A local inverse spectral theorem for Hamiltonian systems

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Abstract

We consider 2×2–Hamiltonian systems of the form $y'(t) = zJH(t)y(t)$, $t \in [s_-, s_+)$. If a system of this form is in the limit point case, an analytic function is associated with it, namely its Titchmarsh–Weyl coefficient q_H. The (global) uniqueness theorem due to L. de Branges says that the Hamiltonian H is (up to reparameterization) uniquely determined by the function q_H. In the present paper we give a local uniqueness theorem: if the Titchmarsh–Weyl coefficients q_{H_1} and q_{H_2} corresponding to two Hamiltonian systems are exponentially close, then the Hamiltonians H_1 and H_2 coincide (up to reparameterization) up to a certain point of their domain, which depends on the quantitative degree of exponential closeness of the Titchmarsh–Weyl coefficients.

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1 Introduction

a. Formulation of the main result.

We consider a 2×2-Hamiltonian system without potential, i.e. a 2×2-system of the form

$$y'(t) = zJH(t)y(t), \quad t \in [s_-, s_+), \tag{1.1}$$

with a function $H(t)\colon [s_-, s_+) \to \mathbb{C}^{2\times 2}$ which is locally integrable on $[s_-, s_+)$, takes non-negative matrices as values, and does not vanish identically on any set of positive measure. Moreover, z is a complex parameter, and J denotes the signature matrix

$$J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The function $H(t)$ is also called the Hamiltonian of the system (1.1).

Equation (1.1) generates in a natural way a differential operator (actually, in some cases it is a linear relation, i.e. a multi-valued operator). It acts in a certain weighted L^2-space $L^2(H)$ whose elements are 2-vector valued functions; see, e.g. [Ka1], [O], [HSW]. The spectral theory of this operator changes tremendously depending whether the integral $\int_{s_-}^{s_+} \operatorname{tr} H(t)\,dt$ is finite or infinite; in the first case one says that the Hamiltonian H (or the system (1.1)) is in Weyl's limit circle case, in the latter one speaks of limit point case. In the limit point case the operator mentioned above has deficiency indices $(1,1)$; in the limit circle case it has deficiency indices $(2,2)$.

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Of course, 'changes of scale' in the equation (1.1) will not affect the spectral theory of the associated differential operator. The notion of 'changes of scale', however, needs to be defined rigorously. Two Hamiltonians H_1 and H_2 defined on respective intervals (s_-^1, s_+^1) and (s_-^2, s_+^2) are called *reparameterizations* of each other if there exists an absolutely continuous, increasing bijection $\varphi\colon (s_-^2, s_+^2) \to (s_-^1, s_+^1)$ such that φ^{-1} is also absolutely continuous and $H_2(t) = H_1(\varphi(t)) \cdot \varphi'(t)$, $t \in (s_-^2, s_+^2)$. In this case, we write $H_1 \sim H_2$. We have the following connection between solutions in this situation: if y_1 is a solution of (1.1) with $H = H_1$, then $y_2(t) = y_1(\varphi(t))$ is a solution of (1.1) with $H = H_2$ and vice versa.

Consider a Hamiltonian H which is in the limit point case and denote by $W(x,z) = (w_{ij}(x,z))_{i,j=1,2}$ the transpose of the fundamental solution of (1.1), i.e. the solution of the initial value problem

$$\frac{d}{dt} W(t,z)J = zW(t,z)H(t), \quad t \in (s_-, s_+), \qquad W(s_-, z) = I, \tag{1.2}$$

where I denotes the 2×2-identity matrix. Then, for each $\tau \in \mathbb{R} \cup \{\infty\}$, the limit

$$\lim_{t \nearrow s_+} \frac{w_{11}(t,z)\tau + w_{12}(t,z)}{w_{21}(t,z)\tau + w_{22}(t,z)} =: q_H(z)$$

exists locally uniformly on $\mathbb{C} \setminus \mathbb{R}$ and does not depend on τ. For $\tau = \infty$, the quotient is understood as $w_{11}(t,z)/w_{21}(t,z)$. The function q_H is called the *Titchmarsh–Weyl coefficient* (in some areas better known as the *Weyl m-function*) associated with the Hamiltonian H. It belongs to the Nevanlinna class $\mathcal{N}_0$, which means that

$$q_H \text{ is analytic on } \mathbb{C} \setminus \mathbb{R}, \qquad q_H(\overline{z}) = \overline{q_H(z)},\ z \in \mathbb{C} \setminus \mathbb{R},$$
$$\operatorname{Im} q_H(z) \geq 0 \quad \text{for } \operatorname{Im} z > 0.$$

This construction is vital for the spectral theory of the equation (1.1); for example the function q_H can be used to construct a Fourier transform of the space $L^2(H)$ onto a certain L^2-space of scalar valued functions, namely the space $L^2(\mu)$ where μ is the measure in the Herglotz integral representation of the function q_H (appropriately including a possible point mass at infinity). The name Titchmarsh–Weyl *coefficient* comes from the fact that the function

$$y(t) = \begin{pmatrix} w_{11}(t,z) \\ w_{12}(t,z) \end{pmatrix} - q_H(z) \begin{pmatrix} w_{21}(t,z) \\ w_{22}(t,z) \end{pmatrix} \tag{1.3}$$

is the unique (up to scalar multiples) solution of (1.1) that is in $L^2(H)$.

It is a fundamental result due to L.de Branges that for each function $q \in \mathcal{N}_0$ there exists (up to reparameterization) one and only one Hamiltonian H such that $q = q_H$, cf. [dB], [W1]. The uniqueness part of this result may be formulated in the following way.

1.1. *Uniqueness theorem* ([dB]). *Let H_1 and H_2 be Hamiltonians defined on intervals $[s_-^1, s_+^1)$ and $[s_-^2, s_+^2)$, respectively. Assume that H_1 and H_2 are in the limit point case and denote by q_{H_1} and q_{H_2} their respective Titchmarsh–Weyl coefficients. Then the following are equivalent.*

(i) $H_1 \sim H_2$.

(ii) $q_{H_1} = q_{H_2}$.

Our aim in this paper is to prove a refinement of this theorem, namely the following local version.

1.2. *Local uniqueness theorem.* *Let H_1 and H_2 be Hamiltonians defined on intervals $[s^1_-, s^1_+)$ and $[s^2_-, s^2_+)$, respectively. Assume that H_1 and H_2 are in the limit point case, and denote by q_{H_1} and q_{H_2} their respective Titchmarsh–Weyl coefficients. Moreover, let $a > 0$ and set*

$$s^j_a := \sup\Big\{t \in [s^j_-, s^j_+)\colon \int_{s^j_-}^t \sqrt{\det H_j(x)}\,dx < a\Big\}, \quad j = 1, 2\,.$$

Then the following are equivalent.

(i) $H_1\big|_{[s^1_-, s^1_a)} \sim H_2\big|_{[s^2_-, s^2_a)}$.

(ii) *There exists $\theta \in (0, \pi)$ such that for every $\varepsilon > 0$,*

$$q_{H_1}(re^{i\theta}) - q_{H_2}(re^{i\theta}) = O\big(e^{(-2a+\varepsilon)r\sin\theta}\big), \quad r \to +\infty\,.$$

(iii) *Denote by Γ_α, $\alpha \in (0, \frac{\pi}{2})$, the Stolz angle $\Gamma_\alpha := \{z \in \mathbb{C}\colon \alpha \le \arg z \le \pi - \alpha\}$. For every $\alpha \in (0, \frac{\pi}{2})$ we have*[1]

$$q_{H_1}(z) - q_{H_2}(z) = O\big((\operatorname{Im} z)^3 e^{-2a\operatorname{Im} z}\big), \quad |z| \to \infty, \;\; z \in \Gamma_\alpha\,.$$

We should say it very explicitly that, although the local theorem 1.2 is a refinement of de Branges' global uniqueness theorem 1.1, we do not obtain a new proof of Theorem 1.1 since this result enters in our proof of Theorem 1.2.

Moreover, let us note that Hamiltonian systems in the limit circle case are also covered by our uniqueness result; simply by prolonging them in an appropriate way, e.g., by putting an additional boundary condition at the right endpoint. However, if it is a priori known that both Hamiltonians under consideration are in the limit circle case, then one could use an alternative, simpler, argument for the proof, being based on classical bounded type theory and Phragmén–Lindelöf principles.

1.3 Remark. The local uniqueness theorem analogous to 1.2 remains true in the setting of indefinite Hamiltonian systems as introduced and studied in [KW/IV]–[KW/VI] [2]. The proof is word by word the same, only in some places one has to refer to Pontryagin space theory instead of classical Hilbert space results. In order to avoid the somewhat tedious introduction of these notions, we decided not to formulate the general 'indefinite' result. We will content ourselves with indicating the proper references in footnotes.

This observation becomes valuable when considering local uniqueness theorems for Sturm–Liouville equations with singular potentials; some examples of such equations have been studied, e.g. in [LLS], [HM], [FL]. However, in the present paper, we will not touch upon this topic. //

[1] Maybe the power $(\operatorname{Im} z)^3$ is not the best growth estimate one can get. However, it is not far from optimal; it is easy to construct examples with $q_{H_1}(z) - q_{H_2}(z) \asymp e^{-2a\operatorname{Im} z}$.

[2] Only the power in (iii) has to be adapted.

The following fact, although a simple consequence of 1.2, is worth being stated separately.

1.4 Corollary. *Let H_1 and H_2 be Hamiltonians defined on intervals $[s_-^1, s_+^1)$ and $[s_-^2, s_+^2)$, respectively. Each of the following two conditions implies that $H_1 \sim H_2$.*

(i) *Assume that $\alpha := \int_{s_-^1}^{s_+^1} \sqrt{\det H_1(x)}\, dx < \infty$ and that for some $\theta \in (0, \pi)$ and some $\beta > \alpha$*

$$q_{H_1}(re^{i\theta}) - q_{H_2}(re^{i\theta}) = O\big(e^{-2\beta r \sin\theta}\big), \quad r \to +\infty. \tag{1.4}$$

(ii) *Let $a > 0$, and assume that $H_1|_{(s_a^1, s_+^1)} \sim H_2|_{(s_a^2, s_+^2)}$, and that for some $\theta \in (0, \pi)$ and all $\varepsilon > 0$*

$$q_{H_1}(re^{i\theta}) - q_{H_2}(re^{i\theta}) = O\big(e^{(-2a+\varepsilon) r \sin\theta}\big), \quad r \to +\infty.$$

We can also deduce a local Borg-Marchenko uniqueness result for Sturm–Liouville equations without potential from Theorem 1.2. Because of its relevance for applications, let us state this fact explicitly.

1.5 Proposition. *Let $b_1, b_2 > 0$ and let p_j, w_j be measurable functions defined on $[0, b_j)$, $j = 1, 2$, such that $p_j(x) > 0$, $w_j(x) > 0$ almost everywhere and $\frac{1}{p_j}, w_j \in L^1_{\mathrm{loc}}([0, b_j))$ for $j = 1, 2$. Moreover, denote by m_j the Titchmarsh–Weyl coefficient for the Sturm–Liouville equation*

$$-(p_j y')' = \lambda w_j y.$$

Let $a > 0$ and set

$$s_a^j := \int_0^a \sqrt{\frac{w_j(x)}{p_j(x)}}\, dx.$$

Then the following statements are equivalent.

(i) *There exists an absolutely continuous, increasing function $\varphi\colon (0, s_a^2) \to (0, s_a^1)$ such that φ^{-1} is also absolutely continuous and*

$$\left.\begin{aligned} w_2(t) &= w_1\big(\varphi(t)\big)\varphi'(t) \\ p_2(t) &= p_1\big(\varphi(t)\big)\frac{1}{\varphi'(t)} \end{aligned}\right\} \quad \text{for almost all } t \in (0, s_a^2). \tag{1.5}$$

(ii) *There exists $\theta \in (0, 2\pi)$ such that for every $\varepsilon > 0$,*

$$m_1(re^{i\theta}) - m_2(re^{i\theta}) = O\big(e^{(-2a+\varepsilon)\sqrt{r}\sin\frac{\theta}{2}}\big), \quad r \to +\infty.$$

(iii) *For every $\alpha \in (0, \pi)$,*

$$m_1(\lambda) - m_2(\lambda) = O\Big(|\lambda|^2 e^{-2a \operatorname{Im}\sqrt{\lambda}}\Big),$$

$$|\lambda| \to \infty,\ \lambda \in \{z \in \mathbb{C}\colon \alpha \le \arg z \le 2\pi - \alpha\}.$$

Let us note that, similar as in Theorem 1.2/Corollary 1.4, the corresponding corollary of Proposition 1.5 holds true.

b. History and relation to previous work.

Generally speaking, an inverse problem is the task to find a property of a system or a medium from its response to a probing signal. Solutions to inverse problems enable us to remotely sense or non-destructively evaluate the system. For example one could think of the task to find caves by collecting scattering data of acoustic waves sent from the surface of the earth, of the task to determine the structure of a DNA from x-ray measurements, or of the task to reconstruct the shape of a membrane from its natural frequencies of vibration, see, e.g. [Si2], [CCPR], [R], [K].

Let us turn to inverse problems for the Sturm–Liouville equation

$$-\big(p(x)y'(x)\big)' + q(x)y(x) = \lambda w(x)y(x) \tag{1.6}$$

for an unknown function $y(x)$. Equations of this kind arise from many partial differential equations important in physical context, for example: In quantum mechanics from the time-independent Schrödinger equation (where (1.6) is in potential form, i.e. with $p = r = 1$), from the Helmholtz equation when studying acoustic scattering in waveguides (where (1.6) is in its general form, i.e. with all coefficients being present), cf. [CDCO], or in geophysics from an elliptic equation modelling the propagation of waves in the earth's crust (where (1.6) is in potentialless form, i.e. with $q = 0$), cf. [BB], [McL].

Questions: Is p, q, w uniquely determined by data accessible to measurement, for example eigenfrequencies, impedance function, reflexion coefficients, etc.? If this is not the case, can at least parts of p, q, w be recovered? Having available only noisy data, how much can still be said?

Answers #1; Global uniqueness: As a starting point of modern theory one may regard the paper [Am]. There the equation

$$-y''(x) + \big(q(x) - \lambda\big)y(x) = 0, \quad x \in [0,1]\,, \tag{1.7}$$

is studied, and it is shown that the potential q must vanish identically, when the eigenvalues of the selfadjoint operator obtained by imposing Dirichlet boundary conditions at both endpoints are equal to $n^2\pi$, $n \in \mathbb{N}$. The definite result in this direction was proved later, in [Bor1], where it is shown that two spectra of (1.7) are always sufficient to determine the potential uniquely. Equivalently, one may say that the impedance function (Weyl m-function) fully determines the potential. This fact was recognised as an important result, and various proofs, variations, generalisations, reconstruction methods, etc., were given. As examples, let us mention [M1], [GL], [Bor2], [M2], [Sak], [GS1], [BW], [BBW]. Among them, in some sense as a generalisation, there appears the above mentioned Theorem 1.1 of L.de Branges, cf. [dB]. This result is particular, since it can be viewed a posteriori as the mother of many different inverse theorems which have been established over the years. Moreover, up to the present day it gives rise to interesting new implications.

Answers #2; Local Uniqueness: Local theorems seek to recover a part of the potential from asymptotic knowledge of spectral or scattering data. Results of this type came up only comparatively recently, starting with [Si1] who studied the equation (1.7) on the half-line. This paper has initiated growing interest in local versions of inverse spectral results, and has been extended over the past ten years in several directions; as examples let us mention [GS2], [Be], [CG], [GKM], [S1], [CGR], [S2], [GZ], [CGZ], [S3].

Our main results Theorem 1.2 and Proposition 1.5 fall in this category; asymptotic knowledge on the Titchmarsh–Weyl coefficient partially determines the Hamiltonian.

Answers #3; Reconstruction from incomplete or mixed data: Unlike in the case treated by Ambartsumyan, in general one spectrum of (1.7) is not enough to recover the potential. Similarly, knowledge of the reflexion coefficients of a full–line problem will not suffice. Still, many things can be said. For example, if one spectrum is completely known but the second is missing some information, then q can be recovered modulo a sum of eigenfunctions from the missing part, cf. [H]. Or, if the potential is known on half of the interval, then one spectrum is enough to recover q, cf. [HL]. Or, if in the full–line problem it is a priori known that the potential vanishes on the negative half-line or decays sufficiently fast, then the reflexion coefficients will be enough, cf. [R, §3.7]. A variety of other results in this direction can be found in the literature, see, e.g., [An1], [An2], [LPR].

The above stated Corollary 1.4, as well as the corresponding corollary of Proposition 1.5, fall in this category; the Hamiltonian is fully determined by some a priori knowledge on H_i (integrable square root of determinant or equality from some point on) and some partial information on the Titchmarsh–Weyl coefficients (sufficiently fast exponential closeness).

Relation to previous work: Roughly speaking, previous papers on local uniqueness deal with equations of the form

$$J_m\Psi'(z,t) = (zH + V(t))\Psi(z,t)$$

where $m \geq 1$, Ψ is m-vector valued,

$$J_m := \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}$$

V is a $2m \times 2m$-matrix valued function (possessing specific properties varying from case to case), and where H is a constant $2m \times 2m$-matrix (of a particular form varying from case to case).

In contrast to this setup, we consider the situation where the coefficient H of the spectral parameter may depend on t, but no potential V is present and $m = 1$. Allowing H to be time–dependent changes the face of affairs dramatically. For example, if $m > 1$, not even a global uniqueness result holds true. If H is diagonal and has sufficiently smooth (e.g. twice continuously differentiable) entries, a Liouville transformation can be applied to reduce to a Sturm–Liouville problem in potential form, see, e.g., [CCPR, §3.2], and for equations of this kind a vast body of results is available. Other cases, non–smooth coefficients or presence of non–diagonal entries, can up to our knowledge not be reduced to what is already known.

The methods employed in previous work in order to investigate local inverse problems are either similar to the classical Borg-Marchenko approach, to Simon's method, use Sakhnovich's method of operator identities, or rely on exact asymptotics of Weyl-functions and complex analysis methods. Our method for proving the present local uniqueness theorem 1.2 is based on the relationship between de Branges' theory of Hilbert spaces of entire functions, cf. [dB], and the theory of positive definite functions, cf. e.g. [Sa], [AG]. The interaction between these notions has been revealed in [K]. Moreover, we invoke some classical results of complex analysis, cf. e.g. [Ko]. The closest relative to our method among the above cited papers is probably [Be]. Attempts were undertaken to generalize the approach of Bennewitz to the equation (1.1), but, up to our knowledge, did not succeed.

c. Some applications.

We close this introduction with two examples from physics, where the present results can be applied.

1.6 Example (An inverse problem of reflection seismology). In exploration seismology an impulsive or vibrating force is applied at the ground level. This launches elastic waves into the earth's interior which are partially reflected by inhomogenities like caves or interfaces between geological strata. It is required to reconstruct the earths desity profile from measurement of this reflection data. In an idealized setup one assumes that the equations of isotropic elasticity hold, and that the density depends only on the depth measured from the surface, cf. [BB], [McL].

The linear spectral problem corresponding to the associated hyperbolic system of the form

$$-\big(p(x)y'(x)\big)' = \omega^2\rho(x)y(x), \quad x \geq 0\,, \tag{1.8}$$

where x measures the depth from the surface, $\rho(x)$ is the density of the media, $p(x) = \lambda(x) + 2\mu(x)$ with the Lamé parameters λ, μ, and ω is the eigenvalue parameter. The Weyl m–function associated with the equation (1.8) is determined, e.g., by two sets of eigenfrequencies (its poles and zeros), or its poles and residues.

The global uniqueness theorem now tells us that the density profile is uniquely determined by the Weyl m–function. Contrasting this, the local uniqueness theorem says that approximate knowledge of the Weyl m–function (in the sense of exponential asymptotics) determines the density $\rho(x)$ up to a certain depth.

If the density profile were sufficiently smooth, one could transform (1.8) to an equation in potential form and apply previous results to obtain this local knowledge on ρ (at least on some transform of ρ). However, since we deal with a layered media (thinking of caves or different layers of soil), the density profile is not even continuous. //

1.7 Example (Propagation of shallow water waves). In [CH] the Camassa–Holm equation

$$u_t - u_{xxt} = -3uu_x + 2u_xu_{xx} + uu_{xxx} \tag{1.9}$$

was derived from the Green–Naghdi equation as a model for shallow water waves. The Green–Naghdi equation itself derives from Euler's equations for an inviscid incompressible fluid with uniform density, cf. [GN]. Camassa and Holm constructed a class of special solutions, the multi–peakon solutions, which interact similar as soliton solutions of the Korteweg–de Vries equation, and are shown to be of particular importance.

The linear spectral problem associated with the Camassa–Holm equation (1.9) can be transformed to a Sturm–Liouville equation in potentialless form with a simple substitution. This fact can, for example, be exploited to relate multi–peakon solutions with classical moment problems and Stieltjes' theory of continued fractions and thereby obtain direct and inverse spectral results, cf. [BSS1], [BSS2], and [BSS3], see also the review article [BSS4].

The coefficients of the occuring Sturm–Liouville equations are in general not smooth; for example the coefficients in equations giving rise to multi–peakon solutions are step functions. //

2 Proof of the local uniqueness theorem

We need to recall the relations among Hamiltonian systems, de Branges spaces and positive definite functions in some detail.

2.1. Some classes of functions. Besides Hamiltonians and the class $\mathcal{N}_0$ of Nevanlinna functions, some other classes of functions have to be specified.

(*i*) A continuous function $f\colon \mathbb{R} \to \mathbb{C}$ is called *positive definite* if $f(-t) = \overline{f(t)}$, $t \in \mathbb{R}$, and if the kernel

$$K_f(s,t) := f(t-s), \qquad t,s \in \mathbb{R},$$

is positive definite, i.e. the matrices $\big(K_f(t_i,t_j)\big)_{i,j=1}^n$ are positive semi-definite for all choices of $n \in \mathbb{N}$ and $t_1,\dots,t_n \in \mathbb{R}$.

(*ii*) An entire functions is said to belong to the *Hermite–Biehler class* $\mathcal{HB}_0$ if $|E(\overline{z})| < |E(z)|$, $z \in \mathbb{C}^+$, where $\mathbb{C}^+$ denotes the upper half-plane. Equivalently, we could require that the kernel

$$K_E(w,z) := \frac{i}{2}\,\frac{E(z)\overline{E(w)} - \overline{E(\overline{z})}E(\overline{w})}{z-\overline{w}}\,, \qquad z,w \in \mathbb{C},$$

is positive definite (for $z = \overline{w}$ this formula has to interpreted appropriately as a derivative).

Each function $E \in \mathcal{HB}_0$ generates a reproducing kernel Hilbert space $\mathcal{H}(E)$ via the kernel K_E whose elements are entire functions. The reproducing kernel property is

$$F(w) = \big(F, K_E(w,\cdot)\big)_{\mathcal{H}(E)}, \qquad F \in \mathcal{H}(E),\ w \in \mathbb{C}.$$

The space $\mathcal{H}(E)$ is called the *de Branges space* generated by E.

(*iii*) An entire 2×2-matrix-valued function $W(z)$ is said to belong to the class $\mathcal{M}_0$ if $W(\overline{z}) = \overline{W(z)}$, $W(0) = I$ and if the kernel

$$K_W(w,z) := \frac{W(z)JW(w)^* - J}{z-\overline{w}}\,, \qquad z,w \in \mathbb{C},$$

is positive definite (for $z = \overline{w}$ this formula again has to interpreted appropriately as a derivative). Each function $W \in \mathcal{M}_0$ generates a reproducing kernel Hilbert space $\mathfrak{K}(W)$ via the kernel K_W whose elements are 2-vector-valued entire functions.

//

2.2. We are going to explain the relations among all these objects according to the following diagram:

$$
\begin{tikzcd}
\big(H(t)\big)_{t\in[s_-,s_+)} \arrow[r, leftrightarrow, "(1)"] \arrow[d, leftrightarrow, "(3)"'] & \big(W(t,z)\big)_{t\in[s_-,s_+)} \arrow[r, leftrightarrow, "(2)"] & \big(E_{W(t,\cdot)}(z)\big)_{t\in[s_-,s_+)} \arrow[d, leftrightarrow, "(4)"] \\
q_H(z) \arrow[r, leftrightarrow, "(5)"] & f_H(x) \arrow[r, leftrightarrow, "(6)"] & \big(\mathcal{H}(E_{W(t,\cdot)})\big)_{t\in[s_-,s_+)}
\end{tikzcd}
\tag{2.1}
$$

Relation (1)*:* A Hamiltonian $H(t)$ gives rise to a family of matrix functions $W(t,z)$ via (1.2). These matrices have the property that $W(s,z)^{-1}W(t,z) \in \mathcal{M}_0$ whenever $s,t \in [s_-,s_+)$, $s \leq t$; in particular, $W(t,z) \in \mathcal{M}_0$, $t \in [s_-,s_+)$. Conversely, the fundamental solution of a Hamiltonian system determines its Hamiltonian uniquely. These classical facts can be found, e.g. in [GK], [HSW], [O], [dB] [3].

Note that for two Hamiltonians H_1, H_2 we have $H_1 \sim H_2$ with reparameterization φ, i.e. $H_2(t) = H_1(\varphi(t))\varphi'(t)$ if and only if $W_2(t) = W_1(\varphi(t))$ where W_j is the fundamental solution corresponding to H_j.

Relation (2)*:* If $W(z) = (w_{ij}(z))_{i,j=1,2}$ belongs to the class $\mathcal{M}_0$, then the function

$$E_W(z) := w_{22}(z) + iw_{21}(z)$$

is a Hermite–Biehler function. It satisfies $E_W(0) = 1$ and the de Branges space generated by E_W is invariant under forming difference quotients, i.e.

$$F \in \mathcal{H}(E_W) \quad \Longrightarrow \quad \frac{F(z)-F(w)}{z-w} \in \mathcal{H}(E_W),\ w \in \mathbb{C}\,.$$

Conversely, if $E \in \mathcal{HB}_0$ possesses these two additional properties, then there exists a function $W \in \mathcal{M}_0$ with $E = E_W$. The function W can be chosen such that the projection π_2 onto the second component is an isometric isomorphism of the space $\mathfrak{K}(W)$ onto $\mathcal{H}(E)$, and with this additional requirement it becomes unique. This can be found in [dB, Theorem 27] [4].

Relation (3)*:* The Hamiltonian H corresponds bijectively (up to reparameterization) to its Titchmarsh–Weyl coefficient q_H via de Branges' inverse spectral theorem, cf. [dB], [W1, Theorem 1] [5].

For our present task it is important to know that the following three conditions are equivalent, see [W2], [dB] [6]:

– The Hamiltonian H satisfies

$$\exists\, \varepsilon > 0: \quad H(t) = h(t)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ t \in [s_-, s_- + \varepsilon] \text{ a.e.}, \tag{2.2}$$

with some scalar integrable function h.

[3] for the indefinite case see [KW/V, Theorem 5.1], [KW/VI, Theorem 1.5, Theorem 1.6]
[4] [KW/I, Proposition 8.3, Lemma 8.6], [KW/I, Proposition 10.3, Corollary 10.4]
[5] [KW/VI, Theorem 1.4]
[6] [KW/II, Proof of Theorem 7.1], [K, Proposition 5.3]

– For one, and hence for all, values $t \in (s_-, s_+)$ the reproducing kernel space $\mathfrak{K}(W(t,\cdot))$ contains the constant function $\binom{0}{1}$.

– The Titchmarsh–Weyl coefficient q_H of H satisfies

$$\lim_{y\to+\infty} q_H(iy) = 0 \quad \text{and} \quad \lim_{y\to+\infty} y|\operatorname{Im} q_H(iy)| < \infty.$$

Relation (4)*:* A function $E \in \mathcal{HB}_0$ gives rise to the de Branges space $\mathcal{H}(E)$. Two different functions E_1 and E_2 of Hermite–Biehler class may induce the same Hilbert space, meaning that

$$F \in \mathcal{H}(E_1) \Leftrightarrow F \in \mathcal{H}(E_2) \quad \text{and} \quad [F,G]_{\mathcal{H}(E_1)} = [F,G]_{\mathcal{H}(E_2)},\ F,G \in \mathcal{H}(E_1).$$

In fact, this is the case if and only if there exists a matrix $U \in \mathbb{R}^{2\times 2}$ with $\det U = 1$, such that

$$(A_2, B_2) = (A_1, B_1)U \tag{2.3}$$

where $A_j(z) := \frac{1}{2}(E_j(z) + \overline{E_j(\overline{z})})$ and $B_j(z) := \frac{i}{2}(E_j(z) - \overline{E_j(\overline{z})})$. This fact is shown in [dB] [7].

It is clear that, if besides $\mathcal{H}(E_1) = \mathcal{H}(E_2)$ we require that $E_1(0) = E_2(0) = 1$, then the matrix U in (2.3) must be of the form

$$U = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}.$$

We need the following, more specific and less elementary statement.

2.3 Lemma. Let H_1 and H_2 be Hamiltonians defined on intervals $[s_-^1, s_+^1)$ and $[s_-^2, s_+^2)$, respectively. Assume that both are in limit circle case and satisfy (2.2)*. Denote by $(W_j(t,z))_{t\in[s_-^j, s_+^j)}$, $j = 1,2$, the respective solutions of* (1.2)*, and set $E_j := E_{W_j(s_+^j,\cdot)}$, $j = 1,2$. Then $\mathcal{H}(E_1) = \mathcal{H}(E_2)$ implies that $H_1 \sim H_2$, and thus $E_1 = E_2$.*

Proof. Set $W_k(z) = (w_{ij}^k(z))_{i,j=1,2} := W_k(s_+^k, \cdot)$, $k = 1,2$. Then $E_k = w_{22}^k + iw_{21}^k$, and $A_k = w_{22}^k$, $B_k = -w_{21}^k$. By the above considerations, the relation $\mathcal{H}(E_1) = \mathcal{H}(E_2)$ thus implies that for some $u \in \mathbb{R}$

$$(w_{21}^2, w_{22}^2) = (w_{21}^1, w_{22}^1) \begin{pmatrix} 1 & -u \\ 0 & 1 \end{pmatrix}. \tag{2.4}$$

Set

$$\tilde{W} := \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} W_1 \begin{pmatrix} 1 & -u \\ 0 & 1 \end{pmatrix},$$

then $\tilde{W} \in \mathcal{M}_0$ and the map

$$\begin{pmatrix} f_+ \\ f_- \end{pmatrix} \mapsto \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} f_+ \\ f_- \end{pmatrix}$$

[7] [KW/I, Corollary 6.2]

is an isometric isomorphism from $\mathfrak{K}(W_1)$ onto $\mathfrak{K}(\tilde{W})$; see, e.g. [Aro] [8]. Since we assume that H_1 satisfies (2.2), it follows that

$$\begin{pmatrix} u \\ 1 \end{pmatrix} \in \mathfrak{K}(\tilde{W}) .$$

Since the space $\mathfrak{K}(\tilde{W})$ can contain at most one constant, in particular, the constant $\binom{1}{0}$ does not belong to the space $\mathfrak{K}(\tilde{W})$. Since also H_2 satisfies (2.2), we also have $\binom{1}{0} \notin \mathfrak{K}(W_2)$.

By (2.4) we have $(0,1)W_2 = (0,1)\tilde{W}$, and [dB, Theorem 27][9] implies that for some $v \in \mathbb{R}$

$$W_2 = \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \tilde{W} \begin{pmatrix} 1 & -v \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & u+v \\ 0 & 1 \end{pmatrix} W_1 \begin{pmatrix} 1 & -(u+v) \\ 0 & 1 \end{pmatrix} .$$

Hence the map

$$\begin{pmatrix} f_+ \\ f_- \end{pmatrix} \mapsto \begin{pmatrix} 1 & u+v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} f_+ \\ f_- \end{pmatrix}$$

is an isometric isomorphism from $\mathfrak{K}(W_1)$ onto $\mathfrak{K}(W_2)$, and we conclude that $\binom{u+v}{1} \in \mathfrak{K}(W_2)$. Again using that H_2 satisfies (2.2), it follows that $u+v=0$, i.e. $W_2 = W_1$ and hence also $E_1 = E_2$. By de Branges' Inverse Spectral Theorem this even implies that $H_1 \sim H_2$. ❑

Relation (5)*:* The class $\mathcal{P}_0$ of positive definite functions corresponds bijectively to the subclass of $\mathcal{N}_0$ which contains all functions $q \in \mathcal{N}_0$ with

$$\lim_{y\to+\infty} q(iy) = 0, \qquad \lim_{y\to+\infty} y|\operatorname{Im} q(iy)| < \infty. \tag{2.5}$$

This bijection is established by the one-sided Fourier transform ($f \in \mathcal{P}_0$)

$$q(z) = i\int_0^\infty f(t)e^{izt}\,dt, \quad \operatorname{Im} z > 0. \tag{2.6}$$

This relationship is best understood via the measures appearing in the Herglotz integral representation of q and in the integral representation of f as a Fourier transform by Bochner's Theorem. In fact, a function q belongs to $\mathcal{N}_0$ and satisfies (2.5) if and only if it can be written as

$$q(z) = \int_{\mathbb{R}} \frac{d\mu_q(t)}{t-z}\,, \quad \operatorname{Im} z > 0,$$

with a finite positive Borel measure μ_q on $\mathbb{R}$, cf. [KK1, Theorem S1.4.1][10]. On the other hand, by Bochner's Theorem, a positive definite function f can be represented as

$$f(x) = \int_{\mathbb{R}} e^{-itx}\,d\mu_f(t), \quad x \in \mathbb{R},$$

[8] [ADSR, Theorem 1.5.3]

[9] [KW/I, Corollary 9.8]

[10] In the indefinite case the measure μ_q has to be replaced by some distribution; see [JLT], [KWW1].

with some finite positive Borel measure μ_f on $\mathbb{R}$; see, e.g. [AG]. An application of Fubini's Theorem gives that the functions q and f are related by (2.6) if and only if $\mu_q = \mu_f$ [11].

Relation (6)*:* Let H be a Hamiltonian that satisfies (2.2). Going counterclockwise in the scheme (2.1) along (3) and (5), let $q_H \in \mathcal{N}_0$ be its Titchmarsh–Weyl coefficient and $f_H \in \mathcal{P}_0$ be the Fourier transform of q_H. Going clockwise along (1), (2) and (4), let $W(t,\cdot)$ be the fundamental solution of the Hamiltonian system with Hamiltonian H, and let $E_{W(t,\cdot)}$ and $\mathcal{H}(E_{W(t,\cdot)})$ be the Hermite–Biehler function generated by $W(t,\cdot)$ and the corresponding de Branges space, respectively. Moreover, set

$$s_a := \sup\Big\{t\in[s_-,s_+)\colon \int_{s_-}^t \sqrt{\det H(x)}\,dx < a\Big\},\quad a>0.$$

Then, for each $a\in(0,\int_{s_-}^{s_+}\sqrt{\det H(t)}\,dt)$, we have

$$\mathcal{H}(E_{W(s_a,\cdot)}) = \text{c.l.s.}\big\{e^{-itz}\colon |t|\le a\big\},$$
$$[e^{-itz},e^{-iuz}]_{\mathcal{H}(E_{W(s_a,\cdot)})} = f(u-t),\quad |t|,|u|\le a,$$

where c.l.s. stands for 'closed linear span'. See [K, Lemma 5.8] [12] [13] .

We have collected everything that is needed for the proof of the local uniqueness theorem 1.2. //

Proof (of the local uniqueness theorem 1.2).
Step 1: the case that H_1 and H_2 satisfy (2.2)*.*
By Relation (5) there exist positive definite functions f_1 and f_2 with

$$q_{H_j}(z) = i\int_0^\infty f_j(t)e^{itz}\,dt,\qquad \operatorname{Im} z>0,\ j=1,2.$$

Set $f:=f_1-f_2$; then ($\operatorname{Im} z>0$)

$$q_{H_1}(z)-q_{H_2}(z) = i\int_0^\infty f(t)e^{itz}\,dt = \underbrace{i\int_0^{2a} f(t)e^{itz}\,dt}_{=:F_1(z)} + \underbrace{i\int_{2a}^\infty f(t)e^{itz}\,dt}_{=:F_2(z)}.$$

First we estimate F_2. Since f_j is positive definite, we have $f_j(0)\ge 0$ and $|f_j(t)|\le f_j(0)$, $t\in\mathbb{R}$. This allows us to compute ($\operatorname{Im} z>0$)

$$|F_2(z)| = \Big|ie^{2aiz}\int_{2a}^\infty f(t)e^{i(t-2a)z}\,dt\Big| \le e^{-2a\operatorname{Im} z}\cdot\tfrac{f_1(0)+f_2(0)}{\operatorname{Im} z}.$$

Secondly, if $f|_{[0,2a]}\ne 0$, we set $m:=\min(\operatorname{supp} f\cap[0,2a])$. Then, by the Theorem of Paley–Wiener on Fourier transforms of functions with compact support

[11] [KL, Satz 5.3]
[12] We do not know a reference which deals with the positive definite case only.
[13] Density of exponentials depends on de Branges' Ordering Theorem.

combined with the knowledge on regular asymptotic behaviour of functions of bounded type, we have

$$\limsup_{r\to\infty} \frac{\log|F_1(re^{i\theta})|}{r} = -m\sin\theta, \quad \theta\in(0,\pi);$$

see, e.g. [Ko, §3.D, Scholium p. 35], [Boa, §7.2].

Next, note that by Lemma 2.3, the condition $H_1|_{[s_-^1,s_a^1)} \sim H_2|_{[s_-^2,s_a^2)}$ is equivalent to the relation $\mathcal{H}(E_{W_1(s_a^1,\cdot)}) = \mathcal{H}(E_{W_2(s_a^2,\cdot)})$, which by Relation (6) is equivalent to $f_1|_{[0,2a]} = f_2|_{[0,2a]}$, i.e. to $f|_{[0,2a]} = 0$.

Now we are in position to establish the equivalences asserted in 1.2. Assume that (i) of 1.2 holds. Then the function F_1 vanishes identically, and hence for each fixed $\alpha\in(0,\pi)$

$$q_{H_1}(z) - q_{H_2}(z) = O\bigg(\frac{e^{-2a\operatorname{Im} z}}{\operatorname{Im} z}\bigg), \quad |z|\to\infty, \ z\in\Gamma_\alpha. \tag{2.7}$$

In particular, (iii) holds. Trivially, (iii) implies (ii). Assume next that (ii) holds and $f|_{[0,2a]}\neq 0$. Then $m<2a$, and hence the summand F_1 dominates the asymptotic behaviour of $q_{H_1}-q_{H_2}$. We obtain that for each $\theta\in(0,\pi)$

$$\limsup_{r\to\infty} \frac{\log|q_{H_1}(re^{i\theta}) - q_{H_2}(re^{i\theta})|}{r} = -m\sin\theta\,.$$

However, by (ii), there exists at least one value of θ such that this limit superior does not exceed $-2a\sin\theta$. It follows that $-2a\geq -m$, and we have reached a contradiction.

Step 2: reduction of the general case to the previous one.
Let H_j, $j=1,2$, be arbitrary Hamiltonians defined on respective intervals $[s_-^j, s_+^j)$. Define Hamiltonians $\tilde{H}_j$ on the intervals $[s_-^j-1, s_+^j)$ by

$$\tilde{H}_j(t) := \begin{cases} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, & t\in[s_-^j-1, s_-^j), \\ \\ H_j(t), & t\in[s_-^j, s_+^j). \end{cases}$$

The fundamental matrix solutions of the canonical systems with Hamiltonians $\tilde{H}_j$ are given by

$$\tilde{W}_j(t,z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ -(t-s_-^j+1)z & 1 \end{pmatrix}, & t\in[s_-^j-1, s_-^j), \\ \\ \begin{pmatrix} 1 & 0 \\ -z & 1 \end{pmatrix} W_j(t,z), & t\in[s_-^j, s_+^j), \end{cases}$$

and hence

$$q_{\tilde{H}_j}(z) = \frac{q_{H_j}(z)}{-zq_{H_j}(z)+1} = \frac{-1}{z-\frac{1}{q_{H_j}(z)}}\,, \qquad j=1,2.$$

It follows that

$$q_{\tilde{H}_1}(z) - q_{\tilde{H}_2}(z) = \frac{q_{H_1}(z) - q_{H_2}(z)}{q_{H_1}(z) q_{H_2}(z) \big(z - \frac{1}{q_{H_1}(z)}\big)\big(z - \frac{1}{q_{H_2}(z)}\big)} \,.$$

Denote by $(\tilde{i}), (\tilde{ii}), (\tilde{iii})$ the respective conditions of 1.2 for the pair $(\tilde{H}_1, \tilde{H}_2)$. Since $\det \tilde{H}_j(t) = 0$, $t \in [s_-^j - 1, s_-^j)$, the value of s_a^j remains the same whether computed for H_j or $\tilde{H}_j$. Thus we have $(i) \Leftrightarrow (\tilde{i})$. Next, remember that each Nevanlinna function q satisfies above and below polynomial estimates, in fact there exist constants $\gamma_\pm > 0$ such that[14]

$$\frac{\gamma_-}{|z|} \leq |q(z)| \leq \gamma_+ |z|, \quad z \in \Gamma_\alpha, \, \operatorname{Im} z \geq 1 \,.$$

With q_{H_j} also the function $z - \frac{1}{q_{H_1}(z)}$ is a Nevanlinna function. Hence $(ii) \Leftrightarrow (\tilde{ii})$, and (2.7) for $\tilde{H}_j$ instead of H_j implies (iii). The implication $(iii) \Rightarrow (ii)$ is of course trivial. Together with what we showed in Step 1, we have by now established the following implications

$$\begin{tikzcd}
(\tilde{i}) \arrow[r, Rightarrow] \arrow[d, Leftrightarrow] & (2.7) \arrow[r, Rightarrow] \arrow[d, Rightarrow] & (\tilde{iii}) \arrow[r, Rightarrow] & (\tilde{ii}) \arrow[lll, Rightarrow, bend right] \arrow[d, Leftrightarrow] \\
(i) & (iii) \arrow[rr, Rightarrow] & & (ii)
\end{tikzcd}$$

This gives the equivalence of (i), (ii) and (iii). ☺

3 Proof of mixed data results and the Sturm–Liouville situation

The mixed data results stated as Corollary 1.4 are easily deduced from Theorem 1.2.

Proof (of Corollary 1.4). Assume first that the hypothesis (i) holds true. We apply Theorem 1.2 with H_1, H_2, and a number a which is arbitrarily chosen in (α, β). Since $a < \beta$, the present assumption (1.4) implies that the condition (ii) of 1.2 is satisfied. Hence $H_1|_{[s_-^1, s_a^1)} \sim H_2|_{[s_-^2, s_a^2)}$. However, since $a > \alpha$, we have $s_a^1 = s_+^1$. It follows that $H_2|_{[s_-^2, s_a^2)}$ is in the limit point case, and hence that $s_a^2 = s_+^2$. We see that $H_1 \sim H_2$.

Next, assume that the hypothesis (ii) holds true. Then Theorem 1.2 gives equality of H_1 and H_2 (up to reparameterization) up to the points s_a^1, s_a^2. The remaining parts are equal by assumption. ❑

Next, we turn to Sturm–Liouville equations without potential term. Thus, we consider an equation of the form

$$-(p(x)y'(x))' = \lambda w(x) y(x) \tag{3.1}$$

[14] In the indefinite case one has $\gamma_- |z|^{-N} \leq |q(z)| \leq \gamma_+ |z|^N$ with some appropriate $N \in \mathbb{N}$.

on the interval $[0,b)$ with $0<b\leq\infty$, where p and w are measurable functions, $p(x)>0$, $w(x)>0$ almost everywhere, and $\frac{1}{p}, w\in L^1_{\text{loc}}([0,b))$. Note that here y denotes a scalar function. Equation (3.1) can be written as a Hamiltonian system (1.1) with

$$H(x)=\begin{pmatrix} w(x) & 0 \\ 0 & \dfrac{1}{p(x)}\end{pmatrix} \tag{3.2}$$

and $\lambda=z^2$ because y is a solution of (3.1) if and only if $\binom{y_1}{y_2}$ with

$$\begin{aligned} y_1(x) &= y(x), \\ y_2(x) &= -\frac{1}{z}p(x)y'(x) \end{aligned}$$

is a solution of (1.1) with H from (3.2).

We say that (3.1) is in limit point case if (3.1) for $\lambda\in\mathbb{R}\setminus\mathbb{C}$ has only one linearly independent solution in $L^2_w(0,b)$, where $L^2_w(0,b)$ is the L^2 space with weight w. In this case also H in (3.2) is in limit point case because otherwise every solution of (1.1) is in $L^2(H)$ and hence the first component in $L^2_w(0,b)$. The converse (i.e. H in (3.2) is in limit point case $\Rightarrow$ (3.1) is in limit point case) is not true as the example $p(x)=1$, $w(x)=(1+x)^{-4}$ shows.

The Titchmarsh–Weyl coefficient of (3.1) is defined as follows. Assume that (3.1) is in limit point case. Let $\theta(x,\lambda)$, $\phi(x,\lambda)$ be solutions of (3.1) that satisfy the initial conditions

$$\begin{aligned} \theta(0,\lambda) &= 1, \qquad & p(x)\theta'(x,\lambda)|_{x=0} &= 0, \\ \phi(0,\lambda) &= 0, \qquad & p(x)\phi'(x,\lambda)|_{x=0} &= 1. \end{aligned}$$

Since we have limit point case, there exists a unique number $m(\lambda)$ for every $\lambda\in\mathbb{C}\setminus\mathbb{R}$ such that

$$\theta(x,\lambda)+m(\lambda)\phi(x,\lambda) \tag{3.3}$$

is in $L^2_w(0,b)$. It is easily seen that

$$\theta(x,z^2)=w_{11}(x,z), \qquad \phi(x,z^2)=-\frac{1}{z}w_{21}(x,z).$$

Comparing (3.3) and (1.3) we obtain

$$m(z^2)=zq_H(z). \tag{3.4}$$

The proof of Proposition 1.5 is now easily obtained.

Proof (of Proposition 1.5). We can apply the local uniqueness theorem 1.2 to the Hamiltonians

$$H_j=\begin{pmatrix} w_j & 0 \\ 0 & \frac{1}{p_j}\end{pmatrix}, \qquad j=1,2.$$

Observing (3.4) one can easily see that the conditions $(i)-(iii)$ directly correspond to the conditions $(i)-(iii)$ in Theorem 1.2. Note that in (iii) in Theorem 1.2, $(\operatorname{Im} z)^3$ can be replaced by $|z|^3$. ❑

To close the paper, let us mention the particular case of Proposition 1.5 when the considered Sturm–Liouville equation is in impedance form, i.e. that $p = w$. Such equations sometimes appear in applications and have been investigated previously see, e.g. [An1] or [AHM] and the reference therein. The observation we can make is that for equations in impedance form reparameterization is not necessary.

3.1 Corollary. *Let $b_1, b_2 > 0$ and let p_j be measurable functions defined on $[0, b_j)$, $j = 1, 2$, such that $p_j(x) > 0$ almost everywhere and $p_j, \frac{1}{p_j} \in L^1_{\text{loc}}([0, b_j))$ for $j = 1, 2$. Moreover, denote by m_j the Titchmarsh–Weyl coefficient for*

$$-(p_j y')' = \lambda p_j y.$$

For $a > 0$ the following statements are equivalent.

(*i*) *$p_1(x) = p_2(x)$ almost everywhere on $(0, a)$.*

(*ii*) *There exists $\theta \in (0, 2\pi)$ such that for every $\varepsilon > 0$,*

$$m_1(re^{i\theta}) - m_2(re^{i\theta}) = O\big(e^{(-2a+\varepsilon)r \sin \frac{\theta}{2}}\big), \quad r \to +\infty\,.$$

(*iii*) *For every $\alpha \in (0, \pi)$,*

$$m_1(\lambda) - m_2(\lambda) = O\Big(|\lambda|^2 e^{-2a \operatorname{Im} \sqrt{\lambda}}\Big),$$
$$|\lambda| \to \infty,\ \lambda \in \{z \in \mathbb{C} \colon \alpha \leq \arg z \leq 2\pi - \alpha\}\,.$$

Proof. If (1.5) holds for $w_j = p_j$, then $\varphi'(t) = 1$ for almost all $t \in (0, a)$ and hence $\varphi(t) = t$, $t \in (0, a)$. ❑

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