An Inverse Spectral Theorem for Kreĭn strings with a negative eigenvalue

Harald Woracek

Abstract

A string is a pair $(L, \mathfrak{m})$ where $L \in [0, \infty]$ and $\mathfrak{m}$ is a positive, possibly unbounded, Borel measure supported on $[0, L]$; we think of L as the length of the string and of $\mathfrak{m}$ as its mass density. To each string a differential operator acting in the space $L^2(\mathfrak{m})$ is associated. Namely, the Kreĭn-Feller differential operator $-D_{\mathfrak{m}} D_x$; its eigenvalue equation can be written, e.g., as

$$f'(x) + z \int_0^L f(y)\, d\mathfrak{m}(y) = 0,\ x \in \mathbb{R}, \qquad f'(0-) = 0\,.$$

A positive Borel measure τ on $\mathbb{R}$ is called a (canonical) spectral measure of the string $\mathrm{S}[L, \mathfrak{m}]$, if there exists an appropriately normalized Fourier transform of $L^2(\mathfrak{m})$ onto $L^2(\tau)$.

In order that a given positive Borel measure τ is a spectral measure of some string, it is necessary that:

- $\int_{\mathbb{R}} \frac{d\tau(\lambda)}{1+|\lambda|} < \infty$.
- Either $\operatorname{supp} \tau \subseteq [0, \infty)$, or τ is discrete and has exactly one point mass in $(-\infty, 0)$.

It is a deep result, going back to M.G.Kreĭn in the 1950's, that each measure with $\int_{\mathbb{R}} \frac{d\tau(\lambda)}{1+|\lambda|} < \infty$ and $\operatorname{supp} \tau \subseteq [0, \infty)$ is a spectral measure of some string, and that this string is uniquely determined by τ. The question remained open, which conditions characterize whether a measure τ with $\operatorname{supp} \tau \not\subseteq [0, \infty)$ is a spectral measure of some string. In the present paper, we answer this question. Interestingly, the solution is much more involved than the first guess might suggest.

AMS MSC 2010: Primary: 34L40, 34A55. Secondary: 46C20, 34L20, 37J99
Keywords: string, spectral measure, direct and inverse spectral theorem

1 Introduction and main result

A string is a pair $(L, \mathfrak{m})$ where $L \in [0, +\infty]$ and $\mathfrak{m}$ is a positive (possibly unbounded) Borel measure on $\mathbb{R} \cup \{+\infty\}$ with

$$\begin{aligned} &\operatorname{supp} \mathfrak{m} \subseteq [0, L], \quad \mathfrak{m}([0, x]) < \infty,\ x \in [0, L)\,, \\ &\mathfrak{m}(\{L\}) < \infty, \quad \mathfrak{m}(\{L\}) = 0 \text{ if } L + \mathfrak{m}([0, L]) = +\infty\,. \end{aligned}$$

We refer to L as the length of the string, to the function $m(x) := \mathfrak{m}((-\infty, x))$, $x \in \mathbb{R}$, as its mass distribution function, to the number $M := \mathfrak{m}([0, L])$ as its total mass, and denote the string given by L and $\mathfrak{m}$ as $\mathrm{S}[L, \mathfrak{m}]$. Throughout this paper we assume in addition that

$$\inf \operatorname{supp} \mathfrak{m} = 0, \quad \sup \operatorname{supp} \mathfrak{m} = L\,,$$

meaning that the string has two heavy endpoints, i.e. cannot start or end with an interval free of mass.

Given a string $\mathrm{S}[L,\mathfrak{m}]$, we consider the eigenvalue equation of the Kreĭn-Feller differential operator $-D_{\mathfrak{m}}D_x$. Written in the form of an integral boundary value problem it reads as

$$\begin{cases} f(x) - f(0) + z \displaystyle\int_{[0,x]} (x-y) f(y)\, d\mathfrak{m}(y) = 0, \quad x \in \mathbb{R}\,, \\ f'(0-) = 0 \end{cases} \tag{1.1}$$

see, e.g., [7, §1]. Thereby, $z \in \mathbb{C}$ is the eigenvalue parameter.

The Kreĭn-Feller differential operator arises when Fourier's method is applied to the partial differential equation

$$\frac{\partial}{\partial m(s)}\left(\frac{\partial v(s,t)}{\partial s}\right) - \frac{\partial^2}{\partial t^2} v(s,t) = 0\,,$$

which describes the vibrations of a string with a free left endpoint, which is stretched with unit tension on the interval $[0,L)$, and whose total mass on the interval $[0,x]$ equals $\mathfrak{m}([0,x])$. If the distribution function is sufficiently smooth, the boundary value problem (1.1) can be rewritten as a Sturm-Liouville equation. Conversely, for most potentials, the one-dimensional Schrödinger operator on a finite interval or on the half-line can be rewritten as a string equation.

Let a string $\mathrm{S}[L,\mathfrak{m}]$ be given, and denote by $\varphi(x,z)$ the unique solution of the integral equation in (1.1) with boundary values

$$\varphi(0,z) = 1,\ \varphi'(0-,z) = 0\,.$$

We call a positive Borel measure τ on $\mathbb{R}$ a spectral measure of $\mathrm{S}[L,\mathfrak{m}]$, if the map

$$f(x) \mapsto \hat{f}(\lambda) := \int_{[0,L]} f(x)\varphi(x,\lambda)\, d\mathfrak{m}(x)$$

is a unitary operator of the space $L^2(\mathfrak{m})$ onto the space $L^2(\tau)$. In this case, one speaks of this map as the Fourier transform corresponding to the spectral measure τ. Sometimes, one calls measures τ with this property also orthogonal (or canonical) spectral measures; speaking of just a 'spectral measure' then means that the map $\hat{.}$ is an isometry of $L^2(\mathfrak{m})$ onto some closed subspace of $L^2(\tau)$. However, we do not deal with non-orthogonal spectral measures, and hence there is no need to distinguish notationally.

It is a fundamental problem to describe the totality $\mathfrak{T}_{\mathrm{S}[L,\mathfrak{m}]}$ of all spectral measures of a given string. It turns out that the set $\mathfrak{T}_{\mathrm{S}[L,\mathfrak{m}]}$ either contains exactly one or infinitely many elements. From the viewpoint of mechanical interpretation, spectral measures being supported on the nonnegative semi-axis are of particular importance; denote the totality of all such measures by $\mathfrak{T}^+_{\mathrm{S}[L,\mathfrak{m}]}$. It turns out that $\mathfrak{T}^+_{\mathrm{S}[L,\mathfrak{m}]}$ is always nonempty and contains either exactly one or infinitely many elements. The measures belonging to $\mathfrak{T}_{\mathrm{S}[L,\mathfrak{m}]}$ or $\mathfrak{T}^+_{\mathrm{S}[L,\mathfrak{m}]}$, respectively, can be described in terms of their Cauchy transform and the fundamental solutions of the integral equation in (1.1); we recall the details in the next section, cf. 2.13–2.15.

As we just said, a description of the totality of all spectral measures (spectral measures with nonnegative support, respectively) of *one given* string is available. The question arises to describe the totality of all measures (measures with nonnegative support, respectively) which are a spectral measure of *some* string. Some necessary conditions are easily found, cf. [7].

1.1. *Necessary conditions:* *Let τ be a positive Borel measure on $\mathbb{R}$, and assume that τ is the spectral measure of some string* $\mathrm{S}[L,\mathfrak{m}]$. *Then the following hold:*

(*i*) *We have*
$$\int_{\mathbb{R}} \frac{d\tau(\lambda)}{1+|\lambda|} < \infty\,. \tag{1.2}$$

(*ii*) *Either* $\operatorname{supp}\tau \subseteq [0,\infty)$, *or the set* $(-\infty,0)\cap\operatorname{supp}\tau$ *contains exactly one point. In the second case,* τ *must be a discrete measure having no mass at the point* 0.

The following deep result goes back to M.G.Kreĭn in the 1950's, cf. [19], [20]. A proof was published only much later in the paper [7] by I.S.Kac and M.G.Kreĭn.

1.2. *Inverse Spectral Theorem:* *Let τ be a positive Borel measure on $\mathbb{R}$. If τ satisfies* (1.2) *and* $\operatorname{supp}\tau\subseteq[0,\infty)$, *then there exists a unique string* $\mathrm{S}[L,\mathfrak{m}]$ *such that τ is a spectral measure of* $\mathrm{S}[L,\mathfrak{m}]$.

It is our aim in this paper to complete the picture by giving a characterization of all positive Borel measures on $\mathbb{R}$ with $\operatorname{supp}\tau\not\subseteq[0,\infty)$ which are the spectral measure of some string, cf. Theorem 1.5. Interestingly, the situation turns out much more complicated than the first guess might suggest.

To establish this characterization, we use Pontryagin space theory. Very roughly speaking, the following two observations lead to the desired result: One, a measure τ whose support intersects $(-\infty,0)$ is a spectral function of some string if and only if the indefinite canonical system whose Weyl coefficient is the generalized Nevanlinna function
$$Q(z) := z\int_{\mathbb{R}} \frac{d\tau(\lambda)}{\lambda - z^2}$$
has a certain, very particular, structure. This can be shown using the theory developed in [9] and [16]. Two, appearance of the relevant structure is characterized by asymptotic conditions on locations and weights of the point masses of τ. This can be done using some results given in [26].

We find the present result a particularly appealing demonstration of the power of indefinite methods; we start with a 'classical question' (about the spectral theory of a differential operator in a Hilbert space), and arrive at a 'classical answer' (a characterization in terms of the asymptotics of point masses). Our proof, however, relies heavily on Pontryagin space methods, and a proof without making a detour through the indefinite world is, at present, not known to us.

The content of this paper is arranged in five sections. In the second part of this introductory section, we formulate and discuss our main result, the characterization Theorem 1.5. In Section 2, we briefly recall some known facts about strings and indefinite canonical systems, and provide some preliminary results which are used later on. The next two sections are devoted to the proof of Theorem 1.5; in Section 3 we investigate the direct spectral problem, in Section 4 we settle the inverse problem. Understanding the direct problem is already a major effort; all phenomena and ideas become clear, and all required technique has to be employed. For the inverse problem, we show that the arguments used to settle the direct problem can be reversed. Finally, in Section 5, we discuss the interaction of the various asymptotic conditions appearing in Theorem 1.5, providing examples for different possible behaviour.

Our main reference concerning the theory of strings is [7]; for a vast collection of literature, we refer to [5]. Much of the theory of strings can be deduced from the theory of two-dimensional canonical systems, since the string equation (1.1) can be rewritten as a canonical system. This fact is also a starting point for our present considerations. A detailed treatment of the relation between strings and canonical systems is given in [10]. Let us also refer to [22], where this relation is investigated starting from a one-dimensional Schrödinger operator. For the classical spectral theory of canoncial systems see e.g. [2] or [4]. Finally, let us mention that strings are also often viewed from a more probabilistic viewpoint, see [3].

Formulation of the main result.
In order to state the characterization we are aiming for, we have to introduce one notation.

1.3 Definition. Let $\xi_1, \xi_2, \xi_3, \ldots$ be a (finite or infinite, but nonempty) sequence of real numbers with

$$0 < \xi_1 < \xi_2 < \xi_3 < \ldots ,$$

and assume that

$$\sum_k \frac{1}{\xi_k} < \infty . \tag{1.3}$$

Then we denote by $\Gamma : \mathbb{C} \to \mathbb{C}$ the function

$$\Gamma(z) := \prod_k \Big(1 - \frac{z}{\xi_k}\Big) .$$

Due to (1.3), this product converges and represents an entire function.

Let, in addition to the points ξ_k, a (accordingly finite or infinite) sequence of positive real numbers $\sigma_1, \sigma_2, \sigma_3, \ldots$ be given, and assume that

$$\sum_k \xi_k^{-3} \frac{1}{\Gamma'(\xi_k)^2 \sigma_k} < \infty . \tag{1.4}$$

Then we denote by $\Xi : (-\infty, 0) \to (0, \infty)$ the function

$$\Xi(x) := \Big[\sum_k \frac{(-x)\Gamma(x)^2}{(1 - \frac{x}{\xi_k})^2} \cdot \xi_k^{-3} \frac{1}{\Gamma'(\xi_k)^2 \sigma_k} \Big]^{-1} . \tag{1.5}$$

//

Since, for each fixed $x < 0$,

$$\frac{(-x)\Gamma(x)^2}{(1 - \frac{x}{\xi_k})^2} < (-x)\Gamma(x)^2 ,$$

and the series (1.4) converges, also the series in the definition (1.5) of Ξ converges. Clearly, its sum is positive, and hence Ξ is well-defined.

Let us note some simple properties of the function Ξ.

1.4 Remark. The function Ξ is a continuous and strictly increasing bijection of $(-\infty,0)$ onto $(0,\infty)$.

To see this, note that for $y<x<0$ always

$$0<\frac{(-x)\Gamma(x)^2}{(1-\frac{x}{\xi_k})^2}=(-x)\prod_{n\neq k}\Big(1-\frac{x}{\xi_n}\Big)^2<(-y)\prod_{n\neq k}\Big(1-\frac{y}{\xi_n}\Big)^2=\frac{(-y)\Gamma(y)^2}{(1-\frac{y}{\xi_k})^2}\,.$$

This already shows that $\Xi(x)$ is strictly increasing. Moreover, for each fixed y, we have obtained a uniform bound for $(-x)\Gamma(x)^2(1-\frac{x}{\xi_k})^{-2}$, $x\in[y,0)$. The bounded convergence theorem thus applies and shows that Ξ is continuous and satisfies

$$\lim_{x\nearrow 0}\Xi(x)=+\infty\,.$$

Finally, since $\frac{(-x)\Gamma(x)^2}{(1-\frac{x}{\xi_k})^2}\geq -x$, each summand of series in (1.5) tends to $+\infty$ if $x\to-\infty$. All summands are nonnegative, and it follows that

$$\lim_{x\searrow-\infty}\Xi(x)=0\,.$$

//

1.5 Theorem. *Let τ be a positive Borel measure on $\mathbb{R}$ with $\operatorname{supp}\tau\not\subseteq[0,\infty)$ and $\operatorname{supp}\tau\cap[0,\infty)\neq\emptyset$.*

If τ is a spectral measure of some string $\mathrm{S}[L,\mathfrak{m}]$, then the following conditions (SM_1)–(SM_7) *hold.*

($\mathbf{SM_1}$) *The set $\operatorname{supp}\tau\cap(-\infty,0)\neq\emptyset$ contains exactly one point.*

($\mathbf{SM_2}$) *The measure τ is discrete, and has no point mass at* 0.

Write

$$\operatorname{supp}\tau=\{\xi\}\cup\{\xi_1,\xi_2,\xi_3,\dots\}$$

with $\xi<0<\xi_1<\xi_2<\xi_3<\dots$, and denote by σ and $\sigma_1,\sigma_2,\sigma_3,\dots$ the weights of the point masses of τ at the points ξ and $\xi_1,\xi_2,\xi_3,\dots$, respectively.

($\mathbf{SM_3}$)

$$\sum_k\frac{\sigma_k}{\xi_k}<\infty\,.$$

($\mathbf{SM_4}$) *The limit $\lim_{k\to\infty}\frac{k^2}{\xi_k}$ exists in $[0,\infty)$.*

($\mathbf{SM_5}$)

$$\sum_k\xi_k^{-3}\frac{1}{\Gamma'(\xi_k)^2\sigma_k}<\infty\,.$$

($\mathbf{SM_6}$) $0<\sigma\leq\Xi(\xi)$.

($\mathbf{SM_7}$) *If $\sigma=\Xi(\xi)$, then*

$$\sum_k\xi_k^{-2}\frac{1}{\Gamma'(\xi_k)^2\sigma_k}=\infty\,.$$

Thereby, the second inequality in (SM_6) *is strict if and only if* $\mathrm{S}[L,\mathfrak{m}]$ *is regular.*

Conversely, if τ satisfies (SM_1)–(SM_7)*, then there exists a unique string* $\mathrm{S}[L,\mathfrak{m}]$*, such that τ is a spectral measure of this string.*

1.6 Remark. In the above theorem we have excluded the case that $\operatorname{supp}\tau \cap [0,\infty) = \emptyset$. This case, however, is easily settled by explicit computation. It turns out that a measure τ with $\operatorname{supp}\tau \cap [0,\infty) = \emptyset$ is a spectral measure of a string, if and only if $\operatorname{supp}\tau$ consists of only one point.

If τ has this property, say $\operatorname{supp}\tau = \{\xi\}$ with some $\xi < 0$, then τ is the spectral measure of the string $\mathrm{S}[L,\mathfrak{m}]$ where $L := 0$ and $\mathfrak{m}(\{L\}) := \tau(\{\xi\})^{-1}$. It is, in the description 2.13, induced by the parameter $\gamma := \xi \cdot \tau(\{\xi\})^{-1}$. //

In view of this remark, we exclude once and for all measures τ with $\operatorname{supp}\tau \cap [0,\infty) = \emptyset$ from our considerations.

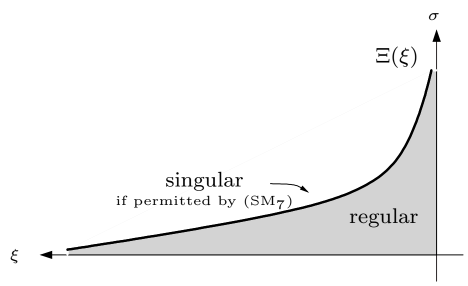
Let us point out some, rather astonishing, differences between the situations '$\operatorname{supp}\tau \subseteq [0,\infty)$' and $\operatorname{supp}\tau \not\subseteq [0,\infty)$'.

1.7 Remark.
Assume that a discrete measure τ with $\operatorname{supp}\tau \subseteq [0,\infty)$ is given, which is a spectral measure of some string.

(i^+) Changing the weights of a finite number of point masses of τ, produces a measure τ' which is again a spectral measure of some string.

Assume that a discrete measure τ with $\operatorname{supp}\tau \not\subseteq [0,\infty)$ is given, which is a spectral measure of some string. Contrasting the above property (i^+), we have:

(i^-) Keeping locations and weights of all point masses on the nonnegative real semi-axis fixed, the weight σ of the point mass located at $\xi \in (-\infty, 0)$ cannot be changed arbitrarily so that the new measure will still be a spectral measure of some string. By (SM_6), the weight σ must not exceed $\Xi(\xi)$. We can picture the situation as



Keeping location and weight of all point masses fixed with one exception ξ_k on the positive real semi-axis, the situation is similar; the weight σ_k may only vary in a certain (finite or infinite) interval.

//

When thinking in terms of the asymptotics of the sequence of weights while location of point masses is fixed, we observe the following –intuitively formulated– phenomena.

1.8 Remark. Let a sequence $(\xi_k)_{k\in\mathbb{N}}$ of real numbers with $0 < \xi_1 < \xi_2 < \dots$ be given. Consider the band of 'allowed asymptotics' for the sequence of weights σ_k, such that the measure τ with

$$\operatorname{supp}\tau = \{\xi_1, \xi_2, \xi_3, \dots\} \subseteq [0,\infty),\ \tau(\{\xi_k\}) = \sigma_k\,,$$

is a spectral measure of some string. Then

(ii^+) The allowed band is bounded above by means of the convergence condition

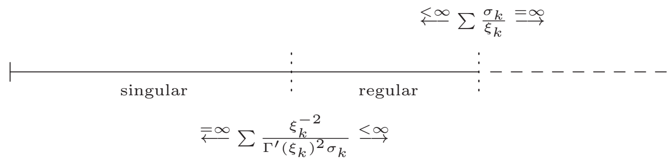
$$\sum_{k\in\mathbb{N}} \frac{\sigma_k}{\xi_k} < \infty\,. \tag{1.6}$$

It is not bounded away from zero.

(iii^+) The allowed band is divided in two parts by means of divergence/convergence of the series

$$\sum_{k\in\mathbb{N}} \xi_k^{-2} \frac{1}{\Gamma'(\xi_k)^2\sigma_k}\,. \tag{1.7}$$

The lower half (small growth of weights) corresponds exclusively to singular strings, the upper half (fast growth of weights) corresponds exclusively to regular strings.



Contrasting this, when asking for spectral measures of strings supported at the points ξ_k plus one point $\xi \in (-\infty, 0)$, we may say: Consider the band of 'allowed asymptotics' for the sequence of weights σ_k, such that there exists a spectral measure of some string with

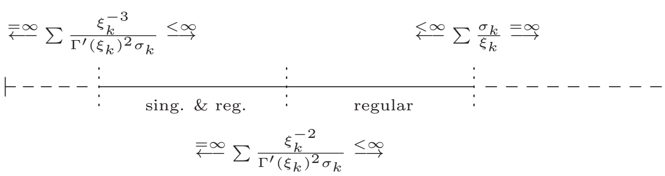
$$\operatorname{supp}\tau \not\subseteq [0,\infty), \qquad \operatorname{supp}\tau \cap [0,\infty) = \{\xi_1, \xi_2, \xi_3, \dots\},\ \tau(\{\xi_k\}) = \sigma_k\,.$$

Then

(ii^-) The allowed band is bounded above by the convergence condition (1.6), and bounded away from zero by the convergence condition

$$\sum_{k\in\mathbb{N}} \xi_k^{-3} \frac{1}{\Gamma'(\xi_k)^2\sigma_k} < \infty\,.$$

(iii^-) The allowed band is divided in two parts by means of divergence/convergence of the series (1.7). Each point in the lower half carries singular and regular strings, on the upper half we find only regular strings.



//

The division into two parts of the allowed band in (iii^+) and (iii^-) may degenerate in the sense that convergence of (1.6) implies divergence of (1.7). Moreover, and this is yet another peculiarity of the case '$\operatorname{supp}\tau \not\subseteq [0,\infty)$', the allowed band in ($ii^-$) may be empty. We provide some quantitative statements concerning this phenomenon later on in Section 5.

2 Preliminaries

This section is divided into four subsections. In the first two, we recall some definition and facts about matrix functions of the class $\mathcal{M}_{<\infty}$, generalized Nevanlinna functions, and maximal chains of matrices, especially the notion of their Weyl coefficients. Then we turn to canonical systems, recall their definition and their relation with maximal chains. Finally, we recall some results about strings, among them the precise description of the sets $\mathfrak{T}_{\mathrm{S}[L,\mathfrak{m}]}$ and $\mathfrak{T}^+_{\mathrm{S}[L,\mathfrak{m}]}$, and provide some preliminary statements which are needed in the following sections.

a. The classes $\mathcal{M}_{<\infty}$ and $\mathcal{N}_{<\infty}$.
If W is an entire 2×2-matrix valued function which satisfies $W(z)JW(\overline{z})^* = J$, $z\in\mathbb{C}$, let a kernel H_W be defined as

$$H_W(w,z) := \frac{W(z)JW(w)^* - J}{z-\overline{w}}, \quad z,w\in\mathbb{C}\,,$$

where $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. For $z=\overline{w}$ this formula has to be interpreted appropriately as a derivative, which is possible by analyticity.

2.1 Definition. Let $W=(w_{ij})_{i,j=1}^2$ be a 2×2-matrix valued function, and let $\kappa\in\mathbb{N}_0$. We write $W\in\mathcal{M}_\kappa$, if

- **(M1)** The entries w_{ij} of W are entire functions which take real values along the real line.
- **(M2)** We have $\det W(z)=1$, $z\in\mathbb{C}$, and $W(0)=I$.
- **(M3)** The kernel H_W has κ negative squares on $\mathbb{C}$.

Note here that the conditions (M1) and (M2) together imply $W(z)JW(\overline{z})^* = J$. //

We use the notation

$$\mathcal{M}_{<\infty} := \bigcup_{\nu\in\mathbb{N}\cup\{0\}} \mathcal{M}_\nu\,,$$

and write $\operatorname{ind}_- W=\kappa$ to express that $W\in\mathcal{M}_\kappa$. Each matrix $W\in\mathcal{M}_{<\infty}$ generates by means of the kernel H_W a reproducing kernel Pontryagin space whose elements are 2-vector valued entire functions, and we denote this space by $\mathfrak{K}(W)$.

A matrix of the class $\mathcal{M}_{<\infty}$ gives rise to a de Branges Pontragin space. Let $W=(w_{ij})_{i,j=1}^2\in\mathcal{M}_{<\infty}$ be given, and assume that the constant $(0,1)^T$ does not belong to $\mathfrak{K}(W)$. Then the projection of $\mathfrak{K}(W)$ onto its first component is an isometric isomorphism of $\mathfrak{K}(W)$ onto the de Branges-Pontryagin space $\mathfrak{P}(w_{11}-iw_{12})$, cf. [11, §8, §9].

Let us turn to the class $\mathcal{N}_{<\infty}$ of generalized Nevanlinna functions. If $q: D\to\mathbb{C}$ is an analytic function defined on some open subset D of the complex plane, we define a kernel N_q as

$$N_q(w,z) := \frac{q(z)-\overline{q(w)}}{z-\overline{w}}, \quad z,w\in D\,.$$

Again, for $z=\overline{w}$, this formula has to be interpreted appropriately.

2.2 Definition. Let q be a complex-valued function, and let $\kappa \in \mathbb{N}_0$. We write $q \in \mathcal{N}_\kappa$, if

(N1) q is real and meromorphic on $\mathbb{C} \setminus \mathbb{R}$.

(N2) The kernel N_q has κ negative squares on the domain of holomorphy of q.

//

Once more, we set $\mathcal{N}_{<\infty} := \bigcup_{\kappa \in \mathbb{N} \cup \{0\}} \mathcal{N}_\kappa$, and write $\operatorname{ind}_- q = \kappa$ to express that $q \in \mathcal{N}_{<\infty}$ belongs to $\mathcal{N}_\kappa$.

Matrices of the class $\mathcal{M}_{<\infty}$ give rise to generalized Nevanlinna functions by the following construction: For a 2×2-matrix valued function $W(z) = (w_{ij}(z))_{i,j=1}^2$ and a scalar function $\tau(z)$, we denote by $W \star \tau$ the scalar function

$$(W \star \tau)(z) := \frac{w_{11}(z)\tau(z) + w_{12}(z)}{w_{21}(z)\tau(z) + w_{22}(z)},$$

wherever this expression is defined. For the parameter $\tau = \infty$, we set $W \star \tau := w_{21}^{-1} w_{11}$. A straightforward computation shows that

$$(W_1 W_2) \star \tau = W_1 \star (W_2 \star \tau).$$

Rewriting the kernel $N_{W \star \tau}$ shows that $W \star \tau \in \mathcal{N}_{<\infty}$ whenever $W \in \mathcal{M}_{<\infty}$ and $\tau \in \mathcal{N}_{<\infty}$. In fact, $\operatorname{ind}_- W \star \tau \leq \operatorname{ind}_- W + \operatorname{ind}_- \tau$, cf. [15, §2e].

b. Maximal chains of matrices.

The aim of the notion of maximal chains of matrices is to model 'fundamental solutions of indefinite canonical systems'. However, maximal chains are first introduced axiomatically as objects of their own right, cf. [13].

2.3 Definition. Let $I \subseteq \mathbb{R}$ and $\omega : I \to \mathcal{M}_{<\infty}$. Then we call ω a maximal chain of matrices if the following axioms are satisfied:

(W1) The set I is of the form $\bigcup_{i=0}^n (\sigma_i, \sigma_{i+1})$ for some numbers $n \in \mathbb{N} \cup \{0\}$ and $\sigma_0, \ldots, \sigma_{n+1} \in \mathbb{R} \cup \{+\infty\}$ with $\sigma_0 < \sigma_1 < \ldots < \sigma_{n+1}$.

(W2) The function ω is not constant on any interval contained in I.

(W3) For all $s, t \in I$, $s \leq t$, the transfer matrix $\omega(s,.)^{-1}\omega(t,.)$ belongs to $\mathcal{M}_{<\infty}$, and

$$\operatorname{ind}_- \omega(t,.) = \operatorname{ind}_- \omega(s,.) + \operatorname{ind}_- \big[\omega(s,.)^{-1}\omega(t,.)\big].$$

(W4) Let $t \in I$ and $W \in \mathcal{M}_{<\infty}$, $W \neq I$. If $W^{-1}\omega(t,.) \in \mathcal{M}_{<\infty}$ and $\operatorname{ind}_- \omega(t,.) = \operatorname{ind}_- W + \operatorname{ind}_- W^{-1}\omega(t,.)$, then there exists a number $s \in I$ such that $W = \omega(s,.)$.

(W5) We have

$$\lim_{t \nearrow \sigma_{n+1}} \operatorname{tr} \frac{\partial}{\partial z} \omega(t,z) J \big|_{z=0} = +\infty.$$

If I is not connected, i.e. $n > 0$, there exist numbers $s, t \in (\sigma_n, \sigma_{n+1})$ such that $\omega(s,.)^{-1}\omega(t,.)$ is not a linear polynomial.

The points $\sigma_1, \ldots, \sigma_n$ are called the singularities of ω.

We call ω a finite maximal chain of matrices, if it satisfies

(W1$_f$) the set I is of the form $I = [\sigma_0, \sigma_{n+1}] \setminus \{\sigma_1, \ldots, \sigma_n\}$ for some numbers $n \in \mathbb{N} \cup \{0\}$ and $\sigma_0, \ldots, \sigma_{n+1} \in \mathbb{R}$ with $\sigma_0 < \sigma_1 < \ldots < \sigma_n < \sigma_{n+1}$,

and the conditions (W2)–(W4) above. //

The reader will recognise that the limit condition in (W5) means in some sense that at the endpoint σ_{n+1} Weyl's limit point case prevails; for a maximal chain ω the limit $\lim_{t \nearrow \sigma_{n+1}} \omega(t,.)$ does not exist. Contrasting this, for a finite maximal chain, $\lim_{t \nearrow \sigma_{n+1}} \omega(t,.) = \omega(\sigma_{n+1},.)$ locally uniformly on $\mathbb{C}$. We see that for finite maximal chains in some sense limit circle case takes place.

If ω is a maximal chain, at the left endpoint one has $\lim_{x \searrow \sigma_0} \omega(x,.) = I$, cf. [13, Lemma 3.5, (v)]. In view of this fact, we always extend ω to $I \cup \{\inf I\}$ by setting $\omega(\inf I) := I$. For finite maximal chains one always has $\lim_{x \searrow \sigma_0} \omega(x,.) = \omega(\sigma_0,.) = I$.

The function $t \mapsto \operatorname{ind}_- \omega(t,.)$ is constant on connected components of I and takes different values on different components, cf. [13, Lemma 3.5]. We denote

$$\operatorname{ind}_- \omega := \max_{t \in I} \operatorname{ind}_- \omega(t,.) \,.$$

It is obvious from (W5) that intervals where the transfer matrices $\omega(s,t)$ are linear polynomials play a special role. For $l, \phi \in \mathbb{R}$, set

$$W_{(l,\phi)}(z) := \begin{pmatrix} 1 - lz \sin\phi\cos\phi & lz\cos^2\phi \\ -lz\sin^2\phi & 1 + lz\sin\phi\cos\phi \end{pmatrix}$$

Let $\omega = (\omega(x,z))_{x \in I}$ be a (finite) maximal chain. Then a nonempty interval $(s,t) \subseteq I$ is called indivisible of type $\phi \in [0,\pi)$, if for all $s', t' \in (s,t)$

$$\omega(s',.)^{-1}\omega(t',.) = W_{(l(s',t'),\phi)} \,.$$

The number $\sup\{l(s',t') :\, s',t' \in (s,t), s' < t'\} \in (0,\infty]$ is called the length of the indivisible interval (s,t). If the intersection of two indivisible intervals is nonempty, then their types coincide and their union is again indivisible. Hence, each indivisible interval is contained in a maximal indivisible interval.

Let ω be a maximal chain. Then the limit ($\tau \in \mathbb{R} \cup \infty$)

$$q_\omega(z) := \lim_{t \nearrow \sigma_{n+1}} \omega(t,z) \star \tau$$

exists locally uniformly on $\mathbb{C} \setminus \mathbb{R}$ with respect to the chordal metric, and does not depend on the value of τ, cf. [12, Lemma 8.2]. We have $q_\omega \in \mathcal{N}_{<\infty}$, and $\operatorname{ind}_- q_\omega = \operatorname{ind}_- \omega$, cf. [12, Lemma 8.5]. The function q_ω is called the Weyl coefficient of the maximal chain ω.

It is an important fact that the assignment $\omega \mapsto q_\omega$ is in essence bijective (we do not go into details what 'in essence' means), cf. [12, Theorem 8.7].

2.4. Inverse Spectral Theorem/maximal chains: Let $q \in \mathcal{N}_{<\infty}$ be given. Then there exists a maximal chain of matrices ω, such that $q = q_\omega$. This chain is (essentially) unique. //

An analogue for finite maximal chains was shown in [12, Theorem 7.1].

2.5. Inverse Spectral Theorem/finite maximal chains: Let $W \in \mathcal{M}_{<\infty}$ be given. Then there exists a finite maximal chain $\omega : I \to \mathcal{M}_{<\infty}$, such that $W = W(\max I)$. This chain is (essentially) unique. //

c. Positive definite canonical systems.

A two-dimensional canonical (or Hamiltonian) system is an initial value problem of the form

$$y'(t) = zJH(t)y(t), \quad t \in [s_-, s_+),\ y(s_-) = y_0\,, \tag{2.1}$$

where z is a complex parameter and H is a 2×2-matrix valued function with $H(t) \geq 0$ for $t \in (s_-, s_+)$ a.e., which is locally integrable and does not vanish identically on any set of positive measure. The function H is called the Hamiltonian of the system (2.1). A Hamiltonian H is called regular, if $\int_{s_-}^{s_+} \operatorname{tr} H(t)\, dt < \infty$, and singular otherwise. One also speaks of Weyl's limit circle or limit point case instead of regular or singular, respectively.

To a Hamiltonian a chain of matrices is associated. Namely, the (transposed of) the fundamental solution of the canonical differential equation (2.1). Explicitly, this is the solution $\omega_H(t,z)$ of the initial value problem

$$\frac{\partial}{\partial t}\omega_H(t,z)J = z\omega_H(t,z)H(x),\ t \in [s_-, s_+), \quad \omega_H(s_-,z) = I\,.$$

If H is regular, ω_H is a finite maximal chain, if H is singular, ω_H is a maximal chain. In any case, $\operatorname{ind}_- \omega_H = 0$. We refer to the function $q_H := q_{\omega_H}$ as the Weyl coefficient of the singular Hamiltonian H, and to $W_H := \omega_H(\max I)$ as the monodromy matrix of the regular Hamiltonian H.

The following two facts are cornerstones in the spectral theory of canonical systems. For a proof see, e.g., [2] or [4].

2.6. Inverse Spectral Theorem/singular Hamiltonians: Let $q \in \mathcal{N}_0$ be given. Then there exists a singular Hamiltonian whose Weyl coefficient equals q. This Hamiltonian is (essentially) unique. //

2.7. Inverse Spectral Theorem/regular Hamiltonians: Let $W \in \mathcal{M}_0$ be given. Then there exists a regular Hamiltonian whose monodromy matrix equals W. This Hamiltonian is (essentially) unique. //

d. The indefinite analogue of canonical systems.

Corresponding to the indefinite generalizations $\mathcal{N}_{<\infty}$ and $\mathcal{M}_{<\infty}$ of the classes $\mathcal{N}_0$ of Nevanlinna functions and $\mathcal{M}_0$ of J-contractive matrix functions, respectively, the notion of a canonical system can be generalized to the indefinite (Pontryagin space) setting, cf. [14]. In order to state the definition of a general Hamiltonian, we first have to introduce some preliminary notation.

An interval (α, β) is called H-indivisible of type ϕ if ($\xi_\phi := (\cos\phi, \sin\phi)^T$)

$$H(t) = h(t)\xi_\phi\xi_\phi^T, \quad t \in (\alpha, \beta)\,,$$

where $h(t)$ is some scalar function that is positive almost everywhere.

With any Hamiltonian H a number $\Delta(H) \in \mathbb{N} \cup \{0, \infty\}$ is associated, cf. [14, Definition 3.1], which in some sense measures the growth of H towards L. For example, $\Delta(H) = 0$ means that $\int_0^L \operatorname{tr} H(t)\, dt < \infty$; or if $\int_0^L \operatorname{tr} H(t)\, dt = \infty$ and the interval (L_1, L) is H-indivisible for some $L_1 < L$, then $\Delta(H) = 1$.

Assume that $\int_0^L \operatorname{tr} H(t)\,dt = \infty$. The Hamiltonian H is said to satisfy the condition (HS) if the resolvents of one and hence of all self-adjoint extensions of the minimal operator $T_{\min}(H)$ associated with H on $[0, L)$ are Hilbert–Schmidt operators. In this case, the growth of H towards L, as measured by $\Delta(H)$, is extremal in one direction ξ_ϕ in the sense that, for a unique angle $\phi \in [0, \pi)$, we have

$$\int_0^L \xi_\phi^T H(t) \xi_\phi \, dt < \infty \,,$$

cf. [17, Theorem 2.4]. This angle will be denoted by $\phi(H)$.

Let H be a function defined on an interval (L_-, L_+) which takes real and non-negative 2×2-matrices as values, is locally integrable on (L_-, L_+) and does not vanish on any set of positive measure. Fix $\alpha \in (L_-, L_+)$, and put $H_+(t) := H(\alpha + t)$, $t \in [0, L_+ - \alpha)$, and $H_-(t) := H(\alpha - t)$, $t \in [0, \alpha - L_-)$. Then $H_\pm$ are Hamiltonians. We say that H is in the limit point/circle case at L_+ or L_-, if H_+ or H_-, respectively, has this property. The conditions (HS_+) and (HS_-) and the numbers $\Delta_\pm(H)$ and $\phi_\pm(H)$ are defined similarly. These numbers do not depend on the choice of α. In the following we also call such a function H defined on an open interval (L_-, L_+) a Hamiltonian.

2.8 Definition. A general Hamiltonian $\mathfrak{h}$ is a collection of data of the following kind:

- (i) $n \in \mathbb{N} \cup \{0\}$, $\sigma_0, \dots, \sigma_{n+1} \in \mathbb{R} \cup \{\pm\infty\}$ with $\sigma_0 < \sigma_1 < \ldots < \sigma_{n+1}$,
- (ii) Hamiltonians H_i, $i = 0, \dots, n$, defined on the respective intervals (σ_i, σ_{i+1}),
- (iii) numbers $\ddot{o}_1, \dots, \ddot{o}_n \in \mathbb{N} \cup \{0\}$ and $b_{i,1}, \dots, b_{i,\ddot{o}_i+1} \in \mathbb{R}$, $i = 1, \dots, n$, with $b_{i,1} \neq 0$ in the case $\ddot{o}_i \geq 1$,
- (iv) numbers $d_{i,0}, \dots, d_{i,2\Delta_i - 1} \in \mathbb{R}$, $i = 1, \dots, n$, where $\Delta_i := \max\{\Delta_+(H_{i-1}), \Delta_-(H_i)\}$,
- (v) a finite subset E of $\{\sigma_0, \sigma_{n+1}\} \cup \bigcup_{i=0}^n (\sigma_i, \sigma_{i+1})$,

which is assumed to be subject to the following conditions:

(H1) H_0 is in the limit circle case at σ_0 and, if $n \geq 1$, in the limit point case at σ_1. H_i is in the limit point case at both endpoints σ_i and σ_{i+1}, $i = 1, \dots, n-1$. If $n \geq 1$, then H_n is in the limit point case at σ_n.

(H2) For $i = 1, \dots, n-1$ the interval (σ_i, σ_{i+1}) is not H_i-indivisible. If H_n is in the limit point case at σ_{n+1}, then also (σ_n, σ_{n+1}) is not H_n-indivisible.

(H3) We have $\Delta_i < \infty$, $i = 1, \dots, n$. Moreover, H_0 satisfies (HS_+), H_i satisfies (HS_-) and (HS_+) for $i = 1, \dots, n-1$, and H_n satisfies (HS_-).

(H4) We have $\phi_+(H_{i-1}) = \phi_-(H_i)$, $i = 1, \dots, n$.

(H5) Let $i \in \{1, \dots, n\}$. If for some $\epsilon > 0$ the interval $(\sigma_i - \epsilon, \sigma_i)$ is H_{i-1}-indivisible and the interval $(\sigma_i, \sigma_i + \epsilon)$ is H_i-indivisible, then $d_1 = 0$. If additionally $b_{i,1} = 0$, then also $d_0 < 0$.

(E1) $\sigma_0, \sigma_{n+1} \in E$, and $E \cap (\sigma_i, \sigma_{i+1}) \neq \varnothing$ for $i = 1, \dots, n-1$. If H_n is in the limit point case at σ_{n+1}, then also $E \cap (\sigma_n, \sigma_{n+1}) \neq \varnothing$. Let $i \in \{0, \dots, n\}$; if (α, σ_{i+1}) or (σ_i, α) is a maximal H_i-indivisible interval, then $\alpha \in E$.

(E2) No point of E is an inner point of an indivisible interval.

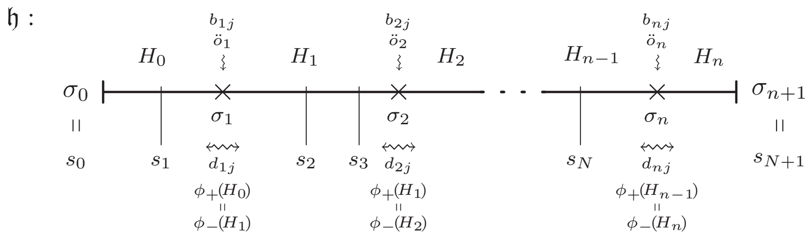
The number

$$\operatorname{ind}_- \mathfrak{h} := \sum_{i=1}^{n} \Big(\Delta_i + \Big[\frac{\ddot{o}_i}{2}\Big] \Big) + \big| \big\{ 1 \leq i \leq n : \ddot{o}_i \text{ odd}, b_{i,1} > 0 \big\} \big|$$

is called the negative index of the general Hamiltonian $\mathfrak{h}$. We say that $\mathfrak{h}$ is singular (or in the limit point case), or regular (or in the limit circle case), if H_n has the respective property at σ_{n+1}. //

Intuitively, this notion can be understood as follows: its purpose is to model an indefinite canonical system. So we deal with the differential equation $f' = zJHf$ given on an interval (σ_0, σ_{n+1}) which involves some kind of singularities located at the points σ_i, $i = 1, \dots, n$. Condition (H1) says that the differential equation is regular at σ_0, so that the initial value problem at σ_0 is well-defined, but that $\sigma_1, \dots, \sigma_n$ actually are singularities. Moreover, and this is the condition (H2), two adjacent singularities σ_i and σ_{i+1} must be separated by more than just a single indivisible interval. The meaning of (H3) is that the growth of H_i towards a singularity is not too fast. Moreover, (H4) is an interface condition at σ_i. A singularity itself contributes to the canonical system in two ways. First, a contribution concentrated inside the singularity; passing the singularity influences the solution. This is modelled by the parameters $\ddot{o}_i, b_{ij}$. Secondly, the numbers $d_{i,0}, \dots, d_{i,2\Delta_i - 1}$ model the part of this singularity which is in interaction with the local behaviour around σ_i. The elements of E in the vicinity of σ_i determine quantitatively what local here means. The freedom of this interaction is, by the first part of (H5), restricted if to both sides of σ_i indivisible intervals adjoin. The possibility that on both sides of σ_i indivisible intervals adjoin and at the same time $b_{i,1} = 0$, can occur by the second part of (H5) only in the case of 'indivisible intervals of negative length', the simplest possible kind of singularity.

We could picture the situation as follows ($E = \{s_0, \dots, s_{N+1}\}$):



2.9 Remark. In our present considerations, only general Hamiltonians with negative index 1 appear. Let us explicitly state which data is needed to obtain an object of this kind. In order to have $\operatorname{ind}_- \mathfrak{h} = 1$, the general Hamiltonian $\mathfrak{h}$ has to consists of: two Hamiltonians H_0 and H_1 defined on intervals (σ_0, σ_1) and (σ_1, σ_2), respectively, which are subject to the conditions of Definition 2.8 and satisfy $\Delta = 1$; a number $\ddot{o} \in \{0,1\}$; a number $b_1 \in \mathbb{R}$ which, if $\ddot{o} = 1$, is negative; in case $\ddot{o} = 1$ another number $b_2 \in \mathbb{R}$; real numbers d_0, d_1; a finite subset E, which can be chosen of the form $\{s_0, s_1\}$ with $s_0 = \sigma_0$, $s_1 \in (\sigma_1, \sigma_2)$ or of the form $\{s_0, s_1, s_2\}$ with $s_0 = \sigma_0$, $s_1 \in (\sigma_0, \sigma_1)$, $s_2 \in (\sigma_1, \sigma_2)$. //

To a (regular) general Hamiltonian $\mathfrak{h}$ a (finite) maximal chain of matrices $\omega_{\mathfrak{h}}$ is associated, cf. [15, Theorem 5.1], and conversely each (finite) maximal chain gives rise to a (regular) general Hamiltonian, cf. [16, Theorems 1.5, 1.6]. If $\mathfrak{h}$ is singular, we call the function $q_{\mathfrak{h}} := q_{\omega_{\mathfrak{h}}}$ the Weyl coefficient of $\mathfrak{h}$. If $\mathfrak{h}$ is regular, we refer to $W_{\mathfrak{h}} := \omega_{\mathfrak{h}}(\max I)$ as the monodromy matrix of $\mathfrak{h}$.

The chain $\omega_{\mathfrak{h}}$ can be considered as the fundamental solution of the indefinite canonical system. The following two statements have been shown as [16, Theorems 1.3, 1.4], and are the indefinite analogues of 2.6 and 2.7.

2.10. Inverse Spectral Theorem/singular general Hamiltonians: Let $q \in \mathcal{N}_{<\infty}$ be given. Then there exists a singular general Hamiltonian whose Weyl coefficient equals q. This general Hamiltonian is (essentially) unique. //

2.11. Inverse Spectral Theorem/regular general Hamiltonians: Let $W \in \mathcal{M}_{<\infty}$ be given. Then there exists a regular general Hamiltonian whose monodromy matrix equals W. This general Hamiltonian is (essentially) unique. //

e. Strings and their spectral measures.

The description of $\mathfrak{T}_{\mathrm{S}[L,\mathfrak{m}]}$ and $\mathfrak{T}^+_{\mathrm{S}[L,\mathfrak{m}]}$ via their Cauchy transforms, which has been established in [7], plays a crucial role in our considerations. Different behaviour shows up, depending on the speed of growth of the mass distribution function m, and whether the length L of the string is finite or infinite.

2.12 Definition. Let a string $\mathrm{S}[L,\mathfrak{m}]$ be given.

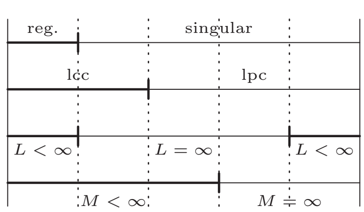
(*i*) The string $\mathrm{S}[L,\mathfrak{m}]$ is called regular, if $L < \infty$ and $M < \infty$. Otherwise, it is called singular.

(*ii*) We say that Weyl's limit circle case takes place for the string $\mathrm{S}[L,\mathfrak{m}]$, if $\int_{[0,L]} x^2\, d\mathfrak{m}(x) < \infty$, i.e. if the string has a finite energy moment. Otherwise, we speak of limit point case.

//

Apparently, these notions are not independent of each other. For example, for a regular string certainly limit circle case prevails. The dependencies being present can be pictured as follows:

$\mathrm{S}[L,\mathfrak{m}]$:



Throughout the following, denote by $\varphi(x,z)$ and $\psi(x,z)$ the unique solutions of the integral equation in (1.1) with boundary values

$$\varphi(0,z) = 1,\ \varphi'(0-,z) = 0, \qquad \psi(0,z) = 0,\ \psi'(0-,z) = 1\,. \tag{2.2}$$

2.13. Description of spectral measures; regular: Let a string $\mathrm{S}[L,\mathfrak{m}]$ be given, and assume that $\mathrm{S}[L,\mathfrak{m}]$ is regular. Then the limits

$$\varphi(L-,z) := \lim_{x\nearrow L} \varphi(x,z), \quad \varphi'(L-,z) := \lim_{x\nearrow L} \varphi'(x,z)\,,$$
$$\psi(L-,z) := \lim_{x\nearrow L} \psi(x,z), \quad \psi'(L-,z) := \lim_{x\nearrow L} \psi'(x,z)\,,$$

exist and represent entire functions which take real values along the real axis. In case $L=0$, we understand these limits as $\varphi(L-,z)=\psi'(L-,z)=1$ and $\varphi'(L-,z)=\psi(L-,z)=0$. The matrix function

$$W_{\mathrm{S}[L,\mathfrak{m}]}(z) := \begin{pmatrix} \psi'(L-,z) & \psi(L-,z)-L\psi'(L-,z) \\ \varphi'(L-,z) & \varphi(L-,z)-L\varphi'(L-,z) \end{pmatrix} \cdot W_{(\mathfrak{m}(\{L\})(1+L^2),\operatorname{Arccot} L)}$$

belongs to the class $\mathcal{M}_0$.

If τ is a spectral measure of $\mathrm{S}[L,\mathfrak{m}]$, then $\int_{\mathbb{R}} \frac{d\tau(\lambda)}{1+|\lambda|} < \infty$ and there exists a unique parameter $\gamma \in \mathbb{R}\cup\{\infty\}$, such that

$$\int_{\mathbb{R}} \frac{d\tau(\lambda)}{\lambda - z} = W_{\mathrm{S}[L,\mathfrak{m}]}(z) \star \gamma\,. \tag{2.3}$$

Conversely, if $\gamma \in \mathbb{R}\cup\{\infty\}$ with $\gamma \neq L$ in case $\mathfrak{m}(\{L\})>0$, then there exists a unique spectral measure τ of $\mathrm{S}[L,\mathfrak{m}]$, such that (2.3) holds.

If τ and γ are related by (2.3), then $\operatorname{supp}\tau \subseteq [0,\infty)$ if and only if $\gamma \geq L$.

//

2.14. Description of spectral measures; singular/lcc: Let a string $\mathrm{S}[L,\mathfrak{m}]$ be given, and assume that $\mathrm{S}[L,\mathfrak{m}]$ is singular, but still in limit circle case. Then the limits

$$\tilde\varphi(z) := \lim_{x\nearrow L} \big[\varphi(x,z) - x\varphi'(x,z)\big], \quad \varphi'(L-,z) := \lim_{x\nearrow L} \varphi'(x,z)$$
$$\tilde\psi(z) := \lim_{x\nearrow L} \big[\psi(x,z) - x\psi'(x,z)\big], \quad \psi'(L-,z) := \lim_{x\nearrow L} \psi'(x,z)$$

exist and represent entire functions which take real values along the real axis. The matrix function

$$W_{\mathrm{S}[L,\mathfrak{m}]}(z) := \begin{pmatrix} \psi'(L-,z) & \tilde\psi(z) \\ \varphi'(L-,z) & \tilde\varphi(z) \end{pmatrix}$$

belongs to the class $\mathcal{M}_0$.

If τ is a spectral measure of $\mathrm{S}[L,\mathfrak{m}]$, then $\int_{\mathbb{R}} \frac{d\tau(\lambda)}{1+|\lambda|} < \infty$ and there exists a unique parameter $\gamma \in \mathbb{R}\cup\{\infty\}$, such that (2.3) holds. Conversely, if $\gamma \in \mathbb{R}\cup\{\infty\}$, then there exists a unique spectral measure τ of $\mathrm{S}[L,\mathfrak{m}]$, such that (2.3) holds.

If τ and γ are related by (2.3), then $\operatorname{supp}\tau \subseteq [0,\infty)$ if and only if $\gamma = \infty$.

//

2.15. Description of spectral measures; lpc: Let a string $\mathrm{S}[L,\mathfrak{m}]$ be given, and assume that limit point case takes place. Then there exists exactly one spectral measure τ of $\mathrm{S}[L,\mathfrak{m}]$. This measure satisfies $\int_{\mathbb{R}} \frac{d\tau(\lambda)}{1+|\lambda|} < \infty$ and $\operatorname{supp}\tau \subseteq [0,\infty)$. Its Cauchy transform can be obtained, independently of $\gamma \in \mathbb{R}\cup\{\infty\}$, as the limit

$$\int_{\mathbb{R}} \frac{d\tau(\lambda)}{\lambda - z} = \lim_{x\nearrow L} \begin{pmatrix} \psi'(x,z) & \psi(x,z) - x\psi'(x,z) \\ \varphi'(x,z) & \varphi(x,z) - x\varphi'(x,z) \end{pmatrix} \star \gamma\,.$$

//

In the proof of Theorem 1.5 we repeatedly use the following fact which can be extracted from [10], [24], and [6]. However, it is not a straight corollary of the results shown in these papers, since we consider not only principal spectral functions, and since we require a string to have heavy endpoints but permit a concentrated mass at the right endpoint. For these reasons, we provide an explicit proof.

2.16 Lemma. *Let $W = (w_{ij})_{i,j=1}^2 \in \mathcal{M}_0$, $W \neq I$, and let H be the regular Hamiltonian with monodromy matrix W. Then the following are equivalent.*

(*i*) *There exists a string* $\mathrm{S}[L, \mathfrak{m}]$ *which is in the limit circle case, such that* $W = W_{\mathrm{S}[L,\mathfrak{m}]}$.

(*ii*) *The Hamiltonian H can be parameterized such that it takes the form*

$$H(t) = \begin{pmatrix} v(t)^2 & v(t) \\ v(t) & 1 \end{pmatrix}, \quad t \in (0, T),$$

with a finite and positive number T, and a nondecreasing, left-continuous, real-valued function v with

$$\int_0^T v(t)^2\, dt < \infty, \qquad \lim_{x \searrow v} v(t) = 0.$$

(*iii*) *We have*

$$z\frac{w_{11}(z^2)}{w_{21}(z^2)} \in \mathcal{N}_0, \qquad \lim_{x\to-\infty} \frac{w_{11}(x)}{w_{21}(x)} = 0, \qquad \lim_{y\to+\infty} \frac{1}{y}\frac{w_{12}(iy)}{w_{11}(iy)} = 0.$$

Proof.
'*(i) ⇒ (ii)*': Consider the function $\hat{m} : (-\infty, \infty] \to [0, \infty]$ which is defined as

$$\hat{m}(t) := \inf\big\{x \in [0, \infty) :\, t \leq m(x)\big\},$$

where the infimum of the empty set is understood as ∞. Clearly, $\hat{m}$ is nondecreasing, and

$$\hat{m}(t) = 0,\ t \leq 0, \qquad \hat{m}(t) = \infty,\ t > M.$$

Since $\mathfrak{m}(\{L\}) > 0$ may only happen if $L < \infty$, we have $\lim_{x\to\infty} m(x) = M$. Thus

$$\hat{m}(t) < \infty, \quad t < M.$$

As it was noted in [10, p.1524], $\hat{m}|_{(0,M)}$ is left-continuous. On the intervals $(-\infty, 0]$ and (M, ∞), trivially $\hat{m}$ is left-continuous. Since $\sup\operatorname{supp}\mathfrak{m} = L$, we have $\hat{m}(M) \geq L$. If $L < \infty$, then for each $x > L$ certainly $m(x) = M$. It follows that $\hat{m}(M) = L$. If $y < L$, then $m(y) < M$, and for each $t \in (m(y), M)$ certainly $\hat{m}(t) \geq y$. This shows that

$$\lim_{x \nearrow M} \hat{m}(x) = L, \tag{2.4}$$

i.e. $\hat{m}$ is also left-continuous at M.

Let $\hat{H}$ be the Hamiltonian defined on the interval $\hat{I} := (0, M)$ as

$$\hat{H}(t) := \begin{pmatrix} \hat{m}(t)^2 & \hat{m}(t) \\ \hat{m}(t) & 1 \end{pmatrix}.$$

Denote by $\hat{I}_{\mathrm{ind}}$ the set of all inner points of $\hat{H}$-indivisible intervals, and write $\hat{I}_{\mathrm{ind}}$ as the disjoint union

$$\hat{I}_{\mathrm{ind}} = \dot{\bigcup_n} (\alpha_n, \beta_n)$$

of all (at most countably many) maximal indivisible intervals. Apparently, an interval (α, β) is $\hat{H}$-indivisible, if and only if $\hat{m}$ is constant on $(\alpha, \beta]$. Moreover, $\hat{H}$ ends with an indivisible interval, if and only if $\mathfrak{m}(\{L\}) > 0$, and in this case, $(m(L), M)$ is maximal indivisible; the value of $\hat{m}$ on this interval being L.

Let $\widehat{W}(t,z)$ be the fundamental solution of the canonical system with Hamiltonian $\hat{H}$, and denote by φ and ψ the unique solutions of the integral equation in (1.1) with boundary values (2.2). By [10, Lemma 4.1], we have

$$\widehat{W}(t,z) = \begin{pmatrix} \psi'(\hat{m}(t),z) & \psi(\hat{m}(t),z) - \hat{m}(t)\psi'(\hat{m}(t)) \\ \varphi'(\hat{m}(t),z) & \varphi(\hat{m}(t),z) - \hat{m}(t)\varphi'(\hat{m}(t)) \end{pmatrix}, \quad t \in \hat{I} \setminus \hat{I}_{\mathrm{ind}}\,. \tag{2.5}$$

Since, by [10, (3.7)],

$$\int_{[0,L]} x^2 \, d\mathfrak{m}(x) = \int_{[0,M]} \hat{m}(t)^2 \, dt\,,$$

and limit circle case prevails for $\mathrm{S}[L, \mathfrak{m}]$, we have

$$\int_{(0,M)} \operatorname{tr} \hat{H}(t) \, dt = \int_{(0,M)} \hat{m}(t)^2 \, dt + M < \infty\,,$$

i.e. limit circle case prevails for $\hat{H}$. Thus the limit $\lim_{t \nearrow M} \widehat{W}(t,z)$ exists.

In order to compute this limit, we first consider the case that $\hat{H}$ does not end with an indivisible interval, i.e. that $\mathfrak{m}(\{L\}) = 0$. Then, by (2.5) and (2.4),

$$\begin{aligned} \lim_{t \nearrow M} \widehat{W}(t,z) &= \lim_{\substack{t \nearrow M \\ t \notin \hat{I}_{\mathrm{ind}}}} \widehat{W}(t,z) = \\ &= \lim_{x \to L} \begin{pmatrix} \psi'(x,z) & \psi(x,z) - x\psi'(x,z) \\ \varphi'(x,z) & \varphi(x,z) - x\varphi'(x,z) \end{pmatrix} = W_{\mathrm{S}[L,\mathfrak{m}]}(z)\,. \end{aligned}$$

Assume now that $\hat{H}$ ends indivisibly, i.e. $\mathfrak{m}(\{L\}) > 0$. Then, in particular, $\mathrm{S}[L, \mathfrak{m}]$ must be regular. Set $l := \hat{m}(m(L))$, then $\mathfrak{m}((l, L)) = 0$, and hence [10, (2.13)–(2.18)] show that for each $x \in (l, L)$

$$\begin{aligned} \varphi(x,z) - x\varphi'(x,z) &= \varphi(l,z) - l\varphi'(x,z), \quad \varphi'(x,z) = \varphi'(l,z)\,, \\ \psi(x,z) - x\psi'(x,z) &= \psi(l,z) - l\psi'(x,z), \quad \psi'(x,z) = \psi'(l,z)\,. \end{aligned}$$

Thus, again referring to (2.5),

$$\begin{aligned} \widehat{W}(m(L),z) &= \begin{pmatrix} \psi'(l,z) & \psi(l,z) - l\psi'(l,z) \\ \varphi'(l,z) & \varphi(l,z) - l\varphi'(l,z) \end{pmatrix} = \\ &= \lim_{x \nearrow L} \begin{pmatrix} \psi'(x,z) & \psi(x,z) - x\psi'(x,z) \\ \varphi'(x,z) & \varphi(x,z) - x\varphi'(x,z) \end{pmatrix}. \end{aligned}$$

Moreover, $\hat{m}(t) = L$ whenever $t \in (m(L), M)$, and it follows that

$$\lim_{t \nearrow M} \widehat{W}(t,z) = \widehat{W}(m(L),z) \cdot W_{(\mathfrak{m}(\{L\})(1+L^2), \operatorname{Arccot} L)} = W_{\mathrm{S}[L,\mathfrak{m}]}(z)\,.$$

Since H and $\hat{H}$ have the same monodromy matrix, H is a reparameterization of $\hat{H}$.

'(ii) $\Rightarrow$ (i)': We extend the function v to a function on $\mathbb{R}$ by setting $v(t) := 0$, $t \leq 0$, and $v(t) = \lim_{t \nearrow T} v(t)$, $t \geq T$. Set

$$L := \lim_{t \nearrow T} v(t) \in [0, \infty], \qquad m(x) := \inf \big\{ t \in [0, \infty) : \, x \leq v(t) \big\}, \; x \in \mathbb{R} \,.$$

Then m is a nondecreasing and left-continuous function on $\mathbb{R}$. Since $v(t) \in [0, L]$ for all $t \in \mathbb{R}$, we have

$$m(x) = 0, \; x < 0, \qquad m(x) = T, \; x > L \,. \tag{2.6}$$

Since $\lim_{t \searrow T} v(t) = L$ and $\lim_{t \searrow 0} v(t) = 0$, we have

$$m(x) > 0, \; x > 0, \qquad m(x) < T, \; x < L \,. \tag{2.7}$$

Let $\mathfrak{m}$ be the positive Borel measure on $\mathbb{R} \cup \{\infty\}$ whose distribution function is equal to m, i.e. $m(x) = \mathfrak{m}([0, x))$, $x \in \mathbb{R}$, and which has no concentrated mass at ∞. Due to (2.6), we have $\operatorname{supp} \mathfrak{m} \subseteq [0, L]$. Moreover, the total mass $M := \mathfrak{m}([0, L])$ is equal to T and hence finite. In particular, $\mathfrak{m}([0, x]) < \infty$, $x \in [0, L)$, and $\mathfrak{m}(\{L\}) < \infty$. By the definition of $\mathfrak{m}$, we may have a concentrated mass at the point L only if $L < \infty$. The relation (2.7) shows that $\inf \operatorname{supp} \mathfrak{m} = 0$ and $\sup \operatorname{supp} \mathfrak{m} = L$. Thus the pair $(L, \mathfrak{m})$ constitutes a string of the considered kind.

We apply the construction made in the proof of '$(i) \Rightarrow (ii)$' to the string $\mathrm{S}[L, \mathfrak{m}]$. By [10, (3.4)], the function $\hat{m}$ is nothing but v, and hence the Hamiltonian H is equal to the Hamiltonian $\hat{H}$ constructed for the string $\mathrm{S}[L, \mathfrak{m}]$. As we showed above, the monodromy matrix of $\hat{H}$ equals $W_{\mathrm{S}[L, \mathfrak{m}]}$.

'(iii) $\Rightarrow$ (ii)': Consider the function $q_0 := W \star \infty$. By the present assumption, $z q_0(z^2) \in \mathcal{N}_0$. This implies that q_0 has an analytic extension to $\mathbb{C} \setminus [0, \infty)$ and $\lim_{y \to +\infty} \frac{1}{y} q_0(iy) = 0$, see, e.g., [6]. By [24, Theorem 4.1], there exists a number $T \in (0, \infty]$ and a nondecreasing function $v : (0, T) \to \mathbb{R}$, such that the Hamiltonian H_0 with Weyl coefficient q_0 is (if appropriately parameterized) of the form

$$H_0(t) = \begin{cases} \begin{pmatrix} v(t)^2 & v(t) \\ v(t) & 1 \end{pmatrix}, & t \in (0, T) \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad, & t \in (T, \infty) \text{ if } T + \int_0^T v(t)^2 \, dt < \infty \end{cases} \tag{2.8}$$

In [24] the function v was chosen to be right-continuous. However, modifying a Hamiltonian on a set of measure zero does not affect its Weyl coefficient. Hence, we may equally well assume that v is left-continuous.

In order to get some more knowledge on H_0, we exploit the fact that $q_0 = W \star \infty$. First, this tells us that H_0 must end with an indivisible interval of type 0 and infinite length. It follows that $T < \infty$ and $\int_0^T v(t)^2 \, dt < \infty$. Next, the matrix W must be a member of the fundamental solution ϖ_0 associated with H_0, i.e. $W = \varpi_0(t_0, .)$ with some $t_0 \in [0, \infty)$. However, after the point t_0 there can be only an indivisible interval of type 0, and it follows that $t_0 \geq T$. By the second limit relation in our present assumption, the regular Hamiltonian with monodromy matrix W does not end with an indivisible interval of type 0, and hence $t_0 \leq T$. Together, $W = \varpi_0(T, .)$, and hence $H = H_0|_{(0,T)}$. Finally, by [25, Lemma 3.1], we have

$$\lim_{t \searrow 0} v(t) = \lim_{x \to -\infty} q_0(x) . \tag{2.9}$$

The first limit relation in our present assumption implies $\lim_{t \searrow 0} v(t) = 0$.

'(ii) ⇒ (iii)': This implication is obtained by reversing the arguments. Again, set $q_0 := W \star \infty$ and let H_0 be the Hamiltonian with Weyl coefficient q_0. Since W is the monodromy matrix of H, the Hamiltonian H_0 evolves from H by appending an indivisible interval of type 0 and infinite length, i.e. H_0 is given by (2.8). Since H does not start with an indivisible interval of type 0, we have $\lim_{y \to +\infty} \frac{1}{y} q_0(iy) = 0$, and by (2.9), we have $\lim_{x \to -\infty} q_0(x) = 0$. Again appealing to [24, Theorem 4.1], it follows that q_0 has an analytic continuation to $\mathbb{C} \setminus [0, \infty)$, and [6] implies that $z q_0(z^2) \in \mathcal{N}_0$. ❑

For further reference, let us explicitly state some facts which were observed in the proof of Lemma 2.16.

2.17 Remark. Let $\mathrm{S}[L, \mathfrak{m}]$ be a string which is in the limit circle case, and let H be the regular Hamiltonian with monodromy matrix $W_{\mathrm{S}[L,\mathfrak{m}]}$. Moreover, set $\hat{m}(t) := \inf\{x \in [0, \infty) : t \leq m(x)\}$.

- (i) An interval (α, β) is H-indivisible, if and only if $\hat{m}$ is constant on $(\alpha, \beta]$. In this case, the type of the indivisible interval (α, β) is equal to $\operatorname{Arccot} \hat{m}(\beta)$ and its length is $(\beta - \alpha)(1 + \hat{m}(\beta)^2)$.
- (ii) The Hamiltonian H ends with an indivisible interval, if and only if $\mathfrak{m}(\{L\}) > 0$. In this case, the maximal indivisible interval at the right end of H has type $\operatorname{Arccot} L$ and length $\mathfrak{m}(\{L\})$.
- (iii) The Hamiltonian H does not have any indivisible intervals of type 0.

//

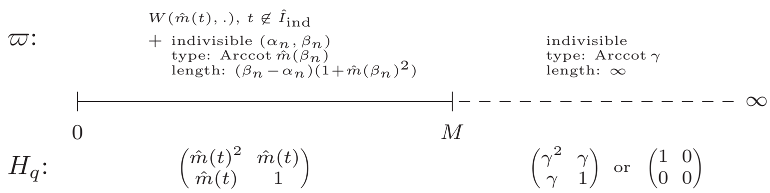
2.18 Remark. Let $\mathrm{S}[L, \mathfrak{m}]$ be a string which is in the limit circle case, let $\gamma \in \mathbb{R} \cup \{\infty\}$, and consider the function

$$q := W_{\mathrm{S}[L,\mathfrak{m}]} \star \gamma .$$

Let H_q be the Hamiltonian with Weyl coefficient q, and let ϖ be its fundamental solution. Moreover, set

$$W(x, z) := \begin{pmatrix} \psi'(x, z) & \psi(x, z) - x\psi'(x, z) \\ \varphi'(x, z) & \varphi(x, z) - x\varphi'(x, z) \end{pmatrix}, \quad x \in [0, L) .$$

Then H_q and ϖ are of the form



//

3 Proof of Theorem 1.5; Direct spectral result

Throughout this section let a string $\mathrm{S}[L,\mathfrak{m}]$ be given, and assume that τ is a spectral measure of this string with $\operatorname{supp}\tau \not\subseteq [0,\infty)$.

Note that for the string $\mathrm{S}[L,\mathfrak{m}]$ limit circle case must take place, since otherwise there could not exist a spectral measure whose support intersects $(-\infty,0)$, cf. 2.15. In particular, we have $M<\infty$, and the equivalence

$$L<\infty \iff \mathrm{S}[L,\mathfrak{m}] \text{ regular}$$

holds.

Moreover, throughout this section, let $\gamma\in\mathbb{R}\cup\{\infty\}$, $\gamma<L$, be the unique parameter such that

$$\int_{\mathbb{R}}\frac{d\tau(\lambda)}{\lambda-z}=W_{\mathrm{S}[L,\mathfrak{m}]}(z)\star\gamma\,,$$

cf. 2.13, 2.14.

Step 1:

Proof of (SM_1)–(SM_3).

The properties (SM_1) and (SM_2) are just what we already know from 1.1, (ii). Since

$$\int_{\mathbb{R}}\frac{d\tau(\lambda)}{1+|\lambda|}=\frac{\sigma}{1+|\xi|}+\sum_k\frac{\sigma_k}{1+\xi_k}\,,$$

the convergence condition required in (SM_3) is nothing but 1.1, (i).

Step 2:

The maximal chain for $z\int_{\mathbb{R}}\frac{d\tau(\lambda)}{\lambda-z^2}$.

Consider the function

$$Q(z):=z\int_{\mathbb{R}}\frac{d\tau(\lambda)}{\lambda-z^2}\,.$$

Explicitly this is

$$\begin{aligned}Q(z)=&\Big(\frac{\frac{\sigma}{2}}{i\sqrt{|\xi|}-z}+\frac{\frac{\sigma}{2}}{-i\sqrt{|\xi|}-z}\Big)+z\int_{(0,\infty)}\frac{d\tau(\lambda)}{\lambda-z^2}=\\ =&\Big(\frac{\frac{\sigma}{2}}{i\sqrt{|\xi|}-z}+\frac{\frac{\sigma}{2}}{-i\sqrt{|\xi|}-z}\Big)+\sum_k\Big(\frac{\frac{\sigma_k}{2}}{\sqrt{\xi_k}-z}+\frac{\frac{\sigma_k}{2}}{-\sqrt{\xi_k}-z}\Big)\,,\end{aligned}\tag{3.1}$$

and it follows that Q belongs to the class $\mathcal{N}_1$, see, e.g., [8, Example 4.5].

Our aim in this step is to determine the general Hamiltonian $\mathfrak{h}$ (equivalently, the maximal chain ω) with Weyl coefficient Q. To achieve this, we apply [9, Theorem 5.10] with the chain ϖ whose Weyl coefficient equals

$$q(z):=\int_{\mathbb{R}}\frac{d\tau(\lambda)}{\lambda-z}\,.$$

The necessary hypothesis for an application of this result is satisfied; we know that $\lim_{x\to-\infty}q(x)=0$.

We have to compute the function $\Lambda(t) := -\lim_{x\to-\infty} \frac{\varpi(t,x)_{22}}{\varpi(t,x)_{21}}$, and the intervals (a_k, b_k) with the properties [9, Lemma 5.9]. First, applying [9, Proposition 7.1] with the function q and its corresponding Hamiltonian H_q, cf. Remark 2.18, gives

$$\Lambda(t) = \begin{cases} \hat{m}(t)\,, & t \in (0, M) \\ \gamma \quad , & t \in (M, \infty) \end{cases} .$$

Second, since $\lim_{t\nearrow M} \hat{m}(t) = L$ and $\gamma < L$, there exist exactly two intervals (a_k, b_k), and these are

$$(a_0, b_0) = (0, M), \quad (a_1, b_1) = (M, \infty)\,.$$

Moreover,

$$-\frac{\partial}{\partial z}(\varpi(t,z)_{21})\big|_{z=0} = \int_{[0,t]} \begin{pmatrix} 0 \\ 1 \end{pmatrix}^* H_q(s) \begin{pmatrix} 0 \\ 1 \end{pmatrix} ds = t\,.$$

The increasing functions $(k = 0, 1)$

$$\tau_k(t) := \Lambda(t) - \frac{\partial}{\partial z}(\varpi(t,z)_{21})|_{z=0} = \begin{cases} \hat{m}(t) + t\,, & k = 0 \\ \gamma + t \quad , & k = 1 \end{cases}$$

have the properties

$$\lim_{t\searrow 0} \tau_0(t) = 0, \quad \lim_{t\nearrow M} \tau_0(t) = L + M\,,$$

$$\lim_{t\searrow M} \tau_1(t) = \gamma + M, \quad \lim_{t\nearrow\infty} \tau_1(t) = \infty\,.$$

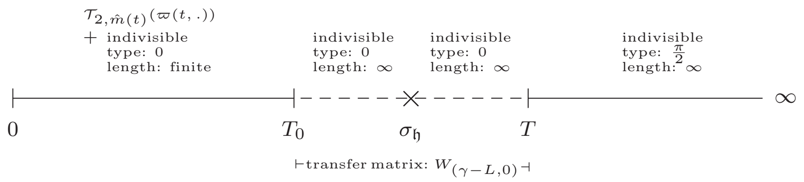
By [9, Theorem 5.10], the chain ω with Weyl coefficient Q is composed out of the matrices $\mathcal{T}_{2,\Lambda(t)}(\varpi(t,.))$, for the precise definition see [9, Definition 3.1], where discontinuities are filled with with indivisible intervals of type 0.

For $t > M$ the respective transfer matrices can be computed easily: Let $M < s < t$, then $\varpi(s,.)^{-1}\varpi(t,.) = W_{(l(s,t),\operatorname{Arccot}\gamma)}$ with some $l(s,t) > 0$. By [9, Proposition 3.6, (ii)], we have (remember $\Lambda(s) = \Lambda(t) = \gamma$)

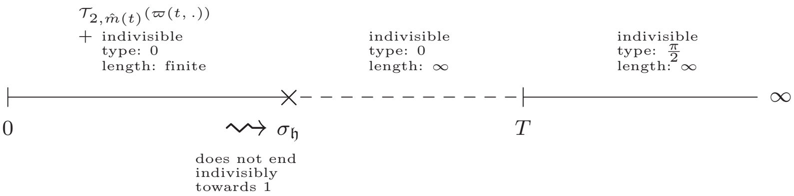
$$\mathcal{T}_{2,\Lambda(s)}\big(\varpi(s,.)\big)^{-1}\mathcal{T}_{2,\Lambda(t)}\big(\varpi(t,.)\big) = W_{(l(s,t)\sin^2[\operatorname{Arccot}\gamma], \frac{\pi}{2})}\,.$$

Referring to [9, Proposition 6.1], we see that ω is of one of the following forms, depending whether $\mathrm{S}[L, \mathfrak{m}]$ is regular (i.e. $L < \infty$) or singular/lcc (i.e. $L = \infty$):

ω (Case $L < \infty$):



ω (Case $L = \infty$):



In order to apply the formulas of [26], which will be done in the next step, we have to rewrite this knowledge about ω in terms of $\mathfrak{h}$.

Case $L < \infty$: The general Hamiltonian $\mathfrak{h}$ is defined on the set

$$I = (0, \sigma_{\mathfrak{h}}) \cup (\sigma_{\mathfrak{h}}, \infty)\,.$$

There exist $T_0 \in (0, \sigma_{\mathfrak{h}})$ and $T \in (\sigma_{\mathfrak{h}}, \infty)$, such that

$$H(t) = \begin{cases} h_-(t) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & t \in (T_0, \sigma_{\mathfrak{h}}) \\ h_+(t) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & t \in (\sigma_{\mathfrak{h}}, T) \\ h(t) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad, & t \in (T, \infty) \end{cases}$$

The values of $H(t)$ on the interval $(0, T_0)$ could be determined explicitly, however, this knowledge is not needed for the present purposes.

From [16, §3.a] we see that the data part of $\mathfrak{h}$ associated with its singularity is

$$\ddot{o} = 0, \qquad b_1 = 0, \qquad d_0 = \gamma - L,\ d_1 = 0\,,$$

where the parameters d_0, d_1 are understood relative to the admissible partition

$$E = \{0, T_0, T, \infty\}\,.$$

Case $L = \infty$: The general Hamiltonian $\mathfrak{h}$ is defined on the set

$$I = (0, \sigma_{\mathfrak{h}}) \cup (\sigma_{\mathfrak{h}}, \infty)\,.$$

There exists $T \in (\sigma_{\mathfrak{h}}, \infty)$, such that

$$H(t) = \begin{cases} h_+(t) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & t \in (\sigma_{\mathfrak{h}}, T) \\ h(t) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad, & t \in (T, \infty) \end{cases}$$

Moreover, no interval $(\sigma_{\mathfrak{h}} - \varepsilon, \sigma_{\mathfrak{h}})$, $\varepsilon > 0$, is H-indivisible. Again, further knowledge on $H(t)$, $t \in (0, \sigma_{\mathfrak{h}})$, could be obtained but is not needed.

Since the Hamiltonian H_q has no indivisible intervals of type 0, we obtain from [9, Proposition 6.1] and [16, Definition 2.6], that

$$\ddot{o} = 0 \,.$$

Remember here that, since $Q \in \mathcal{N}_1$, we know that $\Delta = 1$. The data b_1, and d_0, d_1 with respect to some admissible partition E, could be computed explicitly (requires considerable effort). However, this knowledge is not needed.

Step 3:

Consequences for spectral data.

The Hamiltonian $\mathfrak{h}$ ends with at least two indivisible intervals in the sense of [26, Definition 5.3]. An application of [26, Theorem 5.4] shows that the function Q satisfies the asymptotic conditions (I), (II), (III_0), named in [26, §3, 3.1]. These conditions give information on the distribution of the poles of Q and the sizes of the respective residues. However, the spectral data of Q is known in terms of the measure τ by (3.1). Let us match the present notation with the one of [26]:

- The sequence 'γ_k' in [26] is: $\sqrt{\xi_1}, -\sqrt{\xi_1}, \sqrt{\xi_2}, -\sqrt{\xi_2}, \sqrt{\xi_3}, \ldots$.
  Accordingly the sequences 'γ_k^+' and 'γ_k^-' from [26] are: $\sqrt{\xi_1}, \sqrt{\xi_2}, \sqrt{\xi_3}, \ldots$ and $-\sqrt{\xi_1}, -\sqrt{\xi_2}, -\sqrt{\xi_3}, \ldots$.
- The 'negative residues σ_k' in [26] are: $\frac{\sigma_k}{2}$ for both $\sqrt{\xi_k}$ and $-\sqrt{\xi_k}$.
- The 'remaining poles α_k' in [26] are: $i\sqrt{|\xi|}$ and $-i\sqrt{|\xi|}$, each with multiplicity 1.
- The function 'A_Q' defined as in [26, (3.1)] is:

$$A_Q(z)=\Big(1-\frac{z}{i\sqrt{|\xi|}}\Big)\Big(1-\frac{z}{-i\sqrt{|\xi|}}\Big)\cdot\lim_{r\to\infty}\prod_{\xi_k\le r}\Big(1-\frac{z}{\sqrt{\xi_k}}\Big)\Big(1-\frac{z}{-\sqrt{\xi_k}}\Big)= \\ =\Big(1-\frac{z^2}{\xi}\Big)\lim_{r\to\infty}\prod_{\xi_k\le r}\Big(1-\frac{z^2}{\xi_k}\Big)=\Big(1-\frac{z^2}{\xi}\Big)\Gamma(z^2)\,. \tag{3.2}$$

Condition (II) thus says

$$\lim_{k\to\infty}\frac{k}{\sqrt{\xi_k}} \quad \text{exists in } \mathbb{R}\,,$$

and this is (SM_4). We have

$$A_Q'(z) = -\frac{2z}{\xi}\Gamma(z^2) + \Big(1-\frac{z^2}{\xi}\Big)\cdot 2z\Gamma'(z^2)\,, \tag{3.3}$$

and hence

$$A_Q'(\pm\sqrt{\xi_k}) = \Big(1-\frac{\xi_k}{\xi}\Big)\cdot 2(\pm\sqrt{\xi_k})\Gamma'(\xi_k)\,. \tag{3.4}$$

Thus

$$\sum_k \xi_k^{-3}\frac{1}{\Gamma'(\xi_k)^2\sigma_k} = \sum_k \xi_k^{-3}\cdot\Big[\Big(1-\frac{\xi_k}{\xi}\Big)\cdot 2(\pm\sqrt{\xi_k})\Big]^2\frac{1}{A_Q'(\sqrt{\xi_k})^2\frac{\sigma_k}{2}}\frac{1}{2}\,, \tag{3.5}$$

and the validity of (III_0) implies (SM_5).

Step 4:

The matrix function $\overleftarrow{W}$.

Consider the matrix function

$$\overleftarrow{W} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \omega(T,.)^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \omega(T,.)_{22} & \omega(T,.)_{12} \\ \omega(T,.)_{21} & \omega(T,.)_{11} \end{pmatrix}.$$

By [9, Lemma 2.3] we have $\overleftarrow{W} \in \mathcal{M}_1$.

Set $B_Q := A_Q \cdot Q$. In the first part of this step, we are going to show that

$$(1,0)\overleftarrow{W} = (A_Q, B_Q)\,. \tag{3.6}$$

The function B_Q is real, entire, has no common zeros with A_Q, and $\frac{B_Q}{A_Q} \in \mathcal{N}_1$. Thus we may consider the dB-Pontryagin space $\mathfrak{P}(A_Q - iB_Q)$, which is a Pontryagin space with negativ index 1. Since Q satisfies (I), (II), (III_0), [26, Theorem 3.2] implies $1 \in \mathfrak{P}(A_Q - iB_Q)$. In particular, the functions A_Q and B_Q are of bounded type in the upper and lower half-planes. Also the entries $\overleftarrow{W}_{11}$ and $\overleftarrow{W}_{12}$ of $\overleftarrow{W}$ are real entire functions of bounded type and have no common zeros. Since $(\overleftarrow{W}_{11})^{-1}\overleftarrow{W}_{12} = Q$, thus, the zeros of $\overleftarrow{W}_{11}$ coincide with those of A_Q, and the zeros of $\overleftarrow{W}_{12}$ with those of B_Q. It follows that

$$\overleftarrow{W}_{11} = F \cdot A_Q, \quad \overleftarrow{W}_{12} = F \cdot B_Q$$

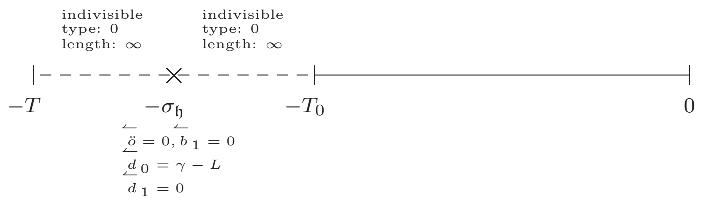
with some real entire and zerofree function F. However, since $F = A_Q^{-1}\overleftarrow{W}_{11}$ is of bounded type, it must be constant. Evaluating at $z = 0$ gives $F = 1$.

In the second part, we determine the regular general Hamiltonian $\overleftarrow{\mathfrak{h}}$ with monodromy matrix $\overleftarrow{W}$. To this end, we only need to refer to [15, Lemma 4.30], which tells how $\overleftarrow{\mathfrak{h}}$ is given in terms of the data of $\mathfrak{h}$.

Case $L < \infty$:

$$\begin{aligned}
&\overleftarrow{I} = (-T, -\sigma_{\mathfrak{h}}) \cup (-\sigma_{\mathfrak{h}}, 0)\,, \\
&\overleftarrow{H}(t) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} H(-t) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad t \in \overleftarrow{I}\,, \\
&\overleftarrow{\ddot{o}} = 0, \qquad \overleftarrow{b}_1 = 0, \qquad \overleftarrow{d}_0 = \gamma - L,\ \overleftarrow{d}_1 = 0\,. \\
&\overleftarrow{E} = \{-T, -T_0, 0\}
\end{aligned}$$

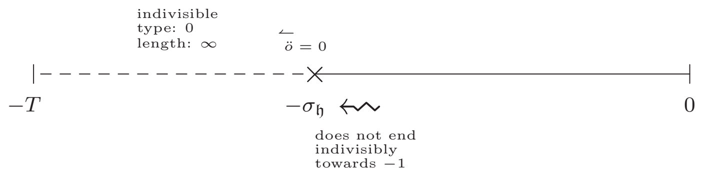
$\overleftarrow{\mathfrak{h}}$:



Case $L = \infty$:

$$\begin{aligned}
&\overleftarrow{I} = (-T, -\sigma_{\mathfrak{h}}) \cup (-\sigma_{\mathfrak{h}}, 0)\,, \\
&\overleftarrow{H}(t) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} H(-t) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad t \in \overleftarrow{I}\,, \\
&\overleftarrow{\ddot{o}} = 0, \qquad \overleftarrow{b}_1 = b_1, \qquad \overleftarrow{d}_0 = d_0,\ \overleftarrow{d}_1 = d_1\,, \\
&\overleftarrow{E} = \{-t : t \in E\}
\end{aligned}$$

$\overleftarrow{\mathfrak{h}}$:



Step 5:

The inequality (SM_6).

We are going to compute the inner product $[1,1]$ in the space $\mathfrak{P}(A_Q - iB_Q)$ in two ways.

On the one hand, let us use our knowledge on $\overleftarrow{\mathfrak{h}}$ to compute this inner product in terms of L and γ. If $L < \infty$, [15, Corollary 4.32] and [15, Proposition 2.8] imply that

$$\begin{pmatrix}1\\0\end{pmatrix} \in \mathfrak{K}(\overleftarrow{W}), \quad \Big[\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\Big]_{\mathfrak{K}(\overleftarrow{W})} = \frac{1}{\gamma - L}\,.$$

If $L = \infty$, [26, Lemma 6.3] gives

$$\begin{pmatrix}1\\0\end{pmatrix} \in \mathfrak{K}(\overleftarrow{W}), \quad \Big[\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\Big]_{\mathfrak{K}(\overleftarrow{W})} = 0\,.$$

Since $\binom{1}{0} \in \mathfrak{K}(\overleftarrow{W})$, we must have $\binom{0}{1} \notin \mathfrak{K}(\overleftarrow{W})$, and hence the projection onto the upper component is an isometric isomorphism of $\mathfrak{K}(\overleftarrow{W})$ onto the dB-Pontryagin space generated by the first row of $\overleftarrow{W}$. By (3.6), this space is nothing but $\mathfrak{P}(A_Q - iB_Q)$. Hence,

$$[1,1] = \begin{cases} \frac{1}{\gamma - L}\,, & L < \infty \\ 0 \quad , & L = \infty \end{cases} \tag{3.7}$$

On the other hand, let us [26, Proposition 2.6] with the angle '$\varphi = \frac{\pi}{2}$' to compute $[1,1]$ from the spectral data of Q. The application of this result is justified, since we have

$$\lim_{y\to+\infty} \frac{1}{y}\frac{B_Q(iy)}{A_Q(iy)} = \lim_{y\to+\infty} \frac{1}{y} Q(iy) = i \lim_{y\to+\infty} q(-y^2) = 0\,,$$

and hence $A_Q \notin \mathfrak{P}(A_Q - iB_Q)$. The space '$\mathcal{X}$', which appears in [26, Proposition 2.6], thus equals the whole space $\mathfrak{P}(A_Q - iB_Q)$, in particular it contains the function 1.

Denote by $K(w,z)$ the reproducing kernel of the space $\mathfrak{P}(A_Q - iB_Q)$. Then we have

$$K(w,z) = -B_Q(\overline{w})\frac{A_Q(z)}{z - \overline{w}}, \quad w \in \mathbb{C},\ A_Q(w) = 0\,. \tag{3.8}$$

Set

$$\mathcal{L}_+ := \operatorname{span}\Big(\big\{K(\sqrt{\xi_k}, .) : \ k = 1,2,\dots\big\} \cup \big\{K(-\sqrt{\xi_k}, .) : \ k = 1,2,\dots\big\}\Big)\,,$$

$$\check{\mathfrak{L}} := \operatorname{span}\big\{K(i\sqrt{|\xi|}, .), K(-i\sqrt{|\xi|}, .)\big\}\,,$$

then $\mathfrak{P}(A_Q - iB_Q) = \mathcal{L}_+[\dot{+}]\check{\mathcal{L}}$. Denote by F_+ and $\check{F}$ the orthogonal projections of the element 1 onto the spaces $\mathcal{L}_+$ and $\check{\mathcal{L}}$, respectively.

By [26, Proposition 2.6], we can compute the inner product $[F_+, F_+]$ as

$$\begin{aligned}[F_+, F_+] = &\sum_k \big|[F_+, K(\sqrt{\xi_k}, .)]\big|^2 \frac{-1}{A_Q'(\sqrt{\xi_k}) B_Q(\sqrt{\xi_k})} + \\ &+ \sum_k \big|[F_+, K(-\sqrt{\xi_k}, .)]\big|^2 \frac{-1}{A_Q'(-\sqrt{\xi_k}) B_Q(-\sqrt{\xi_k})}\end{aligned}$$

However, we have $[\check{F}, K(\pm\sqrt{\xi_k}), .)] = 0$, and hence

$$[F_+, K(\pm\sqrt{\xi_k}), .)] = [1, K(\pm\sqrt{\xi_k}), .)] = 1\,.$$

Moreover,

$$\frac{B_Q(\pm\sqrt{\xi_k})}{A_Q'(\pm\sqrt{\xi_k})} = \operatorname{Res}\big(Q, \pm\sqrt{\xi_k}\big) = -\frac{\sigma_k}{2}\,.$$

Using (3.4) gives

$$[F_+, F_+] = \sum_k \frac{1}{(1 - \frac{\xi_k}{\xi})^2 \xi_k \Gamma'(\xi_k)^2 \sigma_k}\,.$$

Let us next compute $\check{F}$. To this end, note that by (3.8)

$$\begin{aligned}K(i\sqrt{|\xi|}, i\sqrt{|\xi|}) =& K(-i\sqrt{|\xi|}, -i\sqrt{|\xi|}) = 0 \\ K(-i\sqrt{|\xi|}, i\sqrt{|\xi|}) =& - B_Q(i\sqrt{|\xi|}) A_Q'(i\sqrt{|\xi|}) = \\ =& - \underbrace{\frac{B_Q(i\sqrt{|\xi|})}{A_Q'(i\sqrt{|\xi|})}}_{=\operatorname{Res}(Q, i\sqrt{|\xi|}) = -\frac{\sigma}{2}} A_Q'(i\sqrt{|\xi|})^2 = \frac{\sigma}{2} A_Q'(i\sqrt{|\xi|})^2 \\ K(i\sqrt{|\xi|}, -i\sqrt{|\xi|}) =& \overline{K(-i\sqrt{|\xi|}, i\sqrt{|\xi|})} = \frac{\sigma}{2} A_Q'(-i\sqrt{|\xi|})^2\end{aligned}$$

Write $\check{F} = \alpha_+ K(i\sqrt{|\xi|}, .) + \alpha_- K(-i\sqrt{|\xi|}, .)$, then

$$[\check{F}, K(i\sqrt{|\xi|}, .)] = \alpha_- \frac{\sigma}{2} A_Q'(i\sqrt{|\xi|})^2, \quad [\check{F}, K(-i\sqrt{|\xi|}, .)] = \alpha_+ \frac{\sigma}{2} A_Q'(-i\sqrt{|\xi|})^2\,.$$

Since $F_+ \perp K(\pm i\sqrt{|\xi|}, .)$, we have $[\check{F}, K(\pm i\sqrt{|\xi|}, .)] = 1$, and conclude that

$$\alpha_\pm = \frac{1}{A_Q'(\mp i\sqrt{|\xi|})^2 \frac{\sigma}{2}}\,.$$

It follows that

$$[\check{F}, \check{F}] = 2\operatorname{Re}\Big(\alpha_+ \overline{\alpha_-} K(i\sqrt{|\xi|}, -i\sqrt{|\xi|})\Big) = \frac{4}{\sigma} \operatorname{Re} \frac{1}{A_Q'(-i\sqrt{|\xi|})^2}\,.$$

From (3.3) we obtain (note that $(-i\sqrt{|\xi|})^2 = -|\xi| = \xi$)

$$A_Q'(-i\sqrt{|\xi|}) = \frac{2i\sqrt{|\xi|}}{\xi}\Gamma(\xi)\,,$$

and hence $A'_Q(-i\sqrt{|\xi|})^2 = \frac{4}{\xi}\Gamma(\xi)^2$. Thus

$$[\check{F}, \check{F}] = \frac{\xi}{\sigma\Gamma(\xi)^2}\,.$$

Together, it follows that

$$[1,1] = [F_+, F_+] + [\check{F}, \check{F}] = \sum_k \frac{1}{(1-\frac{\xi_k}{\xi})^2 \xi_k \Gamma'(\xi_k)^2 \sigma_k} + \frac{\xi}{\sigma\Gamma(\xi)^2}\,. \tag{3.9}$$

We know from (3.7) that $[1,1] \leq 0$, and hence

$$\frac{1}{\sigma} \geq -\frac{\Gamma(\xi)^2}{\xi} \sum_k \frac{1}{(1-\frac{\xi_k}{\xi})^2 \xi_k \Gamma'(\xi_k)^2 \sigma_k} = \sum_k \underbrace{\frac{-\Gamma(\xi)^2}{\xi(1-\frac{\xi_k}{\xi})^2}}_{=\frac{(-\xi)\Gamma(\xi)^2}{\xi_k^2(1-\frac{\xi}{\xi_k})^2}} \xi_k^{-1} \frac{1}{\Gamma'(\xi_k)^2 \sigma_k} = \frac{1}{\Xi(\xi)}.$$

This is (SM$_6$). We also see that $\sigma = \Xi(\xi)$ if and only if $[1,1] = 0$. Again by (3.7), this happens if and only if $L = \infty$, i.e. if and only if $\mathrm{S}[L, \mathfrak{m}]$ is singular.

Step 6:

The divergence condition (SM$_7$).

Assume that $\sigma = \Xi(\xi)$, i.e. that $L = \infty$. From the form of the general Hamiltonian $\mathfrak{h}$ explained in Step 2, in particular remember that $\ddot{o} = 0$, we see that $\mathfrak{h}$ does not end with at least three indivisible intervals in the sense of [26, Definition 5.3]. Since Q satisfies (I) and (II), [26, Theorem 5.4] implies that Q does not satisfy (III$_{-1}$). Using (3.4), we obtain

$$\begin{aligned}\sum_k \frac{1}{(1-\frac{\xi_k}{\xi})^2 \Gamma'(\xi_k)^2 \sigma_k} &= \sum_k (\sqrt{\xi_k})^2 \frac{1}{A'_Q(\sqrt{\xi_k})^2 \frac{\sigma_k}{2}} + \\ &\quad + \sum_k (-\sqrt{\xi_k})^2 \frac{1}{A'_Q(-\sqrt{\xi_k})^2 \frac{\sigma_k}{2}} = \infty\,,\end{aligned} \tag{3.10}$$

and (SM$_7$) follows.

This finishes the proof of the 'direct part' of Theorem 1.5. ✌

4 Proof of Theorem 1.5; Inverse spectral result

Throughout this section let a positive Borel measure τ on the real line be given, and assume that the conditions (SM$_1$)–(SM$_7$) hold for τ. In essence, we reverse the argument which led to the proof of the direct result.

Step 1:

The functions q and Q.

Due to (SM_3), we have $\int_{\mathbb{R}} \frac{d\tau(\lambda)}{1+|\lambda|} < \infty$, and hence may consider the functions

$$q(z) := \int_{\mathbb{R}} \frac{d\tau(\lambda)}{\lambda - z}, \quad Q(z) := zq(z^2) = z \int_{\mathbb{R}} \frac{d\tau(\lambda)}{\lambda - z^2} .$$

First, note that $q \in \mathcal{N}_0$ and that (SM_1) and (SM_2) imply that $Q \in \mathcal{N}_1$, cf. [8, Example 4.5]. Second, the limit relations

$$\lim_{x\to-\infty} q(x) = 0, \quad \lim_{y\to+\infty} \frac{1}{y} Q(iy) = i \lim_{y\to+\infty} q(-y^2) = 0 , \tag{4.1}$$

hold. Moreover, since τ has no point mass at 0, we have $Q(0) = 0$.

Third, let us show that the function Q satisfies (I), (II), (III_0): The condition (I) is trivially satisfied because of symmetry of poles of Q, and (II) is just (SM_4). Validity of (III_0) follows from (SM_5) by virtue of the computation (3.5).

Finally, if in (SM_6) equality holds, then Q does not satisfy (III_{-1}). This follows from (SM_7) and the computation (3.10).

Step 2:

Structure of $\mathfrak{h}$.

Let $\mathfrak{h}$ be the general Hamiltonian with Weyl coefficient Q, and let ω be its fundamental solution. Then, by [26, Theorem 5.4], $\mathfrak{h}$ ends with at least two indivisible intervals. Moreover, in case $\sigma = \Xi(\xi)$, it does not end with at least three indivsible intervals. Since $Q(0) = 0$, the last indivsible interval of $\mathfrak{h}$ must be of type $\frac{\pi}{2}$. The function Q satisfies $Q(-z) = -Q(z)$, and hence all matrices $\omega(t, .)$ have the property that their diagonal entries are even functions, whereas their off-diagonal entries are odd, cf. [9, Proposition 4.4]. The type of an indivisible interval in the chain ω thus can only be one of 0 or $\frac{\pi}{2}$, cf. [9, Proposition 3.6, (i)]. We conclude that the one but last indivisible interval of $\mathfrak{h}$ must be of type 0. Moreover, notice that, due to the limit relation (4.1), $\mathfrak{h}$ does not start with an indivisible interval of type 0.

Consider the function A_Q defined by (3.2), and set $B_Q := Q \cdot A_Q$. Then, by [26, Theorem 3.2], we have $1 \in \mathfrak{P}(A_Q - iB_Q)$. The computation carried out in the second part of §3, Step 5, leads to the formula (3.9), and we conclude that

$$[1, 1] \begin{cases} < 0 \,, & \sigma < \Xi(\xi) \\ = 0 \,, & \sigma = \Xi(\xi) \end{cases} .$$

Denote by T the left endpoint of the last indivisible interval of $\mathfrak{h}$, let $\overleftarrow{\mathfrak{h}}$ be the regular general Hamiltonian whose monodromy matrix is equal to

$$\overleftarrow{W} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \omega(T, .) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ,$$

and let $\overleftarrow{\omega}$ be the fundamental solution of $\overleftarrow{\mathfrak{h}}$. Then $\mathfrak{h}$ and $\overleftarrow{\mathfrak{h}}$ are related by means of [15, Definition 3.40, Lemma 4.30]. Since Q satisfies (I), (II), (III_0), the argument carried out in the first part of §3, Step 4, can be applied, and gives $(1, 0)\overleftarrow{W} = (A_Q, B_Q)$. It follows that

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathfrak{K}(\overleftarrow{W}), \qquad \Big[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Big]_{\mathfrak{K}(\overleftarrow{W})} = [1, 1] \begin{cases} < 0 \,, & \sigma < \Xi(\xi) \\ = 0 \,, & \sigma = \Xi(\xi) \end{cases}$$

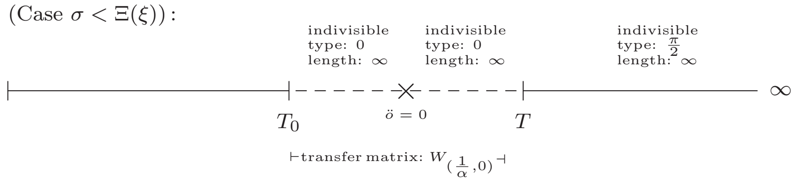
This shows that $\overleftarrow{\mathfrak{h}}$ starts with an indivisible interval of type 0 which has infinite length.

Case $\sigma < \Xi(\xi)$*:* We have $\alpha := \big[\binom{1}{0}, \binom{1}{0}\big]_{\mathfrak{K}(\overleftarrow{W})} < 0$, and hence $\overleftarrow{W}$ can be factorized as

$$\overleftarrow{W} = W_{(\frac{1}{\alpha},0)} W_0$$

with some $W_0 \in \mathcal{M}_0$, cf. [12, Lemma 7.5]. Hence, the singularity of $\overleftarrow{\mathfrak{h}}$ is left endpoint of an indivisible interval. If $-T_0$ denotes the right endpoint of the maximal indivisible interval whose left endpoint is this singularity, the matrix $\overleftarrow{\omega}(-T_0, .)$ equals $W_{(\frac{1}{\alpha},0)}$. Together with what we already know about $\mathfrak{h}$, this implies that $\mathfrak{h}$ is of the form

$\mathfrak{h}$ (Case $\sigma < \Xi(\xi)$) :

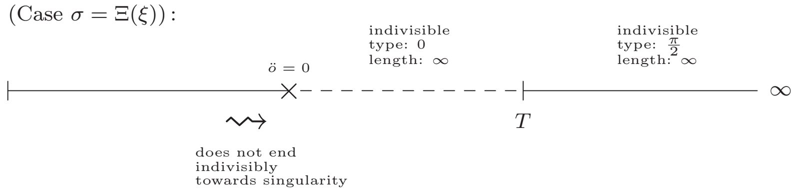


Case $\sigma = \Xi(\xi)$*:* In this case, Q does not satisfy (III_{-1}), and [26, Theorem 3.2] implies that $z \notin \mathfrak{P}(A_Q - iB_Q)$. In turn, we obtain that $\binom{z}{0} \notin \mathfrak{K}(\overleftarrow{W})$. This fact shows that, if π_- denotes the projection onto the lower component in the space $\mathfrak{K}(\overleftarrow{W})$, we have

$$\ker \pi_- = (\ker \pi_-)^\circ = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}.$$

We obtain from [26, Lemma 6.3], together with Subcase 3b of its proof, that the singularity of $\overleftarrow{\mathfrak{h}}$ is not left endpoint of an indivisible interval and that $\overleftarrow{\ddot{o}} = 0$. The general Hamiltonian $\mathfrak{h}$ is therefore of the form

$\mathfrak{h}$ (Case $\sigma = \Xi(\xi)$) :



Step 3:

Applying $\mathcal{T}_{\surd}$*; existence of* $\mathrm{S}[L, \mathfrak{m}]$.

We apply [9, Theorem 5.1] to determine the maximal chain ϖ with Weyl coefficient q, and its corresponding (positive definite) Hamiltonian H_q. The hypothesis necessary for an application of this theorem is satisfied; we know that $\mathfrak{h}$ does not start with an indivisible interval of type 0.

[9, Theorem 5.1] implies that every matrix $\mathcal{T}_{\surd}(\omega(t, .))$ appears as a member of ϖ, in particular we have

$$\mathcal{T}_{\surd}\big(\omega(T, .)\big) = \varpi(x_0, .),$$

with some $x_0 \in \operatorname{dom} \varpi$. Whenever $t > T$, we have $\omega(T,.)^{-1}\omega(t,.) = W_{(l(t),\frac{\pi}{2})}$ with some $l(t) > 0$, and thereby $\lim_{t \nearrow \infty} l(t) = \infty$. By [9, Proposition 3.6],

$$\mathcal{T}_{\surd}\big(\omega(T,.)\big)^{-1}\mathcal{T}_{\surd}\big(\omega(t,.)\big) = W_{(l'(t),\phi)}\,,$$

with

$$l'(t) = l(t)\Big(1 + \Big[\frac{\partial}{\partial z}\big(\omega(T,.)_{21}\big)\big|_{z=0}\Big]^2\Big), \quad \phi = \operatorname{Arccot}\Big[\frac{\partial}{\partial z}\big(\omega(T,.)_{21}\big)\big|_{z=0}\Big]\,.$$

We see that the Hamiltonian H_q consists to the right of x_0 of just one indivisible interval of type ϕ and infinite length.

Let $\sigma_{\mathfrak{h}}$ be the singularity of $\mathfrak{h}$. Our next aim is to show that

$$\lim_{t \nearrow \sigma_{\mathfrak{h}}} \mathcal{T}_{\surd}\big(\omega(t,.)\big) = \lim_{t \searrow \sigma_{\mathfrak{h}}} \mathcal{T}_{\surd}\big(\omega(t,.)\big) = \varpi(x_0,.)\,. \tag{4.2}$$

Notice that, since ϖ has no singularity, these limits certainly exist. Moreover, since $\sigma_{\mathfrak{h}}$ is left endpoint of an indivisible interval of type 0 whose right endpoint is T, the second limit is equal to $\mathcal{T}_{\surd}(\omega(T,.))$. This proves the second equality sign in (4.2).

If $\sigma < \Xi(\xi)$, so that $\sigma_{\mathfrak{h}}$ is the right endpoint of an indivisible interval of type 0 whose left endpoint is T_0, we have

$$\lim_{t \nearrow \sigma_{\mathfrak{h}}} \mathcal{T}_{\surd}\big(\omega(t,.)\big) = \mathcal{T}_{\surd}\big(\omega(T_0,.)\big)\,.$$

However, $\omega(T_0,.)^{-1}\omega(T,.) = W_{(\frac{1}{\alpha},0)}$, and hence $\mathcal{T}_{\surd}(\omega(T_0,.)) = \mathcal{T}_{\surd}(\omega(T,.))$. Thus also the first equality in (4.2) holds.

Consider the case that $\sigma = \Xi(\xi)$. Then $\sigma_{\mathfrak{h}}$ is not right endpoint of an indivisible interval. Assume on the contrary that (4.2) fails. Then, by [9, Theorem 5.1], we have

$$\big[\lim_{t \nearrow \sigma_{\mathfrak{h}}} \mathcal{T}_{\surd}(\omega(t,.))\big]^{-1}\mathcal{T}_{\surd}\big(\omega(T,.)\big) = W_{(l,0)}$$

with some $l > 0$. The chain ω can be reconstructed from ϖ by means of [9, Theorem 5.10]. Hence, the structure of the singularity of $\mathfrak{h}$ can be read off the table in [9, Proposition 6.1]. Since $l > 0$ (and $\Lambda(x) = \cot\phi$, $x > x_0$), we are either in the second or in the fourth case of this table. In both cases, the number $\ddot{o}$ associated with the singularity $\sigma_{\mathfrak{h}}$ is equal to 1, cf. [16, Definition 2.6]. We have reached a contradiction, and conclude that (4.2) also holds in the case $\sigma = \Xi(\xi)$.

Since the transformation $\mathcal{T}_{\surd}$ cannot produce indivisible intervals of type 0, (4.2) implies that ϖ does not contain any such intervals. Applying [12, Theorem 5.10, Lemma 5.2] with the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\varpi(x_0,.)^{-1}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and the parameter '$\tau = 0$', gives

$$\lim_{y \to +\infty} \frac{1}{y}\frac{\varpi(x_0,iy)_{12}}{\varpi(x_0,iy)_{11}} = 0\,.$$

Set $q_0(z) := \varpi(x_0,z) \star \infty$. Then, by (4.2) and [9, Lemma 3.3], we have

$$zq_0(z^2) = \lim_{t \to \sigma_{\mathfrak{h}}} \big[\omega(t,z) \star \infty\big] \in \mathcal{N}_0\,.$$

Since ω does not start with an indivisible interval of type 0, it follows that

$$0 = \lim_{y \to +\infty} \frac{1}{y}(iy)q_0((iy)^2) = i \lim_{x \to -\infty} q_0(x) .$$

Lemma 2.16 implies that there exists a string $\mathrm{S}[L, \mathfrak{m}]$, either regular or singular/lcc, with

$$\varpi(x_0, .) = W_{\mathrm{S}[L,\mathfrak{m}]} .$$

Thus

$$\int_{\mathbb{R}} \frac{d\tau(\lambda)}{\lambda - z} = q(z) = W_{\mathrm{S}[L,\mathfrak{m}]}(z) \star \cot \phi ,$$

and we see that τ is a spectral measure of $\mathrm{S}[L, \mathfrak{m}]$.

Step 4:

Uniqueness of $\mathrm{S}[L, \mathfrak{m}]$.

The fact that the measure τ can be a spectral measure of at most one string can be easily deduced from the uniqueness part in the Inverse Spectral Theorem for (positive definite) canonical systems.

First, we discuss a more general situation. Let τ be any positive Borel measure on the real line, and assume that τ is a spectral measure of two strings $\mathrm{S}[L_1, \mathfrak{m}_1]$ and $\mathrm{S}[L_2, \mathfrak{m}_2]$, both being regular or singular/lcc. Then there exist corresponding parameters $\gamma_1, \gamma_2 \in \mathbb{R} \cup \{\infty\}$ with

$$q(z) := \int_{\mathbb{R}} \frac{d\tau(\lambda)}{\lambda - z} = W_{\mathrm{S}[L_1,\mathfrak{m}_1]}(z) \star \gamma_1 = W_{\mathrm{S}[L_2,\mathfrak{m}_2]}(z) \star \gamma_2 . \tag{4.3}$$

We know from Lemma 2.16 that $W_{\mathrm{S}[L_i,\mathfrak{m}_i]}(z)$ is the monodromy matrix of the canonical system with Hamiltonian

$$H_i(t) := \begin{pmatrix} \hat{m}_i(t)^2 & \hat{m}_i(t) \\ \hat{m}_i(t) & 1 \end{pmatrix} , \quad t \in (0, M_i) , \tag{4.4}$$

where

$$M_i := \mathfrak{m}([0, L_i]), \quad \hat{m}_i(t) := \inf \big\{ x \in [0, \infty) : \, t \leq m_i(x) \big\} .$$

Remember that, since limit circle case takes place, M_i is finite. Let us append an indivsible interval of type $\phi_i := \operatorname{Arccot} \gamma_i$ to H_i, i.e. continue H_i to a Hamiltonian on the whole semiaxis $[0, \infty)$ by defining

$$H_i(t) := \begin{cases} \begin{pmatrix} \gamma_i^2 & \gamma_i \\ \gamma_i & 1 \end{pmatrix} , & \gamma_i \neq \infty \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} , & \gamma_i = \infty \end{cases} , \quad t > M_i . \tag{4.5}$$

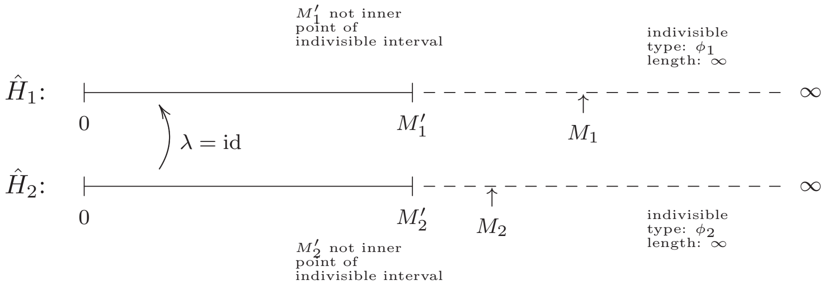
By (4.3), the Weyl coefficient of the extended Hamiltonian H_i is equal to q. By the uniqueness part of the Inverse Spectral Theorem for canonical systems, see [2] or [23], the Hamiltonians H_1 and H_2 must be reparameterizations of each other. This means that there exists an increasing bijection $\lambda : [0, \infty) \to [0, \infty)$ such that λ and λ^{-1} are both absolutely continuous, which relates H_1 and H_2 by means of the formula

$$H_2(x) = H_1\big(\lambda(x)\big) \cdot \lambda'(x) . \tag{4.6}$$

If $\gamma_i \neq \infty$, the right lower corner of H_i in (4.4) and (4.5) is fixed to 1. We see from (4.6) that the reparameterization map λ must be the identity. If $\gamma_i = \infty$, the right lower corner in (4.4) is fixed to 1, whereas the Hamiltonian takes the particular form (4.5) after M_i. Again it follows that λ is the identity. Set

$$M_i' := \begin{cases} M_i & , \quad \gamma_i \neq L_i \\ M_i - \mathfrak{m}(\{L_i\}), & \gamma_i = L_i \end{cases}$$

Then we can picture the situation as follows:



We see that always

$$\gamma_1 = \gamma_2 \ (=: \gamma), \quad M_1' = M_2' \ (=: M'), \quad \hat{m}_1|_{[0,M']} = \hat{m}_2|_{[0,M']} \,.$$

Consider now the case that

$$\gamma \neq L_1 \ \wedge \ \gamma \neq L_2 \tag{4.7}$$

Then we have $M_1 = M_2 = M'$. Due to (2.4), it follows that $L_1 = L_2$ $(=: L)$. The function m_i can be recovered from $\hat{m}_i$ by means of the formula

$$m_i(x) = \inf \big\{ t \in [0, \infty) : \, x \leq \hat{m}_i(t) \big\} \,,$$

cf. [10, (3.4)]. Thus we obtain

$$m_1(x) = m_2(x), \quad x \in [0, L] \,.$$

The point $M' - \mathfrak{m}_i(\{L\})$ is the left endpoint of the maximal indivisible interval with right endpoint M'. We conclude that also $\mathfrak{m}_1(\{L\}) = \mathfrak{m}_2(\{L\})$. Thus we have shown that, in the case that (4.7) holds, the strings $\mathrm{S}[L_1, \mathfrak{m}_1]$ and $\mathrm{S}[L_2, \mathfrak{m}_2]$ are equal.

If (4.7) fails, say $\gamma_1 = L_1$, then we can modify $\mathrm{S}[L_1, \mathfrak{m}_1]$ without affecting τ. Indeed, if we change the weight of the point mass at L_1, this will not change the function $W_{\mathrm{S}[L_1, \mathfrak{m}_1]} \star \gamma_1$. As a side remark, let us notice that '$\gamma = L$' just means that we deal with the principal spectral function in the sense of [7].

Let us now return to the present setting of Theorem 1.5. Since our given measure τ has a point mass in $(-\infty, 0)$, we must have $\gamma_1 < L_1$ and $\gamma_2 < L_2$. In particular, (4.7) does hold. As we have shown above, this gives $\mathrm{S}[L_1, \mathfrak{m}_1] = \mathrm{S}[L_2, \mathfrak{m}_2]$. This finishes the proof of uniqueness.

The proof of Theorem 1.5 is complete. ❑

5 Interaction of the conditions (SM_3)–(SM_7)

In this section we continue the discussion started in Remark 1.8. We think of the sequence $\xi_1, \xi_2, \xi_3, \dots$ as fixed, and ask how the 'allowed band' for asymptotics of the sequence of weights may look like. If the sequence $(\xi_k)_k$ is finite, these matters are of course trivial. Hence, throughout this section, we assume that it is infinite.

Clearly, the band in Remark 1.8, (ii^+), is never void. However, its division into two parts described in (iii^+), the lower one giving rise to singular strings and the upper one to regular strings, may degenerate in the sense that the upper part is void. The band in Remark 1.8, (ii^-), may or may not be void; if it is not, its division into two parts described in (iii^-) may or may not degenerate.

The relevant conditions in this respect are convergence/divergence of the series

$$\sum_{k\in\mathbb{N}} \frac{\sigma_k}{\xi_k}, \quad \sum_{k\in\mathbb{N}} \xi_k^{-2} \frac{1}{\Gamma'(\xi_k)^2 \sigma_k}, \quad \sum_{k\in\mathbb{N}} \xi_k^{-3} \frac{1}{\Gamma'(\xi_k)^2 \sigma_k},$$

which of course depend on the actual values of σ_k. However, the structure of the allowed band (void, nonvoid, etc.) is a property of the sequence $(\xi_k)_{k\in\mathbb{N}}$ alone. It is easy to make this quantitatively precise.

5.1 Lemma. *Let $(\xi_k)_{k\in\mathbb{N}}$ be a sequence of real numbers with $0 < \xi_1 < \xi_2 < \dots$, assume that $\sum_{k\in\mathbb{N}} \frac{1}{\xi_k} < \infty$, and set*

$$\Gamma(z) := \prod_{k\in\mathbb{N}} \Big(1 - \frac{z}{\xi_k}\Big).$$

(i) *There exists a sequence $(\sigma_k)_{k\in\mathbb{N}}$ of positive real numbers, such that*

$$\sum_{k\in\mathbb{N}} \frac{\sigma_k}{\xi_k} < \infty \quad \text{and} \quad \sum_{k\in\mathbb{N}} \xi_k^{-2} \frac{1}{\Gamma'(\xi_k)^2 \sigma_k} < \infty, \tag{5.1}$$

if and only if

$$\bigg(\frac{1}{\xi_k^{\frac{3}{2}} \Gamma'(\xi_k)} \bigg)_{k\in\mathbb{N}} \in \ell^1. \tag{5.2}$$

(ii) *There exists a sequence $(\sigma_k)_{k\in\mathbb{N}}$ of positive real numbers, such that*

$$\sum_{k\in\mathbb{N}} \frac{\sigma_k}{\xi_k} < \infty \quad \text{and} \quad \sum_{k\in\mathbb{N}} \xi_k^{-3} \frac{1}{\Gamma'(\xi_k)^2 \sigma_k} < \infty,$$

if and only if

$$\bigg(\frac{1}{\xi_k^2 \Gamma'(\xi_k)} \bigg)_{k\in\mathbb{N}} \in \ell^1. \tag{5.3}$$

Proof. Assume that $(\sigma_k)_{k\in\mathbb{N}}$ is a sequence with (5.1). Then, by the Cauchy-Schwarz inequality in the space ℓ^2, we have

$$\sum_{k\in\mathbb{N}} \Big(\frac{\sigma_k}{\xi_k}\Big)^{\frac{1}{2}} \cdot \Big(\xi_k^{-2} \frac{1}{\Gamma'(\xi_k)^2 \sigma_k}\Big)^{\frac{1}{2}} \leq \Big(\sum_{k\in\mathbb{N}} \frac{\sigma_k}{\xi_k}\Big)^{\frac{1}{2}} \cdot \Big(\sum_{k\in\mathbb{N}} \xi_k^{-2} \frac{1}{\Gamma'(\xi_k)^2 \sigma_k}\Big)^{\frac{1}{2}} < \infty.$$

However, the series on the left side of this inequality is nothing but $\sum_{k\in\mathbb{N}} \xi_k^{-\frac{3}{2}} \frac{1}{|\Gamma'(\xi_k)|}$.

Conversely, assume that (5.2) holds. Set

$$\sigma_k := \frac{1}{\xi_k^{\frac{1}{2}}|\Gamma'(\xi_k)|}, \quad k \in \mathbb{N},$$

then

$$\sum_{k\in\mathbb{N}} \frac{\sigma_k}{\xi_k} = \frac{1}{\xi_k^{\frac{3}{2}}|\Gamma'(\xi_k)|} < \infty,$$

$$\sum_{k\in\mathbb{N}} \xi_k^{-2}\frac{1}{\Gamma'(\xi_k)^2\sigma_k} = \sum_{k\in\mathbb{N}} \xi_k^{-2}\frac{\xi_k^{\frac{1}{2}}|\Gamma'(\xi_k)|}{\Gamma'(\xi_k)^2} = \frac{1}{\xi_k^{\frac{3}{2}}|\Gamma'(\xi_k)|} < \infty.$$

This completes the proof of (i). Item (ii) is shown in exactly the same way. ❑

Let us note that item (i) gives in particular a new proof of a results of [18, Theorem 3].

The property whether or not the condition (5.2) or (5.3), respectively, holds, depends on the growth of the sequence $(\xi_k)_{k\in\mathbb{N}}$ and on the decay of the sequence $(\Gamma'(\xi_k))_{k\in\mathbb{N}}$. The latter depends on the separation of the points ξ_k, rather than on their speed of growth. It seems that separation is the major necessity for (5.2) or (5.3) to hold, whereas growth plays only a minor role.

It is an involved task to obtain quantitative statements, since they require below estimates for the size of the derivative of the canonical product $\Gamma(z)$. We only show three statements, which give some hints about what might be going on.

Let us recall one more notation: For a sequence $(\alpha_k)_{k\in\mathbb{N}}$ of complex numbers which has no finite accumulation point, denote

$$n_\alpha(r) := \#\big\{k \in \mathbb{N} : |\alpha_k| \le r\big\}, \quad r > 0.$$

5.2 Proposition. *Let $(\zeta_k)_{k\in\mathbb{N}}$ be a sequence of real numbers with $0 < \zeta_1 < \zeta_2 < \dots$ and $\sum_{k\in\mathbb{N}} \frac{1}{\zeta_k} < \infty$. Then there exists a sequence $(\xi_k)_{k\in\mathbb{N}}$, $0 < \xi_1 < \xi_2 < \dots$, with*

$$\lim_{r\to\infty} \frac{n_\xi(r)}{n_\zeta(r)} = 2, \tag{5.4}$$

such that (5.3) *fails.*

Proof. Whenever $(\gamma_k)_{k\in\mathbb{N}}$ is a sequence of real numbers with

$$1 < \gamma_k < \frac{\zeta_{k+1}}{\zeta_k}, \quad k \in \mathbb{N}, \tag{5.5}$$

we define a new sequence $(\xi_k)_{k\in\mathbb{N}}$ as

$$\xi_k := \begin{cases} \zeta_{\frac{k+1}{2}}, & k \text{ odd} \\ \gamma_{\frac{k}{2}}\zeta_{\frac{k}{2}}, & k \text{ even} \end{cases} \tag{5.6}$$

Due to the restriction (5.5) on the size of γ_k, we have

$$\begin{array}{ccccccccc} 0 & < & \xi_1 < \xi_2 & < & \xi_3 < \xi_4 & < & \xi_5 < \xi_6 & < \dots \\ & & \| \quad\;\; & & \| \quad\;\; & & \| \quad\;\; & \\ & & \zeta_1 \quad\;\; & & \zeta_2 \quad\;\; & & \zeta_3 \quad\;\; & \end{array}$$

The idea is to choose γ_k very close to 1, in order to force rapid decay of $|\Gamma'(\xi_{2k-1})|$. Note that γ_k being close to 1 means that the sequence $(\xi_k)_{k\in\mathbb{N}}$ is badly separated.

We construct $(\gamma_k)_{k\in\mathbb{N}}$ inductively as follows:

(1) Choose $\gamma_1 \in \big(1, \frac{\zeta_2}{\zeta_1}\big)$ arbitrarily.

(2) Let $k \geq 2$, assume that $\gamma_1, \ldots, \gamma_{k-1}$ are already chosen, and let $\xi_1, \ldots, \xi_{2k-2}$ be defined by (5.6). Choose $\gamma_k \in \big(1, \frac{\zeta_{k+1}}{\zeta_k}\big)$, such that

$$\zeta_k \prod_{l=1}^{2k-2} \Big|1 - \frac{\zeta_k}{\xi_l}\Big| \cdot \Big(1 - \frac{1}{\gamma_k}\Big) \leq 1 \,.$$

Let us show that indeed, for this choice of $(\gamma_k)_{k\in\mathbb{N}}$, the value of $|\Gamma'(\xi_{2k-1})|$ is small. To this end, write

$$\begin{aligned} \Gamma'(\xi_{2k-1}) &= \Gamma'(\zeta_k) = -\frac{1}{\zeta_k} \prod_{\substack{l\in\mathbb{N} \\ l\neq 2k-1}} \Big(1 - \frac{\zeta_k}{\xi_l}\Big) = \\ &= -\frac{1}{\zeta_k} \prod_{l=1}^{2k-2} \Big(1 - \frac{\zeta_k}{\xi_l}\Big) \cdot \underbrace{\Big(1 - \frac{\zeta_k}{\xi_{2k}}\Big)}_{=1-\frac{1}{\gamma_k}} \cdot \prod_{l\geq 2k+1} \Big(1 - \frac{\zeta_k}{\xi_l}\Big) \,. \end{aligned}$$

Since $\xi_l > \zeta_k$, $l \geq 2k+1$, we have

$$0 < 1 - \frac{\zeta_k}{\xi_l} < 1, \quad l \geq 2k+1 \,,$$

and hence the last product in the above formula is bounded by 1. By the choice of γ_k, it follows that

$$|\Gamma'(\xi_{2k-1})| \leq \frac{1}{\zeta_k^2} = \frac{1}{\xi_{2k-1}^2}, \quad k \in \mathbb{N} \,.$$

This implies that the condition (5.3) fails for the sequence $(\xi_k)_{k\in\mathbb{N}}$.

In order to show (5.4), it is enough to note that

$$n_\xi(r) = \begin{cases} 2n_\zeta(r) - 1 \,, & r \in [\xi_{2k-1}, \xi_{2k}) \\ 2n_\zeta(r) \quad\;\;\,, & r \in [\xi_{2k}, \xi_{2k+1}) \end{cases} , \quad k \in \mathbb{N} \,,$$

and that $\lim_{r\to\infty} n_\zeta(r) = \infty$. ❑

Note that, clearly, failure of (5.3) implies failure of (5.2). Proposition 5.2 thus provides us with examples of sequences $(\xi_k)_{k\in\mathbb{N}}$ which are arbitrarily distributed in the sense of upper density, as far as permitted by the requirement that the limit $\lim_{k\to\infty} \frac{k^2}{\xi_k}$ exists in $[0, \infty)$, and which have the property that the allowed band in (iii^+) is not divided, and the band in (iii^-) is void.

Examples where the band in (iii^-) is not void but still the division into two parts degenerates, are given by fairly well-separated sequences with good asympotics and minimal growth.

5.3 Proposition. *Let* $(\xi_k)_{k\in\mathbb{N}}$ *be a sequence of real numbers with* $0 < \xi_1 < \xi_2 < \dots$, *and assume that*

$$\xi_k = k^2 + k\delta_k, \quad k \in \mathbb{N}, \tag{5.7}$$

with some sequence $(\delta_k)_{k\in\mathbb{N}}$ *belonging to the space* ℓ^p *for some* $p < \infty$. *Then the condition* (5.2) *fails, but* (5.3) *holds true.*

Proof. From (5.7) we obtain

$$\big(\sqrt{\xi_k} - k\big)\big(\sqrt{\xi_k} + k\big) = \xi_k - k^2 = k\delta_k, \quad k \in \mathbb{N},$$

and hence

$$\sqrt{\xi_k} = k + \frac{\delta_k}{1 + \frac{\sqrt{\xi_k}}{k}}, \quad k \in \mathbb{N}.$$

For $k \in \mathbb{Z}$, set

$$\lambda_k := \begin{cases} \sqrt{\xi_k} & , \quad k > 0 \\ 0 & , \quad k = 0 \\ -\sqrt{\xi_k}\,, & \quad k < 0 \end{cases}, \qquad d_k := \begin{cases} (1 + \frac{\sqrt{\xi_k}}{k})^{-1}\delta_k & , \quad k > 0 \\ 0 & , \quad k = 0 \\ -(1 + \frac{\sqrt{\xi_k}}{k})^{-1}\delta_k\,, & \quad k < 0 \end{cases}$$

so that

$$\lambda_k = k + d_k, \quad k \in \mathbb{Z}.$$

Since $\lim_{k\to\infty} \frac{\sqrt{\xi_k}}{k} = 1$, we have $(d_k)_{k\in\mathbb{N}} \in \ell^p$. In particular, we may choose $N \in \mathbb{N}$ with $|d_k| \leq \frac{1}{3}$, $|k| \geq N$. It follows that

$$\big|\lambda_k - \lambda_l\big| \geq \frac{1}{3}, \quad k \neq l,\ |k| \geq N \wedge |l| \geq N.$$

Since the points λ_k are all different from each other, we have $\inf_{\substack{l\in\mathbb{Z}\\ l\neq k}} |\lambda_k - \lambda_l| > 0$, $k \in \mathbb{Z}$. Together, this implies the separation condition

$$\inf_{\substack{k,l\in\mathbb{Z}\\ k\neq l}} \big|\lambda_k - \lambda_l\big| > 0.$$

Let $m \in \mathbb{Z}$ be given. Then we can apply Hölder's inequality ($\frac{1}{p} + \frac{1}{q} = 1$) to estimate

$$\sum_{k\in\mathbb{Z}} |d_{k+m}| \frac{k}{k^2+1} \leq \underbrace{\|(d_{k+m})_{k\in\mathbb{Z}}\|_p}_{=\|(d_k)_{k\in\mathbb{Z}}\|_p} \cdot \Big\| \Big(\frac{k}{k^2+1} \Big)_{k\in\mathbb{Z}} \Big\|_q.$$

It follows that the sums

$$\sum_{k\in\mathbb{Z}} \big(d_{k+m} - d_k\big) \frac{k}{k^2+1}, \quad m \in \mathbb{Z},$$

are uniformly bounded with respect to $m \in \mathbb{Z}$. By [21, §21.1, Theorem 2], there exists an entire function G of exponential type at most π, which is bounded along the real line and solves the interpolation problem

$$G(k) = (-1)^k d_k, \quad k \in \mathbb{Z}.$$

Thus, the conditions of [21, §22.2, p.168, Corollary] are satisfied, and we conclude that the product

$$P(z) = z \lim_{R\to\infty} \prod_{0<\lambda_k\leq R} \Big(1 - \frac{z}{\lambda_k}\Big)$$

is a function of sine type. By [21, §22.1, Lemma 2], there exist constants $0 < c < C$, such that

$$c \leq |P'(\lambda_k)| \leq C, \quad k \in \mathbb{Z}\,.$$

However, by symmetry of the sequence $(\lambda_k)_{k\in\mathbb{Z}}$,

$$P(z) = z \prod_{k\in\mathbb{N}} \Big(1 - \frac{z^2}{\lambda_k^2}\Big) = z\Gamma(z^2)\,.$$

Thus $P'(z) = \Gamma(z^2) + 2z^2\Gamma'(z^2)$, and, in particular,

$$P'(\lambda_k) = 2\xi_k\Gamma'(\xi_k), \quad k \in \mathbb{N}\,.$$

It follows that $\frac{c}{2\xi_k} \leq |\Gamma'(\xi_k)| \leq \frac{C}{2\xi_k}$, $k \in \mathbb{N}$, and hence

$$\frac{2}{C\xi_k} \leq \frac{1}{\xi_k^2|\Gamma'(\xi_k)|} \leq \frac{2}{c\xi_k}, \quad k \in \mathbb{N}\,, \tag{5.8}$$

$$\frac{2}{C\sqrt{\xi_k}} \leq \frac{1}{\xi_k^{\frac{3}{2}}|\Gamma'(\xi_k)|} \leq \frac{2}{c\sqrt{\xi_k}}, \quad k \in \mathbb{N}\,, \tag{5.9}$$

Since $\lim_{k\to\infty} \frac{\xi_k}{k^2} = 1$, the estimate from above in (5.8) shows that (5.3) holds true, whereas the estimate from below in (5.9) shows that (5.2) fails. ❑

Finally, in the third statement, we provide examples where the conditions (5.2) and (5.3) both hold true, i.e. where the band in (iii^-) is nonvoid, and the division into two parts is proper. In order to obtain sequences with this property, it is enough to slightly stretch out a sequence with the asymptotics (5.7).

5.4 Proposition. *Let $(\gamma_k)_{k\in\mathbb{N}}$ a sequence of real numbers with*

$$0 < \gamma_1 \leq \gamma_2 \leq \dots \quad \text{and} \quad \sum_{k\in\mathbb{N}} \frac{1}{k\gamma_k^{\frac{1}{2}}} < \infty\,,$$

and let $(\zeta_k)_{k\in\mathbb{N}}$, $0 < \zeta_1 < \zeta_2 < \dots$, be a sequence with the asymptotics (5.7). *Then, for the sequence $(\xi_k)_{k\in\mathbb{N}}$ defined as*

$$\xi_k := \gamma_k\zeta_k, \quad k \in \mathbb{N}\,,$$

the condition (5.2) *holds true.*

Proof. Set $\tilde\Gamma(z) := \prod_{k\in\mathbb{N}}\big(1 - \frac{z}{\zeta_k}\big)$. Then the estimate from above in (5.9) applies to $\tilde\Gamma$. Together with the fact that $\lim_{k\to\infty} \frac{\zeta_k}{k^2} = 1$, this furnishes us with a constant $\tilde C > 0$ such that

$$\frac{1}{\zeta_k|\tilde\Gamma'(\zeta_k)|} \leq \frac{\tilde C}{k}, \quad k \in \mathbb{N}\,.$$

Using monotonicity of the sequence $(\gamma_k)_{k\in\mathbb{N}}$ and the definition of ξ_k, we can estimate

$$|\Gamma'(\xi_k)| \geq \frac{1}{\gamma_k}|\tilde{\Gamma}'(\zeta_k)|, \quad k \in \mathbb{N},$$

cf. [1, Proof of Lemma 3.3]. It follows that

$$\frac{1}{\xi_k^{\frac{3}{2}}|\Gamma'(\xi_k)|} \leq \frac{1}{(\gamma_k\zeta_k)^{\frac{3}{2}}| \cdot \frac{1}{\gamma_k}|\tilde{\Gamma}'(\zeta_k)|} = \frac{1}{\gamma_k^{\frac{1}{2}}\zeta_k^{\frac{3}{2}}|\tilde{\Gamma}'(\zeta_k)|} \leq \frac{\tilde{C}}{\gamma_k^{\frac{1}{2}} \cdot k}, \quad k \in \mathbb{N}.$$

The assumption of the present proposition thus ensures that (5.2) holds. ❑

References

[1] A.Baranov, H.Woracek: *Subspaces of de Branges spaces with prescribed growth,* Algebra and Analysis 18(5) (2006), 23–45 / St.Petersburg Math. J. 18 (2007), 699-716.

[2] L.de Branges: *Hilbert spaces of entire functions,* Prentice-Hall, London 1968.

[3] H. Dym, H. McKean: *Gaussian processes, function theory, and the inverse spectral problem,* Academic Press, New York, 1976.

[4] I.Gohberg, M.G.Krein: *Theory and applications of Volterra operators in Hilbert space (Russian),* Izdat. 'Nauka', Moscow 1967. English translation: Translations of Mathematical Monographs, Amer. Math. Soc., Providence, Rhode Island, 1970.

[5] I.S.Kac: *Spectral theory of a string,* Ukrainian Math. J. 46(3) (1994), 159–182.

[6] I.S.Kac and M.G.Krein: *R-functions–analytic functions mapping the upper halfplane into itself (Russian),* in F.V.Atkinson, Discrete and Continuous Boundary Problems (Russian translation), Moscow, Mir, 1968, 648–737 (Addition II). English translation: Amer. Math. Soc. Transl. (2) 103 (1974), 1–18.

[7] I.S.Kac, M.G.Kreĭn: *On spectral functions of a string (Russian),* in F.V.Atkinson, Discrete and Continuous Boundary Problems (Russian translation), Moscow, Mir, 1968, 648–737 (Addition II). English translation: Amer. Math. Soc. Transl. (2) 103 (1974), 19–102.

[8] M.Kaltenbäck, H.Winkler, H.Woracek: *Generalized Nevanlinna functions with essentially positive spectrum,* J. Oper. Theory 55(1) (2006), 101–132.

[9] M.Kaltenbäck, H.Winkler, H.Woracek: *Singularities of generalized strings,* Oper. Theory Adv. Appl. 163 (2006), 191–248.

[10] M.Kaltenbäck, H.Winkler, H.Woracek: *Strings, dual strings and related canonical systems,* Math. Nachr. 280 (13/14) (2007), 1518–1536.

[11] M.Kaltenbäck, H.Woracek: *Pontryagin spaces of entire functions I,* Integral Equations Operator Theory 33 (1999), 34–97.

[12] M.Kaltenbäck, H.Woracek: *Pontryagin spaces of entire functions II,* Integral Equations Operator Theory 33 (1999), 305–380.

[13] M.Kaltenbäck, H.Woracek: *Pontryagin spaces of entire functions III,* Acta Sci.Math. (Szeged) 69 (2003), 241–310.

[14] M.Kaltenbäck, H.Woracek: *Pontryagin spaces of entire functions IV,* Acta Sci. Math. (Szeged) 72 (2006), 709–835.

[15] M.Kaltenbäck, H.Woracek: *Pontryagin spaces of entire functions V,* Acta Sci.Math. (Szeged), to appear. Preprint available online as ASC Preprint Series 21/2009, http://asc.tuwien.ac.at.

[16] M.Kaltenbäck, H.Woracek: *Pontryagin spaces of entire functions VI,* Acta Sci.Math. (Szeged), to appear. Preprint available online as ASC Preprint Series 22/2009, http://asc.tuwien.ac.at.

[17] M.Kaltenbäck, H.Woracek: *Canonical differential equations of Hilbert–Schmidt type,* Oper. Theory Adv. Appl. 175 (2007), 159–168.

[18] M.G.Kreĭn: *On inverse problems for a nonhomogeneous string (Russian),* Doklady Akad. Nauk SSSR (N.S.) 82 (1952), 669–672.

[19] M.G.Kreĭn: *On a generalization of investigations of Stieltjes (Russian),* Doklady Akad. Nauk SSSR (N.S.) 87 (1952), 881–884.

[20] M.G.Kreĭn: *On some cases of effective determination of the density of an inhomogeneous cord from its spectral function (Russian),* Doklady Akad. Nauk SSSR (N.S.) 93 (1953), 617–620.

[21] B.Ya.Levin: *Lectures on entire functions,* Translated from the Russian manuscript by V.Tkachenko. Translations of Mathematical Monographs 150. Amer. Math. Soc., Providence, RI, 1996.

[22] C.Remling: *Schrödinger operators and de Branges spaces,* J. Funct. Anal. 196(2) (2002), 323–394.

[23] H.Winkler: *The inverse spectral problem for canonical systems,* Integral Equations Operator Theory, 22 (1995), 360-374.

[24] H.Winkler: *Canonical systems with a semibounded spectrum,* Oper. Theory Adv. Appl. 106 (1998), 397–417.

[25] H.Winkler, H.Woracek: *On semibounded canonical systems,* Lin. Alg. Appl. 429 (2008), 1082–1092.

[26] H.Woracek: *Existence of zerofree functions N-associated to a de Branges Pontryagin space,* Monatsh. Math., to appear (DOI: 10.1007/s00605-010-0203-2). Preprint available online as ASC Preprint Series 23/2009, http://asc.tuwien.ac.at.

H. Woracek
Institut für Analysis und Scientific Computing
Technische Universität Wien
Wiedner Hauptstraße. 8–10/101
A–1040 Wien
AUSTRIA
email: harald.woracek@tuwien.ac.at