think of the Hamburger- or Stieltjes power moment problems or the continuation problem for a positive definite function on an interval. In either problem the solutions are given via their Cauchy transforms as expressions of the form (1.1).

In view of the above facts it is interesting to observe that to a large extent it is possible to pass from an arbitrary symmetry S to a minimal one S_1, and from an arbitrary selfadjoint extension A to a minimal one A_1. This process is simple. Concerning minimality of the symmetry, set

$$\mathcal{C} := \bigcap_{w\in\gamma(S)} \operatorname{ran}(S-w), \tag{1.2}$$

then $S|_{\mathcal{C}}$ is selfadjoint. Let S_1 be the restriction of S to $\mathcal{H}[-]\mathcal{C}$. Then S_1 is minimal and the families of Q-functions of S and S_1 coincide. The selfadjoint extensions of S are related with those of S_1 in the obvious way. Namely, if A acts in $\widetilde{\mathcal{H}}$ and extends S, then the restriction of A to $\widetilde{\mathcal{H}}[-]\mathcal{C}$ extends S_1. As for the converse: if A_1 acts in a space $\widetilde{\mathcal{H}}_1 \subseteq \mathcal{H}[-]\mathcal{C}$ and extends S_1, then the diagonal operator $A_1 \times S|_{\mathcal{C}}$ acting in the direct product $\widetilde{\mathcal{H}} := \widetilde{\mathcal{H}}_1 \times \mathcal{C}$ extends S. Concerning the minimality of selfadjoint extensions A, set

$$\widetilde{\mathcal{H}}_L := \operatorname{cls}\Big(L \cup \bigcup_{w\in\rho(A)} (A-w)^{-1}L\Big), \tag{1.3}$$

and let A_1 be the restriction of A to this space. Then the families of functions (1.1) for A and A_1, respectively, coincide (in the case that $L = \mathcal{H}$, the compressed resolvents R_A and R_{A_1} coincide). As for the converse: each compressed resolvent of S can be realized with a minimal extension of S (and similar for L-resolvents (1.1)).

Now return to the extension theory in spaces with an indefinite inner product, more specifically, in Pontryagin spaces or almost Pontryagin spaces.

In this case the subspaces (1.2) and (1.3) need not be orthocomplemented; they may degenerate and they may intersect the isotropic part of the spaces $\mathcal{H}$ or $\widetilde{\mathcal{H}}$, respectively. Our aim in the present paper is to show to what extent it is possible to pass to operators constructed by restriction and factorization while keeping track of extensions and compressed resolvents. Conditions are given which ensure that the families of compressed resolvents of the original and the restricted-factorized operator coincide. Thereby, the difficult part is what was mentioned above as "the converse".

The problems which are addressed in this paper have been noted in [KL73] in the context of Pontryagin spaces. For a symmetric operator (with equal defect numbers) in a Pontryagin space there is a decomposition of the space which leads to a (nondiagonal) matrix decomposition of the operator involving a simple symmetric part and various other components due to the indefiniteness of the space, see [KL73, Satz 1.1]. In the present paper, in addition to the indefiniteness, also the degeneracy of the indefinite space contributes to the difficulties.

For technical reasons, the case of isometric and homeomorphic operators between closed subspaces of an almost Pontryagin space is dealt with first. After establishing the required knowledge for this situation, the Cayley transform is used to pass to the case of symmetric operators or relations.

At this point the reader is recommended to go to Section 4 directly; there the core problem is illustrated in detail and an example shows the difficulties ahead. Furthermore, in Section 4 one may find statements of the main extension results for spaces (Theorem 4.2) and for isometric operators (Theorem 4.3).

A systematic study of compressed resolvents and Q-functions in almost Pontryagin spaces, where the present results are applied to discuss minimality issues, will be undertaken in the forthcoming work [SW].

The paper is organised as follows.

- Section 2 contains preliminary material. We set up our notation, and recall from the literature several facts concerning almost Pontryagin spaces and isometric and symmetric linear relations.
- In Section 3 we define the main player of the paper, a restriction-factorization operator acting on subspaces and linear relations. We show how sets of points of regular type, ranges, and resolvent operators transform under the action of such an operator.
- Section 4 is, as already mentioned, devoted to a detailed description of the core problem, the difficulties which occur, and to the formulation of the extension theorems 4.2 and 4.3.
- The proof of Theorem 4.2 is carried out in Section 5. It settles the extension problem for spaces.
- The proof of Theorem 4.3 is carried out in Section 6. It settles the extension problem for maps.

- Finally, in Section 7 we deduce from Theorem 4.3 the corresponding result Theorem 7.1 about symmetric relations by applying Cayley transforms.

2. Preliminaries

2.1. Almost Pontryagin spaces

An *almost Pontryagin space* is a triple $\langle\mathcal{A},[.,.],\mathcal{T}\rangle$ consisting of a linear space $\mathcal{A}$, an inner product $[.,.]$ on $\mathcal{A}$, and a topology $\mathcal{T}$ on $\mathcal{A}$, such that

(aPs1) $\mathcal{T}$ is a Hilbert space topology on $\mathcal{A}$;

(aPs2) $[.,.]:\mathcal{A}\times\mathcal{A}\to\mathbb{C}$ is $\mathcal{T}\times\mathcal{T}$-continuous;

(aPs3) There exists a $\mathcal{T}$-closed linear subspace $\mathcal{M}$ of $\mathcal{A}$ with finite codimension such that $\langle\mathcal{M},[.,.]\rangle$ is a Hilbert space.

We often suppress explicit notation of the inner product $[.,.]$ and the topology $\mathcal{T}$, and shortly speak of an almost Pontryagin space $\mathcal{A}$.

The negative index of an inner product space $\mathcal{L}$ is defined as

$$\operatorname{ind}_-\mathcal{L}:=\sup\big\{\dim\mathcal{N}:\ \mathcal{N}\text{ negative subspace of }\mathcal{L}\big\}\in\mathbb{N}_0\cup\{\infty\},$$

where a subspace $\mathcal{N}$ of $\mathcal{L}$ is called negative, if $[x,x]<0$, $x\in\mathcal{N}\setminus\{0\}$. Moreover, $\mathcal{L}^\circ$ denotes the isotropic part of $\mathcal{L}$, i.e. $\mathcal{L}^\circ:=\mathcal{L}\cap\mathcal{L}^\perp$, and $\operatorname{ind}_0\mathcal{L}:=\dim\mathcal{L}^\circ$ is called the degree of degeneracy of $\mathcal{L}$. The inner product space $\mathcal{L}$ is called nondegenerated if $\operatorname{ind}_0\mathcal{L}=0$; otherwise $\mathcal{L}$ is called degenerated.

For the basics about the geometry of almost Pontryagin spaces we refer the reader [KWW05], for more specific properties we shall provide precise references in course of the presentation. Some further literature dealing with almost Pontryagin spaces and operators therein is [SW12], [Wor14], [PT09].

2.2. Linear relations

Let $\mathcal{A}$ be an almost Pontryagin space. A linear subspace T of $\mathcal{A}^2=\mathcal{A}\times\mathcal{A}$ is called a *linear relation* in $\mathcal{A}$. We say that T is a closed linear relation, if T is closed in the product topology of $\mathcal{A}^2$. For a linear relation T we denote

$$\begin{aligned}
\operatorname{dom}T&:=\big\{x\in\mathcal{A}:\ \exists y\in\mathcal{A}\text{ such that }(x,y)\in T\big\},\\
\operatorname{ran}T&:=\big\{y\in\mathcal{A}:\ \exists x\in\mathcal{A}\text{ such that }(x,y)\in T\big\},\\
\ker T&:=\big\{x\in\mathcal{A}:\ (x,0)\in T\big\},\\
\operatorname{mul}T&:=\big\{y\in\mathcal{A}:\ (0,y)\in T\big\}.
\end{aligned}$$

A linear operator between subspaces of $\mathcal{A}$ is identified with a linear relation via its graph. We refer to a linear relation T as an operator if $\operatorname{mul}T=\{0\}$, since this property characterizes that T is the graph of some linear operator.

We use the following algebraic operations with linear relations:

$$\begin{aligned}
T+z&:=\big\{(x,y+zx):(x,y)\in T\big\},\quad z\in\mathbb{C},\\
zT&:=\big\{(x,zy):(x,y)\in T\big\},\quad z\in\mathbb{C},\\
T^{-1}&:=\big\{(y,x):(x,y)\in T\big\}.
\end{aligned}$$

Let $\mathcal{A}$ be an almost Pontryagin space and let T be a linear relation in $\mathcal{A}$. The *point spectrum* $\sigma_p(T)$ of T is defined by

$$\sigma_p(T) := \big\{ z \in \mathbb{C} : \ker(T-z) \neq \{0\} \big\} \underbrace{\cup \ \{\infty\}}_{\text{if } \operatorname{mul} T \neq \{0\}}$$

The set $\gamma(T)$ of *points of regular type* of T is defined as

$$\gamma(T) := \big\{ z \in \mathbb{C} : (T-z)^{-1} \text{ is bounded operator} \big\}.$$

The set $\gamma(T)$ is open. If T is closed we have

$$\gamma(T) = \big\{ z \in \mathbb{C} : \ker(T-z) = \{0\}, \operatorname{ran}(T-z) \text{ closed} \big\}.$$

The *resolvent set* $\rho(T)$ of T is defined as

$$\rho(T) := \big\{ z \in \gamma(T) : \operatorname{ran}(T-z) \text{ is dense in } \mathcal{A} \big\}.$$

The set $\rho(T)$ is open. If T is closed we have

$$\begin{aligned} \rho(T) :=& \big\{ z \in \mathbb{C} : (T-z)^{-1} \text{ is bounded everywhere defined operator} \big\} \\ =& \big\{ z \in \mathbb{C} : \ker(T-z) = \{0\}, \operatorname{ran}(T-z) = \mathcal{A} \big\}. \end{aligned}$$

for details see [DS87a, Proposition 2.3].

2.3. Symmetric and isometric linear relations

The adjoint T^* of a linear relation T is defined as

$$T^* := \big\{ (x,y) \in \mathcal{A}^2 : [y,a] - [x,b] = 0, (a,b) \in T \big\}. \tag{2.1}$$

Clearly, T^* is a linear relation in $\mathcal{A}$. Since the inner product is continuous, T^* is closed.

Definition 2.1. A linear relation T in $\mathcal{A}$ is called *isometric* if $T^{-1} \subseteq T^*$, i.e.,

$$[x_1, x_2] = [y_1, y_2], \quad (x_1, y_1), (x_2, y_2) \in T.$$

Likewise, a linear relation T in $\mathcal{A}$ is called *symmetric* if $T \subseteq T^*$, i.e.,

$$[y_1, x_2] = [x_1, y_2], \quad (x_1, y_1), (x_2, y_2) \in T.$$

The following identity, which holds for isometric relations, is often practical (the proof is by computation).

Lemma 2.2. *Let $\mathcal{A}$ be an almost Pontryagin space, let T be an isometric linear relation in $\mathcal{A}$, and let $w \in \mathbb{C}$. Then*

$$[x,y] + w[x,v] + \frac{1}{\overline{w}}[y,u] = 0, \quad (x,y) \in T - w, (u,v) \in T - \frac{1}{\overline{w}}. \tag{2.2}$$

□

Lemma 2.3. *Let $\mathcal{A}$ be an almost Pontryagin space, and let $T : \mathcal{A} \to \mathcal{A}$ be an isometric and bijective linear map. Then the following statements hold.*

(i) *We have $T(\mathcal{A}^\circ) = \mathcal{A}^\circ$.*

(ii) *Let $w \in \mathbb{C} \setminus \{0\}$. If $\frac{1}{\overline{w}} \notin \sigma_p(T)$ and $w \notin \sigma_p(T|_{\mathcal{A}^\circ})$, then* $\operatorname{ran}(T-w)$ *is dense in $\mathcal{A}$.*

Proof. Item (i) is simple. Let $x \in \mathcal{A}^\circ$ and $y \in \mathcal{A}$. Then we have

$$[Tx, y] = [Tx, T(T^{-1}y)] = [x, T^{-1}y] = 0.$$

This shows that $Tx \in \mathcal{A}^\circ$. Since $\mathcal{A}^\circ$ is finite-dimensional and T is injective, it follows that $T(\mathcal{A}^\circ) = \mathcal{A}^\circ$.

For the proof of (ii), we first determine the orthogonal complement of $\operatorname{ran}(T - w)$. Assume that $x[\perp]\operatorname{ran}(T - w)$. Then, for all $y \in \mathcal{A}$, we have

$$0 = [x, (T - w)y] = [(T^{-1} - \overline{w})x, y],$$

and hence $(T^{-1} - \overline{w})x \in \mathcal{A}^\circ$. In turn, also $(T - \frac{1}{\overline{w}})x \in \mathcal{A}^\circ$. By our assumption $T - \frac{1}{\overline{w}}$ is injective. In particular, $(T - \frac{1}{\overline{w}})(\mathcal{A}^\circ) = \mathcal{A}^\circ$. Together it follows that $x \in \mathcal{A}^\circ$, and we conclude that

$$\big(\operatorname{ran}(T - w)\big)^{[\perp]} = \mathcal{A}^\circ. \tag{2.3}$$

Since $w \notin \sigma_p(T|_{\mathcal{A}^\circ})$, we have $(T - w)(\mathcal{A}^\circ) = \mathcal{A}^\circ$, and hence

$$\mathcal{A}^\circ \subseteq \operatorname{ran}(T - w). \tag{2.4}$$

The relations (2.3) and (2.4) together imply that $\operatorname{ran}(T - w)$ is dense in $\mathcal{A}$, see e.g. [Wor14, Lemma A.6,(iii)]. ☐

2.4. The Cayley transform

The Cayley transform is a particular instance of fractional linear transforms of linear relations which were studied [DS87b, §2]. Some algebraic properties of this particular transform were collected in [DS87a, Proposition 2.1].

Let T a linear relation in an almost Pontryagin space $\mathcal{A}$. For some base point $\mu \in \mathbb{C} \setminus \mathbb{R}$ we define the *Cayley transform* $C_\mu(T)$ of T as

$$C_\mu(T) := \big\{(g - \mu f, g - \overline{\mu} f) : (f, g) \in T\big\},$$

and the *inverse Cayley transform* $F_\mu(T)$ of T as

$$F_\mu(T) := \big\{(g - f, \mu g - \overline{\mu} f) : (f, g) \in T\big\}.$$

It is immediate from the definitions that

$$\begin{aligned} \operatorname{dom} C_\mu(T) &= \operatorname{ran}(T - \mu), & \operatorname{dom} F_\mu(T) &= \operatorname{ran}(T - 1), \\ \operatorname{ran} C_\mu(T) &= \operatorname{ran}(T - \overline{\mu}), & \operatorname{ran} F_\mu(T) &= \operatorname{ran}(T - \tfrac{\overline{\mu}}{\mu}), \\ \operatorname{mul} C_\mu(T) &= \ker(T - \mu), & \operatorname{mul} F_\mu(T) &= \ker(T - 1), \end{aligned}$$

and that

$$C_\mu(T) = I + (\mu - \overline{\mu})(T - \mu)^{-1}, \quad F_\mu(T) = \mu + (\mu - \overline{\mu})(T - 1)^{-1}.$$

Observe that, in particular, $C_\mu(T)$ is an operator if and only if $\mu \notin \sigma_p(T)$ (and analogous for F_μ). The following property of the Cayley transform explains the term inverse transform:

$$F_\mu(C_\mu(T)) = C_\mu(F_\mu(T)) = T.$$

Clearly, the relation T is closed if and only if $C_\mu(T)$ is closed. With the corresponding scalar fractional linear transforms[1]

$$c_\mu(z) := \frac{z-\overline{\mu}}{z-\mu} \text{ and } f_\mu(z) := \frac{\overline{\mu}z-\mu}{z-1},$$

one sees the identities

$$\begin{aligned} &\operatorname{ran}\big(C_\mu(T)-c_\mu(z)\big) = \operatorname{ran}(T-z), &&\operatorname{ran}\big(F_\mu(T)-f_\mu(z)\big) = \operatorname{ran}(T-z), \\ &\ker\big(C_\mu(T)-c_\mu(z)\big) = \ker(T-z), &&\ker\big(F_\mu(T)-f_\mu(z)\big) = \ker(T-z). \end{aligned}$$

These relations yield that

$$\sigma_p(C_\mu(T)) = c_\mu(\sigma_p(T)), \quad \sigma_p(F_\mu(T)) = f_\mu(\sigma_p(T)).$$

Formal resolvents of the Cayley transforms $C_\mu(T)$ and $F_\mu(T)$ can be related to the formal resolvent $(T-z)^{-1}$:

$$\begin{aligned} \big(C_\mu(T)-c_\mu(z)\big)^{-1} &= \frac{z-\mu}{\overline{\mu}-\mu} + \frac{(z-\mu)^2}{\overline{\mu}-\mu}(T-z)^{-1}, \\ \big(F_\mu(T)-f_\mu(z)\big)^{-1} &= \frac{z-1}{\mu-\overline{\mu}} + \frac{(z-1)^2}{\mu-\overline{\mu}}(T-z)^{-1}. \end{aligned}$$

It follows that

$$\begin{aligned} \gamma(C_\mu(T))\setminus\{1\} &= c_\mu(\gamma(T)\setminus\{\mu\}), \quad \gamma(F_\mu(T))\setminus\{\mu\} = f_\mu(\gamma(T)\setminus\{1\}), \\ \rho(C_\mu(T))\setminus\{1\} &= c_\mu(\rho(T)\setminus\{\mu\}), \quad \rho(F_\mu(T))\setminus\{\mu\} = f_\mu(\rho(T)\setminus\{1\}). \end{aligned}$$

The next property is easy to see and is stated here for later reference (its proof is straightforward).

Lemma 2.4. *Let T be a linear relation in an almost Pontryagin space $\mathcal{A}$ and let $\mu \in \mathbb{C}\setminus(\mathbb{R}\cup\sigma_p(T))$. Assume that $\mathcal{D}$ is a closed linear subspace of $\mathcal{A}$, which satisfies*

$$(T-\mu)^{-1}(\mathcal{D}) \subseteq \mathcal{D}.$$

Then

$$C_\mu\big(T\cap(\mathcal{D}\times\mathcal{D})\big) = C_\mu(T)|_{\mathcal{D}}, \quad \operatorname{ran}\big(T\cap(\mathcal{D}\times\mathcal{D})\big) = \mathcal{D}.$$

Here the expression $C_\mu(T)|_{\mathcal{D}}$ is understood as the graph of the restriction of the map $C_\mu(T)$ to $\mathcal{D}$. □

The Cayley transform can be used to switch between isometric and symmetric relations. Let T be a linear relation in $\mathcal{A}$ and $\mu \in \mathbb{C}\setminus\mathbb{R}$, then

$$T \text{ is symmetric if and only if } C_\mu(T) \text{ is isometric.}$$

This fact makes it possible to translate results on isometric relations to results on symmetric relations, and vice versa. In particular the following observation is important.

Lemma 2.5. *Let T be a linear relation in the almost Pontryagin space $\mathcal{A}$ and let $\mu \in \mathbb{C}\setminus\mathbb{R}$. Then T is a closed symmetric relation and $\mu,\overline{\mu} \in \gamma(T)$ if and only if $C_\mu(T)$ is an isometric homeomorphism between two closed subspaces of $\mathcal{A}$.* □

[1]We consider fractional linear transforms as acting on the Riemann sphere in the usual way.

3. Restriction and factorization in an almost Pontryagin space

Let $\mathcal{A}$ be an almost Pontryagin space with inner product $[.,.]$ and topology $\mathcal{O}$, and let $\mathcal{D}$ and $\mathcal{B}$ be closed linear subspaces of $\mathcal{A}$ such that

$$\mathcal{B} \subseteq \mathcal{D}^{\circ}.$$

Then the factor space $\mathcal{D}/\mathcal{B}$ becomes an almost Pontryagin space when endowed with the inner product and topology naturally inherited from $\mathcal{A}$, cf. [KWW05, Propositions 3.1 and 3.5].

The canonical projection $\pi : \mathcal{D} \to \mathcal{D}/\mathcal{B}$ is linear, isometric and continuous. Its kernel equals $\mathcal{B}$ and hence is, as a neutral subspace of an almost Pontryagin space, finite-dimensional. It follows that π maps closed subspaces of $\mathcal{D}$ to closed subspaces of $\mathcal{D}/\mathcal{B}$, cf. [KWW05, Proof of Proposition 3.5].

3.1. The restriction-factorization operator

Passing from $\mathcal{A}$ to $\mathcal{D}/\mathcal{B}$ may be viewed as a restriction-factorization process. This process extends to subspaces of $\mathcal{A}$ and to linear relations in $\mathcal{A}$.

Definition 3.1. Let $\mathcal{A}$ be an almost Pontryagin space and let $\mathcal{D}$ and $\mathcal{B}$ be closed linear subspaces of $\mathcal{A}$ with $\mathcal{B} \subseteq \mathcal{D}^{\circ}$. We define the *restriction-factorization operator* $\mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{D}}$ to act on linear subspaces $\mathcal{L}$ of $\mathcal{A}$ as

$$\mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{D}}(\mathcal{L}) := \pi(\mathcal{L} \cap \mathcal{D}) = \big\{\pi x : x \in \mathcal{L} \cap \mathcal{D}\big\}, \tag{3.1}$$

and on linear relations T in $\mathcal{A}$ as

$$\mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{D}}(T) := (\pi \times \pi)(T \cap \mathcal{D}^2) = \big\{(\pi g, \pi g) : (f, g) \in T \cap \mathcal{D}^2\big\}. \tag{3.2}$$

Thus $\mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{D}}$ maps linear subspaces of $\mathcal{A}$ to linear subspaces of $\mathcal{D}/\mathcal{B}$ and linear relations in $\mathcal{A}$ to linear relations in $\mathcal{D}/\mathcal{B}$.

Let us point out that a linear relation T in $\mathcal{A}$ is nothing but a linear subspace of $\mathcal{A}^2$. Via the canonical identification one has

$$\mathcal{D}^2/\mathcal{B}^2 \cong \mathcal{D}/\mathcal{B} \times \mathcal{D}/\mathcal{B}.$$

Hence we can write $\mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{D}}(T) = \mathfrak{F}^{\mathcal{B}^2}_{\mathcal{A}^2|\mathcal{D}^2}(T)$ for a linear relation T in $\mathcal{A}$, where the left side is understood in the sense of (3.2) and the right side as in (3.1).

Observe that $\mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{D}}(\mathcal{L})$ is a closed linear subspace of $\mathcal{D}/\mathcal{B}$ whenever $\mathcal{L}$ is a closed linear subspace of $\mathcal{A}$ since π maps closed subspaces to closed subspaces. Likewise, $\mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{D}}(T)$ is a closed linear relation in $\mathcal{D}/\mathcal{B}$ whenever T is a closed linear relation in $\mathcal{A}$. However, note that in general $\mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{D}}$ may transform (the graph of) a linear operator T into a linear relation $\mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{D}}(T)$ which is not necessarily (the graph of) a linear operator.

It is important to understand in detail the connection between the resolvents of the relations T and $\mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{D}}(T)$. The next proposition is basic.

Proposition 3.2. *Let $\mathcal{A}$ be an almost Pontryagin space and let $\mathcal{D}$ and $\mathcal{B}$ be closed linear subspaces of $\mathcal{A}$ with $\mathcal{B} \subseteq \mathcal{D}^{\circ}$. Let T be a closed linear relation* *in $\mathcal{A}$, and denote by $\Gamma(T)$ the set of all points $w \in \mathbb{C} \setminus \sigma_p(T)$ for which*

$$(T-w)^{-1}\big(\mathcal{D} \cap \operatorname{ran}(T-w)\big) \subseteq \mathcal{D}, \tag{3.3}$$

$$(T-w)^{-1}\big(\mathcal{B} \cap \operatorname{ran}(T-w)\big) \subseteq \mathcal{B}. \tag{3.4}$$

Then

$$\operatorname{ran}\big(\mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{D}}(T) - w\big) = \pi\big(\mathcal{D} \cap \operatorname{ran}(T-w)\big), \quad w \in \Gamma(T), \tag{3.5}$$

$$\Gamma(T) \subseteq \mathbb{C} \setminus \sigma_p\big(\mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{D}}(T)\big), \tag{3.6}$$

$$\Gamma(T) \cap \gamma(T) \subseteq \gamma\big(\mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{D}}(T)\big),\ \Gamma(T) \cap \rho(T) \subseteq \rho\big(\mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{D}}(T)\big), \tag{3.7}$$

$$\big(\mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{D}}(T) - w\big)^{-1} \circ \pi|_{\mathcal{D}\cap\operatorname{ran}(T-w)} = \pi \circ (T-w)^{-1}|_{\mathcal{D}\cap\operatorname{ran}(T-w)},\ w \in \Gamma(T). \tag{3.8}$$

In particular, it holds that

$$\big[\big(\mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{D}}(T) - w\big)^{-1}\pi x, \pi y\big] = \big[(T-w)^{-1}x, y\big],$$
$$x \in \mathcal{D} \cap \operatorname{ran}(T-w), y \in \mathcal{D},\ w \in \Gamma(T). \tag{3.9}$$

Proof. Using the notation of linear relations, we have for each $w \in \mathbb{C}$

$$\begin{aligned}
(\mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{D}}(T) - w)^{-1} &= \big\{\, (b - wa, a) : (a,b) \in \mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{D}}(T) \big\} \\
&= \big\{\, (\pi g - w\pi f, \pi f) : (f,g) \in T \cap \mathcal{D}^2 \big\} \\
&= (\pi \times \pi)\Big(\big\{\, (g - wf, f) : (f,g) \in T \cap \mathcal{D}^2 \big\}\Big) \\
&= (\pi \times \pi)\Big(\big\{\, (g - wf, f) : (f,g) \in T \big\} \cap \mathcal{D}^2\Big) \\
&= (\pi \times \pi)\Big((T-w)^{-1} \cap \mathcal{D}^2\Big).
\end{aligned} \tag{3.10}$$

Let $w \in \mathbb{C} \setminus \sigma_p(T)$ be given. Then $(T-w)^{-1}$ is the graph of a linear operator with domain $\mathcal{D} \cap \operatorname{ran}(T-w)$. Consider the graph of the domain restriction

$$(T-w)^{-1}|_{\mathcal{D}\cap\operatorname{ran}(T-w)}$$

of $(T-w)^{-1}$.

Assume that w satisfies (3.3). Then

$$(T-w)^{-1}|_{\mathcal{D}\cap\operatorname{ran}(T-w)} = \big\{\, (g - wf, f) : (f,g) \in T, g - wf \in \mathcal{D} \big\} =$$
$$= \big\{\, (g - wf, f) : (f,g) \in T, g - wf \in \mathcal{D}, f \in \mathcal{D} \big\} = (T-w)^{-1} \cap \mathcal{D}^2.$$

Putting together with (3.10), thus

$$(\mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{D}}(T) - w)^{-1} = (\pi \times \pi)\Big((T-w)^{-1}|_{\mathcal{D}\cap\operatorname{ran}(T-w)}\Big). \tag{3.11}$$

The relation (3.11) implies that

$$\operatorname{ran}(\mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{D}}(T) - w) = \operatorname{dom}(\mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{D}}(T) - w)^{-1} =$$
$$= \operatorname{dom}\Big((\pi \times \pi)\big((T-w)^{-1}|_{\mathcal{D}\cap\operatorname{ran}(T-w)}\big)\Big) = \pi\big(\mathcal{D} \cap \operatorname{ran}(T-w)\big),$$

and this is (3.5).

For the proof (3.6), let $w \in \Gamma(T)$ be given. Let $y \in \ker(\mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{D}}(T) - w)$, so that $(0, y) \in (\mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{D}}(T) - w)^{-1}$, and choose $x \in \mathcal{D} \cap \operatorname{ran}(T - w)$ with

$$(0, y) = (\pi x, \pi(T - w)^{-1}x).$$

Comparing the first entries, it follows that $x \in \mathcal{B}$. By (3.4) also $(T-w)^{-1}x \in \mathcal{B}$, and therefore $y = 0$. We conclude that $\ker(\mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{D}}(T) - w) = \{0\}$, i.e., $w \notin \sigma_p(\mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{D}}(T))$.

The inclusions in (3.7) follow using (3.5), (3.6), and that

$$\begin{aligned} w \in \gamma(T) \quad &\Longleftrightarrow \quad w \notin \sigma_p(T) \wedge \operatorname{ran}(T - w) \text{ closed}, \\ w \in \rho(T) \quad &\Longleftrightarrow \quad w \notin \sigma_p(T) \wedge \operatorname{ran}(T - w) = \mathcal{A}, \end{aligned}$$

and the same for $\mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{D}}(T)$.

The required relation (3.8) is just a reformulation of (3.11) in terms of operators instead of relations, and (3.9) is a consequence of (3.8) since π is isometric. □

Since the projection $\pi : \mathcal{D} \to \mathcal{D}/\mathcal{B}$ is isometric, it follows that the restriction-factorization operator transforms isometric relations into isometric ones.

Lemma 3.3. *Let $\mathcal{A}$ be an almost Pontryagin space and let $\mathcal{D}$ and $\mathcal{B}$ be closed linear subspaces of $\mathcal{A}$ with $\mathcal{B} \subseteq \mathcal{D}^\circ$. If T is an isometric linear relation in $\mathcal{A}$, then $\mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{D}}(T)$ is an isometric linear relation in $\mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{D}}(\mathcal{A})$.*

Proof. For (f, g) and (h, k) in $T \cap \mathcal{D}^2$ one has

$$[\pi f, \pi h] - [\pi g, \pi k] = [f, h] - [g, k] = 0.$$

Hence it follows that the linear relation $\mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{D}}(T)$ is isometric in the space $\mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{D}}(\mathcal{A})$. □

The focus in the present considerations lies on isometric homeomorphisms between closed subspaces of an almost Pontryagin space.

Lemma 3.4. *Let $\mathcal{A}$ be an almost Pontryagin space and let $\mathcal{D}$ and $\mathcal{B}$ be closed linear subspaces of $\mathcal{A}$ with $\mathcal{B} \subseteq \mathcal{D}^\circ$. Let R and R' be closed linear subspaces of $\mathcal{A}$, let $\beta : R \to R'$ be an isometric homeomorphism (of R onto R') with*

$$\beta(\mathcal{B}) \subseteq \mathcal{B}. \tag{3.12}$$

Then

$$\beta_1 := \mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{D}}(\beta)$$

is an isometric homeomorphism between the closed linear subspaces

$$\pi\big(\mathcal{D} \cap \beta^{-1}(\mathcal{D} \cap R')\big) \quad \text{and} \quad \pi\big(\mathcal{D} \cap \beta(\mathcal{D} \cap R)\big)$$

of $\mathcal{A}_1 := \mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{D}}(\mathcal{A})$. It holds that

$$\begin{aligned} &\operatorname{ran}(\beta_1 - w) = \pi\big(\mathcal{D} \cap (\beta - w)(\mathcal{D} \cap R)\big), \quad w \in \mathbb{C}, \\ &\sigma_p(\beta_1) \subseteq \sigma_p(\beta), \quad \gamma(\beta_1) \supseteq \gamma(\beta). \end{aligned}$$

□

Notice that (3.12) is equivalent to

$$\beta(\mathcal{B}) = \mathcal{B}.$$

This follows since $\mathcal{B}$ is a neutral subspace of $\mathcal{A}$, hence finite-dimensional, and β is injective.

Proof of Lemma 3.4. To show that β_1 is (the graph of) a map, consider an element $(0, y) \in \beta_1$. Choose $x \in R \cap (\mathcal{A}[-]\mathcal{C})$ with $(0, y) = (\pi x, \pi\beta x)$. Then $x \in \ker \pi = \mathcal{B}$, and hence also $\beta x \in \mathcal{B}$. This implies that $y = \pi\beta x = 0$. To show that β_1 is injective, consider an element $(y, 0) \in \beta_1$. Choose $x \in R \cap (\mathcal{A}[-]\mathcal{C})$ with $(y, 0) = (\pi x, \pi\beta x)$. Then $\beta x \in \ker \pi = \mathcal{B}$, and hence also $x \in \mathcal{B}$. This implies that $y = \pi\beta x = 0$.

To show that the domain of β_1 contains $\pi(\mathcal{D} \cap \beta^{-1}(\mathcal{D} \cap R'))$, let y in this space be given. Choose $x \in \mathcal{D} \cap \beta^{-1}(\mathcal{D} \cap R')$ with $\pi x = y$, then $\beta x \in \mathcal{D}$ and hence $(\pi x, \pi\beta x) \in \beta_1$. Thus $\pi(\mathcal{D} \cap \beta^{-1}(\mathcal{D} \cap R')) \subseteq \operatorname{dom}\beta_1$. The reverse inclusion is obvious.

To determine the range of $\beta_1 - w$, let first $y \in \pi(\mathcal{D} \cap (\beta - w)(\mathcal{D} \cap R))$ be given. Choose $x \in \mathcal{D} \cap (\beta - w)(\mathcal{D} \cap R)$ with $\pi x = y$ and $z \in \mathcal{D} \cap R$ with $x = (\beta - w)z$. Then $(z, x + wz) \in \beta \cap \mathcal{D}^2$, and hence $\{\pi z, \pi x + w\pi z\} \in \beta_1$. From this we see that $y = \pi x \in \operatorname{ran}(\beta_1 - w)$. For the reverse inclusion, let $y_1 \in \operatorname{ran}(\beta_1 - w)$ be given. Choose $(x, z) \in \beta \cap \mathcal{D}^2$ such that $y_1 = \pi z - w\pi x$. Then $z - wx \in \mathcal{D} \cap (\beta - w)(\mathcal{D} \cap R)$, and we see that $y_1 \in \pi(\mathcal{D} \cap (\beta - w)(\mathcal{D} \cap R))$.

As we already observed R_1 and R_1' are closed as the projection $\pi : \mathcal{A} \to \mathcal{A}_1$ maps closed subspaces to closed subspaces. Since β is continuous and the domain of β is closed, the graph of β is closed. It follows that also the graph of β_1 is closed. The closed graph theorem applies with β_1 and β_1^{-1}, and shows that β_1 is a homeomorphism. Moreover, β_1 is isometric, remember Lemma 3.3.

Next we show that $\sigma_p(\beta_1) \subseteq \sigma_p(\beta)$. Assume that $w \in \sigma_p(\mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{D}}(\beta))$, and choose $x \in \mathcal{D} \cap \operatorname{dom}\beta$ with

$$\pi x \in \ker\big(\mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{D}}(\beta) - w\big) \setminus \{0\}.$$

Then $x \notin \mathcal{B}$ while

$$\pi((\beta - w)x) = \big(\mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{D}}(\beta) - w\big)\pi x = 0,$$

i.e., $(\beta - w)x \in \mathcal{B}$. Assume now on the contrary that $w \notin \sigma_p(\beta)$. In particular, then $w \notin \sigma_p(\beta|_{\mathcal{B}})$, and we can choose $y \in \mathcal{B}$ with $(\beta - w)y = (\beta - w)x$. It follows that

$$x - y \in \ker(\beta - w) \setminus \{0\},$$

and we reached a contradiction.

Finally, assume that $w \in \gamma(\beta)$. Then $w \notin \sigma_p(\beta)$ and hence also $w \notin \sigma_p(\beta_1)$. Moreover, $\beta - w$ is a homeomorphism of R onto $(\beta - w)(R)$ and hence maps the closed subspaces to closed subspaces. The already proved equality of ranges, together with the fact that π maps closed subspaces to closed subspaces, implies that $\operatorname{ran}(\beta_1 - w)$ is closed. $\square$

Let us turn our attention to symmetric relations. Again, since $\pi : \mathcal{D} \to \mathcal{D}/\mathcal{B}$ is isometric, a restriction factorization operator preserves symmetry. In this context it is practical to observe that restriction-factorization operators are compatible with Cayley transforms (the proof is straightforward).

Lemma 3.5. *Let $\mathcal{A}$ be an almost Pontryagin space and let $\mathcal{D}$ and $\mathcal{B}$ be closed linear subspaces of $\mathcal{A}$ with $\mathcal{D} \subseteq \mathcal{C}^\circ$. Let T be a linear relation in $\mathcal{A}$, then*

$$\mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{D}}\big(C_\mu(T)\big) = C_\mu\big(\mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{D}}(T)\big), \quad \mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{D}}\big(F_\mu(T)\big) = F_\mu\big(\mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{D}}(T)\big).$$

□

3.2. Restriction and factorization for isometric homeomorphisms

We will consider restriction to the orthogonal complement of a given space $\mathcal{C}$ rather than restriction to a given space $\mathcal{D}$. That means, we use $\mathcal{D} := \mathcal{A}[-]\mathcal{C}$ and consider $\mathcal{C}$ as the given data. For the motivation to take this viewpoint see the first paragraph of §4.1.

Lemma 3.6. *Let $\mathcal{A}$ be an almost Pontryagin space and let $\mathcal{C}$ and $\mathcal{B}$ be closed linear subspaces of $\mathcal{A}$ with $\mathcal{B} \subseteq \mathcal{C}^\circ$. Let R and R' be closed linear subspaces of $\mathcal{A}$, let $\beta : R \to R'$ be an isometric homeomorphism, and assume that*

$$\mathcal{C} \subseteq R, \ \beta(\mathcal{C}) = \mathcal{C}, \quad \beta(\mathcal{B}) \subseteq \mathcal{B}.$$

Then $\beta_1 := \mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{A}[-]\mathcal{C}}(\beta)$ is an isometric homeomorphism between the closed linear subspaces

$$R_1 := \mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{A}[-]\mathcal{C}}(R) \quad \textit{and} \quad R_1' := \mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{A}[-]\mathcal{C}}(R')$$

of $\mathcal{A}_1 := \mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{A}[-]\mathcal{C}}(\mathcal{A})$.

Proof. Lemma 3.4 applied with the map β and the spaces

$$\mathcal{D} := \mathcal{A}[-]\mathcal{C} \quad \text{and} \quad \mathcal{B}$$

yields that β_1 is an isometric homeomorphism between

$$\pi\big((\mathcal{A}[-]\mathcal{C}) \cap \beta^{-1}\big((\mathcal{A}[-]\mathcal{C}) \cap R'\big)\big) \quad \text{and} \quad \pi\big((\mathcal{A}[-]\mathcal{C}) \cap \beta\big((\mathcal{A}[-]\mathcal{C}) \cap R\big)\big).$$

We are going to show that

$$\beta\big((\mathcal{A}[-]\mathcal{C}) \cap R\big) = (\mathcal{A}[-]\mathcal{C}) \cap R'. \tag{3.13}$$

From this it follows that

$$\pi\big((\mathcal{A}[-]\mathcal{C}) \cap \beta^{-1}\big((\mathcal{A}[-]\mathcal{C}) \cap R'\big)\big) = \pi\big((\mathcal{A}[-]\mathcal{C}) \cap R\big) = R_1,$$
$$\pi\big((\mathcal{A}[-]\mathcal{C}) \cap \beta\big((\mathcal{A}[-]\mathcal{C}) \cap R\big)\big) = \pi\big((\mathcal{A}[-]\mathcal{C}) \cap R'\big) = R_1',$$

To show (3.13) let $x \in R \cap (\mathcal{A}[-]\mathcal{C})$ and $y \in \mathcal{C}$. Since $\beta(\mathcal{C}) = \mathcal{C}$ we have $\beta^{-1}y \in \mathcal{C}$, and it follows that

$$[\beta x, y] = [x, \beta^{-1}y] = 0.$$

This yields $\beta(R \cap (\mathcal{A}[-]\mathcal{C})) \subseteq \mathcal{A}[-]\mathcal{C}$. Conversely, if $x \in R' \cap (\mathcal{A}[-]\mathcal{C})$ and $y \in \mathcal{C}$, then $\beta y \in \mathcal{C}$ and hence

$$[\beta^{-1}x, y] = [x, \beta y] = 0,$$

and hence $\beta^{-1}(R' \cap (\mathcal{A}[-]\mathcal{C})) \subseteq \mathcal{A}[-]\mathcal{C}$. □

4. Behaviour of isometric extensions in a restriction-factorization process

4.1. An extension problem

Let $\mathcal{A}$ be an almost Pontryagin space, let $\mathcal{C}$ and $\mathcal{B}$ be closed subspaces of $\mathcal{A}$ with $\mathcal{B} \subseteq \mathcal{C}^\circ$, and let β be an isometric homeomorphism between closed subspaces R and R' of $\mathcal{A}$. Assume that

$$\mathcal{C} \subseteq R, \quad \beta(\mathcal{C}) = \mathcal{C}, \quad \beta(\mathcal{B}) \subseteq \mathcal{B}.$$

Then also every extension $\tilde{\beta}$ of β fulfills these conditions. Moreover, for every almost Pontryagin space $\tilde{\mathcal{A}} \supseteq \mathcal{A}$ we have $\mathcal{C}^\circ \subseteq (\tilde{\mathcal{A}}[-]\mathcal{C})^\circ$. Hence, we can apply Lemma 3.6 for restriction to the orthogonal complement of $\mathcal{C}$ followed by factorization of $\mathcal{B}$ with every extension $\tilde{\beta}$ acting between some closed subspaces $\tilde{R} \supseteq R$ and $\tilde{R}' \supseteq R'$ of some almost Pontryagin space $\tilde{\mathcal{A}} \supseteq \mathcal{A}$.

Clearly, the restriction-factorization[2] of $\tilde{\beta}$ will be an extension of the restriction-factorization of β. Thus our given data $\mathcal{A}, R, R', \beta, \mathcal{C}, \mathcal{B}$ gives rise to a procedure assigning to each extension $\tilde{\beta}$ of β the extension

$$\tilde{\beta}_1 := \mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{\beta}) \quad \text{of} \quad \beta_1 := \mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{A}[-]\mathcal{C}}(\beta),$$

where $\tilde{\beta}_1$ acts in the space

$$\tilde{\mathcal{A}}_1 := \mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{\mathcal{A}}) \quad \text{which extends} \quad \mathcal{A}_1 := \mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\mathcal{A}).$$

Thus the situation is

$$\begin{tikzcd}
\mathcal{A}: & R \arrow[r, "\beta"] \arrow[rrr, phantom, bend left, dashed, "\mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{A}[-]\mathcal{C}}"] & R' & R_1 \arrow[r, dashed, "\beta_1"] & R_1' & :\mathcal{A}_1 \\
\cap| & \cap| & \cap| & \cap| & \cap| & \cap| \\
\tilde{\mathcal{A}}: & \tilde{R} \arrow[r, "\tilde{\beta}"'] \arrow[rrr, phantom, bend right, dashed, "\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}"'] & \tilde{R}' & \tilde{R}_1 \arrow[r, dashed, "\tilde{\beta}_1"'] & \tilde{R}_1' & :\tilde{\mathcal{A}}_1
\end{tikzcd}$$

where

$$R_1 := \mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{A}[-]\mathcal{C}}(R), \quad R_1' := \mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{A}[-]\mathcal{C}}(R'),$$

$$\tilde{R}_1 := \mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{R}), \quad \tilde{R}_1' := \mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{R}').$$

The aim of the present paper is to answer the following question:

> *Is it true that every isometric and homeomorphic extension $\tilde{\beta}_1$ of β_1 arises as, or (at least) is closely related to, the restriction-factorization of some isometric and homeomorphic extension $\tilde{\beta}$ of β ?*

[2]For the definition of the restriction-factorization operators $\mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{D}}$ see Definition 3.1.

Thus we start with a diagram

$$\begin{tikzcd}
\mathcal{A}: & R \arrow[r, "\beta"] & R' \arrow[rr, bend left, "\mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{A}[-]\mathcal{C}}"] & & R_1 \arrow[r, "\beta_1"] \arrow[d, phantom, sloped, "\subseteq"] & R_1' \arrow[d, phantom, sloped, "\subseteq"] & :\mathcal{A}_1 \arrow[d, phantom, sloped, "\subseteq"] \\
& & & & \tilde{R}_1 \arrow[r, "\tilde{\beta}_1"] & \tilde{R}_1' & :\tilde{\mathcal{A}}_1
\end{tikzcd}$$

and aim to complete it to

$$\begin{tikzcd}
\mathcal{A}: \arrow[d, phantom, sloped, "\subseteq"] & R \arrow[r, "\beta"] \arrow[d, phantom, sloped, "\subseteq"] & R' \arrow[d, phantom, sloped, "\subseteq"] \arrow[rr, bend left, "\mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{A}[-]\mathcal{C}}"] & & R_1 \arrow[r, "\beta_1"] \arrow[d, phantom, sloped, "\subseteq"] & R_1' \arrow[d, phantom, sloped, "\subseteq"] & :\mathcal{A}_1 \arrow[d, phantom, sloped, "\subseteq"] \\
\tilde{\mathcal{A}}: & \tilde{R} \arrow[r, dashed, "\tilde{\beta}"'] & \tilde{R}' \arrow[rr, dashed, bend right, "\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}"'] & & \tilde{R}_1 \arrow[r, bend left, "\tilde{\beta}_1"] \arrow[r, dashed, bend right, "\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{\beta})"'] & \tilde{R}_1' & :\tilde{\mathcal{A}}_1
\end{tikzcd} \tag{4.1}$$

in such a way that $\tilde{\beta}_1$ and $\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{\beta})$ are as closely related as possible, preferably coincide.

This extension problem has two aspects. The first is an extension problem for spaces: to determine the existence of an almost Pontryagin space $\tilde{\mathcal{A}}$ such that

$$\tilde{\mathcal{A}} \supseteq \mathcal{A} \quad \text{and} \quad \mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{\mathcal{A}}) = \tilde{\mathcal{A}}_1. \tag{4.2}$$

The second is an extension problem for operators: to determine the existence of closed linear subspaces $\tilde{R}$ and $\tilde{R}'$ in such an almost Pontryagin space $\tilde{\mathcal{A}}$ and of an isometric homeomorphism $\tilde{\beta}$ from $\tilde{R}$ onto $\tilde{R}'$, such that $\tilde{\beta}$ extends the original operator β and, preferably,

$$\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{\beta}) = \tilde{\beta}_1. \tag{4.3}$$

In order to illustrate the difficulties which appear in the extension problem for operators, we elaborate a toy example. Though being very simple, this example already shows that in general we cannot expect to achieve (4.3).

Example 4.1. Let $\mathcal{L}$ be the linear space $\mathbb{C}^4$ endowed with the inner product induced by the Gram-matrix

$$G := \left(\begin{array}{c|c} \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & 0 \\ \hline 0 & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \end{array}\right).$$

Then $\mathcal{L}$ is a Pontryagin space. All considerations will take place within this space.

Denote by e_j, $j = 1, \dots, 4$, the canonical basis vectors of $\mathbb{C}^4$ and define $\mathcal{A}$ as

$$\mathcal{A} := \operatorname{span}\{e_1, e_2, e_3\}.$$

Then $\mathcal{A}$ is an almost Pontryagin space and $\mathcal{A}^\circ = \operatorname{span}\{e_3\}$. Set

$$R := \mathcal{A}, R' := \mathcal{A}, \quad \beta := \operatorname{id}_{\mathcal{A}} : R \to R'.$$

Clearly, β is an isometric bijection of R onto R'.

Now define linear subspaces $\mathcal{B}$ and $\mathcal{C}$ of $\mathcal{A}$ as

$$\mathcal{B} = \{0\}, \quad \mathcal{C} := \operatorname{span}\{e_1\}.$$

Then, trivially, $\mathcal{B} \subseteq \mathcal{C}^\circ$. Moreover, one sees that

$$\mathcal{C}^\circ = (\mathcal{A}[-]\mathcal{C})^\circ = \operatorname{span}\{e_1\}.$$

The restriction operator $\mathfrak{F}^{\{0\}}_{\mathcal{A}|\mathcal{A}[-]\mathcal{C}}$ (since $\mathcal{B} = \{0\}$ factorization is not present) gives rise to the following spaces $\mathcal{A}_1, R_1, R_1'$ and map β_1:

$$\mathcal{A}_1 := \mathfrak{F}^{\{0\}}_{\mathcal{A}|\mathcal{A}[-]\mathcal{C}}(\mathcal{A}) = \operatorname{span}\{e_1, e_3\}, \quad R_1 = R_1' = \mathcal{A}_1, \quad \beta_1 = \operatorname{id}_{\mathcal{A}_1}.$$

Next, consider the space

$$\tilde{\mathcal{A}}_1 := \operatorname{span}\{e_1, e_3, e_4\}.$$

This is an almost Pontryagin space, $(\tilde{\mathcal{A}}_1)^\circ = \operatorname{span}\{e_1\}$, and $\tilde{\mathcal{A}}_1 \supseteq \mathcal{A}_1$. For $\xi \in \mathbb{R}$ and $\eta \in \mathbb{C}$ let $\tilde{\beta}_1^{\xi,\eta}$ be the bijection of $\tilde{R}_1 := \tilde{\mathcal{A}}_1$ onto $\tilde{R}_1' := \tilde{\mathcal{A}}_1$ which acts as

$$\tilde{\beta}_1^{\xi,\eta}(x + \lambda e_4) := x + \lambda(e_4 + i\xi e_3 + \eta e_1), \quad x \in \mathcal{A}_1,\ \lambda \in \mathbb{C}. \tag{4.4}$$

Then $\tilde{\beta}_1^{\xi,\eta}$ is isometric and extends β_1. It is easily checked that the family of all isometric extensions $\tilde{\beta}_1$ of β_1 which act bijectively between subspaces of $\tilde{\mathcal{A}}_1$ is given by

$$\{\tilde{\beta}_1^{\xi,\eta} : \xi \in \mathbb{R}, \eta \in \mathbb{C}\} \cup \{\beta\}.$$

Extension spaces $\tilde{\mathcal{A}}$ of $\mathcal{A}$ with

$$\mathfrak{F}^{\{0\}}_{\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{\mathcal{A}}) = \tilde{\mathcal{A}}_1 \tag{4.5}$$

do exist. For example, the space $\tilde{\mathcal{A}} := \mathcal{L}$ has this property. Let us show that this is the only space $\tilde{\mathcal{A}}$ with (4.5). If $\tilde{\mathcal{A}}$ satisfies (4.5), then $\dim \tilde{\mathcal{A}} = \dim \tilde{\mathcal{A}}_1 + \dim \mathcal{C} = 4$. The subspace $\operatorname{span}\{e_1, e_2\}$ of $\tilde{\mathcal{A}}$ is nondegenerated and its orthogonal complement is two-dimensional and contains the neutral element e_3. Hence, $\operatorname{span}\{e_1, e_2\}^\perp$ is either neutral or nondegenerated with positive and negative index equal to 1. We have $\tilde{\mathcal{A}}[-]\mathcal{C} = \operatorname{span}\{e_1\}[\dot{+}]\operatorname{span}\{e_1, e_2\}^\perp$, hence the first case cannot take place. We see that $\tilde{\mathcal{A}}$ is a 4-dimensional space whose positive and negative indices are equal to 2 and which contains $\mathcal{A}$. Hence, $\tilde{\mathcal{A}}$ is equal to $\mathcal{L}$ (by making an appropriate choice of the basis vector e_4).

For $\xi \in \mathbb{R}$ let $\tilde{\beta}^\xi$ be the bijection of $\tilde{R} := \tilde{\mathcal{A}}$ onto $\tilde{R}' := \tilde{\mathcal{A}}$ which acts as

$$\tilde{\beta}^\xi(x + \lambda e_4) := x + \lambda(e_4 + i\xi e_3), \quad x \in \mathcal{A},\ \lambda \in \mathbb{C}.$$

Then $\tilde{\beta}^\xi$ is isometric and extends β. Again, it is easy to check that the family of all isometric extensions $\tilde{\beta}$ of β which act bijectively between subspaces of $\tilde{\mathcal{A}}$ is

$$\{\tilde{\beta}^\xi : \xi \in \mathbb{R}\} \cup \{\beta\}.$$

Restricting to $\tilde{\mathcal{A}}_1$ gives

$$\mathfrak{F}^{\{0\}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{\beta}^{\xi}) = \tilde{\beta}_1^{\xi,0}, \quad \xi \in \mathbb{R}. \tag{4.6}$$

Thus we have the following diagram.

$$\begin{tikzcd}
\mathcal{A} & : & \mathcal{A} \arrow[r, "\mathrm{id}_{\mathcal{A}}"] & \mathcal{A} \arrow[r, bend left, "\mathfrak{F}^{\{0\}}_{\mathcal{A}|\mathcal{A}[-]\mathcal{C}}"] & \mathcal{A}_1 \arrow[r, "\beta_1"] & \mathcal{A}_1 & : & \mathcal{A}_1 \\
\mathrm{span}\{e_1,e_2,e_3\} & & \rotatebox[origin=c]{90}{$\subseteq$} & \rotatebox[origin=c]{90}{$\subseteq$} & \rotatebox[origin=c]{90}{$\subseteq$} & \rotatebox[origin=c]{90}{$\subseteq$} & & \mathrm{span}\{e_1,e_3\} \\
\tilde{\mathcal{A}} & : & \tilde{\mathcal{A}} \arrow[r, dashed, "\tilde{\beta}^{\xi}"'] & \tilde{\mathcal{A}} \arrow[r, dashed, bend right, "\mathfrak{F}^{\{0\}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}"'] & \tilde{\mathcal{A}}_1 \arrow[r, bend left, "\tilde{\beta}_1^{\xi,\eta}"] \arrow[r, dashed, bend right, "\tilde{\beta}_1^{\xi,0}"'] & \tilde{\mathcal{A}}_1 & : & \tilde{\mathcal{A}}_1 \\
\mathrm{span}\{e_1,e_2,e_3,e_4\} & & & & & & & \mathrm{span}\{e_1,e_3,e_4\}
\end{tikzcd}$$

Observe that not every bijective and isometric extension $\tilde{\beta}_1$ of β_1 can be obtained as a restriction $\mathfrak{F}^{\{0\}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{\beta})$ with some bijective and isometric extension $\tilde{\beta}$ of β. However, the only obstacle is the presence of the summand ηe_1 in (4.4), which belongs to the isotropic part of $\tilde{\mathcal{A}}_1$.

4.2. Two extension theorems

The solutions of the two aspects of the extension problem will now be stated. The first theorem concerns the extension problem for spaces; its proof is carried out in Section 5.

Theorem 4.2. *Let $\mathcal{A}$ be an almost Pontryagin space and let $\mathcal{C}$ and $\mathcal{B}$ be closed linear subspaces of $\mathcal{A}$ with $\mathcal{B} \subseteq \mathcal{C}^{\circ}$. Set*[3]

$$\mathcal{A}_1 := \mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{A}[-]\mathcal{C}}(\mathcal{A}).$$

Let $\tilde{\mathcal{A}}_1$ be an almost Pontryagin space with $\tilde{\mathcal{A}}_1 \supseteq \mathcal{A}_1$. Then an almost Pontryagin space $\tilde{\mathcal{A}}$ with $\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{\mathcal{A}}) = \tilde{\mathcal{A}}_1$ exists if and only if

$$\mathcal{C}^{\circ}/\mathcal{B} \subseteq (\tilde{\mathcal{A}}_1)^{\circ}. \tag{4.7}$$

The second theorem concerns the extension problem for operators; its proof is carried out in Section 6. in general one can construct $\tilde{\beta}$ such that it satisfies a weak version of the equality (4.3): the resolvents of $\tilde{\beta}_1$ and $\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{\beta})$ differ only in isotropic summands and hence their action in terms of inner products coincides. Under an additional geometric condition on the data $\mathcal{C}, \mathcal{B}, R$ it is even possible to exhibit the equality (4.3).

Theorem 4.3. *Let $\mathcal{A}$ be an almost Pontryagin space and let $\mathcal{C}$ and $\mathcal{B}$ be closed linear subspaces of $\mathcal{A}$ with $\mathcal{B} \subseteq \mathcal{C}^{\circ}$. Set*[3]

$$\mathcal{A}_1 := \mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{A}[-]\mathcal{C}}(\mathcal{A}).$$

Let R and R' be closed linear subspaces of $\mathcal{A}$, let $\beta : R \to R'$ be a linear and isometric homeomorphism, assume that

$$\mathcal{C} \subseteq R, \quad \beta(\mathcal{C}) = \mathcal{C}, \quad \beta(\mathcal{B}) \subseteq \mathcal{B}, \tag{4.8}$$

[3] Again, for the notation $\mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{A}[-]\mathcal{C}}$ see Definition 3.1.

and set[4]

$$R_1 := \mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{A}[-]\mathcal{C}}(R),\ R_1' := \mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{A}[-]\mathcal{C}}(R'),\ \beta_1 := \mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{A}[-]\mathcal{C}}(\beta). \tag{4.9}$$

Moreover, denote by $\mathcal{E}$ the exceptional set

$$\mathcal{E} := \sigma_p(\beta|_{\mathcal{C}}) \cup \Big\{w \in \mathbb{C} \setminus \{0\} : \frac{1}{\overline{w}} \in \sigma_p(\beta|_{\mathcal{C}^\circ})\Big\} \cup \{\infty\}. \tag{4.10}$$

Let $\tilde{\mathcal{A}}_1$ be an almost Pontryagin space with $\tilde{\mathcal{A}}_1 \supseteq \mathcal{A}_1$, let $\tilde{R}_1$ and $\tilde{R}_1'$ be closed linear subspaces of $\tilde{\mathcal{A}}_1$ with $\tilde{R}_1 \supseteq R_1$ and $\tilde{R}_1' \supseteq R_1'$, and $\tilde{\beta}_1 : \tilde{R}_1 \to \tilde{R}_1'$ be a linear and isometric homeomorphism with $\tilde{\beta}_1|_{R_1} = \beta_1$. Assume that $\tilde{\mathcal{A}}$ is an almost Pontryagin space with

$$\tilde{\mathcal{A}} \supseteq \mathcal{A} \quad \textit{and} \quad \mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{\mathcal{A}}) = \tilde{\mathcal{A}}_1.$$

Then the following statements hold.

(i) *There exist closed linear subspaces $\tilde{R}$ and $\tilde{R}'$ of $\tilde{\mathcal{A}}$ with $\tilde{R} \supseteq R$ and $\tilde{R}' \supseteq R'$, and a linear and isometric homeomorphism $\tilde{\beta} : \tilde{R} \to \tilde{R}'$ extending β, such that*

$$\sigma_p\big(\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{\beta})\big) \setminus \mathcal{E} = \sigma_p(\tilde{\beta}) \setminus \mathcal{E} = \sigma_p(\tilde{\beta}_1) \setminus \mathcal{E}, \tag{4.11}$$

$$\left.\begin{aligned}&\gamma(\tilde{\beta}) \setminus \mathcal{E} \subseteq \gamma\big(\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{\beta})\big) \setminus \mathcal{E} = \gamma(\tilde{\beta}_1) \setminus \mathcal{E},\\ &\rho(\tilde{\beta}) \setminus \mathcal{E} \subseteq \rho\big(\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{\beta})\big) \setminus \mathcal{E} = \rho(\tilde{\beta}_1) \setminus \mathcal{E}\end{aligned}\right., \tag{4.12}$$

$$\begin{aligned}\operatorname{ran}\big(\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{\beta}) - w\big) &= \pi\big((\tilde{\mathcal{A}}[-]\mathcal{C}) \cap \operatorname{ran}(\tilde{\beta} - w)\big) = \\ &= \operatorname{ran}(\tilde{\beta}_1 - w), \quad w \in \mathbb{C} \setminus (\sigma_p(\tilde{\beta}_1) \cup \mathcal{E}),\end{aligned} \tag{4.13}$$

$$\begin{aligned}\big[(\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{\beta}) - w)^{-1}x_1, y_1\big] &= \big[(\tilde{\beta}_1 - w)^{-1}x_1, y_1\big],\\ x_1 \in \operatorname{ran}(\tilde{\beta}_1 - w),\ y_1 \in \tilde{\mathcal{A}}_1,&\quad w \in \mathbb{C} \setminus (\sigma_p(\tilde{\beta}_1) \cup \mathcal{E}).\end{aligned} \tag{4.14}$$

(ii) *If it holds in addition that*

$$(\mathcal{C}^\circ \cap R^\circ) + \mathcal{B} = \mathcal{C}^\circ, \tag{4.15}$$

then the choice of $\tilde{\beta}$ in (i) *can be made such that $\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{\beta}) = \tilde{\beta}_1$.*

Let us revisit Example 4.1 to illustrate the assertions of the theorem.

Example 4.4. Let notation be as in Example 4.1. First, we have

$$\mathcal{C}^\circ / \mathcal{B} = \operatorname{span}\{e_1\} = (\tilde{\mathcal{A}}_1)^\circ,$$

which reflects the fact that we can find an almost Pontryagin space $\tilde{\mathcal{A}}$ with

$$\mathfrak{F}^{\{0\}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{\mathcal{A}}) = \tilde{\mathcal{A}}_1.$$

Second, we have for $\xi, \eta \in \mathbb{R}$

$$\big[(\tilde{\beta}_1^{\xi,\eta} - w)^{-1}x, y\big] = \big[(\tilde{\beta}_1^{\xi,0} - w)^{-1}x, y\big] = \big[\big(\mathfrak{F}^{\{0\}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{\beta}^{\xi}) - w\big)^{-1}x, y\big],$$

for all $x, y \in \tilde{\mathcal{A}}_1$ and $w \in \mathbb{C}$ which is not an eigenvalue of $\tilde{\beta}_1^{\xi,\eta}$ or $\tilde{\beta}_1^{\xi,0}$.

[4] Revisit the visualization in diagram (4.1).

Finally, the fact that we cannot always achieve that

$$\tilde\beta_1^{\xi,\eta} = \mathfrak{F}^{\{0\}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde\beta)$$

with some extension $\tilde\beta$ of β reflects in the fact that

$$(\mathcal{C}^\circ \cap R^\circ) + \mathcal{B} = \{0\} \neq \operatorname{span}\{e_1\} = \mathcal{C}^\circ.$$

5. An extension problem for an almost Pontryagin space

Let $\mathcal{A}$ be an almost Pontryagin space and let $\mathcal{C}$ and $\mathcal{B}$ be closed subspaces of $\mathcal{A}$ with $\mathcal{B} \subseteq \mathcal{C}^\circ$. Let $\tilde{\mathcal{A}}_1$ be an almost Pontryagin space such that

$$\mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{A}[-]\mathcal{C}}(\mathcal{A}) \subseteq \tilde{\mathcal{A}}_1.$$

The present extension problem is to show when there exists an almost Pontryagin space $\tilde{\mathcal{A}}$ such that

$$\mathcal{A} \subseteq \tilde{\mathcal{A}}, \quad \mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{\mathcal{A}}) = \tilde{\mathcal{A}}_1.$$

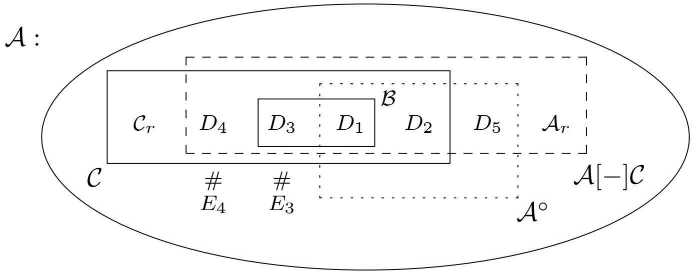
Necessary and sufficient conditions, as well as a uniqeness statement, will be given in Proposition 5.5. The construction involves a number of steps. These are

§5.1 A direct sum decompostion of an almost Pontryagin space.
§5.2 Necessary conditions for the extension of an almost Pontryagin space.
§5.3 Construction of an almost Pontryagin space.
§5.4 Uniqueness of the extension.
§5.5 A characterization of the extension of an almost Pontryagin space.

5.1. A direct sum decompostion of an almost Pontryagin space

Let $\mathcal{A}$ be an almost Pontryagin space and let $\mathcal{C}$ and $\mathcal{B}$ be closed subspaces with $\mathcal{B} \subseteq \mathcal{C}^\circ$. It will be shown that $\mathcal{A}$ has a direct sum decomposition induced by these subspaces.

Lemma 5.1. *Let $\mathcal{A}$ be an almost Pontryagin space and let $\mathcal{C}$ and $\mathcal{B}$ be closed subspaces of $\mathcal{A}$ with $\mathcal{B} \subseteq \mathcal{C}^\circ$. Then there exists a direct sum decomposition of $\mathcal{A}$ into nine closed linear subspaces $D_1, \ldots, D_5$, E_3, E_4, $\mathcal{C}_r$, $\mathcal{A}_r$, such that:*



The columns in this diagram are pairwise orthogonal.

The decompositions indicated in the above diagram mean that:

$$\mathcal{A} = \mathcal{C}_r\,[\dot{+}]\,\big(D_4\dot{+}E_4\big)\,[\dot{+}]\,\big(D_3\dot{+}E_3\big)\,[\dot{+}]\,\underbrace{\big(D_1[\dot{+}]D_2[\dot{+}]D_5\big)}_{=\mathcal{A}^\circ}\,[\dot{+}]\,\mathcal{A}_r,$$
$$\text{with}\quad D_4\# E_4,\ D_3\# E_3,$$

$$\mathcal{C} = \mathcal{C}_r\,[\dot{+}]\,\underbrace{\big(D_4[\dot{+}]D_3[\dot{+}]D_1[\dot{+}]D_2\big)}_{=\mathcal{C}^\circ},$$

$$\mathcal{B} = D_3[\dot{+}]D_1,$$

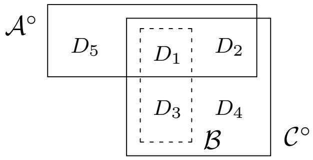
$$\mathcal{A}[-]\mathcal{C} = \underbrace{\big(D_4[\dot{+}]D_3[\dot{+}]D_1[\dot{+}]D_2[\dot{+}]D_5\big)}_{=(\mathcal{A}[-]\mathcal{C})^\circ}\,[\dot{+}]\,\mathcal{A}_r.$$

Note in particular that

$$(D_3\dot{+}D_4)\cap\mathcal{A}^\circ = \{0\},\qquad D_5\cap\mathcal{C} = \{0\}.$$

Proof. First consider the space $\mathcal{A}^\circ + \mathcal{C}^\circ$. Clearly, this space is neutral and it contains the space $\mathcal{B}$. Write it as $\mathcal{A}^\circ + \mathcal{C}^\circ = D_1\dot{+}D_2\dot{+}D_3\dot{+}D_4\dot{+}D_5$ according to the scheme:

$\mathcal{A}^\circ + \mathcal{C}^\circ$:



This can be done by first introducing the subspace $D_1 = \mathcal{B}\cap\mathcal{A}^\circ$ and then by choosing the subspaces D_2, D_3, D_4, and D_5 as follows:

$$\begin{aligned}
&D_2\dot{+}D_1 = \mathcal{C}^\circ\cap\mathcal{A}^\circ,\\
&D_3\dot{+}D_1 = \mathcal{B},\\
&D_4\dot{+}((\mathcal{C}^\circ\cap\mathcal{A}^\circ)+\mathcal{B}) = \mathcal{C}^\circ,\\
&D_5\dot{+}(\mathcal{C}^\circ\cap\mathcal{A}^\circ) = \mathcal{A}^\circ.
\end{aligned}\tag{5.1}$$

In order to obtain a decomposition of $\mathcal{A}$ several summands will be added to this decomposition of $\mathcal{A}^\circ + \mathcal{C}^\circ$. For this purpose choose a closed and nondegenerated subspace $\mathcal{C}_r$ of $\mathcal{C}$ such that $\mathcal{C}$ decomposes as

$$\mathcal{C}_r\,[\dot{+}]\,\mathcal{C}^\circ = \mathcal{C}.\tag{5.2}$$

The sum $\mathcal{C}_r[\dot{+}](D_3\dot{+}D_4)$ is a linear subspace of $\mathcal{A}$, and

$$[\mathcal{C}_r[\dot{+}](D_3\dot{+}D_4)]^\circ = D_3\dot{+}D_4,\quad [\mathcal{C}_r[\dot{+}](D_3\dot{+}D_4)]\cap\mathcal{A}^\circ = \{0\}.$$

Hence there exists a linear subspace of $\mathcal{A}[-]\mathcal{C}_r$ which is skewly linked with $D_3\dot{+}D_4$. Joining bases of D_3 and D_4 to a basis of their sum, and using a skewly linked basis, this skewly linked space can be written as a direct sum $E_3\dot{+}E_4$ with

$$\begin{aligned}
&E_3, E_4 \subseteq \mathcal{A}[-]\mathcal{C}_r,\\
&D_3\dot{+}E_3 \perp D_4\dot{+}E_4,\quad E_3\# D_3,\ E_4\# D_4.
\end{aligned}\tag{5.3}$$

The linear space $\mathcal{C}\dot{+}E_3\dot{+}E_4\dot{+}D_5$ is a closed subspace of $\mathcal{A}$ which contains $\mathcal{A}^\circ$. Hence one may choose a closed and nondegenerated subspace $\mathcal{A}_r$ of $\mathcal{A}$ with

$$\mathcal{A}_r[\dot{+}](\mathcal{C}\dot{+}E_3\dot{+}E_4\dot{+}D_5) = \mathcal{A}. \tag{5.4}$$

This is a decomposition of $\mathcal{A}$ as announced in the lemma. □

5.2. Necessary conditions for the extension of an almost Pontryagin space

The extension problem for an almost Pontryagin space as stated in the beginning of this section will now be taken up. The existence of a solution results in the formulation of necessary conditions.

Lemma 5.2. *Let $\mathcal{A}$ be an almost Pontryagin space, let $\mathcal{C}$ and $\mathcal{B}$ be closed subspaces of $\mathcal{A}$ with $\mathcal{B} \subseteq \mathcal{C}^\circ$, and let $\tilde{\mathcal{A}}_1$ be an almost Pontryagin space with*

$$\mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{A}[-]\mathcal{C}}(\mathcal{A}) \subseteq \tilde{\mathcal{A}}_1. \tag{5.5}$$

Assume that there exists an almost Pontryagin space $\tilde{\mathcal{A}}$ with

$$\mathcal{A} \subseteq \tilde{\mathcal{A}}, \quad \mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{\mathcal{A}}) = \tilde{\mathcal{A}}_1. \tag{5.6}$$

Then

$$\mathcal{C}^\circ/_{\mathcal{B}} \subseteq (\tilde{\mathcal{A}}_1)^\circ. \tag{5.7}$$

Moreover, if $\tilde{\Delta}$ and $\tilde{\delta}$ are defined by

$$\tilde{\Delta} = \operatorname{ind}_0 \tilde{\mathcal{A}}, \quad \tilde{\delta} = \dim\big(\tilde{\mathcal{A}}^\circ \cap \mathcal{B}\big), \tag{5.8}$$

then

$$\dim\, {(\tilde{\mathcal{A}}_1)^\circ}\big/{(\mathcal{C}^\circ/_{\mathcal{B}})} \le \tilde{\Delta} \le \dim\, {(\tilde{\mathcal{A}}_1)^\circ}\big/{(\mathcal{C}^\circ/_{\mathcal{B}})} + \dim\big(\mathcal{A}^\circ \cap \mathcal{C}^\circ\big), \tag{5.9}$$

and

$$\begin{aligned}\tilde{\Delta} - \dim\, {(\tilde{\mathcal{A}}_1)^\circ}\big/{(\mathcal{C}^\circ/_{\mathcal{B}})} - \dim\big(\mathcal{A}^\circ \cap \mathcal{C}^\circ\big) + \dim(\mathcal{A}^\circ \cap \mathcal{B})&\\ \le \tilde{\delta} \le \min\Big\{\tilde{\Delta} - \dim\, {(\tilde{\mathcal{A}}_1)^\circ}\big/{(\mathcal{C}^\circ/_{\mathcal{B}})}, \dim(\mathcal{A}^\circ \cap \mathcal{B})\Big\}&.\end{aligned} \tag{5.10}$$

Proof. Assume that $\tilde{\mathcal{A}}$ is an almost Pontryagin space which satisfies (5.6). Then by definition it follows that $\tilde{\mathcal{A}}_1 = (\tilde{\mathcal{A}}[-]\mathcal{C})/_{\mathcal{B}}$, and hence

$$(\tilde{\mathcal{A}}_1)^\circ = (\tilde{\mathcal{A}}^\circ + \mathcal{C}^\circ)/_{\mathcal{B}}. \tag{5.11}$$

This shows that the inclusion in (5.7) holds. Thus the space $(\tilde{\mathcal{A}}_1)^\circ/(\mathcal{C}^\circ/_{\mathcal{B}})$ is well defined. Let $\tilde{\Delta}$ and $\tilde{\delta}$ be defined by (5.8).

Now (5.9) will be shown. Since $\mathcal{B} \subseteq \mathcal{C}^\circ$, the identity (5.11) implies

$$\dim(\tilde{\mathcal{A}}_1)^\circ = \dim \tilde{\mathcal{A}}^\circ - \dim\big(\tilde{\mathcal{A}}^\circ \cap \mathcal{C}^\circ\big) + \dim\big(\mathcal{C}^\circ/_{\mathcal{B}}\big),$$

i.e.,

$$\tilde{\Delta} = \dim\, {(\tilde{\mathcal{A}}_1)^\circ}\big/{(\mathcal{C}^\circ/_{\mathcal{B}})} + \dim\big(\tilde{\mathcal{A}}^\circ \cap \mathcal{C}^\circ\big). \tag{5.12}$$

This immediately shows that

$$\dim\, {(\tilde{\mathcal{A}}_1)^\circ}\big/{(\mathcal{C}^\circ/_{\mathcal{B}})} \le \tilde{\Delta}.$$

On the other hand, it is clear that $\tilde{\mathcal{A}}^\circ \cap \mathcal{C}^\circ \subseteq \mathcal{A}^\circ \cap \mathcal{C}^\circ$; hence (5.12) implies

$$\tilde{\Delta} \leq \dim {}^{(\tilde{\mathcal{A}}_1)^\circ}\Big/_{(\mathcal{C}^\circ/_{\mathcal{B}})} + \dim\big(\mathcal{A}^\circ \cap \mathcal{C}^\circ\big).$$

Therefore, (5.9) has been shown.

It remains to show (5.10). To see the estimate from above, note that

$$\tilde{\mathcal{A}}^\circ \cap \mathcal{B} \subseteq \tilde{\mathcal{A}}^\circ \cap \mathcal{C}^\circ \quad \text{and} \quad \tilde{\mathcal{A}}^\circ \cap \mathcal{B} \subseteq \mathcal{A}^\circ \cap \mathcal{B}.$$

Hence, once again remembering (5.12), one obtains

$$\begin{aligned}\tilde{\delta} &= \dim(\tilde{\mathcal{A}}^\circ \cap \mathcal{B}) \\ &\leq \min\big\{\dim(\tilde{\mathcal{A}}^\circ \cap \mathcal{C}^\circ), \dim(\mathcal{A}^\circ \cap \mathcal{B})\big\} \\ &= \min\Big\{\tilde{\Delta} - \dim {}^{(\tilde{\mathcal{A}}_1)^\circ}\Big/_{(\mathcal{C}^\circ/_{\mathcal{B}})}, \dim(\mathcal{A}^\circ \cap \mathcal{B})\Big\}.\end{aligned}$$

To see the estimate from below, choose a linear subspace $\mathcal{L}$ of $\tilde{\mathcal{A}}^\circ \cap \mathcal{C}^\circ$ such that $\tilde{\mathcal{A}}^\circ \cap \mathcal{C}^\circ = \mathcal{L} \dot{+} (\tilde{\mathcal{A}}^\circ \cap \mathcal{B})$. Since $\tilde{\mathcal{A}}^\circ \cap \mathcal{A} \subseteq \mathcal{A}^\circ$, one has $\mathcal{L} \subseteq \mathcal{A}^\circ \cap \mathcal{C}^\circ$ and $\mathcal{L} \cap (\mathcal{A}^\circ \cap \mathcal{B}) = \{0\}$. Hence, it holds that

$$\dim\big(\tilde{\mathcal{A}}^\circ \cap \mathcal{C}^\circ\big) - \dim\big(\tilde{\mathcal{A}}^\circ \cap \mathcal{B}\big) = \dim \mathcal{L} \leq \dim\big(\mathcal{A}^\circ \cap \mathcal{C}^\circ\big) - \dim\big(\mathcal{A}^\circ \cap \mathcal{B}\big),$$

which completes the proof of (5.10). □

5.3. Construction of an almost Pontryagin space

It will be shown that the necessary condition (5.7) is also sufficient to construct an almost Pontryagin space $\tilde{\mathcal{A}}$ as in Lemma 5.2.

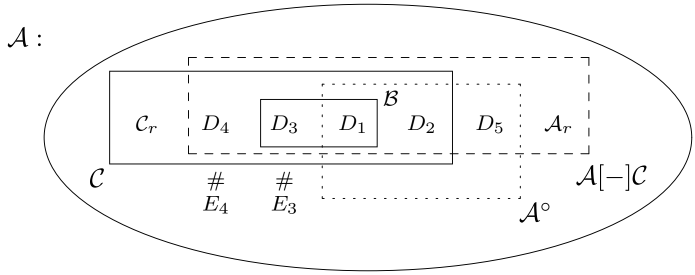
Lemma 5.3. *Let $\mathcal{A}$ be an almost Pontryagin space and let $\mathcal{C}$ and $\mathcal{B}$ be closed subspaces of $\mathcal{A}$ with $\mathcal{B} \subseteq \mathcal{C}^\circ$. Let $\tilde{\mathcal{A}}_1$ be an almost Pontryagin space with* (5.5) *which satisfies* (5.7)*. Then, for each number $\tilde{\Delta}$ belonging to the nonempty interval of integers described by* (5.9)*, the left and right hand sides of* (5.10) *describe a nonempty interval of integers, i.e.,*

$$\begin{aligned}\tilde{\Delta} - \dim {}^{(\tilde{\mathcal{A}}_1)^\circ}\Big/_{(\mathcal{C}^\circ/_{\mathcal{B}})} - \dim\big(\mathcal{A}^\circ \cap \mathcal{C}^\circ\big) + \dim(\mathcal{A}^\circ \cap \mathcal{B}) \\ \leq \min\Big\{\tilde{\Delta} - \dim {}^{(\tilde{\mathcal{A}}_1)^\circ}\Big/_{(\mathcal{C}^\circ/_{\mathcal{B}})}, \dim(\mathcal{A}^\circ \cap \mathcal{B})\Big\}.\end{aligned} \tag{5.13}$$

For each two numbers $\tilde{\Delta}, \tilde{\delta} \in \mathbb{N}_0$ subject to (5.9) *and* (5.10)*, there exists an almost Pontryagin space $\tilde{\mathcal{A}}$ for which* (5.6) *and* (5.8) *hold.*

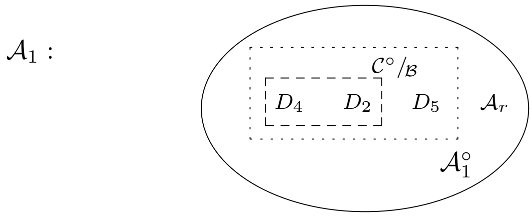
Proof. The construction of the almost Pontryagin space $\tilde{\mathcal{A}}$ will be given in a number of steps.

Step 1. Due to $\mathcal{B} \subseteq \mathcal{C}^\circ$, Lemma 5.1 may be applied with the spaces $\mathcal{A}$, $\mathcal{C}$, and $\mathcal{B}$. This gives the following direct sum decomposition:



(5.14)

From this decomposition it is clear that the space $\mathcal{A}_1 = (\mathcal{A}[-]\mathcal{C})/_{\mathcal{B}}$ is (isomorphic to)

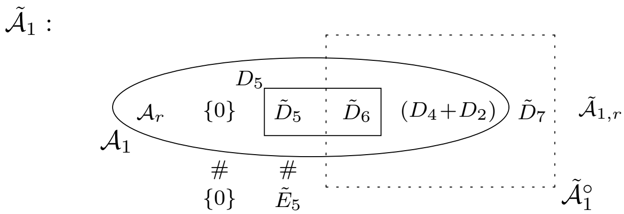


(5.15)

Step 2. Due to $D_5 \subseteq \mathcal{A}_1^\circ$, Lemma 5.1 may be applied with the spaces $\tilde{\mathcal{A}}_1$, $\mathcal{A}_1$, and D_5. Due to the inclusion (5.7) it follows that

$$\mathcal{A}_1^\circ = D_5 + \mathcal{C}^\circ/_{\mathcal{B}} \subseteq D_5 + \big(\mathcal{A}_1^\circ \cap \tilde{\mathcal{A}}_1^\circ\big) \subseteq \mathcal{A}_1^\circ,$$

and hence that $D_5 + \big(\mathcal{A}_1^\circ \cap \tilde{\mathcal{A}}_1^\circ\big) = \mathcal{A}_1^\circ$. Therefore, the decomposition obtained from Lemma 5.1 is of the form



Step 3. Let a number $\tilde{\Delta}$ be given such that (5.9) holds. Then the left side of (5.10) does not exceed $\dim(\mathcal{A}^\circ \cap \mathcal{B})$. Since $\mathcal{B} \subseteq \mathcal{C}^\circ$ it also does not exceed $\tilde{\Delta} - \dim(\tilde{\mathcal{A}}_1)^\circ / (\mathcal{C}^\circ/_{\mathcal{B}})$, and we see that (5.13) holds.

Now let in addition to $\tilde{\Delta}$ a number $\tilde{\delta} \in \mathbb{N}_0$ with (5.10) be given. Let $\tilde{E}_1$ and $\tilde{E}_2$ be linear spaces with dimensions

$$\dim \tilde{E}_1 := \dim(\mathcal{A}^\circ \cap \mathcal{B}) - \tilde{\delta},$$

$$\dim \tilde{E}_2 := \dim \tilde{\mathcal{A}}_1^\circ \Big/ (\mathcal{C}^\circ/_{\mathcal{B}}) + \dim\big(\mathcal{A}^\circ \cap \mathcal{C}^\circ\big) - \tilde{\Delta} - \dim(\mathcal{A}^\circ \cap \mathcal{B}) + \tilde{\delta},$$

and choose subspaces $\tilde{D}_1$ and $\tilde{D}_2$ of D_1 and D_2, respectively, with

$$\dim \tilde{D}_1 = \dim \tilde{E}_1, \quad \dim \tilde{D}_2 = \dim \tilde{E}_2.$$

This is possible, because

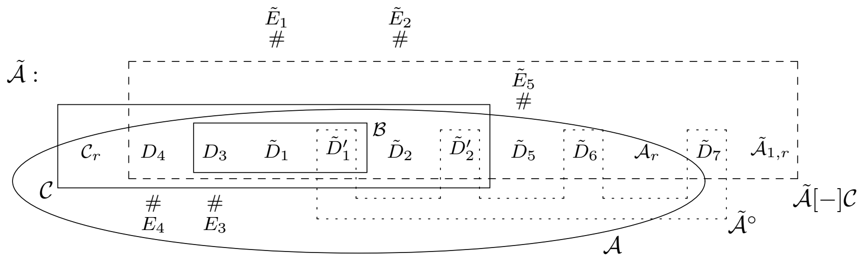
$$0 \overset{(5.10)}{\leq} \dim(\mathcal{A}^\circ \cap \mathcal{B}) - \tilde{\delta} \leq \dim(\mathcal{A}^\circ \cap \mathcal{B}) = \dim D_1,$$

and

$$\begin{aligned} 0 &\overset{(5.10)}{\leq} \dim \tilde{\mathcal{A}}_1^\circ \big/ (\mathcal{C}^\circ/_\mathcal{B}) + \dim\big(\mathcal{A}^\circ \cap \mathcal{C}^\circ\big) - \tilde{\Delta} - \dim(\mathcal{A}^\circ \cap \mathcal{B}) + \tilde{\delta} \\ &\overset{(5.10)}{\leq} \dim\big(\mathcal{A}^\circ \cap \mathcal{C}^\circ\big) - \dim(\mathcal{A}^\circ \cap \mathcal{B}) = \dim D_2, \end{aligned}$$

Finally, choose $\tilde{D}_i'$ be such that $D_i = \tilde{D}_i \dot{+} \tilde{D}_i'$, $i = 1, 2$.

Step 4. Define the space $\tilde{\mathcal{A}}$ as indicated in the following diagram:



(5.16)

where, as usual, the columns are pairwise orthogonal. Then $\tilde{\mathcal{A}}$ is an almost Pontryagin space with the required property:

$$\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{\mathcal{A}}) = \tilde{\mathcal{A}}_1.$$

It remains to show that (5.8) holds.

First note that the isotropic part of $\tilde{\mathcal{A}}$ is given by $\tilde{\mathcal{A}}^\circ = \tilde{D}_6 \dot{+} \tilde{D}_7 \dot{+} \tilde{D}_1' \dot{+} \tilde{D}_2'$ and recall that

$$\tilde{D}_6 \dot{+} \tilde{D}_7 \cong \tilde{\mathcal{A}}_1 \big/ (\mathcal{C}^\circ/_\mathcal{B}).$$

Hence it follows that

$$\begin{aligned} \operatorname{ind}_0 \tilde{\mathcal{A}} &= \dim \tilde{\mathcal{A}}_1^\circ \big/ (\mathcal{C}^\circ/_\mathcal{B}) + \big(\dim D_1 - \dim \tilde{E}_1\big) + \big(\dim D_2 - \dim \tilde{E}_2\big) \\ &= \dim \tilde{\mathcal{A}}_1^\circ \big/ (\mathcal{C}^\circ/_\mathcal{B}) + \dim\big(\mathcal{A}^\circ \cap \mathcal{C}^\circ\big) - \big(\dim \tilde{E}_1 + \dim \tilde{E}_2\big) = \tilde{\Delta}. \end{aligned}$$

Next observe that $\tilde{\mathcal{A}}^\circ \cap \mathcal{B} = \tilde{D}_1'$, which leads to

$$\dim\big(\tilde{\mathcal{A}}^\circ \cap \mathcal{B}\big) = \dim D_1 - \dim \tilde{E}_1 = \dim\big(\mathcal{A}^\circ \cap \mathcal{B}\big) - \dim \tilde{E}_1 = \tilde{\delta}.$$

Together we see that indeed (5.8) is satisfied. □

5.4. Uniqueness of the extension

In the next lemma we show that the numbers $\tilde{\Delta}$ and $\tilde{\delta}$ determine the extension $\tilde{\mathcal{A}}$ uniquely (up to isomorphisms).

Lemma 5.4. *Let $\mathcal{A}$ be an almost Pontryagin space and let $\mathcal{C}$ and $\mathcal{B}$ be closed subspaces of $\mathcal{A}$ with $\mathcal{B} \subseteq \mathcal{C}^\circ$. Let $\tilde{\mathcal{A}}_1$ be an almost Pontryagin space with* (5.5) *which satisfies* (5.7)*. Then for each two almost Pontryagin spaces $\tilde{\mathcal{A}}$ and $\hat{\mathcal{A}}$ with*

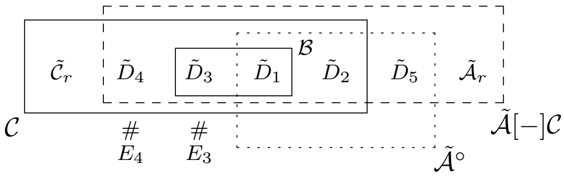
$$\mathcal{A} \subseteq \tilde{\mathcal{A}},\ \mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{\mathcal{A}}) = \tilde{\mathcal{A}}_1, \quad \mathcal{A} \subseteq \hat{\mathcal{A}},\ \mathfrak{F}^{\mathcal{B}}_{\hat{\mathcal{A}}|\hat{\mathcal{A}}[-]\mathcal{C}}(\hat{\mathcal{A}}) = \tilde{\mathcal{A}}_1,$$

$$\operatorname{ind}_0 \tilde{\mathcal{A}} = \operatorname{ind}_0 \hat{\mathcal{A}}, \quad \dim(\tilde{\mathcal{A}}^\circ \cap \mathcal{B}) = \dim(\hat{\mathcal{A}} \cap \mathcal{B}),$$

there exists an isometric homeomorphism φ of $\tilde{\mathcal{A}}$ onto $\hat{\mathcal{A}}$ with $\varphi(\mathcal{C}) = \mathcal{C}$ and $\varphi(\mathcal{B}) = \mathcal{B}$.

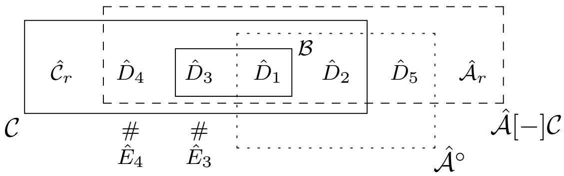
Proof. Apply Lemma 5.1 with the spaces $\tilde{\mathcal{A}}, \mathcal{C}, \mathcal{B}$ and with the spaces $\hat{\mathcal{A}}, \mathcal{C}, \mathcal{B}$, respectively. This gives, similar as in (5.14), the following decompositions:

$\tilde{\mathcal{A}}$:



and

$\hat{\mathcal{A}}$:



Since $\tilde{\mathcal{C}}_r$ and $\hat{\mathcal{C}}_r$ are closed and nondegenerated subspaces of $\mathcal{C}$ which satisfy

$$\tilde{\mathcal{C}}_r[\dot{+}]\mathcal{C}^\circ = \hat{\mathcal{C}}_r[\dot{+}]\mathcal{C}^\circ = \mathcal{C},$$

there exists an isometric homeomorphism between $\tilde{\mathcal{C}}_r$ and $\hat{\mathcal{C}}_r$. Since $\tilde{\mathcal{A}}_r$ and $\hat{\mathcal{A}}_r$ are closed and nondegenerated subspaces of $\tilde{\mathcal{A}}$ and $\hat{\mathcal{A}}$, respectively, which satisfy

$$\tilde{\mathcal{A}}_r[\dot{+}](\tilde{\mathcal{A}}_1)^\circ \cong \tilde{\mathcal{A}}_1 \cong \hat{\mathcal{A}}_r[\dot{+}](\tilde{\mathcal{A}}_1)^\circ,$$

there exists an isometric homeomorphism between $\tilde{\mathcal{A}}_r$ and $\hat{\mathcal{A}}_r$. Since

$$(\mathcal{C}^\circ/_{\mathcal{B}})[\dot{+}]\tilde{D}_5 = (\tilde{\mathcal{A}}_1)^\circ = (\mathcal{C}^\circ/_{\mathcal{B}})[\dot{+}]\hat{D}_5,$$

one has $\dim \tilde{D}_5 = \dim \hat{D}_5$. Next recall (5.12), so that $\dim(\tilde{D}_1 \dot{+} \tilde{D}_2) = \dim(\hat{D}_1 \dot{+} \hat{D}_2)$. This also implies

$$\dim(\tilde{D}_4 \dot{+} \tilde{D}_3) = \dim \mathcal{C}^\circ - \dim(\tilde{D}_1 \dot{+} \tilde{D}_2) = \dim \mathcal{C}^\circ - \dim(\hat{D}_1 \dot{+} \hat{D}_2) = \dim(\hat{D}_4 \dot{+} \hat{D}_3).$$

Finally, one has

$$\dim \tilde{D}_1 = \dim\big(\tilde{\mathcal{A}}^\circ \cap \mathcal{B}\big) = \dim\big(\hat{\mathcal{A}}^\circ \cap \mathcal{B}\big) = \dim \hat{D}_1.$$

Hence also $\dim \tilde{D}_2 = \dim \hat{D}_2$ and

$$\dim \tilde{D}_3 = \dim \mathcal{B} - \dim \tilde{D}_1 = \dim \mathcal{B} - \dim \hat{D}_1 = \dim \hat{D}_3.$$

Thus an isometric homeomorphism $\varphi : \tilde{\mathcal{A}} \to \hat{\mathcal{A}}$ can be defined such that $\varphi(\mathcal{C}) = \mathcal{C}$ and $\varphi(\mathcal{B}) = \mathcal{B}$. □

5.5. A characterization of the extension of an almost Pontryagin space

A combination of Lemma 5.2, Lemma 5.3, and Lemma 5.4 gives rise to the following main result about the characterization of the existence of a unique solution of the extension problem. Theorem 4.2 is an immediate consequence of this more refined assertion.

Proposition 5.5. *Let $\mathcal{A}$ be an almost Pontryagin space and let $\mathcal{C}$ and $\mathcal{B}$ be closed subspaces of $\mathcal{A}$ with $\mathcal{B} \subseteq \mathcal{C}^\circ$. Let $\tilde{\mathcal{A}}_1$ be an almost Pontryagin space with*

$$\mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{A}[-]\mathcal{C}}(\mathcal{A}) \subseteq \tilde{\mathcal{A}}_1$$

and let $\tilde{\Delta}, \tilde{\delta} \in \mathbb{N}_0$. Then the following statements are equivalent:

(i) *There exists an almost Pontryagin space $\tilde{\mathcal{A}}$ with*

$$\begin{aligned} &\mathcal{A} \subseteq \tilde{\mathcal{A}}, \quad \mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{\mathcal{A}}) = \tilde{\mathcal{A}}_1, \\ &\operatorname{ind}_0 \tilde{\mathcal{A}} = \tilde{\Delta}, \ \dim\big(\tilde{\mathcal{A}}^\circ \cap \mathcal{B}\big) = \tilde{\delta}. \end{aligned} \tag{5.17}$$

(ii) *The inclusion $\mathcal{C}^\circ/_{\mathcal{B}} \subseteq \tilde{\mathcal{A}}_1^\circ$ holds and*

$$\dim \tilde{\mathcal{A}}_1^\circ \big/ (\mathcal{C}^\circ/_{\mathcal{B}}) \leq \tilde{\Delta} \leq \dim \tilde{\mathcal{A}}_1^\circ \big/ (\mathcal{C}^\circ/_{\mathcal{B}}) + \dim\big(\mathcal{A}^\circ \cap \mathcal{C}^\circ\big), \tag{5.18}$$

$$\begin{aligned} &\tilde{\Delta} - \dim \tilde{\mathcal{A}}_1^\circ \big/ (\mathcal{C}^\circ/_{\mathcal{B}}) - \dim\big(\mathcal{A}^\circ \cap \mathcal{C}^\circ\big) + \dim(\mathcal{A}^\circ \cap \mathcal{B}) \\ &\qquad \leq \tilde{\delta} \leq \min\Big\{\tilde{\Delta} - \dim \tilde{\mathcal{A}}_1^\circ \big/ (\mathcal{C}^\circ/_{\mathcal{B}}), \dim(\mathcal{A}^\circ \cap \mathcal{B})\Big\}. \end{aligned} \tag{5.19}$$

If either statement holds, then

$$\operatorname{ind}_- \tilde{\mathcal{A}} + \tilde{\Delta} = \operatorname{ind}_- \tilde{\mathcal{A}}_1 + \operatorname{ind}_0 \tilde{\mathcal{A}}_1 + \operatorname{ind}_- \mathcal{C} + \dim \mathcal{B}. \tag{5.20}$$

Moreover, $\tilde{\mathcal{A}}$ is uniquely determined by (5.17) *up to isometric isomorphisms φ with $\varphi(\mathcal{C}) = \mathcal{C}$ and $\varphi(\mathcal{B}) = \mathcal{B}$.*

If $\mathcal{C}^\circ/_{\mathcal{B}} \subseteq \tilde{\mathcal{A}}_1^\circ$ and $\tilde{\Delta}$ satisfies (5.18)*, then numbers $\tilde{\delta}$ with* (5.19) *exist.*

Proof. The implication “(i)⇒(ii)” has been proved in Lemma 5.2, and the converse implication “(ii)⇒(i)” in Lemma 5.3. The uniqueness statement was shown in Lemma 5.4, and the last addition is included in the assertion of Lemma 5.3.

It remains to show the identity (5.20). Due to the essential uniqueness of the spaces, it suffices to consider the spaces constructed in the proof of

Lemma 5.3. The defining diagram (5.16) for $\tilde{\mathcal{A}}$ easily reveals the following identities for the negative indices:

$$\begin{aligned}
\operatorname{ind}_- \tilde{\mathcal{A}} &= \operatorname{ind}_- \mathcal{C}_r + \dim(D_4 \dot{+} D_3 \dot{+} \tilde{D}_1 \dot{+} \tilde{D}_2 \dot{+} \tilde{D}_5) + \operatorname{ind}_- \mathcal{A}_r + \operatorname{ind}_- \tilde{\mathcal{A}}_{1,r} \\
&= \operatorname{ind}_- \tilde{\mathcal{A}}_1 + \operatorname{ind}_- \mathcal{C} + \dim(D_4 \dot{+} D_3) + \dim(\tilde{E}_1 \dot{+} \tilde{E}_2) \\
&= \operatorname{ind}_- \tilde{\mathcal{A}}_1 + \operatorname{ind}_- \mathcal{C} + \dim(D_4 \dot{+} D_3) \\
&\quad + \dim \tilde{\mathcal{A}}_1^\circ \big/ (\mathcal{C}^\circ /_{\mathcal{B}}) + \dim\big(\mathcal{A}^\circ \cap \mathcal{C}^\circ\big) - \tilde{\Delta} \\
&= \operatorname{ind}_- \tilde{\mathcal{A}}_1 + \operatorname{ind}_- \mathcal{C} + \dim(D_4 \dot{+} D_3) \\
&\quad + \dim \tilde{\mathcal{A}}_1^\circ - \dim(D_4 \dot{+} D_2) + \dim(D_1 \dot{+} D_2) - \tilde{\Delta} \\
&= \operatorname{ind}_- \tilde{\mathcal{A}}_1 + \operatorname{ind}_- \mathcal{C} + \dim \tilde{\mathcal{A}}_1^\circ + \dim \mathcal{B} - \tilde{\Delta}.
\end{aligned}$$

Hence, all assertions of Proposition 5.5 have been proved. □

6. An extension problem for a class of isometric homeomorphisms in an almost Pontryagin space

Throughout this section we fix data as in Theorem 4.3. That is (revisit the visualization in diagram (4.1)):

(1) An almost Pontryagin space $\mathcal{A}$ and closed subspaces $\mathcal{C}$ and $\mathcal{B}$ of $\mathcal{A}$ with $\mathcal{B} \subseteq \mathcal{C}^\circ$.

(2) An isometric homeomorphism β between closed subspaces R and R' of $\mathcal{A}$ with

$$\mathcal{C} \subseteq R, \quad \beta(\mathcal{C}) = \mathcal{C}, \quad \beta(\mathcal{B}) \subseteq \mathcal{B}.$$

(3) An almost Pontryagin space $\tilde{\mathcal{A}}_1$ with

$$\tilde{\mathcal{A}}_1 \supseteq \mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{A}[-]\mathcal{C}}(\mathcal{A}) =: \mathcal{A}_1,$$

and an almost Pontryagin space $\tilde{\mathcal{A}}$ with

$$\tilde{\mathcal{A}} \supseteq \mathcal{A} \quad \text{and} \quad \mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{\mathcal{A}}) = \tilde{\mathcal{A}}_1.$$

(4) An isometric homeomorphism $\tilde{\beta}_1$ between closed subspaces $\tilde{R}_1$ and $\tilde{R}_1'$ of $\tilde{\mathcal{A}}_1$ with

$$\tilde{R}_1 \supseteq R_1, \ \tilde{R}_1' \supseteq R_1', \quad \tilde{\beta}_1|_{R_1} = \beta_1,$$

where R_1, R_1', β_1 are

$$R_1 := \mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{A}[-]\mathcal{C}}(R), \ R_1' := \mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{A}[-]\mathcal{C}}(R'), \ \beta_1 := \mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{A}[-]\mathcal{C}}(\beta).$$

It will be shown that there exist closed subspaces $\tilde{R}$ and $\tilde{R}'$ of $\tilde{\mathcal{A}}$ with $\tilde{R} \supseteq R$ and $\tilde{R}' \supseteq R'$ and an isometric homeomorphism $\tilde{\beta}$ from $\tilde{R}$ onto $\tilde{R}'$, such that (4.11), (4.12), (4.13), and (4.14) hold, and that the choice of $\tilde{\beta}$ can be made such that $\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{\beta}) = \tilde{\beta}_1$ provided that $(\mathcal{C}^\circ \cap R^\circ) + \mathcal{B} = \mathcal{C}^\circ$.

The construction of $\tilde{R}, \tilde{R}', \tilde{\beta}$ involves a number of steps. These are

§6.1 The decompositions of an almost Pontryagin space relative to an isometry.

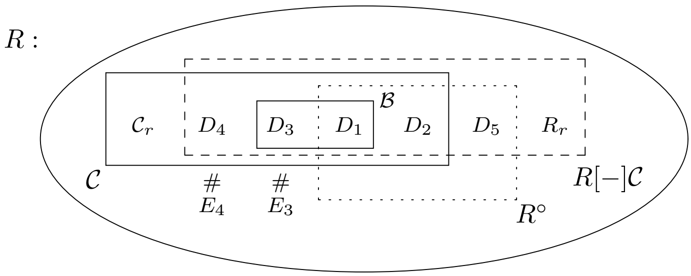
§6.2 The construction of a homeomorphic extension of β.
§6.3 The construction of a homeomorphic and isometric extension of β.
§6.4 Relating $\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{\beta})$ with $\tilde{\beta}_1$.
§6.5 Finishing the proof of Theorem 4.3.

Throughout §6.1-§6.4 we assume in addition that the given almost Pontryagin space $\tilde{\mathcal{A}}$ is nondegenerated, i.e., that $\tilde{\mathcal{A}}^\circ = \{0\}$. This restriction will be lifted in §6.5.

6.1. The decompositions of an almost Pontryagin space relative to an isometry

We assume in this subsection that $\tilde{\mathcal{A}}$ is nondegenerated. The aim is to construct decompositions of $\tilde{\mathcal{A}}$ which are compatible with the action of β. Due to the possible presence of the isotropic parts R° and $\mathcal{C}^\circ$, the geometric configuration is rather complicated.

A decomposition of R. Since $\mathcal{C} \subseteq R$, we can apply Lemma 5.1 with the spaces $R, \mathcal{C}, \mathcal{B}$ (instead of $\mathcal{A}, \mathcal{C}, \mathcal{B}$). This gives the following direct sum decomposition of R (as usual, columns are pairwise orthogonal):



(6.1)

A decomposition of $\tilde{\mathcal{A}}$ relative to R. The decomposition (6.1) of the space R can be completed to a decomposition of $\tilde{\mathcal{A}}$. Since $\tilde{\mathcal{A}}$ is nondegenerated we can choose a linear subspace of $\tilde{\mathcal{A}}$ which is skewly linked to R° and orthogonal to

$$\mathcal{C}_r[\dot{+}](D_3\dot{+}E_3)[\dot{+}](D_4\dot{+}E_4)[\dot{+}]R_r.$$

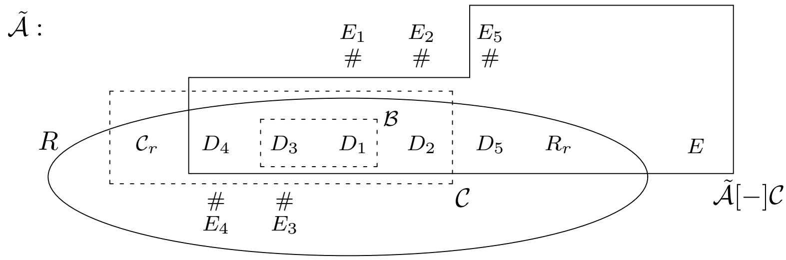
Note here that this space is orthocomplemented since each summand has this property. The space R° is decomposed as $R^\circ = D_1\dot{+}D_2\dot{+}D_5$. Hence, choosing appropriate bases, we can write the constructed skewly linked space as a direct sum $E_1\dot{+}E_2\dot{+}E_5$ with

$$E_1\#D_1,\ E_2\#D_2,\ E_5\#D_5, \quad D_j\dot{+}E_j \perp D_i\dot{+}E_i,\ i \neq j.$$

Since $R\dot{+}(E_1\dot{+}E_2\dot{+}E_5)$ is a closed and nondegenerated subspace of the Pontryagin space $\tilde{\mathcal{A}}$, it is orthocomplemented. Setting

$$E := \big(R\dot{+}(E_1\dot{+}E_2\dot{+}E_5)\big)^{[\perp]},$$

thus leads to the following decomposition of $\tilde{\mathcal{A}}$ (columns are pairwise orthogonal):



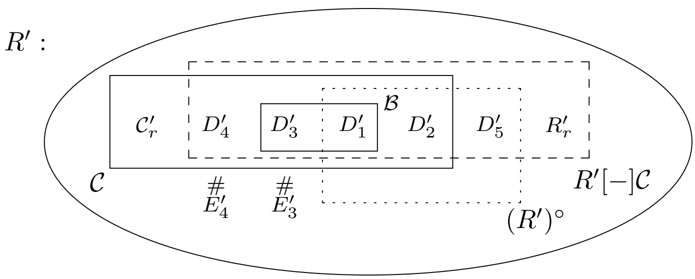
(6.2)

A decomposition of R' compatible with β. Corresponding to the decomposition (6.1) of the space R there is also a similar decomposition of the space R'. In fact, since β is an isometric homeomorphism from R onto R', we can simply transport (6.1) by applying β to each summand.

Define the following image spaces

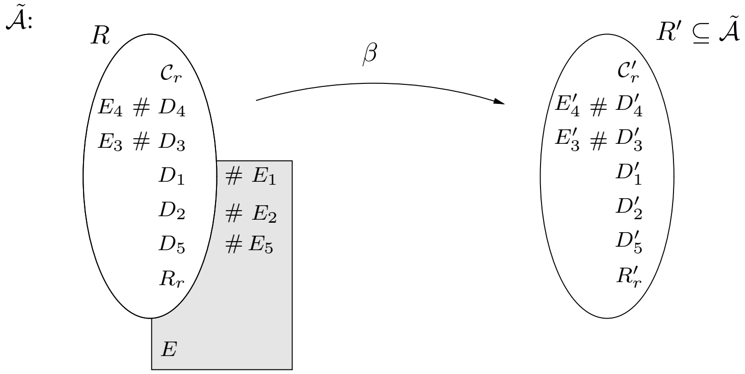
$$\begin{cases} \mathcal{C}_r' & := \beta(\mathcal{C}_r), \quad R_r' := \beta(R_r), \\ D_i' & := \beta(D_i), \quad i = 1, \dots, 5, \\ E_j' & := \beta(E_j), \quad j = 3, 4. \end{cases} \tag{6.3}$$

Since $\beta(R) = R'$, we have $\beta(R^\circ) = (R')^\circ$. Moreover, $\beta(\mathcal{C}) = \mathcal{C}$, $\beta(\mathcal{B}) = \mathcal{B}$, and $\beta(\mathcal{C}^\circ) = \mathcal{C}^\circ$, remember Lemma 2.3, (i). Hence, we obtain the decomposition of R' (columns are pairwise orthogonal):



(6.4)

A decomposition of $\tilde{\mathcal{A}}$ relative to R'. The decomposition (6.4) of the space R' can be completed to a decomposition of $\tilde{\mathcal{A}}$. So far the decomposition of R' acts as a part of a decomposition of $\tilde{\mathcal{A}}$, which fits the action of β (in this picture rows are pairwise orthogonal !):



To complete the decomposition (6.4) of R' to one of $\tilde{\mathcal{A}}$ one has to find appropriate terms E'_1, E'_2, E'_5 and E'. We proceed in the same way as when we constructed (6.2) from (6.1). Since $\tilde{\mathcal{A}}$ is nondegenerated we can choose a linear subspace of $\tilde{\mathcal{A}}$ which is skewly linked to $(R')^\circ$ and orthogonal to (again this sum is orthocomplemented)

$$\mathcal{C}'_r[\dot{+}](D'_3\dot{+}E'_3)[\dot{+}](D'_4\dot{+}E'_4)[\dot{+}]R'_r.$$

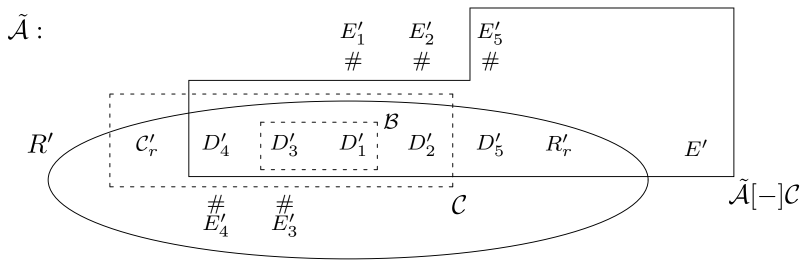
The space $(R')^\circ$ is decomposed as $(R')^\circ = D'_1\dot{+}D'_2\dot{+}D'_5$. Hence, choosing appropriate bases, we can write the constructed skewly linked space as a direct sum $E'_1\dot{+}E'_2\dot{+}E'_5$ with

$$E'_1 \# D'_1,\ E'_2 \# D'_2,\ E'_5 \# D'_5, \quad D'_j\dot{+}E'_j \perp D'_i\dot{+}E'_i,\ i \neq j.$$

Again $R'\dot{+}(E'_1\dot{+}E'_2\dot{+}E'_5)$ is orthocomplemented. Setting

$$E' := \big(R'\dot{+}(E'_1\dot{+}E'_2\dot{+}E'_5)\big)^{[\perp]},$$

thus leads to the following decomposition of $\tilde{\mathcal{A}}$ (columns are pairwise orthogonal):



(6.5)

6.2. The construction of a homeomorphic extension of β

Again, we assume throughout this subsection that $\tilde{\mathcal{A}}$ is nondegenerated.

The following construction is about finding a homeomorphic extension $\check{\beta}$ of β with the property

$$\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\check{\beta}) = \tilde{\beta}_1.$$

This is achieved by pulling back subspaces and operators from $\tilde{\mathcal{A}}_1$ into $\tilde{\mathcal{A}}$ by making use of the two decompositions of $\tilde{\mathcal{A}}$ given in (6.2) and (6.5) above. In general, the extension $\check{\beta}$ will not be isometric. However, we will be able later on to obtain a homeomorphic and isometric extension of β by perturbing $\check{\beta}$.

Let us recall in this place that $\tilde{\mathcal{A}}_1$ is, by definition, equal to

$$\tilde{\mathcal{A}}_1 = (\tilde{\mathcal{A}}[-]\mathcal{C})\big/\mathcal{B}\ .$$

Moreover, π denotes the canonical projection $\pi : \tilde{\mathcal{A}}[-]\mathcal{C} \to \tilde{\mathcal{A}}_1$.

By means of the decompositions (6.2) and (6.5) define subspaces $\tilde{\mathfrak{A}}_1$ and $\tilde{\mathfrak{A}}_1'$ of $\tilde{\mathcal{A}}$ by

$$\begin{aligned} \tilde{\mathfrak{A}}_1 &:= D_4 \dot{+} D_2 \dot{+} D_5 \dot{+} E_5 \dot{+} R_r \dot{+} E, \\ \tilde{\mathfrak{A}}_1' &:= D_4' \dot{+} D_2' \dot{+} D_5' \dot{+} E_5' \dot{+} R_r' \dot{+} E'. \end{aligned} \tag{6.6}$$

Lemma 6.1. *The spaces $\tilde{\mathfrak{A}}_1$ and $\tilde{\mathfrak{A}}_1'$ are closed subspaces of $\tilde{\mathcal{A}}$. We have*

$$\tilde{\mathfrak{A}}_1 \dot{+} \mathcal{B} = \tilde{\mathfrak{A}}_1' \dot{+} \mathcal{B} = \tilde{\mathcal{A}}[-]\mathcal{C},$$

and the restrictions $\pi|_{\tilde{\mathfrak{A}}_1}$ and $\pi|_{\tilde{\mathfrak{A}}_1'}$ are isometric homeomorphisms of $\tilde{\mathfrak{A}}_1$ and $\tilde{\mathfrak{A}}_1'$, respectively, onto $\tilde{\mathcal{A}}_1$:

$$\tilde{\mathcal{A}} \supseteq \tilde{\mathfrak{A}}_1 \xrightarrow[\cong]{\pi|_{\tilde{\mathfrak{A}}_1}} \tilde{\mathcal{A}}_1, \qquad\qquad \tilde{\mathcal{A}} \supseteq \tilde{\mathfrak{A}}_1' \xrightarrow[\cong]{\pi|_{\tilde{\mathfrak{A}}_1'}} \tilde{\mathcal{A}}_1. \tag{6.7}$$

It holds that

$$R \cap \tilde{\mathfrak{A}}_1 = D_4 \dot{+} D_2 \dot{+} D_5 \dot{+} R_r, \qquad R' \cap \tilde{\mathfrak{A}}_1' = D_4' \dot{+} D_2' \dot{+} D_5' \dot{+} R_r'. \tag{6.8}$$

Proof. Both of R_r and E are closed and nondegenerated subspaces of the Pontryagin space $\tilde{\mathcal{A}}$. Hence, both are orthocomplemented. Since $R_r[\perp]E$, also their direct and orthogonal sum $R_r[\dot{+}]E$ is orthocomplemented. In particular $R_r[\dot{+}]E$ is closed. All other summands in the definition of $\tilde{\mathfrak{A}}_1$ are finite-dimensional, and it follows that $\tilde{\mathfrak{A}}_1$ is closed.

The fact that $\tilde{\mathfrak{A}}_1 \dot{+} \mathcal{B} = \tilde{\mathcal{A}}[-]\mathcal{C}$ is obvious from the decomposition (6.2). Thus π maps $\tilde{\mathfrak{A}}_1$ bijectively onto $\tilde{\mathcal{A}}_1$. Clearly, $\pi|_{\tilde{\mathfrak{A}}_1}$ is isometric and continuous. By the open mapping theorem, it is a homeomorphism. The fact that $R \cap \tilde{\mathfrak{A}}_1 = D_4 + D_2 + D_5 + R_r$ is again obvious from (6.2).

The corresponding assertions for $\tilde{\mathfrak{A}}_1'$ follow in the same way. ☐

In view of these facts, we may say that $\tilde{\mathfrak{A}}_1$ and $\tilde{\mathfrak{A}}_1'$ are isomorphic copies of $\tilde{\mathcal{A}}_1$ inside $\tilde{\mathcal{A}}$.

The next step is to pull back the subspaces $\tilde{R}_1 \subseteq \tilde{\mathcal{A}}_1$ and $\tilde{R}_1' \subseteq \tilde{\mathcal{A}}_1$ into $\tilde{\mathcal{A}}$ using the homeomorphisms (6.7): define the spaces

$$\tilde{\mathfrak{R}}_1 := \big(\pi|_{\tilde{\mathfrak{A}}_1}\big)^{-1}(\tilde{R}_1), \qquad \tilde{\mathfrak{R}}_1' := \big(\pi|_{\tilde{\mathfrak{A}}_1'}\big)^{-1}(\tilde{R}_1').$$

Then $\tilde{\mathfrak{R}}_1$ and $\tilde{\mathfrak{R}}_1'$ are isomorphic copies of $\tilde{R}_1$ and $\tilde{R}_1'$, respectively, within $\tilde{\mathcal{A}}$.

Lemma 6.2. *The spaces* $\tilde{\mathfrak{R}}_1$ *and* $\tilde{\mathfrak{R}}_1'$ *are closed subspaces of* $\tilde{\mathcal{A}}$. *We have*

$$\tilde{\mathfrak{R}}_1 = \pi^{-1}(\tilde{R}_1) \cap \tilde{\mathfrak{A}}_1 = \big[R \cap \tilde{\mathfrak{A}}_1\big] \dot{+} \big[\tilde{\mathfrak{R}}_1 \cap (E_5 + E)\big], \tag{6.9}$$

$$\tilde{\mathfrak{R}}_1' = \pi^{-1}(\tilde{R}_1') \cap \tilde{\mathfrak{A}}_1' = \big[R' \cap \tilde{\mathfrak{A}}_1'\big] \dot{+} \big[\tilde{\mathfrak{R}}_1' \cap (E_5' + E')\big], \tag{6.10}$$

$$R \cap \tilde{\mathfrak{R}}_1 = R \cap \tilde{\mathfrak{A}}_1, \;\; R' \cap \tilde{\mathfrak{R}}_1' = R' \cap \tilde{\mathfrak{A}}_1', \tag{6.11}$$

$$(R + \tilde{\mathfrak{R}}_1) \cap (\tilde{\mathcal{A}}[-]\mathcal{C}) = \tilde{\mathfrak{R}}_1 + \mathcal{B}, \;\; (R' + \tilde{\mathfrak{R}}_1') \cap (\tilde{\mathcal{A}}[-]\mathcal{C}) = \tilde{\mathfrak{R}}_1' + \mathcal{B}. \tag{6.12}$$

Proof. The relation $\tilde{\mathfrak{R}}_1 = \pi^{-1}(\tilde{R}_1) \cap \tilde{\mathfrak{A}}_1$ is clear. Since $\tilde{R}_1$ is closed, this readily implies that $\tilde{\mathfrak{R}}_1$ is closed.

From $\pi(R \cap \tilde{\mathfrak{A}}_1) \subseteq R_1 \subseteq \tilde{R}_1$ we see that $R \cap \tilde{\mathfrak{A}}_1 \subseteq \tilde{\mathfrak{R}}_1$. The definition (6.6) of $\tilde{\mathfrak{A}}_1$ and the relation (6.8) now show that

$$\tilde{\mathfrak{R}}_1 = \tilde{\mathfrak{R}}_1 \cap \tilde{\mathfrak{A}}_1 = \tilde{\mathfrak{R}}_1 \cap \big[(R \cap \tilde{\mathfrak{A}}_1) + (E_5 + E)\big] = (R \cap \tilde{\mathfrak{A}}_1) \dot{+} \big(\tilde{\mathfrak{R}}_1 \cap (E_5 + E)\big).$$

Moreover, we have

$$R \cap \tilde{\mathfrak{A}}_1 \subseteq R \cap \tilde{\mathfrak{R}}_1,$$

and the reverse inclusion is obvious. Finally, compute

$$\begin{aligned}(R + \tilde{\mathfrak{R}}_1) \cap (\tilde{\mathcal{A}}[-]\mathcal{C}) &= (R + \tilde{\mathfrak{R}}_1) \cap (\tilde{\mathfrak{A}}_1 + \mathcal{B}) \\ &= \big[(R + \tilde{\mathfrak{R}}_1) \cap \tilde{\mathfrak{A}}_1\big] + \mathcal{B} = \underbrace{(R \cap \tilde{\mathfrak{A}}_1)}_{= R \cap \tilde{\mathfrak{R}}_1 \subseteq \tilde{\mathfrak{R}}_1} + \tilde{\mathfrak{R}}_1 + \mathcal{B} = \tilde{\mathfrak{R}}_1 + \mathcal{B}.\end{aligned}$$

The corresponding assertions for $\tilde{\mathfrak{R}}_1'$ follow in the same way. $\square$

From the decompositions (6.1) and (6.4) of R and R', and the relations (6.8), (6.9), and (6.10) we obtain

$$R + \tilde{\mathfrak{R}}_1 = \big[\mathcal{C}_r + D_1 + D_3 + E_3 + E_4\big] \dot{+} \underbrace{\big[D_4 + D_2 + D_5 + R_r\big]}_{R \cap \tilde{\mathfrak{R}}_1} \dot{+} \big[\tilde{\mathfrak{R}}_1 \cap (E_5 + E)\big],$$

$$R' + \tilde{\mathfrak{R}}_1' = \big[\mathcal{C}_r' + D_1' + D_3' + E_3' + E_4'\big] \dot{+} \underbrace{\big[D_4' + D_2' + D_5' + R_r'\big]}_{R' \cap \tilde{\mathfrak{R}}_1'} \dot{+} \big[\tilde{\mathfrak{R}}_1' \cap (E_5' + E')\big].$$

These identities written in a slightly different way give

$$R + \tilde{\mathfrak{R}}_1 = \overbrace{\Big(\mathcal{C}_r + \mathcal{B} + E_3 + E_4\Big) \dot{+} \Big(R \cap \tilde{\mathfrak{R}}_1}^{=R}\Big) \underbrace{\dot{+} \Big(\tilde{\mathfrak{R}}_1 \cap \big(E_5 + E\big)\Big)}_{=\tilde{\mathfrak{R}}_1}, \tag{6.13}$$

$$R' + \tilde{\mathfrak{R}}_1' = \overbrace{\Big(\mathcal{C}_r' + \mathcal{B} + E_3' + E_4'\Big) \dot{+} \Big(R' \cap \tilde{\mathfrak{R}}_1'}^{=R'}\Big) \underbrace{\dot{+} \Big(\tilde{\mathfrak{R}}_1' \cap \big(E_5' + E'\big)\Big)}_{=\tilde{\mathfrak{R}}_1'}. \tag{6.14}$$

Using these results one can obtain orthogonal sum decompositions for the sum spaces $R + \tilde{\mathfrak{R}}_1$ and $R' + \tilde{\mathfrak{R}}_1'$.

Lemma 6.3. *The spaces $R+\tilde{\mathfrak{R}}_1$ and $R'+\tilde{\mathfrak{R}}_1'$ are closed linear subspaces of $\tilde{\mathcal{A}}$. They admit the following orthogonal sum decompositions*

$$R+\tilde{\mathfrak{R}}_1 = \Big(\big[\mathcal{C}_r+D_1+D_3+E_3+E_4\big]+D_4\Big)[\dot{+}]\Big(D_2+R_r\Big)[\dot{+}]\Big(D_5+\big[\tilde{\mathfrak{R}}_1\cap(E_5+E)\big]\Big),$$

$$R'+\tilde{\mathfrak{R}}_1' = \Big(\big[\mathcal{C}_r'+D_1'+D_3'+E_3'+E_4'\big]+D_4'\Big)[\dot{+}]\Big(D_2'+R_r'\Big)[\dot{+}]\Big(D_5'+\big[\tilde{\mathfrak{R}}_1'\cap(E_5'+E')\big]\Big).$$

Proof. In order to show that $R+\tilde{\mathfrak{R}}_1$ is closed, we reorder the terms in the decomposition (6.13) to produce the above representation for $R+\tilde{\mathfrak{R}}_1$ as a direct and orthogonal sum; cf. (6.2). Each of the three summands is closed since $\mathcal{C}_r, R_r, E, \tilde{\mathfrak{R}}_1$ are closed and all other addends are finite-dimensional. Since $\tilde{\mathcal{A}}$ is nondegenerated, it follows that their direct and orthogonal sum is also closed (this general fact is seen using that the direct and orthogonal sum of orthocomplemented subspaces is orthocomplemented).

The assertions concerning $R'+\tilde{\mathfrak{R}}_1'$ are seen in the same way. $\square$

Having set up the geometric frame for constructing the required extension $\tilde{\beta}$, we can start to define the actual maps. Let $\delta:\tilde{\mathfrak{R}}_1\to\tilde{\mathfrak{R}}_1'$ be defined by

$$\delta := \Big[(\pi|_{\tilde{\mathfrak{A}}_1'})^{-1}\circ\tilde{\beta}_1\circ(\pi|_{\tilde{\mathfrak{A}}_1})\Big]\Big|_{\tilde{\mathfrak{R}}_1}. \tag{6.15}$$

As a composition of isometric homeomorphisms, δ is itself an isometric homeomorphism. Namely, between the closed subspaces $\tilde{\mathfrak{R}}_1$ and $\tilde{\mathfrak{R}}_1'$ of $\tilde{\mathcal{A}}$. By its definition, δ makes the following diagram commute:

$$\begin{tikzcd}
\tilde{\mathfrak{A}}_1 \arrow[d, "\pi"', "\cong"] & \supseteq & \tilde{\mathfrak{R}}_1 \arrow[d, "\pi"', "\cong"] \arrow[rr, bend left, "\delta"] & & \tilde{\mathfrak{R}}_1' \arrow[d, "\pi", "\cong"'] & \subseteq & \tilde{\mathfrak{A}}_1' \arrow[d, "\pi", "\cong"'] \\
\tilde{\mathcal{A}}_1 & \supseteq & \tilde{R}_1 \arrow[rr, "\tilde{\beta}_1"'] & & \tilde{R}_1' & \subseteq & \tilde{\mathcal{A}}_1'
\end{tikzcd} \tag{6.16}$$

The connection with the maps β and β_1 is easy to understand.

Lemma 6.4. *The following diagram commutes:*

$$
\begin{tikzcd}
\tilde{\mathfrak{A}}_1 \arrow[r, phantom, "\supseteq"] \arrow[ddd, "\pi"', "\cong"] & \tilde{\mathfrak{R}}_1 \arrow[rrrrr, bend left, "\delta"] \arrow[ddd, "\pi"', "\cong"] & & R \arrow[r, dotted, "\beta"] & R' & & \tilde{\mathfrak{R}}_1' \arrow[r, phantom, "\subseteq"] \arrow[ddd, "\pi", "\cong"'] & \tilde{\mathfrak{A}}_1' \arrow[ddd, "\pi", "\cong"'] \\
& & R\cap\tilde{\mathfrak{R}}_1 \arrow[ul, phantom, sloped, "\subseteq"] \arrow[ur, phantom, sloped, "\subseteq"] \arrow[rrr, dotted, "\beta|_{R\cap\tilde{\mathfrak{R}}_1}"] \arrow[d, dotted, "\pi"'] & & & R'\cap\tilde{\mathfrak{R}}_1' \arrow[ul, phantom, sloped, "\subseteq"] \arrow[ur, phantom, sloped, "\subseteq"] \arrow[d, dotted, "\pi"] & & \\
& & R_1 \arrow[rrr, dotted, "\beta_1"'] \arrow[dl, phantom, sloped, "\subseteq"] & & & R_1' \arrow[dr, phantom, sloped, "\subseteq"] & & \\
\tilde{\mathcal{A}}_1 \arrow[r, phantom, "\supseteq"] & \tilde{R}_1 \arrow[rrrrr, "\tilde{\beta}_1"'] & & & & & \tilde{R}_1' \arrow[r, phantom, "\subseteq"] & \tilde{\mathcal{A}}_1'
\end{tikzcd}
\tag{6.17}
$$

In particular, $\delta|_{R\cap\tilde{\mathfrak{R}}_1} = \beta|_{R\cap\tilde{\mathfrak{R}}_1}$.

Proof. Commutativity of the left and right trapezoids involving only the map π is trivial. Commutativity of the lower trapezoid is the fact that $\tilde{\beta}_1$ extends β_1, and the upper (dotted) trapezoid is trivial. The middle (dotted) rectangle follows from the definition of β_1 as the image under $\pi\times\pi$ of β; we only have to note that, by the definition of primed spaces and (6.8), $\beta(R\cap\tilde{\mathfrak{R}}_1) = R'\cap\tilde{\mathfrak{R}}_1'$. As already observed in (6.16), the outmost rectangle commutes. Using that the restriction of π to $\tilde{\mathfrak{R}}_1'$ is injective, thus also the upper part of the diagram involving δ and $\beta|_{R\cap\tilde{\mathfrak{R}}_1}$ commutes. $\square$

Since $\beta : R \to R'$ and $\delta : \tilde{\mathfrak{R}}_1 \to \tilde{\mathfrak{R}}_1'$ agree on the intersection of their domains, there exists a unique linear map

$$
\begin{aligned}
&\check{\beta} : R + \tilde{\mathfrak{R}}_1 \to R' + \tilde{\mathfrak{R}}_1' \\
&\text{with} \quad \check{\beta}|_R = \beta \quad \text{and} \quad \check{\beta}|_{\tilde{\mathfrak{R}}_1} = \delta.
\end{aligned}
\tag{6.18}
$$

visualizing again in a diagram, we have completed (6.17) to

$$
\begin{tikzcd}
& & R+\tilde{\mathfrak{R}}_1 \arrow[rrr, dashed, "\check{\beta}"] & & & R'+\tilde{\mathfrak{R}}_1' & & \\
\tilde{\mathfrak{A}}_1 \arrow[r, phantom, "\supseteq"] \arrow[ddd, "\pi"', "\cong"] & \tilde{\mathfrak{R}}_1 \arrow[ur, phantom, sloped, "\subseteq"] \arrow[rrrrr, bend left=40, "\delta"] \arrow[ddd, "\pi"', "\cong"] & & R \arrow[ul, phantom, sloped, "\subseteq"] \arrow[r, dotted, "\beta"] & R' \arrow[ur, phantom, sloped, "\subseteq"] & & \tilde{\mathfrak{R}}_1' \arrow[ul, phantom, sloped, "\subseteq"] \arrow[r, phantom, "\subseteq"] \arrow[ddd, "\pi", "\cong"'] & \tilde{\mathfrak{A}}_1' \arrow[ddd, "\pi", "\cong"'] \\
& & R\cap\tilde{\mathfrak{R}}_1 \arrow[ul, phantom, sloped, "\subseteq"] \arrow[ur, phantom, sloped, "\subseteq"] \arrow[rrr, dotted, "\beta|_{R\cap\tilde{\mathfrak{R}}_1}"] \arrow[d, dotted, "\pi"', "\cong"] & & & R'\cap\tilde{\mathfrak{R}}_1' \arrow[ul, phantom, sloped, "\subseteq"] \arrow[ur, phantom, sloped, "\subseteq"] \arrow[d, dotted, "\pi", "\cong"'] & & \\
& & R_1 \arrow[rrr, dotted, "\beta_1"'] \arrow[dl, phantom, sloped, "\subseteq"] & & & R_1' \arrow[dr, phantom, sloped, "\subseteq"] & & \\
\tilde{\mathcal{A}}_1 \arrow[r, phantom, "\supseteq"] & \tilde{R}_1 \arrow[rrrrr, "\tilde{\beta}_1"'] & & & & & \tilde{R}_1' \arrow[r, phantom, "\subseteq"] & \tilde{\mathcal{A}}_1'
\end{tikzcd}
\tag{6.19}
$$

We can now show that $\check{\beta}$ has all required properties.

Lemma 6.5. *The map $\check{\beta}$ is a homeomorphism from $R+\tilde{\mathfrak{R}}_1$ onto $R'+\tilde{\mathfrak{R}}_1'$. It extends β and satisfies (as usual we consider linear maps as linear relations via their graphs)*

$$\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\check{\beta})=\tilde{\beta}_1. \tag{6.20}$$

Proof. Write the decompositions (6.13) and (6.14) as

$$R+\tilde{\mathfrak{R}}_1=\big(\mathcal{C}_r+\mathcal{B}+E_3+E_4\big)\dot{+}\tilde{\mathfrak{R}}_1, \tag{6.21}$$

$$R'+\tilde{\mathfrak{R}}_1'=\big(\mathcal{C}_r'+\mathcal{B}'+E_3'+E_4'\big)\dot{+}\tilde{\mathfrak{R}}_1'. \tag{6.22}$$

On the first summand in (6.21) the map $\check{\beta}$ coincides with β, and hence maps it bijectively onto the first summand in (6.22). On the second summand in (6.21), $\check{\beta}$ coincides with δ, and hence maps it bijectively onto the second summand of (6.22). Together this shows that $\check{\beta}$ is a bijection of $R+\tilde{\mathfrak{R}}_1$ onto $R'+\tilde{\mathfrak{R}}_1'$. Moreover, since the summands in (6.21) are both closed and β and δ are both continuous, it follows that $\check{\beta}$ is continuous. By the open mapping theorem, $\check{\beta}$ is a homeomorphism.

The fact that $\check{\beta}$ extends β is built in the definition. We need to show (6.20). To this end, observe that

$$\check{\beta}(\tilde{\mathfrak{R}}_1+\mathcal{B})=\delta(\tilde{\mathfrak{R}}_1)+\beta(\mathcal{B})=\tilde{\mathfrak{R}}_1'+\mathcal{B}.$$

The relations (6.12) yield ($\check{\beta}|_{\tilde{\mathfrak{R}}_1+\mathcal{B}}$ denotes the domain restriction of $\check{\beta}$)

$$(\operatorname{graph}\check{\beta})\cap(\tilde{\mathcal{A}}[-]\mathcal{C})^2=\operatorname{graph}\big(\check{\beta}|_{\tilde{\mathfrak{R}}_1+\mathcal{B}}\big).$$

We compute further

$$\begin{aligned}
\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\check{\beta}) &=(\pi\times\pi)\big((\operatorname{graph}\check{\beta})\cap(\tilde{\mathcal{A}}[-]\mathcal{C})^2\big)\\
&=(\pi\times\pi)\big(\big\{\,\{x_1+x_2,\delta(x_1)+\beta(x_2)\}:x_1\in\tilde{\mathfrak{R}}_1,x_2\in\mathcal{B}\big\}\big)\\
&=\big\{\,\{\pi x_1,\pi(\delta x_1)\}:x_1\in\tilde{\mathfrak{R}}_1\big\}\\
&=\operatorname{graph}\Big((\pi|_{\tilde{\mathfrak{R}}_1'})\circ\delta\circ(\pi|_{\tilde{\mathfrak{R}}_1})^{-1}\Big)=\operatorname{graph}\tilde{\beta}_1.
\end{aligned}$$

□

6.3. The construction of a homeomorphic and isometric extension of β

In general, the map $\check{\beta}$ constructed in the previous subsection will not be isometric. However, in order to obtain a homeomorphic and isometric extension of β, it has to be modified only slightly. This subsection is about carrying out this perturbation process. Again we assume throughout this subsection that $\tilde{\mathcal{A}}$ is nondegenerated.

We reorder the summands in the decompositions (6.13) and (6.14), respectively, to obtain the following representations as direct and orthogonal sums:

$$R+\tilde{\mathfrak{R}}_1 = \Big(\overbrace{\mathcal{C}_r+\mathcal{B}+E_3\Big)[\dot{+}]\Big(E_4+D_4}^{=R}\Big)[\dot{+}]\Big(\underbrace{D_2+D_5+R_r+\big[\tilde{\mathfrak{R}}_1\cap(E_5+E)\big]}_{=\tilde{\mathfrak{R}}_1}\Big), \tag{6.23}$$

$$R'+\tilde{\mathfrak{R}}_1' = \Big(\overbrace{\mathcal{C}_r'+\mathcal{B}+E_3'\Big)[\dot{+}]\Big(E_4'+D_4'}^{=R'}\Big)[\dot{+}]\Big(\underbrace{D_2'+D_5'+R_r'+\big[\tilde{\mathfrak{R}}_1'\cap(E_5'+E')\big]}_{=\tilde{\mathfrak{R}}_1'}\Big). \tag{6.24}$$

Denote by $P : R+\tilde{\mathfrak{R}}_1 \to R+\tilde{\mathfrak{R}}_1$ the projection onto the third summand in (6.23) whose kernel equals the sum of the first two summands, and let $P' : R'+\tilde{\mathfrak{R}}_1' \to R'+\tilde{\mathfrak{R}}_1'$ be the correspondingly defined projection using the decomposition (6.24). Then P and P' are orthogonal projections. Since $D_4 \subseteq (\tilde{\mathfrak{R}}_1)^\circ$ and the same for primed spaces, the restrictions $P|_{\tilde{\mathfrak{R}}_1}$ and $P'|_{\tilde{\mathfrak{R}}_1'}$ are isometric. Moreover, since the ranges and kernels of P and P' are closed, P and P' are continuous.

Now we define

$$\begin{aligned} &\tilde{\beta} : R+\tilde{\mathfrak{R}}_1 \to R'+\tilde{\mathfrak{R}}_1' \\ &\text{as} \quad \tilde{\beta} := \beta\circ(I-P) + P'\circ\delta\circ P. \end{aligned} \tag{6.25}$$

Lemma 6.6. *The map $\tilde{\beta}$ is an isometric homeomorphism from $R+\tilde{\mathfrak{R}}_1$ to $R'+\tilde{\mathfrak{R}}_1'$. It extends β.*

Proof. It is clear that $\tilde{\beta}$ is linear and continuous, and that

$$\beta(\ker P) = \ker P', \tag{6.26}$$

cf. (6.3). Using this fact and that β, $\delta, P'|_{\tilde{\mathfrak{R}}_1'}$ are isometric, we can compute

$$\begin{aligned} [\tilde{\beta}x,\tilde{\beta}y] &= [\beta(I-P)x,\beta(I-P)y] + [P'\delta Px, P'\delta Py] = \\ &= [(I-P)x,(I-P)y] + [Px,Py] = [x,y], \quad x,y\in R+\tilde{\mathfrak{R}}_1, \end{aligned}$$

i.e., $\tilde{\beta}$ is isometric.

Next we show that $\tilde{\beta}|_R = \beta$. Observe that, by the definition of P and P',

$$P(R) = D_2 + D_5 + R_r \subseteq R,$$

$$\beta(R\cap \operatorname{ran} P) = \beta(D_2+D_5+R_r) = D_2'+D_5'+R_r' = R'\cap\operatorname{ran} P'.$$

From this we obtain, for each $x\in R$,

$$\tilde{\beta}x = \beta(I-P)x + P'\delta\underbrace{Px}_{\in R} = \beta(I-P)x + P'\underbrace{\beta Px}_{\in \operatorname{ran} P'} = \beta(I-P)x + \beta Px = \beta x.$$

To show that $\tilde{\beta}$ maps $R+\tilde{\mathfrak{R}}_1$ surjectively onto $R'+\tilde{\mathfrak{R}}_1'$, note first that

$$R' = \operatorname{ran}\beta \subseteq \operatorname{ran}\tilde{\beta}.$$

In particular, thus

$$\ker\big(P'|_{\tilde{\mathfrak{R}}_1'}\big) = D_4' \subseteq \operatorname{ran}\tilde{\beta},$$

and it follows, using (6.25), that

$$\begin{aligned}\delta(\operatorname{ran} P) &\subseteq \operatorname{ran}\tilde{\beta} + \ker\big(P'|_{\tilde{\mathfrak{R}}_1'}\big) = \operatorname{ran}\tilde{\beta},\\ \delta(D_4) &= \beta(D_4) = D_4' \subseteq \operatorname{ran}\tilde{\beta}.\end{aligned}$$

Together, thus $\tilde{\mathfrak{R}}_1' = \delta(\tilde{\mathfrak{R}}_1) \subseteq \operatorname{ran}\tilde{\beta}$, and we see that indeed $\operatorname{ran}\tilde{\beta} \supseteq R' + \tilde{\mathfrak{R}}_1'$.

To show that $\tilde{\beta}$ is injective, assume that $x \in R + \tilde{\mathfrak{R}}_1$ with $\tilde{\beta}x = 0$. Remembering (6.26), injectivity of β readily implies that $(I-P)x = 0$. Moreover, it follows that $\delta Px \in \ker(P'|_{\tilde{\mathfrak{R}}_1'}) = D_4'$. Since δ is injective, $\delta|_{D_4} = \beta|_{D_4}$, and $\delta(D_4) = \beta(D_4) = D_4'$, we conclude that $Px \in D_4$. This implies that $Px = 0$, and together with what we saw above thus $x = 0$.

Finally, applying the open mapping theorem yields that $\tilde{\beta}$ is a homeomorphism. □

Next we investigate the connection between $\check{\beta}$ and $\tilde{\beta}$.

Lemma 6.7. *The maps $\check{\beta}$ and $\tilde{\beta}$ are related via*

$$\operatorname{ran}(\check{\beta} - \tilde{\beta}) \subseteq D_4', \tag{6.27}$$

and

$$(\tilde{\beta} - w)^{-1} + \big(\{0\} \times \mathcal{C}^\circ\big) = (\check{\beta} - w)^{-1} + \big(\{0\} \times \mathcal{C}^\circ\big), \quad w \notin \sigma_p(\beta|_{\mathcal{C}^\circ}), \tag{6.28}$$

(where the inverses are interpreted as graphs and the sums are componentwise). It holds that

$$\begin{aligned}\ker(\tilde{\beta} - w)\dot{+}\mathcal{C}^\circ &= \ker(\check{\beta} - w)\dot{+}\mathcal{C}^\circ, \quad && w \notin \sigma_p(\beta|_{\mathcal{C}^\circ}),\\ \operatorname{ran}(\tilde{\beta} - w) &= \operatorname{ran}(\check{\beta} - w), && w \notin \sigma_p(\beta|_{\mathcal{C}^\circ}),\end{aligned} \tag{6.29}$$

and

$$\sigma_p(\tilde{\beta}) = \sigma_p(\check{\beta}), \quad \gamma(\tilde{\beta}) = \gamma(\check{\beta}), \quad \rho(\tilde{\beta}) = \rho(\check{\beta}). \tag{6.30}$$

Moreover, we have

$$\begin{aligned}&\big[(\tilde{\beta} - w)^{-1}x, y\big] = \big[(\check{\beta} - w)^{-1}x, y\big],\\ &\qquad\qquad w \notin \sigma_p(\tilde{\beta}),\ x \in \operatorname{ran}(\tilde{\beta} - w),\ y \in \tilde{\mathcal{A}}[-]\mathcal{C}.\end{aligned} \tag{6.31}$$

Proof. First notice that the definition (6.18) of $\check{\beta}$ together with the facts that $\ker P \subseteq R$ and $\operatorname{ran} P \subseteq \tilde{\mathfrak{R}}_1$ implies that

$$\check{\beta} = \beta \circ (I - P) + \delta \circ P.$$

Comparing with the definition (6.25) of $\tilde{\beta}$ yields

$$\check{\beta} - \tilde{\beta} = (I - P') \circ \delta \circ P.$$

The inclusion (6.27) follows.

To show the identity in (6.28) assume that $w \notin \sigma_p(\beta|_{\mathcal{C}^\circ})$ and note that

$$(\tilde{\beta} - w)^{-1} = \big\{(\tilde{\beta} f - wf, f) \colon f \in R + \tilde{\mathfrak{R}}_1\big\},$$
$$(\check{\beta} - w)^{-1} = \big\{(\check{\beta} f - wf, f) \colon f \in R + \tilde{\mathfrak{R}}_1\big\}.$$

Let an element $(x, y) \in (\tilde{\beta} - w)^{-1}$ be given. Then

$$x = \tilde{\beta} y - wy = (\check{\beta} y - wy) + (\tilde{\beta} y - \check{\beta} y)$$

where $\tilde{\beta} y - \check{\beta} y \in D_4' \subseteq \mathcal{C}^\circ$, cf. (6.27). Since $\beta(\mathcal{C}^\circ) = \mathcal{C}^\circ$, and $w \notin \sigma_p(\beta|_{\mathcal{C}^\circ})$, the mapping $(\beta - w)|_{\mathcal{C}^\circ}$ is a bijection from $\mathcal{C}^\circ$ onto itself.

Set $y_0 := (\beta - w)^{-1}(\tilde{\beta} y - \check{\beta} y)$, and observe that

$$(\tilde{\beta} y - \check{\beta} y, y_0) \in (\beta - w)^{-1} \subseteq (\check{\beta} - w)^{-1},$$

and also that $y_0 \in \mathcal{C}^\circ$. We obtain

$$(x, y) = (\check{\beta} y - wy, y) + (\tilde{\beta} y - \check{\beta} y, y_0) - (0, y_0) \in (\check{\beta} - w)^{-1} + \big(\{0\} \times \mathcal{C}^\circ\big).$$

This shows the inclusion "$\subseteq$" in (6.28). The reverse inclusion is seen by exchanging the roles of $\tilde{\beta}$ and $\check{\beta}$ in the above argument.

The relations (6.29) are a direct consequence of (6.28). We only have to note in addition that

$$\mathcal{C}^\circ \cap \ker(\tilde{\beta} - w) = \mathcal{C}^\circ \cap \ker(\check{\beta} - w) = \ker(\beta|_{\mathcal{C}^\circ} - w), \quad w \in \mathbb{C}. \tag{6.32}$$

This holds since $\tilde{\beta}$ and $\check{\beta}$ are both extensions of β, and shows that the sums in the first line of (6.29) are direct.

To see the inclusion "$\supseteq$" in the first equality in (6.30), assume that $w \notin \sigma_p(\tilde{\beta})$. Then $w \notin \sigma_p(\beta|_{\mathcal{C}^\circ})$, and hence (6.29) is available. The first line of (6.29) implies that $\ker(\check{\beta} - w) \subseteq \mathcal{C}^\circ$, and (6.32) yields

$$\ker(\check{\beta} - w) = \ker\big(\beta|_{\mathcal{C}^\circ} - w\big) = \{0\}.$$

Therefore $w \notin \sigma_p(\check{\beta})$. The reverse inclusion follows in the same way. The second and third identities follow now immediately using the second line of (6.29) since

$$w \in \gamma(\tilde{\beta}) \iff w \notin \sigma_p(\tilde{\beta}) \wedge \operatorname{ran}(\tilde{\beta} - w) \text{ closed},$$
$$w \in \rho(\tilde{\beta}) \iff w \notin \sigma_p(\tilde{\beta}) \wedge \operatorname{ran}(\tilde{\beta} - w) = \tilde{\mathcal{A}},$$

and the same for $\check{\beta}$.

Finally, for (6.31), observe that by (6.28)

$$\operatorname{ran}\Big((\tilde{\beta} - w)^{-1} - (\check{\beta} - w)^{-1}\Big) \subseteq \mathcal{C}^\circ, \quad w \notin \sigma_p(\tilde{\beta}). \tag{6.33}$$

This implies the asserted equality of inner products. ☐

6.4. Relating $\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{\beta})$ with $\tilde{\beta}_1$

Once more we assume throughout this subsection that $\tilde{\mathcal{A}}$ is nondegenerated.

The maps $\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{\beta})$ and $\tilde{\beta}_1$ can be related in a three step procedure, namely

$$\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{\beta}) \overset{\text{Prop.3.2}}{\longleftrightarrow} \tilde{\beta} \overset{\text{Lem.6.7}}{\longleftrightarrow} \check{\beta} \overset{\text{Prop.3.2}}{\longleftrightarrow} \tilde{\beta}_1 = \mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\check{\beta})$$

In order to proceed in this way, we have to make sure that the sets $\Gamma(\tilde{\beta})$ and $\Gamma(\check{\beta})$ which occur in Proposition 3.2 are sufficiently large.

Lemma 6.8. *Let $w \in \mathbb{C}$.*

(*i*) *If $w \neq 0$, assume that*

$$w \notin \sigma_p(\tilde{\beta}), \quad \frac{1}{\overline{w}} \notin \sigma_p(\beta|_{\mathcal{C}^\circ}).$$

Then

$$\begin{aligned} (\tilde{\beta} - w)^{-1}\big((\tilde{\mathcal{A}}[-]\mathcal{C}) \cap \operatorname{ran}(\tilde{\beta} - w)\big) &\subseteq \tilde{\mathcal{A}}[-]\mathcal{C}, \\ (\check{\beta} - w)^{-1}\big((\tilde{\mathcal{A}}[-]\mathcal{C}) \cap \operatorname{ran}(\check{\beta} - w)\big) &\subseteq \tilde{\mathcal{A}}[-]\mathcal{C}. \end{aligned} \tag{6.34}$$

(*ii*) *Assume that $w \notin \sigma_p(\beta|_{\mathcal{B}})$. Then*

$$(\tilde{\beta} - w)^{-1}(\mathcal{B}) = (\check{\beta} - w)^{-1}(\mathcal{B}) = \mathcal{B}, \tag{6.35}$$

Proof. To show (6.34), consider first the case that $w \neq 0$. Assume that $x \in (\tilde{\mathcal{A}}[-]\mathcal{C}) \cap \operatorname{ran}(\tilde{\beta} - w)$. Let $z \in \mathcal{C}$ and set $y := (\beta - \frac{1}{\overline{w}})z$. Then $y \in \mathcal{C}$ and (2.2) implies that

$$\underbrace{[x, y]}_{=0} + w[(\tilde{\beta} - w)^{-1}x, y] + \frac{1}{w}\underbrace{[x, z]}_{=0} = 0.$$

Thus $[(\tilde{\beta} - w)^{-1}x, y] = 0$. By Lemma 2.3 applied with the map $\beta|_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ and the point $\frac{1}{\overline{w}}$, the set $(\beta - \frac{1}{\overline{w}})(\mathcal{C})$ is dense in $\mathcal{C}$, and it follows that

$$(\tilde{\beta} - w)^{-1}x \; [\perp] \; \mathcal{C}.$$

This proves the inclusion required in the first line of (6.34). The inclusion in the second line follows immediately. Let $x \in (\tilde{\mathcal{A}}[-]\mathcal{C}) \cap \operatorname{ran}(\check{\beta} - w)$. Then, by (6.29) and what we already showed above,

$$(\tilde{\beta} - w)^{-1}x \in \tilde{\mathcal{A}}[-]\mathcal{C}.$$

However, by (6.33) we have

$$(\check{\beta} - w)^{-1}x - (\tilde{\beta} - w)^{-1}x \in \mathcal{C}^\circ \subseteq \tilde{\mathcal{A}}[-]\mathcal{C},$$

and thus indeed $(\check{\beta} - w)^{-1}x \in \tilde{\mathcal{A}}[-]\mathcal{C}$.

Now consider the case that $w = 0$. Then for each $x \in (\tilde{\mathcal{A}}[-]\mathcal{C}) \cap \operatorname{ran} \tilde{\beta}$ and $y \in \mathcal{C}$ we have by isometry

$$[\tilde{\beta}^{-1}x, y] = [x, \underbrace{\tilde{\beta}y}_{=\beta y \in \mathcal{C}}] = 0.$$

Again referring to (6.33), we obtain that also that $[\check{\beta}^{-1}x, y] = 0$.

Finally, for (6.35), it is enough to observe that for $w \notin \sigma_p(\beta|_{\mathcal{B}})$ the map $\beta - w$ is a bijection of $\mathcal{B}$ onto itself. Since $\tilde{\beta}$ and $\check{\beta}$ are extension of β, indeed (6.35) follows. $\square$

The crucial step is to determine $\sigma_p(\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{\beta}))$.

Lemma 6.9. *Denote by $\mathcal{E}$ the exceptional set*

$$\mathcal{E} := \sigma_p(\beta|_{\mathcal{C}}) \cup \big\{w \in \mathbb{C} \setminus \{0\} : \frac{1}{w} \in \sigma_p(\beta|_{\mathcal{C}^\circ})\big\} \cup \{\infty\}. \tag{6.36}$$

Then

$$\sigma_p\big(\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{\beta})\big) \setminus \mathcal{E} = \sigma_p(\tilde{\beta}) \setminus \mathcal{E} = \sigma_p(\tilde{\beta}_1) \setminus \mathcal{E}. \tag{6.37}$$

Proof. We first show that

$$\ker(\tilde{\beta} - w) \subseteq \tilde{\mathcal{A}}[-]\mathcal{C}, \quad w \notin \mathcal{E}. \tag{6.38}$$

To see this, let $w \in \mathbb{C} \setminus \mathcal{E}$ and $x \in \ker(\tilde{\beta} - w)$. For $w = 0$ clearly $\ker(\tilde{\beta} - w) = \{0\}$. Hence assume that $w \neq 0$. Then, for each $y \in \mathcal{C}$, we can compute

$$0 = [(\tilde{\beta} - w)x, y] = [\tilde{\beta}x, \tilde{\beta}(\beta^{-1}y)] - w[x, y] =$$
$$= w\big[x, \frac{1}{w}\beta^{-1}y - y\big] = -w\big[x, (\beta - \frac{1}{w})\beta^{-1}y\big].$$

Since $\beta(\mathcal{C}) = \mathcal{C}$, this implies that

$$x\ [\perp]\ \big(\beta - \frac{1}{w}\big)(\mathcal{C}).$$

Since $w \notin \mathcal{E}$, Lemma 2.3 is applicable with the map $\beta|_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ and the point $\frac{1}{w}$, and we conclude that indeed (6.38) holds.

From (6.38) it follows that

$$\pi\big(\ker(\tilde{\beta} - w)\big) \subseteq \ker\big(\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{\beta}) - w\big), \quad w \notin \mathcal{E}. \tag{6.39}$$

Moreover, in conjunction with the first line in (6.29), also

$$\ker(\check{\beta} - w) \subseteq \tilde{\mathcal{A}}[-]\mathcal{C}, \quad w \notin \mathcal{E},$$

and hence also

$$\pi\big(\ker(\check{\beta} - w)\big) \subseteq \ker(\tilde{\beta}_1 - w), \quad w \notin \mathcal{E}. \tag{6.40}$$

The inclusion "$\subseteq$" in the first equality in (6.37) follows from Lemma 3.4 applied with the space $\tilde{\mathcal{A}}, \tilde{\mathcal{A}}[-]\mathcal{C}, \mathcal{B}$ and the map $\tilde{\beta}$. For the reverse inclusion assume that $w \notin \mathcal{E}$ and $w \notin \sigma_p(\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{\beta}))$. Then (6.39) implies that

$$\ker(\tilde{\beta} - w) \subseteq \ker \pi \subseteq \mathcal{C}^\circ.$$

Remembering (6.32), we conclude that $\ker(\tilde{\beta} - w) = \ker(\beta|_{\mathcal{C}^\circ} - w) = \{0\}$, i.e., $w \notin \sigma_p(\tilde{\beta})$.

With the same arguments, referring to (6.40) instead of (6.39), it follows for $w \notin \mathcal{E}$ that $w \in \sigma_p(\tilde{\beta}_1)$ if and only if $w \in \sigma_p(\check{\beta})$. However, we know from Lemma 6.7 that $\sigma_p(\check{\beta}) = \sigma_p(\tilde{\beta})$. This shows the second equality in (6.37). $\square$

It is not anymore difficult to relate the resolvents of $\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{\beta})$ and $\tilde{\beta}_1$.

Lemma 6.10. *Let $\mathcal{E}$ be as in* (6.36)*. Then, for each $w \in \mathbb{C} \setminus (\sigma_p(\tilde{\beta}_1) \cup \mathcal{E})$ it holds that*

$$\operatorname{ran}\big(\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{\beta}) - w\big) = \pi\big((\tilde{\mathcal{A}}[-]\mathcal{C}) \cap \operatorname{ran}(\tilde{\beta} - w)\big) = \operatorname{ran}(\tilde{\beta}_1 - w), \quad (6.41)$$

$$\big[\big(\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{\beta}) - w\big)\pi x, \pi y\big] = [(\tilde{\beta} - w)^{-1}x, y] = [(\tilde{\beta}_1 - w)^{-1}\pi x, \pi y] \quad (6.42)$$

for $x \in (\tilde{\mathcal{A}}[-]\mathcal{C}) \cap \operatorname{ran}(\tilde{\beta} - w), y \in \tilde{\mathcal{A}}[-]\mathcal{C}$.

Moreover,

$$\begin{aligned} \gamma(\tilde{\beta}) \setminus \mathcal{E} &\subseteq \gamma\big(\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{\beta})\big) \setminus \mathcal{E} = \gamma(\tilde{\beta}_1) \setminus \mathcal{E}, \\ \rho(\tilde{\beta}) \setminus \mathcal{E} &\subseteq \rho\big(\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{\beta})\big) \setminus \mathcal{E} = \rho(\tilde{\beta}_1) \setminus \mathcal{E}. \end{aligned} \quad (6.43)$$

Proof. Let $w \in \mathbb{C} \setminus (\sigma_p(\tilde{\beta}_1) \cup \mathcal{E})$. Then $w \notin \sigma_p(\tilde{\beta})$, and Lemma 6.8 shows that

$$w \in \Gamma(\tilde{\beta}) \quad \text{and} \quad w \in \Gamma(\check{\beta}).$$

Applying Proposition 3.2 with $\tilde{\beta}$ and with $\check{\beta}$, and using Lemma 6.7, it follows that

$$\begin{aligned} \operatorname{ran}\big(\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{\beta}) - w\big) &= \pi\big((\tilde{\mathcal{A}}[-]\mathcal{C}) \cap \operatorname{ran}(\tilde{\beta} - w)\big) = \\ &= \pi\big((\tilde{\mathcal{A}}[-]\mathcal{C}) \cap \operatorname{ran}(\check{\beta} - w)\big) = \operatorname{ran}(\tilde{\beta}_1 - w), \end{aligned}$$

and this is (6.41). The same sources imply

$$\begin{aligned} \big[\big(\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{\beta}) - w\big)\pi x, \pi y\big] &= [(\tilde{\beta} - w)^{-1}x, y] = \\ &= [(\check{\beta} - w)^{-1}x, y] = [(\tilde{\beta}_1 - w)^{-1}\pi x, \pi y] \end{aligned}$$

whenever $x \in (\tilde{\mathcal{A}}[-]\mathcal{C}) \cap \operatorname{ran}(\tilde{\beta} - w)$ and $y \in \tilde{\mathcal{A}}[-]\mathcal{C}$, and this is (6.42).

The relations (6.43) are an immediate consequence of (6.37) and (6.41). □

6.5. Finishing the proof of Theorem 4.3

Now we are in position to complete the proof of Theorem 4.3. The case that $\tilde{\mathcal{A}}$ is nondegenerated actually has been settled in the previous subsections §6.1–§6.4. For completeness, let us collect the relevant lemmata.

Proof of Theorem 4.3*; $\tilde{\mathcal{A}}$ nondegenerated.* We use the subspaces

$$\tilde{R} := R + \tilde{\mathfrak{R}}_1, \quad \tilde{R}' := R' + \tilde{\mathfrak{R}}'_1$$

constructed in §6.2, and the map $\tilde{\beta}$ defined in §6.3. Then $\tilde{R}$ and $\tilde{R}'$ are closed, cf. Lemma 6.3, $\tilde{\beta}$ is an isometric homeomorphism of $\tilde{R}$ onto $\tilde{R}'$ which extends β, cf. Lemma 6.6, the relation (4.11) holds, cf. Lemma 6.9, and the relations (4.12), (4.13), (4.14) hold, cf. Lemma 6.10.

We have, from the definition of D_4 in (6.1), (5.1), that

$$(\mathcal{C}^\circ \cap R^\circ) + \mathcal{B} = \mathcal{C}^\circ \quad \Leftrightarrow \quad D_4 = \{0\}$$

Assuming (4.15) ths leads to $D_4' = \beta(D_4) = \{0\}$, and hence to $\tilde\beta = \check\beta$, cf. Lemma 6.7. In turn, it follows that $\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde\beta) = \check\beta_1$, cf. Lemma 6.5. □

The case that $\tilde{\mathcal{A}}$ is degenerated can be reduced to the nondegenerated case using Lemma 3.4.

Proof of Theorem 4.3*; $\tilde{\mathcal{A}}$ degenerated.* Choose a Pontryagin space $\mathcal{X}$ with $\mathcal{X} \supseteq \tilde{\mathcal{A}}$. Set $\mathcal{X}_1 := \mathfrak{F}^{\mathcal{B}}_{\mathcal{X}|\mathcal{X}[-]\mathcal{C}}(\mathcal{X})$, and let $\pi_{\mathcal{X}} : \mathcal{X}[-]\mathcal{C} \to \mathcal{X}_1$ denote the canonical projection. Moreover, denote the canonical projection of $\tilde{\mathcal{A}}$ onto $\tilde{\mathcal{A}}_1$ as $\pi_{\tilde{\mathcal{A}}}$. Then, clearly,

$$\pi_{\mathcal{X}}|_{\tilde{\mathcal{A}}[-]\mathcal{C}} = \pi_{\tilde{\mathcal{A}}}.$$

Moreover, since $\ker \pi_{\mathcal{X}} = \mathcal{B} \subseteq \tilde{\mathcal{A}}$, it holds that

$$\pi_{\mathcal{X}}^{-1}(\tilde{\mathcal{A}}_1) = \tilde{\mathcal{A}}. \tag{6.44}$$

The construction carried out in §6.1-6.4 provides us with closed linear subspaces of $\mathcal{X}$, say S and S', and a homeomorphisms between them, say $\check\xi$ and $\tilde\xi$. As we already saw, $\tilde\xi$ satisfies all the properties required in Theorem 4.3.

Set $\tilde\beta := \tilde\xi \cap (\tilde{\mathcal{A}})^2$. Then, by Lemma 3.4, $\tilde\beta$ is an isometric homeomorphism between closed subspaces of $\tilde{\mathcal{A}}$, call them R and R'. Clearly, $\tilde\beta$ extends β. We show that

$$\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde\beta) = \mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{X}}|\tilde{\mathcal{X}}[-]\mathcal{C}}(\tilde\xi). \tag{6.45}$$

First,

$$\begin{aligned}\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde\beta) &= (\pi_{\tilde{\mathcal{A}}} \times \pi_{\tilde{\mathcal{A}}})\big(\tilde\beta \cap (\tilde{\mathcal{A}}[-]\mathcal{C})^2\big) = (\pi_{\tilde{\mathcal{A}}} \times \pi_{\tilde{\mathcal{A}}})\big(\tilde\xi \cap (\tilde{\mathcal{A}}[-]\mathcal{C})^2\big)\\ &= (\pi_{\tilde{\mathcal{X}}} \times \pi_{\tilde{\mathcal{X}}})\big(\tilde\xi \cap (\tilde{\mathcal{A}}[-]\mathcal{C})^2\big) \subseteq (\pi_{\tilde{\mathcal{X}}} \times \pi_{\tilde{\mathcal{X}}})\big(\tilde\xi \cap (\tilde{\mathcal{X}}[-]\mathcal{C})^2\big),\end{aligned} \tag{6.46}$$

and this is the inclusion "$\subseteq$" in (6.45). To show the reverse inclusion, assume that $(x_1, y_1) \in \mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{X}}|\tilde{\mathcal{X}}[-]\mathcal{C}}(\tilde\xi)$. Choose $(x, y) \in \tilde\xi \cap (\mathcal{X}[-]\mathcal{C})^2$ with $\pi_{\mathcal{X}} x = x_1$ and $\pi_{\mathcal{X}} y = y_1$. Then

$$x \in \operatorname{dom}\tilde\xi = \operatorname{dom}\check\xi,$$

and

$$\check\xi x - \underbrace{\tilde\xi x}_{=y} \in D_4' \subseteq \mathcal{C}^\circ \subseteq \tilde{\mathcal{A}}[-]\mathcal{C} \subseteq \mathcal{X}[-]\mathcal{C}.$$

Setting $z := \check\xi x$, we have $z = (z - y) + y \in \mathcal{X}[-]\mathcal{C}$, and hence

$$(\pi_{\mathcal{X}} \times \pi_{\mathcal{X}})((x, z)) \in (\pi_{\mathcal{X}} \times \pi_{\mathcal{X}})\big(\check\xi \cap (\mathcal{X}[-]\mathcal{C})^2\big) = \tilde\beta_1 \subseteq (\tilde{\mathcal{A}}_1)^2.$$

Remembering (6.44) we conclude that $x, z \in \tilde{\mathcal{A}}[-]\mathcal{C}$ and in turn also that $y \in \tilde{\mathcal{A}}[-]\mathcal{C}$. The chain of equalities written in (6.46) shows that

$$(x, y) \in \mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde\beta).$$

This establishes (6.45), and therefore that $\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde\beta)$ has the properties required in Theorem 4.3, (i). Moreover, if (4.15) holds and we make the choice of $\tilde\xi$ according to Theorem 4.3, (ii), then also $\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde\beta) = \tilde\beta_1$.

We come to the proof of the assertions concerning $\tilde{\beta}$ itself. To determine $\sigma_p(\tilde{\beta})$, we use Lemma 3.4 to compute

$$\sigma_p(\tilde{\beta}) \subseteq \sigma_p(\tilde{\xi}) \subseteq \sigma_p\big(\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{X}}|\tilde{\mathcal{X}}[-]\mathcal{C}}(\tilde{\xi})\big) \cup \mathcal{E} = \sigma_p\big(\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{\beta})\big) \cup \mathcal{E} \subseteq \sigma_p(\tilde{\beta}) \cup \mathcal{E}.$$

To establish the required equality of ranges, compute

$$\begin{aligned} \pi\big((\tilde{\mathcal{A}}[-]\mathcal{C}) \cap \operatorname{ran}(\tilde{\beta}-w)\big) &\subseteq \pi\big((\tilde{\mathcal{X}}[-]\mathcal{C}) \cap \operatorname{ran}(\tilde{\xi}-w)\big) = \operatorname{ran}\big(\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{X}}|\tilde{\mathcal{X}}[-]\mathcal{C}}(\tilde{\xi})-w\big) \\ &= \operatorname{ran}\big(\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{\beta})-w\big) = \pi\big((\tilde{\mathcal{A}}[-]\mathcal{C}) \cap (\tilde{\beta}-w)\big((\tilde{\mathcal{A}}[-]\mathcal{C}) \cap R\big)\big) \\ &\subseteq \pi\big((\tilde{\mathcal{A}}[-]\mathcal{C}) \cap \operatorname{ran}(\tilde{\beta}-w)\big). \end{aligned}$$

The inclusions concerning $\gamma(\tilde{\beta})$ and $\rho(\tilde{\beta})$ follow from the readily proven (4.11) and (4.13). $\square$

7. Closed symmetric extensions of closed symmetric relations

In many applications it is more natural to work with closed symmetric relations or operators and their selfadjoint extensions than with isometric ones and unitary extensions. Let $\mathcal{A}$ be an almost Pontryagin space, let $\mathcal{C}$ and $\mathcal{B}$ be closed linear subspaces of $\mathcal{A}$ with $\mathcal{B} \subseteq \mathcal{C}^\circ$, and set

$$\mathcal{A}_1 := \mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{A}[-]\mathcal{C}}(\mathcal{A}).$$

Let S be a closed symmetric relation in $\mathcal{A}$ and set

$$S_1 := \mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{A}[-]\mathcal{C}}(\mathcal{A}).$$

Let $\tilde{\mathcal{A}}_1$ be an almost Pontryagin space with $\tilde{\mathcal{A}}_1 \supseteq \mathcal{A}_1$ and let $\tilde{S}_1$ be a closed symmetric extension in $\tilde{\mathcal{A}}_1$. The problem is to construct an almost Pontryagin space $\tilde{\mathcal{A}}$ with $\tilde{\mathcal{A}} \supseteq \mathcal{A}$ and $\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{\mathcal{A}}) = \tilde{\mathcal{A}}_1$, and to construct a closed symmetric extension $\tilde{S}$ in $\tilde{\mathcal{A}}$ of S, such that

$$\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{\mathcal{S}}) = \tilde{\mathcal{S}}_1,$$

or a weak version of this identity.

$$\begin{array}{lcccccl}
\mathcal{A}: & \operatorname{dom} S \overset{S}{\text{———}} \operatorname{ran} S & & \xrightarrow{\mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{A}[-]\mathcal{C}}} & & \operatorname{dom} S_1 \overset{S_1}{\text{———}} \operatorname{ran} S_1 & : \mathcal{A}_1 \\
\cap| & \cap| \qquad\quad \cap| & & & & \cap| \qquad\quad \cap| & \cap| \\
\tilde{\mathcal{A}}: & \operatorname{dom} \tilde{S} \underset{\tilde{S}}{\cdots\cdots} \operatorname{ran} \tilde{S} & & \underset{\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}}{\cdots\cdots\cdots} & & \operatorname{dom} \tilde{S}_1 \overset{\tilde{S}_1}{\underset{\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{S})}{\text{———}}} \operatorname{ran} \tilde{S}_1 & : \tilde{\mathcal{A}}_1
\end{array}$$

Again the above extension problem is twofold. The extension problem for spaces was settled in Theorem 4.2: an almost Pontryagin space $\tilde{\mathcal{A}}$ with $\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{\mathcal{A}}) = \tilde{\mathcal{A}}_1$ exists if and only if

$$\mathcal{C}^\circ/\mathcal{B} \subseteq (\tilde{\mathcal{A}}_1)^\circ.$$

The extension problem for symmetric relations is solved via a reduction to the isometric case and an application of Theorem 4.3.

Theorem 7.1. *Let $\mathcal{A}$ be an almost Pontryagin space, let $\mathcal{C}$ and $\mathcal{B}$ be closed linear subspaces of $\mathcal{A}$ with $\mathcal{B} \subseteq \mathcal{C}^\circ$, and let S be a closed symmetric relation in $\mathcal{A}$. Set*

$$\mathcal{A}_1 := \mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{A}[-]\mathcal{C}}(\mathcal{A}), \quad S_1 := \mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{A}[-]\mathcal{C}}(\mathcal{S}).$$

Let $\tilde{\mathcal{A}}_1$ be an almost Pontryagin space and $\tilde{S}_1$ a closed symmetric relation in $\tilde{\mathcal{A}}_1$ with

$$\tilde{\mathcal{A}}_1 \supseteq \mathcal{A}_1, \quad \tilde{S}_1 \supseteq S_1,$$

and let $\tilde{\mathcal{A}}$ be an almost Pontryagin space with

$$\tilde{\mathcal{A}} \supseteq \mathcal{A}, \quad \mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{\mathcal{A}}) = \tilde{\mathcal{A}}_1.$$

Assume that $\mu \in \mathbb{C} \setminus \mathbb{R}$ is given with $\mu, \overline{\mu} \in \gamma(S) \cap \gamma(\tilde{S}_1)$, and

$$\mathcal{C} \subseteq \operatorname{ran}(S-\mu) \cap \operatorname{ran}(S-\overline{\mu}), \tag{7.1}$$

$$(S-\mu)^{-1}(\mathcal{B}) \subseteq \mathcal{B}, \quad (S-\mu)^{-1}(\mathcal{C}) \subseteq \mathcal{C},\ (S-\overline{\mu})^{-1}(\mathcal{C}) \subseteq \mathcal{C}, \tag{7.2}$$

and denote by $\mathcal{E}_s$ the exceptional set

$$\mathcal{E}_s := \sigma_p\big(S \cap (\mathcal{C} \times \mathcal{C})\big) \cup \big\{z \in \mathbb{C} : \overline{z} \in \sigma_p\big(S \cap (\mathcal{C}^\circ \times \mathcal{C}^\circ)\big)\big\} \cup \{\mu\}.$$

Then

(i) *There exists a closed symmetric relation $\tilde{S}$ in $\tilde{\mathcal{A}}$ with $\mu, \overline{\mu} \in \gamma(\tilde{S})$ and $\tilde{S} \supseteq S$, such that*

$$\sigma_p\big(\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{S})\big) \setminus \mathcal{E} = \sigma_p(\tilde{S}) \setminus \mathcal{E} = \sigma_p(\tilde{S}_1) \setminus \mathcal{E}, \tag{7.3}$$

$$\begin{aligned} &\gamma(\tilde{S}) \setminus \mathcal{E} \subseteq \gamma\big(\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{S})\big) \setminus \mathcal{E} = \gamma(\tilde{S}_1) \setminus \mathcal{E}, \\ &\rho(\tilde{S}) \setminus \mathcal{E} \subseteq \rho\big(\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{S})\big) \setminus \mathcal{E} = \rho(\tilde{S}_1) \setminus \mathcal{E}. \end{aligned} \tag{7.4}$$

$$\begin{aligned} \operatorname{ran}\big(\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{S}) - z\big) &= \pi\big((\tilde{\mathcal{A}}[-]\mathcal{C}) \cap \operatorname{ran}(\tilde{S} - z)\big) = \\ &= \operatorname{ran}(\tilde{S}_1 - z), \quad z \in \mathbb{C} \setminus (\sigma_p(\tilde{S}_1) \cup \mathcal{E}), \end{aligned} \tag{7.5}$$

$$\begin{aligned} \big[(\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{S}) - z)^{-1} x_1, y_1\big] &= \big[(\tilde{S}_1 - z)^{-1} x_1, y_1\big], \\ x_1 \in \operatorname{ran}(\tilde{S}_1 - z),\ y_1 &\in \tilde{\mathcal{A}}_1, \quad z \in \mathbb{C} \setminus (\sigma_p(\tilde{S}_1) \cup \mathcal{E}). \end{aligned} \tag{7.6}$$

(ii) *If $\big(\mathcal{C}^\circ \cap \operatorname{ran}(S-\mu)^\circ\big) + \mathcal{B} = \mathcal{C}^\circ$ then the choice of $\tilde{S}$ in (i) can be made such that*

$$\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{S}) = \tilde{S}_1.$$

Proof. The proof is based on a reduction to the situation of Theorem 4.3 by means of the Cayley transform. Thereby we use the given nonreal number μ as a base point. Set

$$\beta := C_\mu(S), \quad R := \operatorname{ran}(S-\mu),\ R' := \operatorname{ran}(S-\overline{\mu}).$$

Then the spaces R and R' are closed and β is a linear and isometric homeomorphism between them, cf. by Lemma 2.5. We are going to check the conditions (4.8). From the present assumptions it is obvious that

$$\mathcal{C} \subseteq R, \quad \beta(\mathcal{C}) \subseteq \mathcal{C}, \quad \beta(\mathcal{B}) \subseteq \mathcal{B}.$$

To see that in the second inclusion equality holds, use Lemma 2.4 which yields (applied first with μ then with $\overline{\mu}$)

$$\beta(\mathcal{C}) = \operatorname{ran} \beta|_{\mathcal{C}} = \operatorname{ran}\big(C_\mu\big(S \cap (\mathcal{C} \times \mathcal{C})\big)\big) = \operatorname{ran}\big(\big[S \cap (\mathcal{C} \times \mathcal{C})\big] - \overline{\mu}\big) = \mathcal{C}.$$

Let R_1, R_1' and β_1 be as in (4.9). The map

$$\tilde{\beta}_1 := C_\mu(\tilde{S}_1)$$

is a linear and isometric homeomorphism between the closed subspaces

$$\tilde{R}_1 := \operatorname{ran}(\tilde{S}_1 - \mu), \quad \tilde{R}_1' := \operatorname{ran}(\tilde{S}_1 - \overline{\mu}).$$

By Lemma 3.5, we have

$$\beta_1 = \mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{A}[-]\mathcal{C}}\big(C_\mu(S)\big) = C_\mu\big(\mathfrak{F}^{\mathcal{B}}_{\mathcal{A}|\mathcal{A}[-]\mathcal{C}}(S)\big) = C_\mu(S_1),$$

and hence $\tilde{\beta}_1$ is an extension of β_1.

Theorem 4.3 provides us with a linear and isometric homeomorphism $\tilde{\beta}$ between some closed subspaces of $\tilde{\mathcal{A}}$ which extends β and satisfies (4.11)–(4.14). Set

$$\tilde{S} := F_\mu(\tilde{\beta}),$$

then $\tilde{S}$ is a closed and symmetric relation in $\tilde{\mathcal{A}}$ with $\mu, \overline{\mu} \in \gamma(\tilde{S})$ and $\tilde{S} \supseteq S$.

We are going to check that $\tilde{S}$ satisfies (7.3)–(7.6). To this end, let $z \in \mathbb{C} \setminus \mathcal{E}_s$ be fixed. The first aim is to show that ($\mathcal{E}$ is the exceptional set (4.10))

$$c_\mu(z) \notin \mathcal{E}. \tag{7.7}$$

If $z = \overline{\mu}$ we have $c_\mu(z) = 0 \notin \mathcal{E}$, hence assume that $z \neq \overline{\mu}$. It holds that

$$c_\mu\big(\sigma_p\big(S \cap (\mathcal{C} \times \mathcal{C})\big)\big) = \sigma_p\big(c_\mu\big(S \cap (\mathcal{C} \times \mathcal{C})\big)\big) = \sigma_p\big(\beta|_{\mathcal{C}}\big),$$

and it follows that $c_\mu(z) \notin \sigma_p(\beta|_{\mathcal{C}})$. In order to use the analogous argument for the relation $S \cap (\mathcal{C}^\circ \times \mathcal{C}^\circ)$, we must make sure that Lemma 2.4 is applicable. Let $x \in \mathcal{C}^\circ$. Then, clearly,

$$(S - \mu)^{-1} x \in (S - \mu)^{-1}(\mathcal{C}) \subseteq \mathcal{C}.$$

Moreover, for each $y \in \mathcal{C}$ we have

$$\big[(S - \mu)^{-1} x, y\big] = \big[x, \underbrace{(S - \overline{\mu})^{-1} y}_{\in \mathcal{C}}\big] = 0.$$

Thus, $(S - \mu)^{-1} x \in \mathcal{C}^\circ$. Using that $c_\mu(\overline{z}) = [\overline{c_\mu(z)}]^{-1}$, it follows now with the same argument as above that

$$\frac{1}{\overline{c_\mu(z)}} = c_\mu(\overline{z}) \notin \sigma_p\big(\beta|_{\mathcal{C}^\circ}\big).$$

This finishes the proof of (7.7).

The relation (4.11) implies that

$$\begin{aligned} c_\mu(z) \in \sigma_p\big(\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(C_\mu(\tilde{S}))\big) &= \sigma_p\big(C_\mu\big(\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{S})\big)\big) \\ \Leftrightarrow \quad & c_\mu(z) \in \sigma_p\big(C_\mu(\tilde{S})\big) \\ \Leftrightarrow \quad & c_\mu(z) \in \sigma_p\big(C_\mu(\tilde{S}_1)\big) \end{aligned}$$

This implies

$$z \in \sigma_p\big(\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{S})\big) \Leftrightarrow z \in \sigma_p(\tilde{S}) \Leftrightarrow z \in \sigma_p(\tilde{S}_1),$$

which is (7.3). The relations in (7.4) follow in the same way.

For the proof of (7.5) and (7.6) assume in addition that $z \notin \sigma_p(\tilde{S}_1)$. Then $c_\mu(z) \notin \sigma_p(\tilde{\beta}_1)$, and it follows from (4.13) that

$$\begin{array}{c} \operatorname{ran}\big(\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{S})-z\big) = \operatorname{ran}\big(C_\mu\big(\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{S})\big)-c_\mu(z)\big) = \operatorname{ran}\big(\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{\beta})-c_\mu(z)\big) \\ \| \ (4.13) \\ \pi\big((\tilde{\mathcal{A}}[-]\mathcal{C}) \cap \operatorname{ran}(\tilde{S}-z)\big) = \pi\big((\tilde{\mathcal{A}}[-]\mathcal{C}) \cap \operatorname{ran}(C_\mu(\tilde{S})-c_\mu(z))\big) = \pi\big((\tilde{\mathcal{A}}[-]\mathcal{C}) \cap \operatorname{ran}(\tilde{\beta}-c_\mu(z))\big) \\ \| \ (4.13) \\ \operatorname{ran}(\tilde{S}_1-z) = \operatorname{ran}\big(C_\mu(\tilde{S}_1)-c_\mu(z)\big) = \operatorname{ran}\big(\tilde{\beta}_1-c_\mu(z)\big) \end{array}$$

Let $x_1 \in \operatorname{ran}(\tilde{S}_1 - z)$ and $y_1 \in \tilde{\mathcal{A}}_1$. Then $x_1 \in \operatorname{ran}(\tilde{\beta}_1 - c_\mu(z))$, and (4.14) yields

$$\begin{aligned} &\frac{z-\mu}{\overline{\mu}-\mu}[x_1,y_1] + \frac{(z-\mu)^2}{\overline{\mu}-\mu}\Big[\big(\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{S}) - z\big)^{-1}x_1,y_1\Big] \\ &\quad = \Big[\big(C_\mu\big(\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{S})\big) - c_\mu(z)\big)^{-1}x_1,y_1\Big] \\ &\quad = \Big[\big(\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{\beta}) - c_\mu(z)\big)^{-1}x_1,y_1\Big] \\ &\quad \overset{(4.14)}{=} \big[(\tilde{\beta}_1 - c_\mu(z))^{-1}x_1,y_1\big] = \frac{z-\mu}{\overline{\mu}-\mu}[x_1,y_1] + \frac{(z-\mu)^2}{\overline{\mu}-\mu}\big[(\tilde{S}_1 - z)^{-1}x_1,y_1\big]. \end{aligned}$$

Thus

$$\Big[\big(\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{S}) - z\big)^{-1}x_1,y_1\Big] = \big[(\tilde{S}_1 - z)^{-1}x_1,y_1\big].$$

We come to the proof of item (ii). Under the assumption stated in item (ii), (4.15) holds and hence the choice of $\tilde{\beta}$ in the above proof can be made such that $\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{\beta}) = \tilde{\beta}_1$. From this it follows that

$$\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{S}) = F_\mu\big(\mathfrak{F}^{\mathcal{B}}_{\tilde{\mathcal{A}}|\tilde{\mathcal{A}}[-]\mathcal{C}}(\tilde{\beta})\big) = F_\mu(\tilde{\beta}_1) = \tilde{S}_1.$$

□

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