Pontryagin spaces of entire functions VI

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Abstract

In the theory of two-dimensional canonical (also called 'Hamiltonian') systems, the notion of the Titchmarsh-Weyl coefficient associated to a Hamiltonian function plays a vital role. A cornerstone in the spectral theory of canonical systems is the Inverse Spectral Theorem due to Louis de Branges which states that the Hamiltonian function of a given system is (up to changes of scale) fully determined by its Titchmarsh-Weyl coefficient. Much (but not all) of this theory can be viewed and explained using the theory of entire operators due to Mark G.Kreĭn.

Motivated from the study of canonical systems or Sturm-Liouville equations with a singular potential, and from other developments in the indefinite world, it was a long standing open problem to find an indefinite (Pontryagin space) analogue of the notion of canonical systems, and to prove a corresponding analogue of de Branges' Inverse Spectral Theorem. We gave a definition of an indefinite analogue of a Hamiltonian function and elaborated the operator theory of such 'indefinite canoncial systems' in previous work. In the present paper we prove the corresponding version of the Inverse Spectral Theorem.

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1 Introduction

A function $H : (s_-, s_+) \to \mathbb{R}^{2\times 2}$ is called a Hamiltonian, if it is locally integrable, $H(t) \geq 0$ for $t \in (s_-, s_+)$ a.e., and if it does not vanish identically on any set of positive measure. A canonical system is a differential equation of the form

$$y'(t) = zJH(t)y(t), \quad t \in (s_-, s_+)\,, \tag{1.1}$$

where H is a Hamiltonian, J denotes the symplectic matrix

$$J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and z is a complex parameter. The spectral theory of canonical systems can be understood with the help of an operator model, which consists of a Hilbert space boundary triplet $(L^2(H), T_{\max}(H), \Gamma(H))$, see e.g. [K], [HSW].

We say that a Hamiltonian H is regular at the endpoint s_- if, for some $\epsilon > 0$,

$$\int_{s_-}^{s_-+\epsilon} \operatorname{tr} H(t)\,dt < \infty\,. \tag{1.2}$$

If $\int_{s_-}^{s_-+\epsilon} \operatorname{tr} H(t)\,dt = \infty$, it is called singular at s_-. The terminology of regular/singular at the endpoint s_+ is defined analogously. Sometimes one also speakes of Weyl's limit circle and limit point case, instead of regular and singular, respectively.

Assume that H regular at s_-. Then each solution of the equation (1.1) possesses a locally absolutely continuous extension to $[s_-, s_+)$. Hence, in this case, it is meaningful to prescribe initial values at s_-. We denote by $W(t,z) = (w_{ij}(t,z))_{i,j=1,2}$ the transposed of the fundamental matrix solution of (1.1), i.e. $W(t,z)$ is the unique 2×2-matrix function which satisfies

$$\frac{\partial}{\partial t}W(t,z)J = zW(t,z)H(x),\ t\in[s_-,s_+),\quad W(s_-,z) = I\,.$$

Depending whether H is regular or singular at s_+, we meet two significantly different situations.

H is regular at s_+: The function $W(t,z)$ admits a continuous extension to s_+. The matrix function $W(s_+,z)$, sometimes also called the monodromy matrix of H, belongs to the class $\mathcal{M}_0$, i.e. the entries of $W(s_+,z)$ are entire functions which are real for real z, $\det W(s_+,z) = 1$, and

$$\frac{W(s_+,z)JW(s_+,z)^* - J}{z-\overline{z}} \geq 0,\quad \operatorname{Im} z > 0\,. \tag{1.3}$$

The chain $\omega_H := (W(t,z))_{t\in[s_-,s_+]}$ is a finite maximal chain going down from $W(s_+,z)$.

H is singular at s_+: We have $\lim_{t\nearrow s_+}\operatorname{tr}(W(t,0)'J) = +\infty$. Write $W(t,z) = (w_{ij}(t,z))_{i,j=1,2}$. Then for each $\tau\in\mathbb{R}\cup\{\infty\}$, the limit

$$q_H(z) := \lim_{t\nearrow s_+}\frac{w_{11}(t,z)\tau + w_{12}(t,z)}{w_{21}(t,z)\tau + w_{22}(t,z)} \tag{1.4}$$

exists locally uniformly on $\mathbb{C}\setminus\mathbb{R}$ and does not depend on τ. The function q_H is called the Titchmarsh-Weyl coefficient associated to the Hamiltonian H. It belongs to the Nevanlinna class $\mathcal{N}_0$, i.e. is analytic on $\mathbb{C}\setminus\mathbb{R}$, satisfies $q_H(\overline{z}) = \overline{q_H(z)}$, $z\in\mathbb{C}\setminus\mathbb{R}$, and

$$\operatorname{Im} q_H(z) \geq 0,\quad \operatorname{Im} z > 0\,. \tag{1.5}$$

The following two results are cornerstones in the study of canonical systems, see [dB],[GK2].

1.1. ***Inverse Spectral Theorem; positive definite, regular:*** *The assignment $H\mapsto W(s_+,z)$ establishes a bijective correspondence between the set of all Hamiltonians (up to changes of scale) which are regular at s_- and s_+, and the set $\mathcal{M}_0$.*

1.2. ***Inverse Spectral Theorem; positive definite, singular:*** *The assignment $H\mapsto q_H$ establishes a bijective correspondence between the set of all Hamiltonians (up to changes of scale) which are regular at s_- and singular at s_+, and the set $\mathcal{N}_0$.*

The notion of the Nevanlinna class $\mathcal{N}_0$ allows a generalization to an indefinite setting, namely the class $\mathcal{N}_{<\infty}$. Thereby, the positivity condition (1.5) is replaced by the condition that the kernel

$$N_q(w,z) := \frac{q(z)-\overline{q(\overline{w})}}{z-\overline{w}}$$

has a finite number of negative squares. The actual number of its negative squares will be denoted by $\operatorname{ind}_- q$. This indefinite analogue of $\mathcal{N}_0$ has a long history and was systematically studied, e.g. in [KL] with help of the theory of selfadjoint operators in Pontryagin spaces.

Also the class $\mathcal{M}_0$ admits a generalization to the indefinite setting, namely the class $\mathcal{M}_{<\infty}$ where the positivity condition (1.3) is replaced by the requirement that the kernel

$$H_W(w,z) := \frac{W(z)JW(w)^* - J}{z - \overline{w}}$$

has a finite number of negative squares. Again we denote the actual number of its negative squares by $\operatorname{ind}_- W$. For more on this class see [KW/V, §2] and the references given there.

Finally, the notion of a Hamiltonian has been generalized to an indefinite setting in our previous work [KW/IV]. The accurate definition of this generalization is a bit lengthy, cf. [KW/IV, Definition 8.1]. Intuitively we can think of it as a Hamiltonian with finitely many singularities which do not behave too badly, plus a contribution to the canonical differential equation which happens inside each singularity, plus interface conditions which relate before and after each singularity. The degree of negativity of a general Hamiltonian $\mathfrak{h}$, denoted by $\operatorname{ind}_- \mathfrak{h}$, is a number which is composed out of measures for the growth of $\mathfrak{h}$ towards its singularities and for the size of the contribution happening inside the singularities. It measures the deviation of $\mathfrak{h}$ from the classical, positive definite, situation. Similar to the distinction between positive definite Hamiltonians being regular at s_- and s_+ or regular at s_- and singular at s_+, one can distinguish cases that a general Hamiltonian $\mathfrak{h}$ is regular or singular.

If $\mathfrak{h}$ is regular, a 2×2-matrix function $\omega(\mathfrak{B}(\mathfrak{h})) \in \mathcal{M}_{<\infty}$ was associated to $\mathfrak{h}$ in [KW/V, §4.e]. This is the analogue of the matrizant of a positive definite regular Hamiltonian. If $\mathfrak{h}$ is singular, a function $q_{\mathfrak{h}} \in \mathcal{N}_{<\infty}$ was associated to $\mathfrak{h}$ in [KW/V, Definition 5.2]. This is the analogue of the Weyl coefficient of a positive definite singular Hamiltonian

Our task in the present paper is to prove the following two results.

1.3. ***Inverse Spectral Theorem; indefinite, regular:*** *The assignment* $\mathfrak{h} \mapsto \omega(\mathfrak{B}(\mathfrak{h}))$ *establishes a bijective correspondence between the set of all regular general Hamiltonians (modulo reparameterization) and the set* $\mathcal{M}_{<\infty}$. *Thereby* $\operatorname{ind}_- \mathfrak{h} = \operatorname{ind}_- \omega(\mathfrak{B}(\mathfrak{h}))$.

1.4. ***Inverse Spectral Theorem; indefinite, singular:*** *The assignment* $\mathfrak{h} \mapsto q_{\mathfrak{h}}$ *establishes a bijective correspondence between the set of all singular general Hamiltonians (modulo reparameterization) and the set* $\mathcal{N}_{<\infty}$ *of all generalized Nevanlinna functions. Thereby* $\operatorname{ind}_- \mathfrak{h} = \operatorname{ind}_- q_{\mathfrak{h}}$.

This is the point of culmination of our series of papers on ‘Pontryagin spaces of entire functions’. With the proof of these theorems we have completed a full indefinite analogue of canonical system, their operator theory, and their Weyl-theory. Also, we see that the notion of general Hamiltonians introduced in [KW/IV] goes exactly as far as Pontryagin space theory might possibly lead.

Our notation in the present paper will follow the terminology introduced in [KW/0]–[KW/V], in particular [KW/V, §3]. We will, without further notice, use the terms introduced there. Moreover, references to [KW/0]–[KW/V] will be given as the following examples indicate: Lemma 0.2.1 refers to Lemma 2.1 of [KW/0], Theorem IV.8.6 to Theorem 8.6 of [KW/IV], or (V.4.1) to equation (4.1) of [KW/V].

In light of our previous work, the Inverse Spectral Theorems 1.3 and 1.4 appear as two-step-results:

1° We have shown in [KW/II], for a more concise formulation see also V.3.6 and V.3.9, that the set of finite maximal chains of matrices (modulo reparameterization) correponds bijectively to $\mathcal{M}_{<\infty}$ and that the set of maximal chains of matrices (modulo reparameterization) correponds bijectively to $\mathcal{N}_{<\infty}$.

2° By Theorem V.5.1 to each general Hamiltonian $\mathfrak{h}$ a chain $\omega_{\mathfrak{h}}$ is associated. This chain is finite maximal or maximal, depending whether $\mathfrak{h}$ is regular or singular.

In order to prove 1.3 and 1.4 it is thus enough to establish the following two theorems.

1.5 Theorem. *Let ω be a maximal chain or a finite maximal chain, and assume that $\mathfrak{t} \circ \omega$ and $(\mathfrak{t} \circ \omega|_J)^{-1}$ are locally absolutely continuous for any maximal interval contained in the domain of ω. Then there exists a general Hamiltonian $\mathfrak{h}(\omega)$ such that $\omega = \omega_{\mathfrak{h}(\omega)}$.*

For each (finite) maximal chain ω there exists a reparameterization which satisfies the above continuity condition.

1.6 Theorem. *Let $\mathfrak{h}$ and $\mathfrak{h}'$ be general Hamiltonians. Then $\omega_{\mathfrak{h}}$ is a reparameterization of $\omega_{\mathfrak{h}'}$ if and only if $\mathfrak{h}$ is a reparameterization of $\mathfrak{h}'$.*

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In Section 2 we carry out the construction of $\mathfrak{h}(\omega)$ for chains of a specific form which, roughly speaking, correspond to elementary indefinite Hamiltonians of kind (A). This is the hardest part of the converse construction. In Section 3 we settle other particular cases and employ the usual splitting-and-pasting technique to complete the proof of Theorem 1.5. Finally, in Section 4, we turn to the proof of Theorem 1.6. We will employ a fairly elementary method, based on a set of recurrance relations for the power series coefficients of the Potapov-Ginzburg transform of $\omega(t)$. In this way we also obtain some knowledge on how the data of $\mathfrak{h}$ concentrated inside a singularity influences $\omega(t)$, a topic we already have touched upon in [LW].

2 The converse construction; core of the argument

As already said, we will in this section deal with chains of a particular form.

2.1. Overall assumption: Throughout this section let ω be a finite maximal chain defined on

$$I = [s_-, \sigma) \cup (\sigma, s_+]$$

and assume that

$$\phi(\sigma) = 0, \qquad \sup\big(I_{\mathrm{reg}} \cap (s_-, \sigma)\big) = \sigma\,, \tag{2.1}$$

$$(\sigma, s_+) \text{ not indivisible } \Rightarrow \inf\big(I_{\mathrm{reg}} \cap (\sigma, s_+)\big) = \sigma\,, \tag{2.2}$$

$$\omega \text{ does not start with an indivisible interval of type 0 at } s_-\,. \tag{2.3}$$

//

2.2 Remark. Let us comment on the role played by the conditions (2.1)–(2.3). The first condition in (2.1) is just a normalization, it corresponds to (I), cf. [KW/IV, p.734]. The second condition in (2.1) is of more intrinsic nature. It means that it cannot happen that an indivisible interval adjoins σ at the left. This case could also be treated explicitly, but this would require development of more, and quite different, tools. Hence, we prefer to reduce it to the case (2.1) with help of the operation rev. The condition (2.2) is of technical nature. It could be avoided on the cost of increasing effort, but there is no need to do so. The condition (2.3) is important. Roughly speaking, its presence allows to identify the chain of spaces $\mathfrak{K}(\omega(t))$ with the chain of dB-subspaces of a de Branges spaces, cf. Corollary I.9.7 together with Proposition I.8.3. Thus it makes it possible to apply the knowledge on the structure of the dB-subspaces of a given de Branges Pontryagin space, especially on the degenerated members of this chain, cf. [KW/III]. //

This section is divided into subsections according to the following schedule:

a. We define Hamiltonians H_- and H_+, and show that H_+ and H_- satisfy (HS) and that $\phi(H_-) = \phi(H_+)$.
b. We define numbers $\ddot{o}(\omega) \in \mathbb{N} \cup \{0\}$ and $b_1(\omega), \dots, b_{\ddot{o}+1}(\omega) \in \mathbb{R}$.
c. We define a map $\psi(\omega) : \mathfrak{K}(\omega(s_+)) \to \mathcal{M}(I)/_{=_H}$.
d. We define numbers $d_j(\omega)$, $j \geq 0$.
e. The so far constructed data constitutes an elementary indefinite Hamiltonian $\mathfrak{h}$ of kind (A). We show that $\omega_{\mathfrak{h}} = \omega$.

a. Construction of the Hamiltonian function.

The definition of the Hamiltonian function of the to-be-constructed elementary indefinite Hamiltonian $\mathfrak{h}(\omega)$ is based on Proposition V.3.23.

2.3 Definition. Let H_- and H_+ denote the Hamiltonians defined on (s_-, σ) and (σ, s_+), respectively, which satisfy

$$\omega_{H_-} = \omega|_{[s_-,\sigma)}, \quad \omega_{\mathrm{rev}\,H_+} = (\mathrm{rev}\,\omega)|_{[-s_+,-\sigma)}\,.$$

Moreover, we will set

$$H(t) := \begin{cases} H_-(t) & ,\ t \in [s_-, \sigma) \\ H_+(t) & ,\ t \in (\sigma, s_+] \end{cases},$$

$$L^2(H) := L^2(H_-) \oplus L^2(H_+)\,,$$

$$T_{\max}(H) := T_{\max}(H_-) \oplus T_{\max}(H_+)\,,$$

$$\Gamma(H) := \big\{ \big(f_- + f_+; g_- + g_+); (a,b) \big) : \\ ((f_-; g_-); a) \in \Gamma(H_-), ((f_+; g_+); b) \in \Gamma(H_+) \big\}\,.$$

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Note that, by the definition of $H_\pm$, the function ω is a solution of the differential equation

$$\frac{\partial}{\partial t}\omega(t)(z)J = z\omega(t)(z)H(t), \quad t \in [s_-, \sigma) \cup (\sigma, s_+]\,.$$

In order to know that $H_\pm$ qualify for constituting the Hamiltonian function of an elementary indefinite Hamiltonian of kind (A), we need to check several properties, cf. (IV.4.1) and Definition IV.4.1.

The condition (A) in Definition IV.4.1 holds because by the second property in (2.1) no interval of the form (s, σ), $s \in [s_-, \sigma)$, is indivisible. The conditions in the first two lines of (IV.4.1) are easy to see, and will be deduced right below. The requirement in the third line of (IV.4.1), however, is much more delicate and will be seen only later, cf. Proposition 2.28.

2.4 Proposition. *The Hamiltonian $H_\pm$ is regular at $s_\pm$ and singular at σ. It satisfies (I) and (HS$_\mp$).*

Proof. It is enough to consider the Hamiltonian H_-, since passing from ω to $\operatorname{rev}\omega$ will exchange the roles of H_- and H_+. The fact that H_- is regular at s_- and singular at σ follows from

$$\lim_{t \searrow s_-} \omega(t) = I, \quad \lim_{t \nearrow \sigma} \mathfrak{t}(\omega(t)) = +\infty\,,$$

since $\mathfrak{t}(H_-) = \mathfrak{t} \circ \omega$ when $\mathfrak{t}(H_-)$ is chosen such that $\mathfrak{t}(H_-)(s_-) = 0$, cf. Proposition V.3.23, (i).

Let q denote the Weyl-coefficient of the positive definite canonical system with Hamiltonian H_-. Then q is an intermediate Weyl-coefficient in the sense of [KW/III, §5], see also Proposition V.3.10. This implies that q is meromorphic in $\mathbb{C}$. Moreover, it is seen from Theorem III.7.4 and Corollary III.7.9, that the sequences (a_k^+) and (a_k^-) of poles of q located in $(0,\infty)$ or $(-\infty, 0)$, respectively, satisfy

$$\lim_{k\to\infty} \frac{k}{a_k^+} = \lim_{k\to\infty} \frac{k}{a_k^-} \in \mathbb{R}\,. \tag{2.4}$$

Since q is meromorphic, the selfadjoint extensions of $T_{\min}(H)$ have discrete spectrum and, hence, compact resolvents. By (2.4), they belong to each Neumann-van Schatten class $\mathfrak{S}_{1+\epsilon}$ with $\epsilon > 0$, cf. [GK1]. In particular, they are Hilbert-Schmidt operators.

By [KW, Theorem 2.4], see also Theorem IV.2.27, there exists a unique angle $\phi_- \in [0, \pi)$ such that $\xi_{\phi_-} \in L^2(H_-)$. However, since $\circlearrowleft_{-\phi_-} \omega|_{[s_-,\sigma)}$ satisfies the canonical differential equation with Hamiltonian $\circlearrowleft_{-\phi_-} H_-$, we have

$$\frac{d}{dz}[\circlearrowleft_{-\phi_-} \omega(t)]_{12}(0) = \int_{s_-}^t \begin{pmatrix} 1 \\ 0 \end{pmatrix}^* \circlearrowleft_{-\phi_-} H_-(s) \begin{pmatrix} 1 \\ 0 \end{pmatrix} ds = \int_{s_-}^t \xi_{\phi_-}^* H_-(s) \xi_{\phi_-}\, ds\,.$$

Letting t tend to σ, we obtain

$$\lim_{t\nearrow\sigma}[\circlearrowleft_{-\phi_-}\ \omega(t)]'_{12}(0)=\|\xi_{\phi_-}\|^2_{L^2(H_-)}<\infty\,,$$

and conclude that $\phi_-=\phi(\sigma)=0$. ❑

2.5 Remark. The construction of $H_\pm$ and the fact that these Hamiltonians satisfy (HS$_\mp$) did not use (2.1)–(2.3). The condition (I) is just the first condition in (2.1). //

b. The elements D_k.

Let us consider the structure of the chain of dB-subspaces of the space $\mathfrak{P}(E_{\omega(s_+)})$, cf. Remark 2.2. By Lemma III.3.15, this chain is equal to

$$\big\{\mathfrak{P}(E_t):\,t\in[s_-,\sigma)\cap I_{\mathrm{reg}}\big\}\cup\{\mathfrak{P}_0,\dots,\mathfrak{P}_a\}\cup\big\{\mathfrak{P}(E_t):\,t\in(\sigma,s_+]\cap I_{\mathrm{reg}}\big\},$$

where $\mathfrak{P}_0,\dots,\mathfrak{P}_a$ are all degenerated dB-subspaces of $\mathfrak{P}(E_{\omega(s_+)})$. We assume that these spaces are enumerated in such a way that $\mathfrak{P}_0\subsetneq\ldots\subsetneq\mathfrak{P}_a$. The singularity σ is not of polynomial type, and hence does certainly not lie inside an indivisible interval with negative length. We conclude from Corollary III.3.17, and the notice after it, that there exists at least one degenerated dB-subspace of $\mathfrak{P}(E_{\omega(s_+)})$, i.e. $a\geq 0$. Moreover, by Remark III.3.13, the second part of (2.1), and (2.3) we have

$$\begin{aligned}
&\mathfrak{P}_0=\operatorname{clos}_{\mathfrak{P}(E_{s_+})}\bigcup_{t\in[s_-,\sigma)\cap I_{\mathrm{reg}}}\mathfrak{P}(E_t)\,,\\
&\dim\Big(\mathfrak{P}_{j+1}\big/\mathfrak{P}_j\Big)=1,\ j=0,\dots,a-1\,,\\
&\dim\Big(\bigcap_{t\in(\sigma,s_+]\cap I_{\mathrm{reg}}}\mathfrak{P}(E_t)\Big/\mathfrak{P}_a\Big)=\begin{cases}1 & ,\ (\sigma,s_+)\text{ indivisible}\\ 0 & ,\ \text{otherwise}\end{cases}
\end{aligned}\tag{2.5}$$

2.6 Definition. Define numbers

$$\begin{aligned}
&\Delta_-(\omega):=\dim\mathfrak{P}_0^\circ,\quad\Delta_+(\omega):=\dim\mathfrak{P}_a^\circ\\
&\ddot{o}(\omega):=a-|\Delta_+(\omega)-\Delta_-(\omega)|
\end{aligned}$$

Moreover, put $\Delta(\omega):=\max\{\Delta_-(\omega),\Delta_+(\omega)\}$. //

2.7 Remark. Note that the symbols $\Delta_\pm$, Δ, were already used to measure the growth of the Hamiltonians $H_\pm$ towards σ, cf. Definition IV.3.1. It will turn out later that actually $\Delta_\pm(\omega)=\Delta(H_\pm)$, cf. Proposition 2.28. However, before knowing this, the explicit notation of the argument ω in $\Delta_\pm$ and Δ is necessary to distinguish these notions and will be dropped only when no confusion is possible. //

We continue with a closer investigation of the structure of the spaces $\mathfrak{P}_j$ as done in [KW/III]. We know from Theorem III.2.3 that there exists an element $F\in\mathfrak{P}(E_{\omega(s_+)})$ such that $\mathfrak{P}_0^\circ=\operatorname{span}\{F(z),\dots,z^{\Delta_-(\omega)-1}F(z)\}$. It is important for our present purposes, that F can be chosen such that $F(0)=-1$. This fact is seen by an analysis of the proof of Theorem III.2.3. In view of our later needs we have to explain the situation in detail.

The proof of Theorem III.2.3 is based on the existence of a dB-Pontryagin space $\tilde{\mathfrak{P}}$ with the following properties:

(i) $\tilde{\mathfrak{P}}$ contains $\mathfrak{P}_a$ isometrically as a dB-subspace,

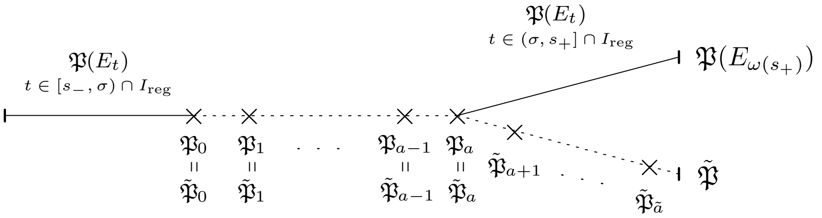
(ii) $\dim \tilde{\mathfrak{P}}/\mathfrak{P}_a < \infty$,

(iii) all dB-subspaces $\mathfrak{L}$ of $\tilde{\mathfrak{P}}$ with $\mathfrak{P}_a \subsetneq \mathfrak{L} \subsetneq \tilde{\mathfrak{P}}$ are degenerated and $\dim \mathfrak{L}^\circ \leq \dim \mathfrak{P}_a^\circ$.

The proof of this fact can be found on [KW/III, p.256]. We will now take a closer look at the chain of dB-subspaces of $\tilde{\mathfrak{P}}$. It is of the form

$$\big\{\mathfrak{P}(E_t) :\, t \in [s_-,\sigma) \cap I_{\text{reg}}\big\} \cup \{\tilde{\mathfrak{P}}_0, \ldots, \tilde{\mathfrak{P}}_{\tilde{a}}\} \cup \{\tilde{\mathfrak{P}}\},$$

where $\tilde{\mathfrak{P}}_0, \ldots, \tilde{\mathfrak{P}}_{\tilde{a}}$ are all degenerated dB-subspaces of $\tilde{\mathfrak{P}}$. We assume that these spaces are enumerated in such a way that $\tilde{\mathfrak{P}}_0 \subsetneq \ldots \subsetneq \tilde{\mathfrak{P}}_{\tilde{a}}$. Note that, clearly, $\tilde{a} \geq a$ and $\tilde{\mathfrak{P}}_j = \mathfrak{P}_j$, $j = 0, \ldots, a$. We can picture the situation as follows:



The spaces $\tilde{\mathfrak{P}}_j$ can be described by means of the operator $\tilde{\mathcal{S}}$ of multiplication by z in the space $\tilde{\mathfrak{P}}$. Write $\tilde{\mathfrak{P}} = \mathfrak{P}(\tilde{E})$ with some function $\tilde{E} \in \mathcal{H}B_{<\infty}$, $\tilde{E}(0) = -i$, and let $\psi \in [0,\pi)$ be the unique number such that $\tilde{S}_\psi \in \mathfrak{P}(\tilde{E})$, cf. Corollary I.6.3. Denote by $\tilde{n}$ the number

$$\tilde{n} := \sup\big\{k \in \mathbb{N}_0 :\, z^k \tilde{S}_\psi(z) \in \tilde{\mathfrak{P}}\big\},$$

which is finite by Lemma I.7.1. By Lemma II.5.19, we have

$$\overline{\operatorname{dom} \tilde{\mathcal{S}}^k} = \operatorname{span}\big\{\tilde{S}_\psi(z), \ldots, z^{k-1}\tilde{S}_\psi(z)\big\}^\perp,\ k = 1, \ldots, \tilde{n}+1, \tag{2.6}$$

and

$$\overline{\operatorname{dom} \tilde{\mathcal{S}}^k}^\circ \neq \{0\},\ k = 1, \ldots, \tilde{n}.$$

Note that the spaces $\overline{\operatorname{dom} \tilde{\mathcal{S}}^k}$ are dB-subspaces of $\tilde{\mathfrak{P}}$, cf. [KW/I]. However, (2.5) implies that there exists no nondegenerated dB-subspace of $\tilde{\mathfrak{P}}$ with finite codimension, hence also $\overline{\operatorname{dom} \tilde{\mathcal{S}}^{\tilde{n}+1}}^\circ \neq \{0\}$. By Lemma II.5.19 this implies $\overline{\operatorname{dom} \tilde{\mathcal{S}}^{\tilde{n}+2}} = \overline{\operatorname{dom} \tilde{\mathcal{S}}^{\tilde{n}+1}}$. Since, by (2.6),

$$\dim\Big(\overline{\operatorname{dom} \tilde{\mathcal{S}}^{k-1}} \Big/ \overline{\operatorname{dom} \tilde{\mathcal{S}}^k}\Big) = 1,\ k = 1, \ldots, \tilde{n}+1,$$

we conclude that

$$\tilde{n} = \tilde{a} \quad \text{and} \quad \tilde{\mathfrak{P}}_j = \overline{\operatorname{dom} \tilde{\mathcal{S}}^{\tilde{a}+1-j}} = \operatorname{span}\big\{\tilde{S}_\psi(z), \ldots, z^{\tilde{a}-j}\tilde{S}_\psi(z)\big\}^\perp, \quad j = 0, \ldots, \tilde{a}. \tag{2.7}$$

Next we describe the isotropic part of $\tilde{\mathfrak{P}}_j$. Set $\tilde{d}_j := \dim \tilde{\mathfrak{P}}_j^\circ$. We obtain from (2.7) that

$$\tilde{\mathfrak{P}}_j^\circ = \operatorname{span}\big\{\tilde{S}_\psi(z), \ldots, z^{\tilde{a}-j}\tilde{S}_\psi(z)\big\}^\circ,\ j = 0, \ldots, \tilde{a}, \tag{2.8}$$

hence $\tilde d_j \le \tilde a + 1 - j$. In particular, we have $\tilde d_{\tilde a} = 1$.

Since $\tilde S$ is symmetric, we have $[z^k \tilde S_\psi, z^l \tilde S_\psi] = \gamma_{k+l}$, $k, l = 0, \dots, \tilde a$, with some numbers $\gamma_0, \dots, \gamma_{2\tilde a}$. The fact that the corresponding Gram-matrix has Hankel-type yields

$$\gamma_0 = \ldots = \gamma_{\tilde a + \tilde d_0 - 1} = 0, \quad \gamma_{\tilde a + \tilde d_0} \neq 0\,, \tag{2.9}$$

and

$$\tilde{\mathfrak{P}}_j^\circ = \operatorname{span}\big\{\tilde S_\psi(z), \dots, z^{\tilde d_j - 1}\tilde S_\psi(z)\big\},\ j = 0, \dots, \tilde a\,. \tag{2.10}$$

Moreover, the dimension $\tilde d_j$ can be computed as

$$\tilde d_j = \begin{cases} \tilde d_0 + j & ,\ j = 0, \dots, [\frac{\tilde a - \tilde d_0}{2}] \\ \tilde a + 1 - j & ,\ j = [\frac{\tilde a - \tilde d_0}{2}] + 1, \dots, \tilde a \end{cases} , \tag{2.11}$$

and we have

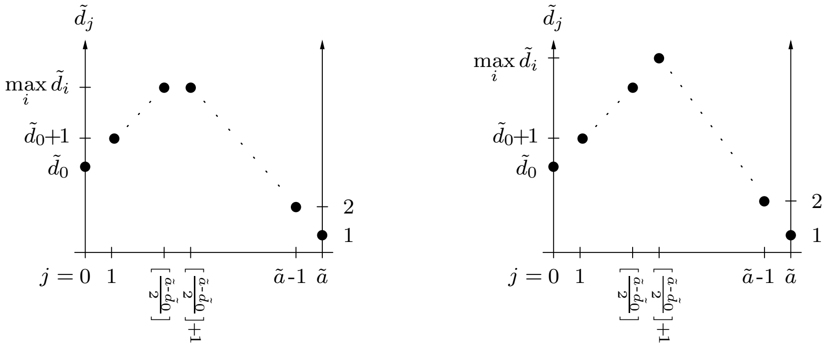
$$\tilde d_{[\frac{\tilde a - \tilde d_0}{2}]+1} = \begin{cases} \tilde d_{[\frac{\tilde a - \tilde d_0}{2}]} & ,\ \tilde a - \tilde d_0 \text{ even} \\ \tilde d_{[\frac{\tilde a - \tilde d_0}{2}]} + 1 & ,\ \tilde a - \tilde d_0 \text{ odd} \end{cases} . \tag{2.12}$$

Note here that $[\frac{\tilde a - \tilde d_0}{2}] \ge -1$. The maximal degree of degeneracy is assumed for $j = [\frac{\tilde a - \tilde d_0}{2}] + 1$, and

$$\max_{j=0,\dots,\tilde a} \tilde d_j = \tilde a - \left[\frac{\tilde a - \tilde d_0}{2}\right] = \left[\frac{\tilde a + \tilde d_0 + 1}{2}\right].$$

We can picture the situation as indicated in the following diagrams:

$\dim \tilde{\mathfrak{P}}_j^\circ$:



$\tilde a - \tilde d_0$ even

$\tilde a - \tilde d_0$ odd

Finally, it also follows that

$$z^{\tilde d_0 + j}\tilde S_\psi(z) \in \overline{\operatorname{dom} \tilde{\mathcal{S}}^{\tilde a - j}} \setminus \overline{\operatorname{dom} \tilde{\mathcal{S}}^{\tilde a - j + 1}},\ j = 0, \dots, \tilde a - \tilde d_0\,,$$

and we conclude that

$$\tilde{\mathfrak{P}}_j = \tilde{\mathfrak{P}}_0 \dot{+} \operatorname{span}\big\{z^{\tilde d_0}\tilde S_\psi(z), \dots, z^{\tilde d_0 + j - 1}\tilde S_\psi(z)\big\},\ j = 1, \dots, \tilde a - \tilde d_0 + 1\,.$$

The numbers $\tilde{a}$ and $\tilde{d}_j$ are in various ways related to the magnitudes $\Delta_\pm$, a, and $\ddot{o}$ associated with ω. First of all, it is trivial that $\tilde{d}_0 = \Delta_-$ and $\tilde{d}_a = \Delta_+$. In conjunction with property (*iii*) of $\tilde{\mathfrak{P}}$, the formulas (2.11) and (2.12) imply

$$a \geq \left[\frac{\tilde{a} - \Delta_-}{2}\right] \quad \text{and} \qquad a = \left[\frac{\tilde{a} - \Delta_-}{2}\right] \iff \tilde{a} - \Delta_- \text{ even}, a = \Delta_+ - \Delta_- \,,$$

$$\tilde{a} - a = \begin{cases} \Delta_+ & , \ \tilde{a} - \Delta_- \text{ even}, a = \Delta_+ - \Delta_- \\ \Delta_+ - 1 & , \ \text{otherwise} \end{cases} \tag{2.13}$$

Since always $|\tilde{d}_j - \tilde{d}_{j+1}| \leq 1$, we see that $\ddot{o} = a - |\tilde{d}_a - \tilde{d}_0| \geq 0$. From (2.11) and (2.12), we conclude that

$$\ddot{o} = 0 \iff \Big(\tilde{a} = \Delta_- - 1 \ \vee \ a = \left[\frac{\tilde{a} - \Delta_-}{2}\right] \ \vee \ a = \left[\frac{\tilde{a} - \Delta_-}{2}\right] + 1, \tilde{a} - \Delta_- \text{ odd}\Big)$$

2.8 Lemma. *Denote* $n := \max\big\{k \in \mathbb{N}_0 : z^k \tilde{S}_\psi(z) \in \mathfrak{P}_a\big\}$. *Then*

$$n = \Delta + \ddot{o} - 1 = \min\{\tilde{a}, \Delta_- + a - 1\} = \begin{cases} \tilde{a} & , \ \Delta_- > \Delta_+ \\ \Delta_- + a - 1 \, , & \Delta_- \leq \Delta_+ \end{cases} \tag{2.14}$$

Moreover, we have

$$\begin{aligned} \mathfrak{P}_a[-]\mathfrak{P}_0 &= \operatorname{span}\big\{\tilde{S}_\psi(z), \ldots, z^n \tilde{S}_\psi(z)\big\} \\ \big(\mathfrak{P}_a[-]\mathfrak{P}_0\big)^\circ &= \operatorname{span}\big\{\tilde{S}_\psi(z), \ldots, z^{\Delta - 1} \tilde{S}_\psi(z)\big\} \end{aligned} \tag{2.15}$$

Proof. Let $k \leq \tilde{a}$. Then we obtain from (2.7) and (2.9) that

$$z^k \tilde{S}_\psi(z) \in \mathfrak{P}_a \iff z^k \tilde{S}_\psi(z) \perp \tilde{S}_\psi(z), \ldots, z^{\tilde{a} - a} \tilde{S}_\psi(z) \iff k < \Delta_- + a \tag{2.16}$$

Thus $z^k \tilde{S}_\psi(z) \in \mathfrak{P}_a$ for all $k \leq \min\{\tilde{a}, \Delta_- + a - 1\}$. On the other hand,

$$z^k \tilde{S}_\psi(z) \ \begin{cases} \notin \tilde{\mathfrak{P}} & , \ \tilde{a} < k \\ \in \tilde{\mathfrak{P}} \setminus \mathfrak{P}_a & , \ \Delta_- + a \leq k \leq \tilde{a} \end{cases}$$

and it follows that $n = \min\{\tilde{a}, \Delta_- + a - 1\}$.

Consider the case that $\Delta_- \leq \Delta_+$. Then $\Delta = \Delta_+$ and $\ddot{o} = a - \Delta_+ + \Delta_-$. By (2.13), we have $\tilde{a} \geq a + \Delta_+ - 1 \geq a + \Delta_- - 1$, and it follows that

$$n = \Delta_- + a - 1 = \Delta + \ddot{o} - 1 \,.$$

If $\Delta_- > \Delta_+$, we have $\Delta = \Delta_-$ and $\ddot{o} = a - \Delta_- + \Delta_+$. Moreover, since $\tilde{d}_a < \max_i \tilde{d}_i$, we must have $[\frac{\tilde{a} - \Delta_-}{2}] + 1 < a$. In particular, in (2.13), we must be in the second case, and it follows that $\tilde{a} = a + \Delta_+ - 1 < \Delta_- + a - 1$. Thus

$$n = \tilde{a} = a + \Delta_+ - 1 = \Delta + \ddot{o} - 1 \,.$$

This proves (2.14). In order to see the first assertion in (2.15), we compute

$$\mathfrak{P}_a[-]\mathfrak{P}_0 = \operatorname{span}\big\{\tilde{S}_\psi(z), \ldots, z^{\tilde{a} - a} \tilde{S}_\psi(z)\big\}^\perp \cap \operatorname{span}\big\{\tilde{S}_\psi(z), \ldots, z^{\tilde{a}} \tilde{S}_\psi(z)\big\} \,.$$

The second equivalence in (2.16) yields the desired formula.

We come to the proof of the second assertion in (2.15). First note that, by (2.8), we have $(\mathfrak{P}_a[-]\mathfrak{P}_0)^\circ = \tilde{\mathfrak{P}}^\circ_{\tilde{a}-n}$. Hence we have to show that $\tilde{d}_{\tilde{a}-n} = \Delta$. We distiguish four cases:
1°*:* $\Delta_- > \Delta_+$. In this case $n = \tilde{a}$, and hence $\tilde{d}_{\tilde{a}-n} = \tilde{d}_0 = \Delta_- = \Delta$.
2°*:* $\Delta_- \leq \Delta_+$ *and* $\tilde{a} - \Delta_-$ *even,* $a = \Delta_+ - \Delta_-$. We have

$$\tilde{a} - n = (a + \Delta_+) - (\Delta_- + a - 1) = \Delta_+ - \Delta_- - 1 = a + 1\,,$$

and $a = [\frac{\tilde{a}-\Delta_-}{2}]$. Hence

$$\tilde{d}_{a+1} = \tilde{d}_a = \Delta_- + a = \Delta_+ = \Delta\,.$$

3°*:* $\Delta_- \leq \Delta_+$ *and* $\tilde{a} - \Delta_-$ *odd,* $a = [\frac{\tilde{a}-\Delta_-}{2}] + 1$. We have

$$\Delta_+ = \tilde{d}_a = \tilde{d}_{a-1} + 1 = \Delta_- + a\,,$$

and

$$\tilde{a} - n = (a + \Delta_+ - 1) - (\Delta_- + a - 1) = \Delta_+ - \Delta_- = a\,.$$

4°*:* $\Delta_- \leq \Delta_+$ *but neither* 2° *nor* 3°*:* Since we are not in case 2°,

$$\tilde{a} - n = (a + \Delta_+ - 1) - (\Delta_- + a - 1) = \Delta_+ - \Delta_-\,,$$

and since we are not in case 3°,

$$\Delta_+ \leq \Delta_- + [\frac{\tilde{a} - \Delta_-}{2}]\,.$$

Hence $\tilde{d}_{\Delta_+ - \Delta_-} = \Delta_- + (\Delta_+ - \Delta_-) = \Delta$. ❑

After having elaborated the geometry of the spaces $\mathfrak{P}_0, \ldots, \mathfrak{P}_a$, we return to our original aim, namely to show that the element F with $\mathfrak{P}_0^\circ = \operatorname{span}\{F(z), \ldots, z^{\Delta_- - 1}F(z)\}$ can be chosen such that $F(0) = -1$. In view of (2.10) this will follow from the next statement.

2.9 Lemma. *We have* $\tilde{S}_\psi(0) = 1$.

Proof. Recall that we have chosen $\tilde{E}$ such that $\tilde{\mathfrak{P}} = \mathfrak{P}(\tilde{E})$ and $\tilde{E}(0) = -i$. By Corollary I.10.4 there exists a matrix $\tilde{W} \in \mathcal{M}_{<\infty}$ with $\mathfrak{K}_-(\tilde{W}) = \mathfrak{K}(\tilde{W})$, such that $E_{\tilde{W}} = \tilde{E}$. Let $\tilde{\omega}$ be the finite maximal chain going downwards from $\tilde{W}$ as constructed in Theorem II.7.1. As we know from the proof of this theorem, the chain $\tilde{\omega}$ is, if parameterized appropriately, given as

$$\tilde{\omega}(t) = \begin{cases} \omega(t) & ,\ t \in [s_-, \sigma) \\ \tilde{W} W_{(\tilde{l}(t),\psi)} & ,\ t \in (\sigma, \sigma + 1) \end{cases} \tag{2.17}$$

where $\tilde{l}(t) := \frac{1}{\sigma - t} + 1$.

By our first assumption in (2.1), the value $\tilde{\omega}(t)'_{12}(0)$ remains bounded if t approaches σ from below, cf. the proof of Proposition 2.4. Hence, by Proposition III.5.8, $\tilde{\omega}(t)'_{12}(0)$ also remains bounded when t approaches σ from above. This implies that $\psi = \frac{\pi}{2}$, i.e. $\tilde{S}_\psi = \frac{i}{2}(\tilde{E} - \tilde{E}^\#)$. In particular, we have $\tilde{S}_\psi(0) = 1$. ❑

2.10 Corollary. *There exists a unique element $D_0 \in \mathfrak{P}(E_{s_+})$ such that*

$$\begin{aligned} \mathfrak{P}_0^\circ &= \operatorname{span}\left\{D_0(z),\dots,z^{\Delta_- -1}D_0(z)\right\} \\ D_0(0) &= -1 \end{aligned} \tag{2.18}$$

Proof. We have already noted that the element $D_0 := -\tilde{S}_\psi$ satisfies (2.18). The uniqueness assertion holds true since any two elements satisfying the equality in the first line of (2.18) must be scalar multiples of each other. ❑

Throughout the following we will denote

$$D_k(z) := -z^k \tilde{S}_\psi(z),\ k = 0,\dots,n\,.$$

and

$$c_j := \begin{cases} [D_{\Delta+j-1}, D_n] = \gamma_{\Delta+j-1+n} & ,\ j = 1,\dots,\ddot{o} \\ 0 & ,\ \text{otherwise} \end{cases}$$

Due to the symmetry of $\tilde{S}$ and (2.15), we have

$$[D_k, D_l] = c_{k+l+2-2\Delta-\ddot{o}}, \quad k,l = 0,\dots,n\,.$$

2.11 Definition. Let $b_1,\dots,b_{\ddot{o}}$ be the unique numbers such that

$$(c_1,\dots,c_{\ddot{o}}) \begin{pmatrix} b_1 & \cdots & b_{\ddot{o}} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & b_1 \end{pmatrix} = (-1,0,\dots,0)\,,$$

and put $b_{\ddot{o}+1} := 0$. Moreover, denote $B(z) := \sum_{l=1}^{\ddot{o}} b_l D_{\Delta+\ddot{o}-l}(z)$. //

2.12 Remark.

- (i) By Proposition I.6.1 and the discussion after it the entire function $\tilde{S}_\psi(z)$ is real. This shows that the numbers c_j, and hence also the numbers b_j, are real.
- (ii) Note that, by our choice of $B(z)$,

$$[B, D_k] = \begin{cases} -1 & ,\ k = \Delta \\ 0 & ,\ \text{otherwise} \end{cases} \tag{2.19}$$

//

If the interval (σ, s_+) is indivisible, then we can choose $\tilde{\mathfrak{P}} := \mathfrak{P}(E_{\omega(s_+)})$ in the above construction. Doing so, things become more explicit.

2.13 Corollary. *Assume that (σ, s_+) is indivisible. Write*

$$\omega(s_+) =: (\omega(s_+)_{ij})_{i,j=1}^2\,,$$

and let $\pi_2 : \mathfrak{K}(\omega(s_+)) \to \mathfrak{P}(E_{\omega(s_+)})$ be the isomorphism of projecting onto the second component. Then

$$\pi_2^{-1} D_0 = -\begin{pmatrix} \omega(s_+)_{12} \\ \omega(s_+)_{22} \end{pmatrix}.$$

Proof. As we saw in Lemma 2.9

$$D_0 = -S_{\frac{\pi}{2}} = -\frac{i}{2}(E_{\omega(s_+)} - E^{\#}_{\omega(s_+)}) = -\omega(s_+)_{22}\,.$$

Corollary II.7.4 implies that

$$\begin{pmatrix}\omega(s_+)_{12}\\ \omega(s_+)_{22}\end{pmatrix} = \omega(s_+)\begin{pmatrix}0\\1\end{pmatrix} \in \mathfrak{K}(\omega(s_+))\,.$$

Clearly, $\pi_2\binom{\omega(s_+)_{12}}{\omega(s_+)_{22}} = \omega(s_+)_{22}$, and thus the assertion follows. ❑

The elements D_k provide examples of elements of $\Gamma(\omega(s_+))$, cf. Lemma 2.15 and Lemma 2.16 below. Thereby, the first one is specific for the case that (σ, s_+) is indivisible. Boundary values will be computed with the help of a simple but useful general fact.

2.14 Lemma. *Let* $(\mathcal{P}_1, T_1, \Gamma_1)$ *and* $(\mathcal{P}_2, T_2, \Gamma_2)$ *be boundary triplets with defect* 2*, and assume that* $\operatorname{mul}\Gamma_1 = \operatorname{mul}\Gamma_2 = \{0\}$*. Let* $((f;g);(a;b)) \in \Gamma_1 \uplus \Gamma_2$*, then*

$$f, g \perp \mathcal{P}_1 \Rightarrow a = 0, \quad f, g \perp \mathcal{P}_2 \Rightarrow b = 0$$

Proof. Assume that $f, g \perp \mathcal{P}_1$. Let $v \in \mathbb{C}^2$ be given. Since Γ_1 is surjective, there exist $F, G \in \mathcal{P}_1$ such that

$$\big((F;G);(v;0)\big) \in \Gamma_1\,.$$

It follows that also $((F;G);(v;0)) \in \Gamma_1 \uplus \Gamma_2$, and hence the abstract Green's idenitity gives

$$0 = [g, F] - [f, G] = v^* J a\,.$$

Since v was arbitrary, this implies that $a = 0$.

The second implication follows in the same way. ❑

In the following we set $\delta(\omega)_k := \pi_2^{-1} D_k$, $k = 0, \dots, \Delta + \ddot{o} - 1$. We will also write δ_k instead of $\delta_k(\omega)$ if no confusion can occur.

2.15 Lemma. *Assume that* (σ, s_+) *is indivisible. Then we have*

$$\big((0;\delta_0);(0;\begin{pmatrix}1\\0\end{pmatrix})\big) \in \Gamma(\omega(s_+))\,. \tag{2.20}$$

Proof. Let us again choose $\tilde{\mathfrak{P}} := \mathfrak{P}(E_{\omega(s_+)})$ in the construction of the element D_0. It follows that D_0 spans the multivalued part of the selfadjoint extension $\mathcal{A}_{-D_0}$ of $\mathcal{S}(E_{\omega(s_+)})$. By Lemma V.2.16, we get $(0;\pi_2^{-1}D_0) \in T(\omega(s_+))$.

Let $a, b \in \mathbb{C}^2$ be such that $((0;\delta_0);(a;b)) \in \Gamma(\omega(s_+))$, and write $b = \binom{b_1}{b_2}$. Choose $r \in (s_-, \sigma) \cap I_{\text{reg}}$ such that (s_-, r) is not indivisible. By Lemma 2.8 $D_0 \perp \mathfrak{P}_0 \supseteq \mathfrak{P}(E_{\omega(r)})$, and hence $\delta_0 \perp \mathfrak{K}(\omega(r))$. By Lemma 2.14, we have $a = 0$. Using the abstract Green's identity with the element

$$\big((H_{\omega(s_+)}(0,.)\begin{pmatrix}0\\1\end{pmatrix};0);(\begin{pmatrix}0\\1\end{pmatrix};\begin{pmatrix}0\\1\end{pmatrix})\big) \in \Gamma(\mathfrak{K}(\omega(s_+)))\,, \tag{2.21}$$

gives

$$-b_1=-\begin{pmatrix}0\\1\end{pmatrix}^*Jb=[\pi_2^{-1}D_0,H_{\omega(s_+)}(0,.)\begin{pmatrix}0\\1\end{pmatrix}]=D_0(0)=-1\,,$$

cf. (2.18). Finally, we use the element

$$\big((H_{\omega(s_+)}(0,.)\begin{pmatrix}1\\0\end{pmatrix};0);(\begin{pmatrix}1\\0\end{pmatrix};\begin{pmatrix}1\\0\end{pmatrix})\big)\in\Gamma(\mathfrak{K}(\omega(s_+)))\,,$$

and remember Corollary 2.13, to obtain

$$b_2=-\begin{pmatrix}1\\0\end{pmatrix}^*Jb=[\pi_2^{-1}D_0,H_{\omega(s_+)}(0,.)\begin{pmatrix}1\\0\end{pmatrix}]=0\,.$$

❑

2.16 Lemma. *We have*

$$\big((\delta_k;\delta_{k+1});(0;0)\big)\in\Gamma(\omega(s_+)),\ \ k=0,\dots,\Delta+\ddot{o}-2\,.$$

Proof. Since $(D_k;D_{k+1})\in\mathcal{S}(E_{\omega(s_+)})$, Lemma V.2.16 implies $(\delta_k;\delta_{k+1})\in T(\omega(s_+))$. Let $a,b\in\mathbb{C}^2$ be such that $((\delta_k;\delta_{k+1});(a;b))\in\Gamma(\omega(s_+))$. The same argument as in the proof of Lemma 2.15 yields that $a=0$.

Assume that (σ,s_+) is not indivsible. Choose $r\in(\sigma,s_+)\cap I_{\text{reg}}$ such that (r,s_+) is not indivisible. Lemma 2.8 tells us that $D_k,D_{k+1}\in\mathfrak{P}_a\subseteq\mathfrak{P}(E_{\omega(r)})$, and hence orthogonal to $\omega(r)\cdot\mathfrak{K}(\omega(r)^{-1}\omega(s_+))$. Lemma 2.14 gives $b=0$.

Finally, assume that (σ,s_+) is indivisible. The abstract Green's identity with the pair (2.21) gives $-b_1=D_{k+1}(0)=0$. Applying it with the pair (2.20) gives

$$0=-\begin{pmatrix}1\\0\end{pmatrix}^*Jb=b_2\,.$$

❑

c. The map $\psi(\omega)$.

We will next define a map $\psi(\omega):\mathfrak{K}(\omega(s_+))\to\mathcal{M}(I)/_{=_H}$ which relates $\mathfrak{B}(\omega(s_+))$ to $L^2(H)$, $T_{\max}(H)$ and $\Gamma(H)$. Here $\mathcal{M}(I)$ denotes the set of all measurable functions of I into $\mathbb{C}^2$ which have the property that, if $(\alpha_-,\alpha_+)\subseteq I$ is an indivisible interval of type ϕ, then $\xi_\phi^T f(t)$ is constant on (α_-,α_+).

The definition of $\psi(\omega)$ is based on the following observation. For $r\in I_{\text{reg}}$ denote by P_r the orthogonal projection of $\mathfrak{K}(\omega(s_+))$ onto

$$\begin{cases}\mathfrak{K}(\omega(r)) & ,\ r\in[s_-,\sigma)\\ \omega(r)\cdot\mathfrak{K}(\omega(r)^{-1}\omega(s_+)) & ,\ r\in(\sigma,s_+]\end{cases}$$

Moreover, recall the notation Θ_{r_-,r_+} from Corollary V.3.27.

2.17 Lemma. *Let $F\in\mathfrak{K}(\omega(s_+))$ be given. Then there exists a unique element $f\in\mathcal{M}(I)/_{=_H}$ such that*

(i) If $r\in[s_-,\sigma)\cap I_{\text{reg}}$ is such that (s_-,r) is not indivisible, then

$$\Theta_{s_-,r}(f|_{(s_-,r)})=P_rF\,.$$

(ii) If $r \in (\sigma, s_+] \cap I_{\mathrm{reg}}$ *is such that* (r, s_+) *is not indivisible, then*

$$\omega(r) \cdot \Theta_{r,s_+}(f|_{(r,s_+)}) = P_r F\,.$$

(iii) If $(\sigma, s^+]$ *is indivisible, then*

$$f|_{(\sigma,s^+]} = -\big[\pi_2 F, D_0\big] \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2.22}$$

Proof. Let $r \in [s_-, \sigma) \cap I_{\mathrm{reg}}$. By Corollary V.3.27, $\Theta_{s_-,r}$ maps $L^2(H|_{(s_-,r)})$ bijectively onto $\mathfrak{K}(\omega(r))$. Hence there exists a unique element $f_r \in L^2(H|_{(s_-,r)})$ such that

$$\Theta_{s_-,r} f_r = P_r F\,. \tag{2.23}$$

Let $r, r' \in [s_-, \sigma) \cap I_{\mathrm{reg}}$, $r' < r$. Then, by Lemma V.3.34, we have

$$\begin{tikzcd}
\mathfrak{K}(\omega(s_+)) \arrow[d, "P_r"] \arrow[dd, bend right=60, "P_{r'}"'] & \\
\mathfrak{K}(\omega(r)) \arrow[d, "P_{r'}"'] & L^2(H|_{(s_-,r)}) \arrow[l, "\Theta_{s_-,r}"'] \arrow[d, "\rho_{s_-,r'}"] \\
\mathfrak{K}(\omega(r')) & L^2(H|_{(s_-,r')}) \arrow[l, "\Theta_{s_-,r'}"']
\end{tikzcd}$$

By uniqueness of the elements satisfying (2.23), it follows that $f_{r'} = \rho_{s_-,r'} f_r$. Since $\sup([s_-, \sigma) \cap I_{reg}) = \sigma$, an element $f_- \in \mathcal{M}((s_-, \sigma))/_{=_H}$ is well-defined by the requirement that always $(f_-)|_{(s_-,r)} = f_r$.

Consider the case that (σ, s_+) is not indivisible. We will argue similarly as in the preceding paragraph. Let $r \in (\sigma, s_+] \cap I_{\mathrm{reg}}$. By Corollary V.3.27, Θ_{r,s_+} maps $L^2(H|_{(r,s_+)})$ bijectively onto $\mathfrak{K}(\omega(r)^{-1}\omega(s_+))$. Hence there exists a unique element $f_r \in L^2(H|_{(r,s_+)})$ such that

$$\omega(r) \cdot \Theta_{s_-,r} f_r = P_r F\,. \tag{2.24}$$

Let $r, r' \in (\sigma, s_+] \cap I_{\mathrm{reg}}$, $r < r'$. Denote by $\tilde{P}_{r'}$ the orthogonal projection of $\mathfrak{K}(\omega(r)^{-1}\omega(s_+))$ onto $\omega(r)^{-1}\omega(r') \cdot \mathfrak{K}(\omega(r')^{-1}\omega(s_+))$. The fact that

$$\mathfrak{K}(\omega(s_+)) = \mathfrak{K}(\omega(r)) \oplus \omega(r)\mathfrak{K}(\omega(r)^{-1}\omega(s_+)) =$$

$$\mathfrak{K}(\omega(r)) \oplus \omega(r)\Big[\mathfrak{K}(\omega(r)^{-1}\omega(r')) \oplus \omega(r)^{-1}\omega(r') \cdot \mathfrak{K}(\omega(r')^{-1}\omega(s_+))\Big] =$$

$$= \mathfrak{K}(\omega(r)) \oplus \omega(r)\mathfrak{K}(\omega(r)^{-1}\omega(r')) \oplus \omega(r') \cdot \mathfrak{K}(\omega(r')^{-1}\omega(s_+))$$

just means that

$$P_{r'} = \underbrace{(\omega(r)\cdot) \circ \tilde{P}_{r'} \circ (\omega(r)^{-1}\cdot)}_{=:\hat{P}_{r'}} \circ P_r\,.$$

Using Lemma V.3.34, we obtain

$$\begin{tikzcd}
\mathfrak{K}(\omega(s_+)) \arrow[d, "P_r"'] \arrow[dd, bend right=60, "P_{r'}"'] & & \\
\omega(r)\cdot\mathfrak{K}(\omega(r)^{-1}\omega(s_+)) \arrow[d, "\hat{P}_{r'}"'] & \mathfrak{K}(\omega(r)^{-1}\omega(s_+)) \arrow[l, "\omega(r)\cdot"'] \arrow[d, "\omega(r')^{-1}\omega(r)\tilde{P}_{r'}"'] & L^2(H|_{(r,s_+)}) \arrow[l, "\Theta_{r,s_+}"'] \arrow[d, "\rho_{r',s_+}"] \\
\omega(r')\cdot\mathfrak{K}(\omega(r')^{-1}\omega(s_+)) & \mathfrak{K}(\omega(r')^{-1}\omega(s_+)) \arrow[l, "\omega(r')\cdot"'] & L^2(H|_{(r',s_+)}) \arrow[l, "\Theta_{r',s_+}"']
\end{tikzcd}$$

By uniqueness of the elements satisfying (2.24), it follows that $f_{r'} = \rho_{r',s_+} f_r$. Since $\inf((\sigma, s_+] \cap I_{reg}) = \sigma$, an element $f_+ \in \mathcal{M}((\sigma, s_+))/_{=_H}$ is well-defined by the requirement that always $(f_+)|_{(r,s_+)} = f_r$. Defining $f \in \mathcal{M}(I)/_{=_H}$ by $f|_{(s_-,\sigma)} = f_-$ and $f|_{(\sigma,s_+)} = f_+$, clearly gives us an element with the desired properties.

If (σ, s_+) is indivisible, we simply take on the right half of the interval the formula (2.22) as a definition, and again obtain an element with the desired properties. ❑

2.18 Definition. Let $\psi(\omega) : \mathfrak{K}(\omega(s_+)) \to \mathcal{M}(I)/_{=_H}$ be the map which assigns to each element $F \in \mathfrak{K}(\omega(s_+))$ the unique element f given by Lemma 2.17. //

2.19 Remark.

- *(i)* Let $r \in (s_-, \sigma)$, then $(\psi F)|_{(s_-,r)} \in L^2(H|_{(s_-,r)})$. Similarly, if $r \in (\sigma, s_+)$, then $(\psi F)|_{(r,s_+)} \in L^2(H|_{(r,s_+)})$.
- *(ii)* Since Θ_{r_-,r_+} and P_r are both compatible with the respective involutions, it follows that $\psi(F^\#) = \overline{\psi(F)}$.

//

In general it will not be possible to compute ψF explicitly. However, for the case of reproducing kernels this can be done.

2.20 Lemma. *We have*

$$\psi\big(H_{\omega(s_+)}(w, .)v\big) \ =_H \ \omega(.)(w)^* v, \ \ w \in \mathbb{C}, v \in \mathbb{C}^2 \, .$$

Proof. Let $r \in [s_-, \sigma) \cap I_{\rm reg}$ be such that (s_-, r) is not indivisible. By (V.3.6), we have

$$\Theta_{s_-,r}\omega(.)(w)^* v = H_{\omega(r)}(w, .)v \, .$$

However, since P_r is the orthogonal projection of $\mathfrak{K}(\omega(s_+))$ onto $\mathfrak{K}(\omega(r))$, we have $P_r H_{\omega(s_+)}(w, .) = H_{\omega(r)}(w, .)$. It follows that $\psi\big(H_{\omega(s_+)}(w, .)v\big)|_{(s_-,\sigma)} =_H \omega(.)(w)^* v|_{(s_-,\sigma)}$.

Assume that (σ, s_+) is not indivisible, and let $r \in (\sigma, s_+) \cap I_{\rm reg}$ be such that (r, s_+) is not indivsible. The relation (V.3.6) applied with $u := \omega(r)^* v$ gives

$$\omega(r)\cdot\Theta_{r,s_+}\omega(.)(w)^* v = \omega(r)\cdot\Theta_{r,s_+}\left([\omega(r)^{-1}\omega(.)](w)^* u\right) =$$

$$= \omega(r)\cdot H_{\omega(r)^{-1}\omega(s_+)}(w, .)u = \omega(r) H_{\omega(r)^{-1}\omega(s_+)}(w, .)\omega(r)^* v \, .$$

Since P_r is the orthogonal projection of $\mathfrak{K}(\omega(s_+))$ onto $\omega(r)\cdot\mathfrak{K}(\omega(r)^{-1}\omega(s_+))$, we have

$$P_r H_{\omega(s_+)}(w,.)v = \omega(r)H_{\omega(r)^{-1}\omega(s_+)}(w,.)\omega(r)^* v\,,$$

cf. (V.2.19). It follows that $\psi\big(H_{\omega(s_+)}(w,.)v\big)|_{(\sigma,s_+)} =_H \omega(.)(w)^*v|_{(\sigma,s_+)}$.

Assume next that (σ,s_+) is indivisible. Write

$$\omega(t) := \begin{pmatrix} w_{11}(t,.) & w_{12}(t,.) \\ w_{21}(t,.) & w_{22}(t,.) \end{pmatrix}$$

then

$$\omega(t)(w)^*v = \begin{pmatrix} (w_{11}(t,\overline{w}),w_{21}(t,\overline{w}))v \\ (w_{12}(t,\overline{w}),w_{22}(t,\overline{w}))v \end{pmatrix}.$$

By the definition of ψ and Corollary 2.13, we have

$$\psi\big(H_{\omega(s_+)}(w,.)v\big) = \big[H_{\omega(s_+)}(w,.)v, \begin{pmatrix} w_{12}(s_+,.) \\ w_{22}(s_+,.) \end{pmatrix}\big]\begin{pmatrix} 0 \\ 1 \end{pmatrix} =$$

$$= \overline{v^*\begin{pmatrix} w_{12}(s_+,w) \\ w_{22}(s_+,w) \end{pmatrix}}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = (w_{12}(s_+,\overline{w}),w_{22}(s_+,\overline{w}))v\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Since we know that the number ψ in (2.17) is equal to $\frac{\pi}{2}$, we have

$$\begin{pmatrix} w_{21}(t,.) \\ w_{22}(t,.) \end{pmatrix} = \begin{pmatrix} w_{21}(s_+,.) \\ w_{22}(s_+,.) \end{pmatrix},\ t\in(\sigma,s_+]\,.$$

Since the Hamiltonian $H|_{(\sigma,s_+)}$ is of the form

$$H(t) = h(t)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ t\in(\sigma,s_+)\,,$$

the asserted equality, i.e. equality modulo $=_H$, follows. ❑

Our next task is to show in detail how ψ relates $\mathfrak{K}(\omega(s_+))$ with $L^2(H)$. In the following denote by $\rho_- : \mathcal{M}(I)\to\mathcal{M}([s_-,\sigma))$ and $\rho_+ : \mathcal{M}(I)\to\mathcal{M}((\sigma,s_+])$ the respective restriction maps, and set

$$\mathfrak{K}^-(\omega) := \pi_2^{-1}(\mathfrak{P}_0),\quad \mathfrak{K}^+(\omega) := \pi_2^{-1}(\mathfrak{P}_a)^\perp\,,$$

$$X^\delta(\omega) := \operatorname{span}\{\delta(\omega)_\Delta,\ldots,\delta(\omega)_{\Delta+\ddot{o}-1}\}\,,$$

$$X_\delta(\omega) := \operatorname{span}\{\delta(\omega)_0,\ldots,\delta(\omega)_{\Delta-1}\}\,.$$

Then, by Lemma 2.8, $X_\delta(\omega)\perp X^\delta(\omega)$, $X^\delta(\omega)$ is nondegenerated, and

$$\begin{gathered}(\mathfrak{K}^-(\omega)+\mathfrak{K}^+(\omega))^\perp = \operatorname{span}\{\delta(\omega)_0,\ldots,\delta(\omega)_{\Delta+\ddot{o}-1}\}\,,\\ (\mathfrak{K}^-(\omega)+\mathfrak{K}^+(\omega))^\circ = \operatorname{span}\{\delta(\omega)_0,\ldots,\delta(\omega)_{\Delta-1}\}\,.\end{gathered}\tag{2.25}$$

The above relations then imply

$$\begin{gathered}(\mathfrak{K}^-(\omega)+\mathfrak{K}^+(\omega)+X^\delta(\omega))^\perp = (\mathfrak{K}^-(\omega)+\mathfrak{K}^+(\omega)+X^\delta(\omega))^\circ = \\ = \operatorname{span}\{\delta(\omega)_0,\ldots,\delta(\omega)_{\Delta-1}\}\,.\end{gathered}\tag{2.26}$$

2.21 Proposition. *We have*

(i)
$$\ker(\rho_- \circ \psi) = (\mathfrak{K}^-)^\perp, \quad \ker(\rho_+ \circ \psi) = (\mathfrak{K}^+)^\perp ,$$
$$\ker \psi = \operatorname{span}\big\{\delta_0, \ldots, \delta_{\Delta+\ddot{o}-1}\big\} .$$

(ii) *The restriction* $\psi|_{\mathfrak{K}^\pm}$ *maps* $\mathfrak{K}^\pm$ *isometrically onto* $L^2(H_\pm)$. *We have*

$$[F,G] = \int_I (\psi G)^* H (\psi F), \quad F \in X_\delta^\perp, G \in (X_\delta + X^\delta)^\perp . \tag{2.27}$$

(iii)
$$(\rho_- \circ \psi)^{-1}(L^2(H_-)) = \operatorname{span}\big\{\delta_0, \ldots, \delta_{\Delta_- -1}\big\}^\perp$$
$$(\rho_+ \circ \psi)^{-1}(L^2(H_+)) = \operatorname{span}\big\{\delta_0, \ldots, \delta_{\Delta_+ -1}\big\}^\perp$$
$$\psi^{-1}(L^2(H)) = \operatorname{span}\big\{\delta_0, \ldots, \delta_{\Delta-1}\big\}^\perp$$

Proof (of Proposition 2.21, (i)). By its definition

$$\mathfrak{P}_0 = \operatorname{cls}\big\{\mathfrak{P}(E_t) : \, t \in [s_-, \sigma) \cap I_{\text{reg}}\big\} ,$$

and hence

$$\mathfrak{K}^- = \operatorname{cls}\big\{\mathfrak{K}(\omega(t)) : \, t \in [s_-, \sigma) \cap I_{\text{reg}}\big\} . \tag{2.28}$$

Let $F \in \mathfrak{K}(\omega(s_+))$, then $F \perp \mathfrak{K}^-$ if and only if $F \perp \mathfrak{K}(\omega(t))$ for all $t \in [s_-, \sigma) \cap I_{\text{reg}}$. This, however, is by the definition of ψ equivalent to $(\psi F)(t) = 0$ for all $t \in [s_-, \sigma)$. Thus, the first part of (i) holds true.

If $(\sigma, s_+]$ is not indivisible, we have

$$\mathfrak{P}_a^\perp = \operatorname{cls}\big\{\mathfrak{P}(E_t)^\perp : \, t \in (\sigma, s_+] \cap I_{\text{reg}}\big\} ,$$

and hence

$$\mathfrak{K}^+ = \operatorname{cls}\big\{\mathfrak{K}(\omega(t))^\perp : \, t \in (\sigma, s_+] \cap I_{\text{reg}}\big\} . \tag{2.29}$$

Since $\mathfrak{K}(\omega(t))^\perp = \omega(t) \cdot \mathfrak{K}(\omega(t)^{-1}\omega(s_+))$, the same argument as in the first paragraph yields that $F \perp \mathfrak{K}^+$ if and only if $(\psi F)(t) = 0$ for all $t \in (\sigma, s_+]$.

Consider the case that (σ, s_+) is indivisible. By (2.22), we have $(\rho_+ \circ \psi)F = 0$ if and only if $F \perp \pi_2^{-1} D_0$. However, $\{D_0\}^\perp = \mathfrak{P}_a$. This completes the proof of the second part of (i).

Finally, from the already shown facts and Lemma 2.8 we obtain

$$\ker \psi = \ker(\rho_- \circ \psi) \cap \ker(\rho_+ \circ \psi) = \pi_2^{-1}\big(\mathfrak{P}_0^\perp \cap \mathfrak{P}_a\big) =$$
$$= \operatorname{span}\{\delta_0, \ldots, \delta_{\Delta+\ddot{o}-1}\} .$$

❑

For the proof of the assertion Proposition 2.21, (ii), we will employ the following elementary geometric statement, which supplements [KWW1].

2.22 Lemma. *Let* $\mathcal{Q}_1$ *and* $\mathcal{Q}_2$ *be almost Pontryagin spaces, and let* $L_1 \subseteq \mathcal{Q}_1$, $L_2 \subseteq \mathcal{Q}_2$, *be dense linear subspaces. If* $\lambda : L_1 \to L_2$ *is a surjective and isometric map, then there exists an isometric isomorphism* $\hat{\lambda} : \mathcal{Q}_1/_{\mathcal{Q}_1^\circ} \to \mathcal{Q}_2/_{\mathcal{Q}_2^\circ}$ *with*

$$\begin{tikzcd}
L_1 \arrow[r, "\lambda"] \arrow[d] & L_2 \arrow[d] \\
\mathcal{Q}_1/_{\mathcal{Q}_1^\circ} \arrow[r, "\hat{\lambda}"'] & \mathcal{Q}_2/_{\mathcal{Q}_2^\circ}
\end{tikzcd}$$

where the downwards arrows are the canonical maps, i.e. inclusion followed by projection.

Proof. Let $x \in L_1 \cap \mathcal{Q}_1^\circ$ and $y \in L_2$. Choose $y_1 \in L_1$ with $\lambda(y_1) = y$, then

$$[\lambda(x), y] = [x, y_1] = 0\,.$$

Since L_2 is dense in $\mathcal{Q}_2$, it follows that $\lambda(x) \in \mathcal{Q}_2^\circ$.

Therefore a map $\tilde{\lambda} : \mathcal{L}_1/_{\mathcal{Q}_1^\circ} \to \mathcal{L}_2/_{\mathcal{Q}_2^\circ}$ with

$$\begin{tikzcd}
L_1 \arrow[r, "\lambda"] \arrow[d] & L_2 \arrow[d] \\
L_1/_{\mathcal{Q}_1^\circ} \arrow[r, "\hat{\lambda}"'] & L_2/_{\mathcal{Q}_2^\circ}
\end{tikzcd}$$

is well-defined. Clearly, $\tilde{\lambda}$ is isometric. However, $\mathcal{L}_j/_{\mathcal{Q}_j^\circ}$ is a dense linear subspace of the Pontryagin space $\mathcal{Q}_j/_{\mathcal{Q}_j^\circ}$, and hence $\tilde{\lambda}$ can be extended by continuity to an isometric isomorphism between $\mathcal{Q}_1/_{\mathcal{Q}_1^\circ}$ and $\mathcal{Q}_2/_{\mathcal{Q}_2^\circ}$. ❑

Proof (of Proposition 2.21, (ii)). Let $r \in [s_-, \sigma) \cap I_{\text{reg}}$ be such that (s_-, r) is not indivisible. By the definition of ψ, we have $\psi|_{\mathfrak{K}(\omega(r))} = \Theta_{s_-,r}^{-1}$, and therefore $\psi|_{\mathfrak{K}(\omega(r))}$ is an isometric isomorphism of $\mathfrak{K}(\omega(r))$ onto $L^2(H_{(s_-,r)})$. It follows that ψ induces a surjective isometry

$$\lambda := \psi|_{\bigcup_r \mathfrak{K}(\omega(r))} : \bigcup_r \mathfrak{K}(\omega(r)) \to \bigcup_r L^2(H_{(s_-,r)})\,,$$

where unions are taken over the range of all values of $r \in [s_-, \sigma) \cap I_{\text{reg}}$ such that (s_-, r) is not indivisible. Lemma 2.22 applied with the map λ and the almost Pontryagin spaces $\mathfrak{K}^-$ and $L^2(H_-)$, gives an isometric isomorphism $\hat{\lambda} : \mathfrak{K}^-/_{(\mathfrak{K}^-)^\circ} \to L^2(H_-)$.

Let $F \in \mathfrak{K}^-$ be given. If $r \in [s_-, \sigma) \cap I_{\text{reg}}$ is such that (s_-, r) is not indivisible, let $g \in L^2(H_{(s_-,r)})$, and choose $G \in \mathfrak{K}(\omega(r))$ such that $g = \lambda(G)$. Then $g = \hat{\lambda}([G]_{(\mathfrak{K}^-)^\circ})$ and

$$(\psi F)|_{(s_-,r)} = (\psi P_r F)|_{(s_-,r)} = \hat{\lambda}([P_r F]_{(\mathfrak{K}^-)^\circ})|_{(s_-,r)}\,.$$

We compute

$$\int_{s_-}^{r} g^* H \hat{\lambda}([F]_{(\mathfrak{K}^-)^\circ}) = \big[\hat{\lambda}([F]_{(\mathfrak{K}^-)^\circ}), \hat{\lambda}([G]_{(\mathfrak{K}^-)^\circ})\big] = [F, G] = [F, P_r G] =$$

$$= [P_r F, G] = \big[\hat{\lambda}([P_r F]_{(\mathfrak{K}^-)^\circ}), \hat{\lambda}([G]_{(\mathfrak{K}^-)^\circ})\big] = \int_{s_-}^{r} g^* H (\psi F)\,.$$

and get that $\psi F = \hat{\lambda}([F]_{(\mathfrak{K}^-)^\circ})$. Since $\hat{\lambda}$ is isometric and onto $L^2(H_-)$, so is $\psi|_{\mathfrak{K}^-}$.

Assume that (σ, s_+) is not indivisible. Let $r \in (\sigma, s_+] \cap I_{\text{reg}}$ be such that (r, s_+) is not indivisible, then

$$\psi|_{\omega(r)\cdot\mathfrak{K}(\omega(r)^{-1}\omega(s_+))} = (\omega(r) \cdot \Theta_{s_-,r})^{-1}\,,$$

and therefore the map $\psi|_{\omega(r)\cdot\mathfrak{K}(\omega(r)^{-1}\omega(s_+))}$ is an isometric isomorphism of $\omega(r)\cdot\mathfrak{K}(\omega(r)^{-1}\omega(s_+))$ onto $L^2(H_{(r,s_+)})$. Hence

$$\psi|_{\bigcup_r \omega(r)\cdot\mathfrak{K}(\omega(r)^{-1}\omega(s_+))} : \bigcup_r \omega(r)\cdot\mathfrak{K}(\omega(r)^{-1}\omega(s_+)) \to \bigcup_r L^2(H_{(r,s_+)})$$

is a surjective isometry between dense subspaces of the almost Pontryagin spaces $\mathfrak{K}^+$ and $L^2(H_+)$. The desired conclusion follows in exactly the same way as in the above paragraph.

If (σ, s_+) is indivisible, matters are trivial. Actually, in this case, $L^2(H_+) = \{0\}$ and $\mathfrak{K}^+ = \operatorname{span}\{\delta_0\}$. However, by definition $\psi\delta_0 = 0$. This completes the proof of the first assertion in *(ii)*.

In order to show (2.27), let $F \perp X_\delta$ and $G \perp (X_\delta + X^\delta)$. Due to (2.25) and (2.26) we can decompose F as $F = F_- + F_+ + F_1$ with $F_\pm \in \mathfrak{K}^\pm$, $F_1 \in X^\delta$, and $G = G_- + G_+$ with $G_\pm \in \mathfrak{K}^\pm$. It follows that

$$[F,G] = [F_-,G_-] + [F_+,G_+] = \int_{s_-}^{\sigma} (\psi G_-)^* H(\psi F_-) + \int_{s_-}^{\sigma} (\psi G_+)^* H(\psi F_+)\,.$$

❑

The assertions in *(iii)* are obtained as a corollary of *(ii)*.

Proof (of Proposition 2.21, (iii)). Since ψ maps $\mathfrak{K}^-$ onto $L^2(H_-)$, we have

$$(\rho_- \circ \psi)^{-1}(L^2(H_-)) = \mathfrak{K}^- + \ker(\rho_- \circ \psi) = \mathfrak{K}^- + (\mathfrak{K}^-)^\perp\,.$$

However, since $\mathfrak{K}(\omega(s_+))$ is a Pontryagin space, the sum $\mathfrak{K}^- + (\mathfrak{K}^-)^\perp$ is closed, and we may compute further

$$\mathfrak{K}^- + (\mathfrak{K}^-)^\perp = \big((\mathfrak{K}^-)^\perp \cap \mathfrak{K}^-\big)^\perp = \operatorname{span}\{\delta_0,\ldots,\delta_{\Delta_- -1}\}^\perp\,.$$

Similarly,

$$(\rho_+ \circ \psi)^{-1}(L^2(H_+)) = \mathfrak{K}^+ + \ker(\rho_+ \circ \psi) = \mathfrak{K}^- + (\mathfrak{K}^+)^\perp =$$

$$= \big((\mathfrak{K}^+)^\perp \cap \mathfrak{K}^+\big)^\perp = \operatorname{span}\{\delta_0,\ldots,\delta_{\Delta_+ -1}\}^\perp\,.$$

The last assertion in *(iii)* is now obvious. ❑

The next step in the investigation of ψ is to show how ψ relates $T(\omega(s_+))$ with differentiation. This is a crucial fact.

2.23 Proposition. *Let $((F;G);(a;b)) \in \Gamma(\omega(s_+))$ be given. Then there exists a locally absolutely continuous representant f of ψF, such that (g is any representant of ψG)*

$$f' = JHg, \quad f(s_-) = a,\ f(s_+) = b\,.$$

Proof. Let $r \in (s_-,\sigma) \cap I_{\text{reg}}$. By Proposition V.2.11 and Corollary V.3.27, we have

$$\big((P_r \boxtimes P_r) \boxtimes \pi_l\big)\Gamma(\omega(s_+)) \subseteq \big((\mathrm{id} \boxtimes \mathrm{id}) \boxtimes \pi_l\big)\Gamma(\omega(r)) =$$

$$= \big((\mathrm{id} \boxtimes \mathrm{id}) \boxtimes \pi_l\big)\big((\Theta_{s_-,r} \boxtimes \Theta_{s_-,r}) \boxtimes \mathrm{id}_{\mathbb{C}^2\times\mathbb{C}^2}\big)\Gamma(H|_{(s_-,r)}) =$$

$$= \big((\Theta_{s_-,r} \boxtimes \Theta_{s_-,r}) \boxtimes \pi_l\big)\Gamma(H|_{(s_-,r)})$$

Hence, there exists $((f_r; g_r); (a_r; b_r)) \in \Gamma(H|_{(s_-,r)})$, such that

$$\Theta_{s_-,r} f_r = P_r F,\ \Theta_{s_-,r} g_r = P_r G, \quad a_r = a\,.$$

In view of the definition of ψ, we must have $f_r = (\psi F)|_{(s_-,r)}$ and $g_r = (\psi G)|_{(s_-,r)}$. It follows that there exists a unique locally absolutely continuous representant $\tilde{f}_r$ of $(\psi F)|_{(s_-,r)}$ satisfying $\tilde{f}_r' = JH(\psi G)|_{(s_-,r)}$, cf. [HSW]. Moreover, it has the boundary value $\tilde{f}_r(s_-) = a$.

If $r, r' \in (s_-, \sigma) \cap I_{\mathrm{reg}}$, $r' < r$, clearly, $(\tilde{f}_r|_{(s_-,r')})' = JH(\psi G)|_{(s_-,r')}$. By uniqueness, $\tilde{f}_{r'} = \tilde{f}_r|_{(s_-,r')}$. Hence, a locally absolutely continuous function f_- on (s_-,σ) is well-defined by the requirement that always $(f_-)|_{(s_-,r)} = \tilde{f}_r$. Apparently, f_- is a representant of $\psi F|_{(s_-,\sigma)}$, satisfies $(f_-)' = JH(\psi G)|_{(s_-,\sigma)}$ and $f_-(s_-) = a$.

Consider the case that (σ, s_+) is not indivisible. We proceed similar as in the preceding paragraph. Let $r \in (\sigma, s_+) \cap I_{\mathrm{reg}}$. By Proposition V.2.11 and Corollary V.3.27,

$$\big((P_r \boxtimes P_r) \boxtimes \pi_r\big)\Gamma(\omega(s_+)) \subseteq \big((\mathrm{id} \boxtimes \mathrm{id}) \boxtimes \pi_r\big)\Gamma(\mathfrak{B}_{\omega(r)}(\omega(r)^{-1}\omega(s_+))) =$$

$$= \big((\mathrm{id} \boxtimes \mathrm{id}) \boxtimes \pi_r\big)\big((\omega(r)\cdot\Theta_{r,s_+} \boxtimes \omega(r)\cdot\Theta_{r,s_+}) \boxtimes \mathrm{id}_{\mathbb{C}^2\times\mathbb{C}^2}\big)\Gamma(H|_{(r,s_+)}) =$$

$$= \big((\omega(r)\cdot\Theta_{r,s_+} \boxtimes \omega(r)\cdot\Theta_{r,s_+}) \boxtimes \pi_r\big)\Gamma(H|_{(r,s_+)})\,.$$

Again we see that $f_r = (\psi F)|_{(r,s_+)}$ and $g_r = (\psi G)|_{(r,s_+)}$ for some element $((f_r; g_r); (a_r; b)) \in \Gamma(H|_{(r,s_+)})$. Hence, the same argument as above will provide us with a locally absolutely continuous representant f_+ of $\psi F|_{(\sigma,s_+)}$, such that $(f_+)' = JH(\psi G)|_{(\sigma,s_+)}$ and $f_+(s_+) = b$. Defining f by putting together these two functions, the assertion follows.

Consider the case that (σ, s_+) is indivisible. Then, by the abstract Green's identity and Lemma 2.15, we have ($b = \binom{b_1}{b_2}$)

$$-[F, \pi_2^{-1} D_0] = -\begin{pmatrix}1\\0\end{pmatrix}^* Jb = b_2\,. \tag{2.30}$$

By the definition (2.22) of ψ, we have

$$\psi F|_{(\sigma,s_+)} = -[\pi_2 F, D_0]\begin{pmatrix}0\\1\end{pmatrix},\ \psi G|_{(\sigma,s_+)} = -[\pi_2 G, D_0]\begin{pmatrix}0\\1\end{pmatrix}.$$

Since the Hamiltonian H is, on the interval (σ, s_+), of the form

$$H(t) = h(t)\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}, \quad t \in (\sigma, s_+)\,,$$

it follows that the function

$$f_+(t) = \begin{pmatrix} b_1 + [\pi_2 G, D_0] \int_{s_+}^t h \\ -[\pi_2 F, D_0] \end{pmatrix}$$

is a locally absolutely continuous representant of $\psi F|_{(\sigma,s_+)}$ and satisfies $(f_+)' = JH\psi G|_{(\sigma,s_+)}$. Moreover, by (2.30), we have $f_+(s_+) = \binom{b_1}{-[\pi_2 F, D_0]} = b$. Proceeding similarly as in the above paragraph completes the proof. ❑

2.24 Corollary. *Let $(F;G) \in T(\omega(s_+))$, and assume that f is a locally absolutely continuous representant of ψF with $f' = JH(\psi g)$. Then there exists a constant $\gamma \in \mathbb{C}$ such that*

$$\big((F;G);(f(s_-);f(s_+)+\gamma\begin{pmatrix}1\\0\end{pmatrix})\big) \in \Gamma(\omega(s_+))\,.$$

If (σ, s_+) is not indivisible, then $\gamma = 0$.

Proof. Choose a locally absolutely continuous representant f_1 of ψF according to Proposition 2.23, so that $((F;G);(f_1(s_-);f_1(s_+)) \in \Gamma(\omega(s_+))$.

Let $r \in (s_-,\sigma) \cap I_{\text{reg}}$ be such that (s_-,r) is not indivisible, then $f, f_1, \psi g \in L^2(H|_{(s_-,r)})$, and

$$f' = f_1' = JHg,\ t \in (s_-,r)\,.$$

Since $f_1 =_H f$, it follows that $f|_{(s_-,\sigma)} = f_1|_{(s_-,\sigma)}$, cf. [HSW, Lemma 3.5]. In particular, boundary values at s_- coincide.

The same argument will work for boundary values at s_+ if (σ, s_+) is not indivsible. If (σ, s_+) is indivisible, we only obtain $\pi_2 f_1(s_+) = \pi_2 f(s_+)$, cf. (2.22). Still, due to Lemma 2.15 this also yields the present assertion. ❑

2.25 Corollary. *Let $((F_1;G_1);(a_1;b_1)),((F_2;G_2);(a_2;b_2)) \in \Gamma(\omega(s_+))$.*

- *(i) If $(\psi F_1)|_{(s_-,r)} = (\psi F_2)|_{(s_-,r)}$ and $(\psi G_1)|_{(s_-,r)} = (\psi G_2)|_{(s_-,r)}$ for some $r \in (s_-,\sigma) \cap I_{\text{reg}}$ such that (s_-,r) is not indivisible, then $a_1 = a_2$.*
- *(ii) If $(\psi F_1)|_{(r,s_+)} = (\psi F_2)|_{(r,s_+)}$ and $(\psi G_1)|_{(r,s_+)} = (\psi G_2)|_{(r,s_+)}$ for some $r \in (\sigma,s_+) \cap I_{\text{reg}}$ such that (r,s_+) is not indivisible, then $b_1 = b_2$.*
- *(iii) Assume that (σ,s_+) is indivisible. If $(\psi F_1)|_{(\sigma,s_+)} = (\psi F_2)|_{(\sigma,s_+)}$, then $\pi_2 b_1 = \pi_2 b_2$.*

  For each prescribed value $\beta \in \mathbb{C}$, there exists $((F;G);(a;b)) \in \Gamma(\omega(s_+))$ such that $(\psi F_1)|_{(\sigma,s_+)} = (\psi F)|_{(\sigma,s_+)}$, $(\psi G_1)|_{(\sigma,s_+)} = (\psi G)|_{(\sigma,s_+)}$, and $\pi_1 b = \beta$.

Proof. Assume we are in the situation of (i). Then

$$\big((\psi F_1)|_{(s_-,r)};(\psi G_1)|_{(s_-,r)}\big) = \big((\psi F_2)|_{(s_-,r)};(\psi G_2)|_{(s_-,r)}\big) \in T_{\max}(H|_{(s_-,r)})\,.$$

Choose locally absolutely continuous representants f_1 and f_2 according to Proposition 2.23 applied with $((F_1;G_1);(a_1;b_1))$ or $((F_2;G_2);(a_2;b_2))$, respectively. Since (s_-,r) is not indivisible, we must have $f_1|_{(s_-,r)} = f_2|_{(s_-,r)}$. It follows that $a_1 = a_2$.

The assertion (ii) is proved completely similar. Let us investigate the situation that (σ,s_+) is indivisible. Then $(\psi F_1)|_{(\sigma,s_+)} = (\psi F_2)|_{(\sigma,s_+)}$ just means $[F_1,\delta_0] = [F_2,\delta_0]$. Using the abstract Green's identity with the element (2.20), yields $\pi_2 b_1 = \pi_2 b_2$. The second part of the assertion in (iii) follows, since we may add arbitrary scalar multiples of the element (2.20) to a given element $((F_1;G_1);(a_1;b_1))$ of $\Gamma(\omega(s_+))$ without altering ψF_1 and ψG_1. ❑

d. The elements P_j.

Let D_0 be the element constructed in Subsection b., let $\mathcal{B} := \mathcal{B}_{-D_0}$ be the bounded selfadjoint operator as introduced in Lemma V.2.15, and recall the notation $B(z)$ from Definition 2.11. Note that $\mathcal{B}D_k = D_{k-1}$, $k = 1,\dots,n$.

2.26 Definition. Let elements $P_j \in \mathfrak{P}(E_{s_+})$, $j \in \mathbb{N}\cup\{0\}$, be defined inductively by

$$P_0(z) := K_{E_{s_+}}(0,z)\,,$$

$$P_j(z) := \mathcal{B}P_{j-1}(z) - \begin{cases} B(z) & , \ j = \Delta \\ 0 & , \ \text{otherwise} \end{cases}$$

Moreover, set $d_j(\omega) := P_j(0)$. //

2.27 Lemma. *We have*

$$[P_j, D_k] = \begin{cases} -1 & , \ j = k \in \{0,\dots,\Delta-1\} \\ 0 & , \ \textit{otherwise} \end{cases}$$

and

$$[P_k, P_l] = d_{k+l},\ \ k,l \in \mathbb{N}_0\,.$$

Proof. Our definitions together with Lemma V.2.15 give

$$[P_0, D_0] = -1, \quad [P_0, D_k] = [P_k, D_0] = 0,\ \ k \geq 1\,. \tag{2.31}$$

We will show the recurrance relation

$$[P_j, D_k] = [P_{j-1}, D_{k-1}] + \begin{cases} 1 & , \ j = k = \Delta \\ 0 & , \ \text{otherwise} \end{cases}, \quad j \geq 1,\ k = 1,\dots,\Delta+\ddot{o}-1\,. \tag{2.32}$$

Together with (2.31), this will imply the first assertion of the lemma.

In order to see (2.32) we use the selfadjointness of $\mathcal{B}$, and obtain together with (2.19), (2.31) and $\mathcal{B}D_k(z) = D_{k-1}(z)$, that for $j \geq 1$, $k = 1,\dots,\Delta+\ddot{o}-1$,

$$[P_j, D_k] = \big[\mathcal{B}P_{j-1} \underbrace{-B}_{\substack{\text{only if} \\ j=\Delta}}, D_k\big] = [P_{j-1}, D_{k-1}] + \left\{ \begin{array}{ll} 1 & , \ j = k = \Delta \\ 0 & , \ \text{otherwise} \end{array} \right. .$$

In order to see the second assertion, note that by the already proved part of the present assertion $B \perp P_k$, $k \in \mathbb{N}\cup\{0\}$. Since $\mathcal{B}$ is selfadjoint,

$$[P_k, P_{l+1}] = [P_k, \mathcal{B}P_l] = [\mathcal{B}P_k, P_l] = [P_{k+1}, P_l],\ \ k,l \in \mathbb{N}\cup\{0\}\,.$$

The asserted formula follows inductively from the definition $d_k = [P_k, P_0]$. ❑

We are now in position to establish the condition $\Delta(H) < \infty$. From now on let $\mathfrak{b}(\omega) := \pi_2^{-1}B(z)$. Again, we will drop the argument ω if no confusion may occur.

2.28 Proposition. *We have* $\Delta(H_-) = \Delta_-(\omega)$ *and* $\Delta(H_+) = \Delta_+(\omega)$. *Let functions* $\mathfrak{w}_k$, $k \in \mathbb{N}_0$, *be defined as in [KW/IV, §4.1], then*

$$\mathfrak{w}_k =_H \psi(\pi_2^{-1}P_k),\ \ k \in \mathbb{N}_0\,,$$

and

$$\big((\pi_2^{-1}P_k \underbrace{+\mathfrak{b}}_{\substack{\textit{only if} \\ k=\Delta}}; \pi_2^{-1}P_{k-1} + d_{k-1}\delta_0); (\mathfrak{w}_k(s_-); \mathfrak{w}_k(s_+))\big) \in \Gamma(\omega(s_+))\,.$$

Proof. Since $P_0 = K_{E_{\omega(s_+)}}(0,.) = \pi_2 H_{\omega(s_+)}(0,.)\binom{0}{1}$, we know from Lemma 2.20 that

$$\psi(\pi_2^{-1}P_0) =_H \begin{pmatrix}0\\1\end{pmatrix}.$$

Moreover, by definition of $\Gamma(\omega(s_+))$,

$$\big((\pi_2^{-1}P_0;0);(\begin{pmatrix}0\\1\end{pmatrix};\begin{pmatrix}0\\1\end{pmatrix})\big) \in \Gamma(\omega(s_+))\,. \tag{2.33}$$

By Lemma V.2.15, the pair

$$(P_k \underbrace{+B}_{\substack{\text{only if}\\k=\Delta}}; P_{k-1} + d_{k-1}D_0)$$

belongs to some selfadjoint extension of $\mathcal{S}(E_{\omega(s_+)})$. Thus Lemma V.2.16 gives

$$(\pi_2^{-1}P_k \underbrace{+\mathfrak{b}}_{\substack{\text{only if}\\k=\Delta}}; \pi_2^{-1}P_{k-1} + d_{k-1}\delta_0) \in T_1(\omega(s_+))\,.$$

Let $a_k, b_k \in \mathbb{C}^2$ be such that

$$\big((\pi_2^{-1}P_k \underbrace{+\mathfrak{b}}_{\substack{\text{only if}\\k=\Delta}}; \pi_2^{-1}P_{k-1} + d_{k-1}\delta_0);(a_k;b_k)\big) \in \Gamma(\omega(s_+))\,,$$

and put $w_0 := \binom{0}{1}$. We have $\mathfrak{b}$ $(= \pi_2^{-1}B(z)), \delta_0 \in \ker\psi$. Hence, by Proposition 2.23, there exist locally absolutely continuous representants w_k of $\psi(\pi_2^{-1}P_k)$, $k \in \mathbb{N}$, such that

$$w_{k+1}' = JHw_k,\ w_k(s_-) = a_k, w_k(s_+) = b_k, \quad k \in \mathbb{N}_0\,.$$

By the definition of $T_1(\omega(s_+))$, we have $\pi_1 a_k = 0$. Using the Green's identity with the pair (2.33), it follows that $\pi_1 b_k = 0$. By Proposition 2.21, (*iii*), and Lemma 2.27, we have

$$\rho_- w_k \in L^2(H_-) \iff k \geq \Delta_-(\omega)$$

$$\rho_+ w_k \in L^2(H_+) \iff k \geq \Delta_+(\omega)$$

We conclude that $\Delta(H_-) = \Delta_-(\omega)$, $\Delta(H_+) = \Delta_+(\omega)$, and that $\mathfrak{w}_k = w_k = \psi P_k$. ❑

We set $p_j := \pi_2^{-1}P_j$.

2.29 Corollary. *The elements $p_0,\dots,p_{\Delta-1}$ are linearly independent modulo $(\mathfrak{K}^- + \mathfrak{K}^+)\dot{+}X^\delta$, and together with this space these elements span all of $\mathfrak{K}(\omega(s_+))$.*

Proof. The image of $p_0,\dots,p_{\Delta-1}$ under the linear map ψ is linearly independent modulo $L^2(H)$, and

$$\psi^{-1}(L^2(H)) = \operatorname{span}\{\delta_0,\dots,\delta_{\Delta-1}\}^\perp =$$

$$= \mathfrak{K}^- + \mathfrak{K}^+ + \operatorname{span}\{\delta_\Delta,\dots\delta_{\Delta+\ddot{o}-1}\}\,.$$

Since, by (2.26), $\dim(\mathfrak{K}(\omega(s_+))/(\mathfrak{K}^- + \mathfrak{K}^+ + X^\delta)) = \Delta$, the last assertion follows.

❑

e. Realization as model of $\mathfrak{h}$.

We have by the time collected all the necessary ingredients to define an elementary Hamiltonian $\mathfrak{h}$ of kind (A): Let $\mathfrak{h}$ be given by the data

$$H_-(\omega), H_+(\omega),\ \ddot{o}(\omega), b_0(\omega), \dots, b_{\ddot{o}+1}(\omega),\ d_0(\omega), \dots, d_{2\Delta(\omega)-1}(\omega)\,.$$

Our aim is to prove that $\omega_{\mathfrak{h}} = \omega$. Since both, $\omega_{\mathfrak{h}}|_{[s_-,\sigma)}$ and $\omega|_{[s_-,\sigma)}$, are solutions of the initial value problem (V.3.5) for the Hamiltonian $H_-(\omega)$, we already know that $\omega_{\mathfrak{h}}(t) = \omega(t)$, for $t \in [s_-,\sigma)$. In order to establish equality also on the interval $(\sigma, s_+]$, it is enough to show

$$\omega(\mathfrak{B}(\mathfrak{h})) = \omega(s_+)\,. \tag{2.34}$$

In fact, once having establish this equality, we may argue that both, $\omega_{\mathfrak{h}}|_{(\sigma,s_+]}$ and $\omega|_{(\sigma,s_+]}$, are solutions of the differential equation in (V.3.5) for the Hamiltonian $H_+(\omega)$ with the same initial value at the regular endpoint s_+. Thus equality on the whole interval $(\sigma, s_+]$ follows from (2.34).

2.30 Remark. Note that, by the uniqueness part of V.3.9, the relation (2.34) immediately implies that ω and $\omega_{\mathfrak{h}}$ are reparameterizations of each other. However, the above argument ensures actual equality. //

Below we will show that there exists an isomorphism $(\varpi; \mathrm{id})$ between the boundary triplets $\mathfrak{B}(\omega(s_+))$ and $\mathfrak{B}(\mathfrak{h})$. Remembering Proposition V.4.7 and Proposition V.4.27, this is sufficient to obtain (2.34).

Our first task is to construct the map ϖ. We need the following result, which supplements to Proposition IV.4.14.

2.31 Lemma. *Let notation be as in Proposition IV.4.14. The linear space* $\bigcup_J \operatorname{ran} \iota_J$*, where the union is taken over all sets of the form* (IV.4.20)*, is dense in* $\mathcal{C}$*.*

Proof. Let $y \in \mathfrak{P}(\mathfrak{h})$ be given, and assume that $y \perp \bigcup_J \operatorname{ran} \iota_J$. Then, by (IV.4.21), we have $\psi y = 0$. However, $\ker \psi = \mathcal{C}^\perp$. ❑

Consider the linear space

$$\mathcal{L} := \bigcup_{r_-, r_+} \Big(L^2(H|_{(s_-,r_-)}) \oplus L^2(H|_{(r_+,s_+)}) \Big)$$

where the union is taken over all $r_- \in (s_-,\sigma) \cap I_{\mathrm{reg}}$ such that (s_-, r) is not indivisible and all $r_+ \in (\sigma, s_+) \cap I_{reg}$ such that (r_+, s_+) is not indivisible, and let $\mathcal{L}$ be endowed with the $L^2(H)$-inner product. Note here that, if (σ, s_+) is indivisible, this definition, as well as the following considerations, are understood in such a way that the summands $L^2(H|_{(r_+,s_+)})$ are not present.

By Proposition IV.4.14 and Lemma V.3.34 we have isometric maps $\iota : \mathcal{L} \to \mathcal{C}(\mathfrak{h})$ and $\theta : \mathcal{L} \to \mathfrak{K}^-(\omega) + \mathfrak{K}^+(\omega)$ which are defined such that for all admissible values of r_-, r_+ the following diagram commutes

$$\begin{tikzcd}
 & L^2(H|_{(s_-,r_-)}) \oplus L^2(H|_{(r_+,s_+)}) \arrow[ddl, bend right, "\substack{\theta_{s_-,r_-} \oplus \\ \omega(r_+)\cdot\theta_{r_+,s_+}}"'] \arrow[d, "\subseteq"] \arrow[ddr, bend left, "\substack{\iota_{(s_-,r_-)} \\ \oplus \iota_{(r_+,s_+)}}"] & \\
 & \mathcal{L} \arrow[dl, "\theta"] \arrow[dr, "\iota"'] & \\
\mathfrak{K}^-(\omega) + \mathfrak{K}^+(\omega) & & \mathcal{C}(\mathfrak{h})
\end{tikzcd}$$

The ranges $\theta(\mathcal{L})$ and $\iota(\mathcal{L})$ are dense in the respective almost Pontryagin spaces $\mathfrak{K}^-(\omega)+\mathfrak{K}^+(\omega)$ and $\mathcal{C}(\mathfrak{h})$. For the map θ this follows from (2.28), (2.29) and, in case that (σ, s_+) is indivisible, because of $\mathfrak{K}^+(\omega) = \operatorname{span}\{\delta_0(\omega)\} \subseteq \mathfrak{K}^-(\omega)$. For the map ι, this is Lemma 2.31. Thus

$$\big(\mathfrak{K}^-(\omega)+\mathfrak{K}^+(\omega), \theta\big) \quad \text{and} \quad \big(\mathcal{C}(\mathfrak{h}), \iota\big)$$

are both almost Pontryagin space completions of $\mathcal{L}$ in the sense of [KWW1, Definition 4.2].

2.32 Lemma. *The completions $(\mathfrak{K}^-(\omega)+\mathfrak{K}^+(\omega), \theta)$ and $(\mathcal{C}(\mathfrak{h}), \iota)$ are isomorphic, i.e. there exists an isometric and bicontinuous isomorphism $\varpi : \mathfrak{K}^-(\omega)+\mathfrak{K}^+(\omega) \to \mathcal{C}(\mathfrak{h})$ such that*

$$\begin{tikzcd}
& \mathcal{L} \arrow[dl, "\theta"'] \arrow[dr, "\iota"] & \\
\mathfrak{K}^-(\omega)+\mathfrak{K}^+(\omega) \arrow[rr, "\varpi"'] & & \mathcal{C}(\mathfrak{h})
\end{tikzcd}$$

Proof. We are going to apply the uniqueness part of [KWW1, Proposition 4.4]. Let $f_1, \dots, f_\Delta : \mathcal{L} \to \mathbb{C}$ be the linear functionals

$$f_k(F) := \int\limits_{(s_-,\sigma)\cup(\sigma,s_+)} \mathfrak{w}_{k-1}^* HF, \quad F \in \mathcal{L}.$$

Since no nontrivial linear combination of $\mathfrak{w}_0, \dots, \mathfrak{w}_{\Delta-1}$ belongs to $L^2(H)$, no nontrivial linear combination of $f_1, \dots, f_\Delta$ is continuous with respect to the topology induced by the $L^2(H)$-inner product given on $\mathcal{L}$. However, we have

$$\begin{tikzcd}
& \mathcal{L} \arrow[dl, "\theta"'] \arrow[dr, "\iota"] \arrow[dd, "f_k"] & \\
\mathfrak{K}^-(\omega)+\mathfrak{K}^+(\omega) \arrow[dr, "{[.,p(\omega)_{k-1}]}"'] & & \mathcal{C}(\mathfrak{h}) \arrow[dl, "{[.,p(\mathfrak{h})_{k-1}]}"] \\
& \mathbb{C} &
\end{tikzcd} \tag{2.35}$$

Thereby, the left side of the diagram commutes by (2.27) which is applicable since each element of $\operatorname{ran}\theta$ has compact support in I. The right side of the diagram follows from (IV.4.21). Moreover, by (2.25), we have

$$\dim(\mathfrak{K}^-(\omega)+\mathfrak{K}^+(\omega))^\circ = \Delta = \dim \mathcal{C}(\mathfrak{h})^\circ .$$

Hence we may apply [KWW1, Proposition 4.4], and obtain an isomorphism ϖ with the desired properties. ❑

2.33 Definition. Define an extension of ϖ to a map from $\mathfrak{K}(\omega(s_+))$ to $\mathcal{P}(\mathfrak{h})$ by linearity and

$$\begin{aligned} p(\omega)_k &\mapsto p(\mathfrak{h})_k, \quad k = 0, \dots, \Delta - 1, \\ \delta(\omega)_k &\mapsto \delta(\mathfrak{h})_k, \quad k = \Delta, \dots, \Delta + \ddot{o} - 1. \end{aligned}$$

We will denote this map again by ϖ. //

First of all note that, due to Corollary 2.29, the map ϖ is well-defined and bijective.

2.34 Lemma. *The map* $\varpi : \mathfrak{K}(\omega(s_+)) \to \mathcal{P}(\mathfrak{h})$ *is an isometric isomorphism.*

Proof. The restriction $\varpi|_{\mathfrak{K}^-(\omega)\dot{+}\mathfrak{K}^+(\omega)}$ is isometric by Lemma 2.32, the restriction $\varpi|_{\operatorname{span}\{P_0,\ldots,P_{\Delta-1},\delta_\Delta,\ldots,\delta_{\Delta+\ddot{o}-1}\}}$ by the definition of $b_j(\mathfrak{h})$, $d_j(\mathfrak{h})$ and Lemma 2.27. Moreover, $\mathfrak{K}^-(\omega)\dot{+}\mathfrak{K}^+(\omega) \perp X^\delta(\omega)$ and $\mathcal{C}(\mathfrak{h}) \perp X^\delta(\mathfrak{h})$.

It remains to show that $[\varpi F, p(\mathfrak{h})_k] = [F, p(\omega)_k]$, $F \in \mathfrak{K}^-(\omega)\dot{+}\mathfrak{K}^+(\omega)$, $k = 0, \ldots, \Delta - 1$. However, the definition of ϖ and (2.35) implies

$$[\theta f, p(\omega)_k] = f_k(f) = [\iota f, p(\mathfrak{h})_k] = [\varpi(\theta f), p(\mathfrak{h})_k],\ f \in \mathcal{L}\,.$$

By continuity, the desired relation follows for all $F \in \mathfrak{K}^-(\omega)\dot{+}\mathfrak{K}^+(\omega)$. ❑

2.35 Corollary. *We have*

$$\varpi\delta(\omega)_k = \delta(\mathfrak{h})_k,\ k = 0, \ldots, \Delta - 1\,,$$

$$\varpi p(\omega)_k = p(\mathfrak{h})_k,\ k \geq \Delta\,.$$

Proof. From the definition of ϖ it is clear that

$$\varpi(\mathfrak{K}^-(\omega) + \mathfrak{K}^+(\omega) + X^\delta(\omega)) \subseteq \mathcal{C}(\mathfrak{h}) + X^\delta(\mathfrak{h})\,.$$

Passing to orthogonal complements, gives

$$\varpi(X_\delta(\omega)) = X_\delta(\mathfrak{h})\,.$$

However, if $k, l \in \{0, \ldots, \Delta\}$, we obtain from Lemma 2.27

$$[\varpi\delta(\omega)_k, p(\mathfrak{h})_l] = [\varpi\delta(\omega)_k, \varpi p(\omega)_l] = [\delta(\omega)_k, p(\omega)_l] = [\delta(\mathfrak{h})_k, p(\mathfrak{h})_l]\,.$$

It follows that $\varpi\delta(\omega)_k = \delta(\mathfrak{h})_k$.

For $k \geq \Delta$ we have $\mathfrak{w}_k \in L^2(H)$. Hence, by Proposition 2.28 and Proposition 2.21 we have $p(\omega)_k \in X_\delta(\omega)^\perp$. For each $g \in \mathcal{C}(\mathfrak{h})$ we have $G := \varpi^{-1}g \in \mathfrak{K}^-(\omega) + \mathfrak{K}^+(\omega)$ and, hence, by (2.27)

$$[\varpi p(\omega)_k, g] = [p(\omega)_k, G] = \int_I (\psi(\omega)G)^* H\mathfrak{w}_k = \int_I (\psi(\mathfrak{h})g)^* H\mathfrak{w}_k = [p(\mathfrak{h}), g]\,.$$

Moreover, by Lemma 2.27,

$$[\varpi p(\omega)_k, \delta(\mathfrak{h})_l] = [p(\omega)_k, \delta(\omega)_l] = [p(\mathfrak{h})_k, \delta(\mathfrak{h})_l]\,,$$

$$[\varpi p(\omega)_k, p(\mathfrak{h})_l] = [p(\omega)_k, p(\omega)_l] = [p(\mathfrak{h})_k, p(\mathfrak{h})_l]\,.$$

Together, these relations imply $\varpi p(\omega)_k = p(\mathfrak{h})_k$. ❑

2.36 Proposition. *The pair* $(\varpi; \mathrm{id})$ *is an isomorphism between the boundary triplets* $\mathfrak{B}(\omega(s_+))$ *and* $\mathfrak{B}(\mathfrak{h})$*.*

In the proof of this result, we will employ the following lemma.

2.37 Lemma. *Denote*

$$\mathcal{F}(H) := L^2(H)\dot{+}\operatorname{span}\{\mathfrak{w}_0,\dots,\mathfrak{w}_{\Delta-1}\}\,,$$

$$T_{\mathcal{F}(H)} := \big\{(f;g)\in\mathcal{F}(H)^2 : \exists\tilde f \textit{ abs.cont.}, [\tilde f]_{=_H} = f,\ \tilde f' = JHg\big\}$$

Then

$$T_{\mathcal{F}(H)} = T_{\max}(H)\dot{+}\operatorname{span}\{(\mathfrak{w}_k;\mathfrak{w}_{k-1}) : k=0,\dots,\Delta\}$$

where we have set $\mathfrak{w}_{-1} := 0$.

Proof. The inclusion '$\supseteq$' is obvious. Assume that $(f;g)\in T_{\mathcal{F}(H)}$, and write

$$f = f_1 + \sum_{j=0}^{\Delta-1}\eta_j\mathfrak{w}_j,\ \ g = g_1 + \sum_{j=0}^{\Delta-1}\mu_j\mathfrak{w}_j\,.$$

Consider the element $(\hat f;\hat g) := (f_1 - \mu_{\Delta-1}\mathfrak{w}_\Delta; g_1 + \sum_{j=0}^{\Delta-2}(\mu_j-\eta_{j+1})\mathfrak{w}_j)$, then

$$(\hat f;\hat g) = (f;g) - \sum_{j=0}^{\Delta-1}\eta_j(\mathfrak{w}_j;\mathfrak{w}_{j-1}) - \mu_{\Delta-1}(\mathfrak{w}_\Delta;\mathfrak{w}_{\Delta-1}) \in T_{\mathcal{F}(H)}$$

and $\hat f\in L^2(H)$. Let $B_\pm$ denote the operators from Section 4.1 of [KW/IV], see also (IV.2.22). It follows that for some constants $c_1^-, c_1^+, c_2\in\mathbb{C}$,

$$\hat f|_{(s_-,s)} = B_-g_1|_{(s_-,s)} + \begin{pmatrix} c_1^- \\ 0\end{pmatrix} + c_2\mathfrak{w}_0|_{(s_-,s)} + \sum_{j=0}^{\Delta-2}(\mu_j-\eta_{j+1})\mathfrak{w}_{j+1}|_{(s_-,s)}\,,$$

$$\hat f|_{(s,s_+)} = B_+g_1|_{(s,s_+)} + \begin{pmatrix} c_1^+ \\ 0\end{pmatrix} + c_2\mathfrak{w}_0|_{(s,s_+)} + \sum_{j=0}^{\Delta-2}(\mu_j-\eta_{j+1})\mathfrak{w}_{j+1}|_{(s,s_+)}\,.$$

As $\mathfrak{w}_0,\dots,\mathfrak{w}_{\Delta-1}$ are linearly independent modulo $L^2(H)$, we obtain $c_2=0$ and $\mu_j-\eta_{j+1}=0$, $j=0,\dots,\Delta-2$, in particular $\hat g\in L^2(H)$. Thus $(\hat f;\hat g)\in L^2(H)$, and the assertion follows. ❑

Proof (of Proposition 2.36).
Step 1: We show

$$\big((\varpi\boxtimes\varpi)\boxtimes\mathrm{id}\big)\Big(\Gamma(\omega(s_+))\cap\big((X_\delta^\perp)^2\times(\mathbb{C}^2)^2\big)\Big)\subseteq\Gamma(\mathfrak{h})\,. \tag{2.36}$$

To this end, let $((F;G);(a;b))\in\Gamma(\omega(s_+))\cap\big((X_\delta^\perp)^2\times(\mathbb{C}^2)^2\big)$ be given. According to Proposition 2.23 $(\psi F;\psi G)\in T_{\max}(H)$. By Proposition IV.4.17 there exists an element $(f;g)\in T(\mathfrak{h})\cap\mathcal{C}(\mathfrak{h})^2$, such that

$$\psi(\mathfrak{h})f = \psi(\omega)F,\ \ \psi(\mathfrak{h})g = \psi(\omega)G\,.$$

Let $\tilde f := \pi_l\Psi^{ac}(f;g)$, then $\tilde f$ is a locally absolutely continuous representant of ψf with $\tilde f' = JH(\psi g)$. By Corollary 2.24, we have $((F;G);(\tilde f(s_-),\tilde f(s_+)))\in\Gamma(\omega(s_+))$, and Corollary 2.25 implies that

$$a = \tilde f(s_-)\quad\text{and}\quad\begin{cases} b = \tilde f(s_+) & ,\ (\sigma,s_+)\text{ not indivisible}\\ \pi_2 b = \pi_2\tilde f(s_+) & ,\ (\sigma,s_+)\text{ indivisible}\end{cases}$$

By adding an appropriate linear combination of $(\delta(\mathfrak{h})_k;\delta(\mathfrak{h})_{k+1})$, $k=0,\dots,\Delta-2$, and a multiple of $(0;\delta_0(\mathfrak{h}))$ if (σ,s_+) is indivisible (remember Lemma 2.15 and Lemma 2.16), we see that the elements f and g can be chosen such that

$$\pi_1 b=\pi_1\tilde{f}(s_+) \text{ also if } (\sigma,s_+) \text{ is indivisible}$$

$$[f,p(\mathfrak{h})_k]=[\varpi F,p(\mathfrak{h})_k],\ k=0,\dots,\Delta-2\,.$$

Case $\ddot{o}=0$: We use the Green's identity with the pairs $(\mathfrak{w}_\Delta;p(\mathfrak{h})_{\Delta-1}+d_{\Delta-1}\delta(\mathfrak{h})_0)$ and $(p(\omega)_\Delta;p(\omega)_{\Delta-1}+d_{\Delta-1}\delta(\omega)_0)$ to obtain

$$[f,p(\mathfrak{h})_{\Delta-1}]=[f,p(\mathfrak{h})_{\Delta-1}+d_{\Delta-1}\delta(\mathfrak{h})_0]=$$

$$=[g,\mathfrak{w}_\Delta]-\mathfrak{w}_\Delta(s_-)^*J\tilde{f}(s_-)+\mathfrak{w}_\Delta(s_+)^*J\tilde{f}(s_+)=$$

$$=\int_I(\mathfrak{w}_\Delta)^*J(\psi g)-\mathfrak{w}_\Delta(s_-)^*Ja+\mathfrak{w}_\Delta(s_+)^*Jb=$$

$$=[G,p(\omega)_\Delta]-\mathfrak{w}_\Delta(s_-)^*Ja+\mathfrak{w}_\Delta(s_+)^*Jb=$$

$$=[F,p(\omega)_{\Delta-1}+d_{\Delta-1}\delta(\omega)_0]=[F,p(\omega)_{\Delta-1}]=[\varpi F,p(\mathfrak{h})_{\Delta-1}]$$

Thus f and ϖF are elements of $\operatorname{span}\{\delta(\mathfrak{h})_0,\dots,\delta(\mathfrak{h})_{\Delta-1}\}^\perp$ whose images under $\psi(\mathfrak{h})$ and whose inner products with $p(\mathfrak{h})_k$, $k=0,\dots,\Delta-1$, coincide. This implies their equality.

Using the Green's idenitity with the pairs $(p_0;0)$ and $(p_k;p_{k-1}+d_{k-1}\delta_0)$, $k=1,\dots,\Delta-1$, yields $[g,p(\mathfrak{h})_k]=[\varpi G,p(\mathfrak{h})_k]$, $k=0,\dots,\Delta-1$. Thus also $\varpi G=g$. Altogether, we obtain

$$\big((\varpi F;\varpi G);(a;b)\big)\in\Gamma(\mathfrak{h})\,.$$

Case $\ddot{o}>0$: By adding an appropriate linear combination of $(\delta(\mathfrak{h})_k;\delta(\mathfrak{h})_{k+1})$, $k=\Delta-1,\dots,\Delta+\ddot{o}-2$, we see that the elements f and g can be choosen in $\mathcal{C}(\mathfrak{h})+X^\delta$ such that

$$[f,p(\mathfrak{h})_{\Delta-1}]=[\varpi F,p(\mathfrak{h})_{\Delta-1}]\,,$$

$$[f,\delta(\mathfrak{h})_k]=[\varpi F,\delta(\mathfrak{h})_k],\ k=\Delta+1,\dots,\Delta+\ddot{o}-1\,.$$

Applying the Green's identity with the pair $(\delta(.)_{\Delta-1};\delta(.)_\Delta)$ gives

$$[f,\delta(\mathfrak{h})_\Delta]=[g,\delta(\mathfrak{h})_{\Delta-1}]=0=[G,\delta(\omega)_{\Delta-1}]=$$

$$=[F,\delta(\omega)_\Delta]=[\varpi F,\delta(\mathfrak{h})_\Delta]\,.$$

Again it follows that $\varpi F=f$. In the same way as above, we obtain that also $\varpi G=g$.

Step 2: Finish of proof. Let $((F;G);(a;b))\in\Gamma(\omega(s_+))$ be given. Applying Ψ and using Lemma 2.37, we find constants $\lambda_j\in\mathbb{C}$ such that

$$\big((\hat{F};\hat{G});(\hat{a};\hat{b})\big):=((F;G);(a;b))+$$

$$+\sum_{k=0}^{\Delta-1}\lambda_k\big((p(\omega)_k;p(\omega)_{k-1}+d_{k-1}\delta(\omega)_0);(\mathfrak{w}_k(s_-);\mathfrak{w}_k(s_+))\big)+$$

$$+\lambda_k\big((p(\omega)_\Delta+\mathfrak{b}(\omega);p(\omega)_{\Delta-1}+d_{\Delta-1}\delta(\omega)_0);(\mathfrak{w}_\Delta(s_-);\mathfrak{w}_\Delta(s_+))\big)$$

belongs to $\Gamma(\omega(s_+))\cap\big((X_\delta^\perp)^2\times(\mathbb{C}^2)^2\big)$. By Step 1 of this proof, it follows that $((\varpi\hat{F};\varpi\hat{G});(a;b))\in\Gamma(\mathfrak{h})$. By the definition of ϖ and Corollary 2.35 it follows that also $((F;G);(a;b))\in\Gamma(\mathfrak{h})$.

An application of Lemma V.2.12 completes the proof. ❑

As we have already remarked, Proposition 2.36 is sufficient to obtain $\omega_\mathfrak{h}=\omega$. Hence, we have realized each finite maximal chain ω subject to the conditions (2.1)–(2.3) as $\omega_\mathfrak{h}$ with an elementary indefinite Hamiltonian of kind (A). ☺

3 Completion of the converse construction

In the previous section we have treated chains which correspond to elementary indefinite Hamiltonians of kind (A). In the present short section we will settle the other cases, namely chains which correspond to positive definite or elementary indefinite Hamiltonians of kind (B) or (C). Moreover, we provide the necessary splitting-and-pasting argument to glue together all these 'peacewise' results.

We divide this section into two subsections.

a. Let ω be a positive definite maximal or finite maximal chain, or let ω be a finite maximal chain defined on

$$I=[s_-,\sigma)\cup(\sigma,s_+]\,,$$

such that

$$(s_-,\sigma),\ (\sigma,s_+)\quad\text{indivisible of type }\frac{\pi}{2}\,. \tag{3.1}$$

We construct a Hamiltonian $\mathfrak{h}$, which is positive definite or elementary indefinite of kind (B) and (C), such that $\omega=\omega_\mathfrak{h}$. This is done by references to the classical theory and explicit computations, respectively.

b. The splitting-and-pasting technique is applied, and the proof of Theorem 1.5 is completed.

a. Positive definite and elementary indefinite kind (B), (C).

Let $\omega\in\mathfrak{M}_0^f\cup\mathfrak{M}_0$. By Proposition V.3.23, *(ii)*, we have $\omega=\omega_H$ for some positive definite Hamiltonian H. Define $\mathfrak{h}$ to be positive definite and given by this Hamiltonian function H. By Remark V.5.7, *(iii)*, we have $\omega_\mathfrak{h}=\omega_H$.

Let $\omega:[s_-,\sigma)\cup(\sigma,s_+]\to\mathcal{M}_{<\infty}$ be a finite maximal chain, and assume that (s_-,σ) and (σ,s_+) are indivisible of type $\frac{\pi}{2}$. Then we have, cf. Proposition V.3.10,

$$0=\lim_{t\nearrow\sigma}\omega(t)\star 0=\lim_{t\searrow\sigma}\omega(t)\star 0=\omega(s_+)\star 0\,.$$

It follows from [KWW2, Lemma 5.3] that $\omega(s_+)$ is a polynomial matrix of the form

$$\omega(s_+)=\begin{pmatrix}1&0\\-p(z)&1\end{pmatrix}\quad\text{where}\ \ p\in\mathbb{R}[z],\ p(0)=0\,. \tag{3.2}$$

The case that $p(z)=a_1z$ with $a_1\geq 0$ is excluded since, by the existence of a singularity, certainly $\operatorname{ind}_-\omega(s_+)>0$. The needed data to compose a general Hamiltonian $\mathfrak{h}$ with the property that $\omega_\mathfrak{h}=\omega$ can be immediately read off from the formula given in Proposition V.4.31:

Case $\deg p = 1$*:* $\mathfrak{h}$ is elementary indefinite of kind (C) and given by the data

$$\begin{aligned}
&s_-, \sigma, s_+ \,,\\
&H(t) := h(t)\xi_{\frac{\pi}{2}}\xi_{\frac{\pi}{2}}^T \text{ where } h(t) := (\mathfrak{t}\circ\omega)'(t)\,,\\
&\ddot{o} := 0, \quad b_1 := 0, \quad d_0 := a_1, d_1 := 0\,.
\end{aligned}$$

Case $\deg p > 1$*:* $\mathfrak{h}$ is elementary indefinite of kind (B) and given by the data

$$\begin{aligned}
&s_-, \sigma, s_+ \,,\\
&H(t) := h(t)\xi_{\frac{\pi}{2}}\xi_{\frac{\pi}{2}}^T \text{ where } h(t) := (\mathfrak{t}\circ\omega)'(t)\,,\\
&\ddot{o} := \deg p - 2, \quad b_1 := -a_{\deg p}, \ldots, b_{\ddot{o}_1+1} := -a_2, \quad d_0 := a_1, d_1 := 0\,.
\end{aligned}$$

We see from Proposition V.4.31 and Corollary V.5.6 that $\omega_{\mathfrak{h}} = \omega$.

b. Splitting-and-pasting.

Let ω be a maximal or finite maximal chain defined on

$$I = \bigcup_{j=0}^{n}(\sigma_j, \sigma_{j+1}) \underbrace{\cup\{\sigma_0, \sigma_{n+1}\}}_{\text{if 'finite'}},$$

and assume that $\mathfrak{t}\circ\omega$ and $(\mathfrak{t}\circ\omega|_J)^{-1}$ are locally absolutely continuous for any maximal interval contained in the domain of ω. Choose a set $F = \{r_0, \ldots, r_{m+1}\}$ which is suitable for splitting ω, cf. Lemma V.3.19, and which has the following additional properties:

- *(i)* F intersects each component of I.
- *(ii)* F contains all endpoints of indivisible intervals with infinite length.
- *(iii)* If ω does not end with an indivisible interval towards σ_{i+1}, then $\max(F \cap [\sigma_0, \sigma_{i+1}))$ is not left endpoint of an indivisible interval of type ϕ_{i+1}.
- *(iv)* If ω does not start with an indivisible interval upwards from σ_{i+1}, then $\min(F \cap (\sigma_{i+1}, \sigma_{n+1}])$ is not right endpoint of an indivisible interval of type ϕ_{i+1}.

The existence of a set F with these properties is obvious.

Define chains ω_i, $i = 0, \ldots, m$, as

$$\omega_i := \begin{cases} \omega_{r_i\leftrightarrow r_{i+1}} & , \ (r_i, r_{i+1}) \text{ contains no singularity}\\ \circlearrowleft_{\phi(\sigma_j)} \omega_{r_i\leftrightarrow r_{i+1}} & , \ \sigma_j \in (r_i, r_{i+1}), \ (r_i, \sigma_j) \text{ not indivisible}\\ \operatorname{rev} \circlearrowleft_{\phi(\sigma_j)} \omega_{r_i\leftrightarrow r_{i+1}} & , \ \sigma_j \in (r_i, r_{i+1}), \ (r_i, \sigma_j) \text{ indivisible} \end{cases}$$

From the results we have established so far, we obtain the following corollary.

3.1 Corollary. *For each chain* ω_i *there exists a general Hamiltonian* $\mathfrak{h}_i$ *such that* $\omega_{\mathfrak{h}_i} = \omega_i$.

Proof. If (r_i, r_{i+1}) contains no singularity, then $\omega_{r_i \leftrightarrow r_{i+1}} \in \mathfrak{M}_0^f \cup \mathfrak{M}_0$. Hence, by the first paragraph of previous subsection, there exists a positive definite Hamiltonian $\mathfrak{h}_i$ with $\omega_{\mathfrak{h}_i} = \omega_i$.

Assume next that $\sigma_j \in (r_i, r_{i+1})$ and that (r_i, σ_j) is not indivisible. By property *(ii)* of the splitting set F, this implies that $\sup(I_{\text{reg}} \cap [\sigma_0, \sigma_j)) = \sigma_j$. Similarly, if (σ_j, r_{j+1}) is not indivisible, then $\inf(I_{\text{reg}} \cap (\sigma_j, \sigma_{n+1}]) = \sigma_j$. Clearly, since we have rotated $\omega_{r_i \leftrightarrow r_{i+1}}$ by the angle $\phi(\omega, \sigma_j) = \phi(\omega_{r_i \leftrightarrow r_{i+1}}, \sigma_j)$, we have $\phi(\omega_i, \sigma_j) = 0$. Finally, property *(iii)* of F ensures that ω_i does not start with an indivisible interval of type 0. Altogether, ω_i is a finite maximal chain defined on $[r_i, \sigma_j) \cup (\sigma_j, r_{i+1}]$ which satisfies (2.1)–(2.3). Hence, by what we have proved in the previous section, there exists an elementary indefinite Hamiltonian $\mathfrak{h}_i$ of kind (A), such that $\omega_{\mathfrak{h}_i} = \omega_i$.

Finally, consider the case that $\sigma_j \in (r_i, r_{i+1})$ and that (r_i, σ_j) is indivisible. Clearly, we again have $\phi(\omega_i, \sigma_j) = 0$. Thus the interval $(-\sigma_j, -r_i)$ is indivisible of type $\frac{\pi}{2}$ in the chain ω_i.

Case (σ_j, r_{i+1}) indivisible in ω: The interval $(-r_{i+1}, -\sigma_j)$ is indivisible of type $\frac{\pi}{2}$ in ω_i. Hence, ω_i is a finite maximal chain defined on $[-r_{i+1}, -\sigma_j) \cup (-\sigma_j, -r_i]$ which satisifes (3.1). As we saw in the previous subsection, there exists an elementary indefinite Hamiltonian $\mathfrak{h}_i$ of kind (B) or (C), such that $\omega_{\mathfrak{h}_i} = \omega_i$.

Case (σ_j, r_{i+1}) not indivisible in ω: By property *(ii)* of F, the chain ω_i satisfies $\sup(I(\omega_i)_{\text{reg}} \cap [-r_{i+1}, -\sigma_j)) = -\sigma_j$. Property *(iv)* of F yields that it does not start with an indivisible interval of type 0. Hence, ω_i is a finite maximal chain defined on $[-r_{i+1}, -\sigma_j) \cup (-\sigma_j, -r_i]$ and satisfies (2.1)–(2.3). By the results of the previous section, there exists an elementary indefinite Hamiltonian $\mathfrak{h}_i$ of kind (A) with $\omega_{\mathfrak{h}_i} = \omega_i$. ❑

With the above notation, define general Hamiltonians $\overset{\infty}{\mathfrak{h}}_i$ as

$$\overset{\infty}{\mathfrak{h}}_i := \begin{cases} \mathfrak{h}_i & , \ (r_i, r_{i+1}) \text{ contains no singularity} \\ \circlearrowleft_{-\phi(\sigma_j)} \mathfrak{h}_i & , \ \sigma_j \in (r_i, r_{i+1}),\ (r_i, \sigma_j) \text{ not indivisible} \\ \circlearrowleft_{-\phi(\sigma_j)} \operatorname{rev} \mathfrak{h}_i & , \ \sigma_j \in (r_i, r_{i+1}),\ (r_i, \sigma_j) \text{ indivisible} \end{cases}$$

and set

$$\mathfrak{h} := \biguplus_{i=0}^{m} \overset{\infty}{\mathfrak{h}}_i \,.$$

By Lemma V.5.14 and Lemma V.5.15, we have $\omega_{\overset{\infty}{\mathfrak{h}}_i} = \omega_{r_i \leftrightarrow r_{i+1}}$, $i = 0, \ldots, m$. Lemma V.5.18 implies that

$$\omega_{\mathfrak{h}} = \omega \,.$$

In order to complete the proof of Theorem 1.5, it remains to note:

3.2 Lemma. *Let ω be a (finite) maximal chain. Then there exists a reparameterization $\tilde{\omega}$ of ω, such that $\mathfrak{t} \circ \tilde{\omega}$ and $(\mathfrak{t} \circ \tilde{\omega}|_J)^{-1}$ are locally absolutely continuous for any maximal interval contained in the domain of ω.*

Proof. We know from Proposition V.3.2, that if ω is a maximal chain defined on $\bigcup_{i=0}^{n} (\sigma_i, \sigma_{i+1})$ and if we consider ω to be continuously continued to $[\sigma_0, \sigma_1) \cup \bigcup_{i=1}^{n} (\sigma_i, \sigma_{i+1})$, then $\mathfrak{t} \circ \omega$ maps $[\sigma_0, \sigma_1)$ bijectively onto $[0, +\infty)$ and (σ_i, σ_{i+1}) bijectively onto $(-\infty, +\infty)$ for $i = 1, \ldots, n$. Hence, $\beta : t \in (\sigma_i, \sigma_{i+1}) \mapsto i \cdot \pi +$

$\arctan\circ\mathfrak{t}\circ\omega(t)$ sets up a bijection from $\bigcup_{i=0}^{n}(\sigma_i,\sigma_{i+1})$ onto $(0,\frac{\pi}{2})\cup\bigcup_{i=1}^{n}(i\pi-\frac{\pi}{2},i\pi+\frac{\pi}{2})$, and $\tilde{\omega}:=\omega\circ\beta^{-1}$ is a reparametrization of ω such that

$$\mathfrak{t}\circ\tilde{\omega}(s)=\mathfrak{t}\circ\omega\circ\beta^{-1}(s)=\tan(i\cdot\pi+\arctan\circ\mathfrak{t}\circ\omega\circ\beta^{-1}(s))=\tan(s)\ .$$

Therefore, $\mathfrak{t}\circ\tilde{\omega}$ is locally absolutely continuous. The inverse of its restriction to a maximal interval contained in $(0,\frac{\pi}{2})\cup\bigcup_{i=1}^{n}(i\pi-\frac{\pi}{2},i\pi+\frac{\pi}{2})$ is arctan, and hence also locally absolutely continuous.

If ω is a finite maximal chains, one can argue in the same way. ❑

3.3 Remark. As we know from Corollary V.5.6 together with Remark V.3.22, the continuity condition on ω is necessary in order that $\omega=\omega_{\mathfrak{h}}$ for some general Hamiltonian $\mathfrak{h}$. //

4 Bijectivity modulo reparameterization

In this section we show that a general Hamiltonian $\mathfrak{h}$ can, up to reparameterization, be fully recovered from the chain $\omega_{\mathfrak{h}}$. This fact could, with considerable effort, be established by tracing back the constructions carried out in the previous section and in [KW/V]. However, we prefer to use a different, more elementary, method. This method will enable us to recover $\mathfrak{h}$ from $\omega_{\mathfrak{h}}$ as explicit as possible, and also, conversely, provide an algorithm how to compute $\omega_{\mathfrak{h}}$ for a given general Hamiltonian $\mathfrak{h}$.

In order to show that $\mathfrak{h}$ is determined by $\omega_{\mathfrak{h}}$, we will see with the usual splitting-and-pasting procedure that it is enough to consider elementary indefinite Hamiltonians. The cases of kind (B) and kind (C) are then trivial; the core of the problem is – once more – to understand elementary indefinite Hamiltonians of kind (A). Matching this situation, the present section is divided into two subsections:

a. We immediately attack the hard part of the problem, and show how to recover an elementary indefinite Hamiltonian $\mathfrak{h}$ of kind (A) from the chain $\omega_{\mathfrak{h}}$. Our approach is based on a set of recurrance relations which relate the parameters of $\mathfrak{h}$ with $\omega(\mathfrak{B}(\mathfrak{h}))$, and which can easily be solved for the parameters of $\mathfrak{h}$.
b. Here we carry out the 'usual' techniques in order to finish the proof of Theorem 1.6.

a. The recurrance relations.

We introduce a set of recurrence relations which involves some parameters. Let

$$\Delta\in\mathbb{N},\ \ddot{o}\in\mathbb{N}_0,\ \epsilon\in\{0,1\}\,,$$

$$U_k,U_k^-,V_k,X_k\in\mathbb{C},\ k\in\mathbb{N}\,,$$

$$Y_k\in\mathbb{C},\ k\in\{1,\dots,\ddot{o}\}\,,$$

be given. For notational convenience, set $Y_k:=0$ for $k\in\mathbb{Z}\setminus\{1,\dots,\ddot{o}\}$, and $\alpha_k:=0$ for $k\in\mathbb{Z}\setminus\mathbb{N}_0$.

Sequences $(\alpha_n)_{n\in\mathbb{N}_0}$, $(\beta_n)_{n\in\mathbb{N}_0}$, and $(\gamma_{n,l})_{n\in\mathbb{N}_0}$, $l\in\mathbb{N}_0$, can be defined from given inital values α_0, β_0, and $\gamma_{0,l}$, $l\in\mathbb{N}_0$, by the recurrence relations ($n\in\mathbb{N}$)

$$\begin{aligned} \alpha_n &= -\sum_{j=0}^{n-1}\big[\alpha_j U_{n-j}+\beta_j V_{n-j}\big]\,, \\ \beta_n &= \gamma_{n-1,0}-\sum_{j=0}^{n-1}\big[\alpha_j X_{n-j}+\beta_j U_{n-j}\big]+\epsilon U_n^-\,, \\ \gamma_{n,l} &= \gamma_{n-1,l+1}+\alpha_{n-\Delta}Y_{\Delta+ö-l},\ l\in\mathbb{N}_0\,. \end{aligned} \tag{4.1}$$

Our interest in these recurrence relations origins in the following result. We will write a_2 for the second component of a vector $a\in\mathbb{C}^2$.

4.1 Proposition. *Let $\mathfrak{h}$ be an elementary indefinite Hamiltonian of kind (A) which is defined on $I=[s_-,\sigma)\cup(\sigma,s_+]$ and given by the data H, $d_0,\dots,d_{2\Delta-1}$, $ö$, $b_1,\dots,b_{ö+1}$. Assume that $b_{ö+1}=0$. Write $\omega(\mathfrak{B}(\mathfrak{h}))=:(w_{ij}(z))_{i,j=1}^2$ and specify the parameters*

$$\Delta:=\max\big\{\Delta(H|_{[s_-,\sigma)}),\Delta(H|_{(\sigma,s_+])})\big\}\,,$$

$$U_k:=\mathfrak{w}_k(s_+)_2,\ U_k^-:=\mathfrak{w}_k(s_-)_2,\ V_k:=\big(B^k\chi_+\binom{1}{0}\big)(s_+)_2,\quad k\in\mathbb{N}\,,$$

$$Y_k:=b_k,\ k\in\{1,\dots,ö\},\quad X_k:=\begin{cases} d_{k-1} & ,\ k\in\{1,\dots,2\Delta\} \\ (\mathfrak{w}_{k-\Delta-1},\mathfrak{w}_\Delta)_{L^2(H)} & ,\ k>2\Delta \end{cases}$$

Let $(\alpha_n)_{n\in\mathbb{N}_0}$, $(\beta_n)_{n\in\mathbb{N}_0}$, and $(\gamma_{n,l})_{n\in\mathbb{N}_0}$, $l\in\mathbb{N}_0$, be the sequences defined by (4.1) *using the specified parameters, $\epsilon:=0$, and the initial values*

$$\alpha_0:=1,\ \beta_0:=0,\quad \gamma_{0,l}:=0,\ l\in\mathbb{N}_0\,.$$

Let $(\hat\alpha_n)_{n\in\mathbb{N}_0}$, $(\hat\beta_n)_{n\in\mathbb{N}_0}$, $(\hat\gamma_{n,l})_{n\in\mathbb{N}_0}$, $l\in\mathbb{N}_0$, be defined from (4.1) *using the same specified parameters, but $\epsilon:=1$, and the initial values*

$$\hat\alpha_0:=0,\ \hat\beta_0:=1,\quad \hat\gamma_{0,l}:=0,\ l\in\mathbb{N}_0\,.$$

Then

$$\begin{gathered} \frac{w_{21}(z)}{w_{22}(z)}=\sum_{n=0}^\infty\beta_n z^n,\quad -\frac{w_{12}(z)}{w_{22}(z)}=\sum_{n=1}^\infty\Big(\hat\alpha_n+\sum_{j=1}^{n-1}\hat\alpha_j U_{n-j}^-+U_n^-\Big)z^n \\ \frac{1}{w_{22}(z)}=\sum_{n=0}^\infty\hat\beta_n z^n=\sum_{n=0}^\infty\Big(\alpha_n+\sum_{j=0}^{n-1}\alpha_j U_{n-j}^-\Big)z^n \end{gathered} \tag{4.2}$$

The proof of this result is based on the following observation. For notational convenience set

$$p_j:=\mathfrak{w}_j,\ j\geq\Delta,\ b_j:=0,\ j\notin\{1,\dots,ö+1\}\,.$$

4.2 Lemma. *Let $\mathfrak{h}$ be an elementary indefinite Hamiltonian of kind (A) with $b_{ö+1}=0$. Let $x\in\mathcal{P}(\mathfrak{h})$ be of the form*

$$x=\sum_{j=0}^{n-1}\Big[\alpha_j p_{n-1-j}+\beta_j B^{n-1-j}\chi_+\binom{1}{0}+\beta_j' B^{n-1-j}\chi_-\binom{1}{0}\Big]+\sum_{l=0}^{\Delta+ö-1}\gamma_{n-1,l}\delta_l\,,$$

with some $n \in \mathbb{N}$ *and numbers* $\alpha_j, \beta_j, \beta'_j$, $j = 0, \ldots, n-1$, *and* $\gamma_{n-1,l} \in \mathbb{C}$, $l = 0, \ldots, \Delta + \ddot{o} - 1$. *Moreover, let* $A := A(0, \frac{\pi}{2})$ *be the operator defined as in Corollary IV.5.6, and set* $\alpha_j := 0$ *for* $j < 0$, *and* $\gamma_{n,\Delta+\ddot{o}} := 0$. *Then*

$$A^{-1}x = \sum_{j=0}^{n} \Big[\alpha_j p_{n-j} + \beta_j B^{n-j} \chi_+ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta'_j B^{n-j} \chi_- \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Big] + \sum_{l=0}^{\Delta+\ddot{o}-1} \gamma_{n,l} \delta_l \,,$$

with

$$\begin{aligned} \alpha_n &:= -\sum_{j=0}^{n-1} \big[\alpha_j \mathfrak{w}_{n-j}(s_+)_2 + \beta_j \big(B^{n-j} \chi_+ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \big)(s_+)_2 \big] \,, \\ \beta_n &:= \gamma_{n-1,0} - \sum_{j=0}^{n-1} \Big[\alpha_j d_{n-j-1} + \beta_j \mathfrak{w}_{n-j}(s_+)_2 - \beta'_j \mathfrak{w}_{n-j}(s_-)_2 \Big] \,, \\ \beta'_n &:= 0 \,, \\ \gamma_{n,l} &:= \gamma_{n-1,l+1} + \alpha_{n-\Delta} b_{\Delta+\ddot{o}-l}, \; l = 0, \ldots, \Delta + \ddot{o} - 1 \,. \end{aligned} \tag{4.3}$$

Note here that, by Proposition IV.5.9, we have $0 \in \rho(A)$.

Proof. For any choice of numbers α, β the element

$$(\hat{y}; \hat{x}) :=$$

$$\begin{aligned} = \sum_{\substack{j=0 \\ j \neq n-\Delta}}^{n-1} & \alpha_j \big(p_{n-j}; p_{n-j-1} + d_{n-j-1} \delta_0 \big) + \alpha_{n-\Delta} \big(p_\Delta + \mathfrak{b}; p_{\Delta-1} + d_{\Delta-1} \delta_0 \big) + \\ & + \sum_{j=0}^{n-1} \beta_j \big(B^{n-j} \chi_+ \begin{pmatrix} 1 \\ 0 \end{pmatrix}; B^{n-j-1} \chi_+ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathfrak{w}_{n-j}(s_+)_2 \delta_0 \big) + \\ & + \sum_{j=0}^{n-1} \beta'_j \big(B^{n-j} \chi_- \begin{pmatrix} 1 \\ 0 \end{pmatrix}; B^{n-j-1} \chi_- \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \mathfrak{w}_{n-j}(s_-)_2 \delta_0 \big) + \\ & + \sum_{l=1}^{\Delta+\ddot{o}-1} \gamma_{n-1,l} (\delta_{l-1}; \delta_l) + \alpha(p_0; 0) + \beta \big(\chi_+ \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \delta_0 \big) \end{aligned}$$

belongs to $T(\mathfrak{h})$ and satisfies $\pi_{l,1}\Gamma(\mathfrak{h})(\hat{y}; \hat{x}) = 0$. By choosing $\beta = \beta_n$ as defined in (4.3), we achieve that $\hat{x} = x$. By taking $\alpha = \alpha_n$, we achieve that $\pi_{r,2}\Gamma(\mathfrak{h})(\hat{y}; \hat{x}) = 0$. Thus we have found an element in $A(0, \frac{\pi}{2})$ whose second component is equal to x, and conclude that $A^{-1}x = \hat{y}$. The asserted formulas for $A^{-1}x$ and for $\gamma_{n,l}$ follow by inspection of $\hat{y}$. ❑

Proof (of Proposition 4.1). Let $\gamma(z)$ and $\varphi(z)$ be defined by

$$\gamma(z) := \big(I + z(A - z)^{-1} \big) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \varphi(z) := \big(I + z(A - z)^{-1} \big) p_0 \,.$$

Then, by the construction of $\omega(\mathfrak{B}(\mathfrak{h}))$, cf. (V.4.19), (V.4.20), and (V.4.21), we have

$$[\gamma(z), \gamma(0)] = \frac{1}{z} \frac{w_{12}(z)}{w_{22}(z)}, \qquad [\varphi(z), \varphi(0)] = -\frac{1}{z} \frac{w_{21}(z)}{w_{22}(z)} \,,$$

$$[\varphi(z),\gamma(0)] = [\gamma(z),\varphi(0)] = \frac{1}{z}\Big(\frac{-1}{w_{22}(z)} + 1\Big)\,,$$

Moreover, for $|z| < \|A^{-1}\|$, we have $I + z(A-z)^{-1} = \sum_{n=0}^{\infty} A^{-n} z^n$, and hence

$$[\gamma(z),\gamma(0)] = \sum_{n=0}^{\infty} \big[A^{-n}\binom{1}{0},\binom{1}{0}\big]\, z^n,\ \ [\varphi(z),\varphi(0)] = \sum_{n=0}^{\infty} \big[A^{-n}p_0, p_0\big]\, z^n\,,$$

$$[\varphi(z),\gamma(0)] = \sum_{n=0}^{\infty} \big[A^{-n}p_0,\binom{1}{0}\big]\, z^n,\ \ [\gamma(z),\varphi(0)] = \sum_{n=0}^{\infty} \big[A^{-n}\binom{1}{0}, p_0\big]\, z^n\,.$$

An inductive application of Lemma 4.2 gives

$$A^{-n}p_0 = \sum_{j=0}^{n} \Big[\alpha_j p_{n-j} + \beta_j B^{n-j}\chi_+\binom{1}{0}\Big] + \sum_{l=0}^{\Delta+\ddot{o}-1} \gamma_{n,l}\delta_l$$

$$A^{-n}\binom{1}{0} = \sum_{j=0}^{n} \Big[\hat{\alpha}_j p_{n-j} + \hat{\beta}_j B^{n-j}\chi_+\binom{1}{0}\Big] + \sum_{l=0}^{\Delta+\ddot{o}-1} \hat{\gamma}_{n,l}\delta_l + B^n\chi_-\binom{1}{0}\,.$$

We compute

$$[A^{-n}p_0, p_0] =$$
$$= \sum_{j=0}^{n} \Big[\alpha_j d_{n-j} + \beta_j \mathfrak{w}_{n-j+1}(s_+)_2\Big] - \gamma_{n,0} = -\beta_{n+1}\,,$$

$$[A^{-n}p_0, \binom{1}{0}] =$$
$$= \sum_{j=0}^{n} \Big[\alpha_j\big(\mathfrak{w}_{n-j+1}(s_+)_2 - \mathfrak{w}_{n-j+1}(s_-)_2\big) + \beta_j\big(B^{n-j+1}\chi_+\binom{1}{0}\big)(s_+)_2\Big] =$$
$$= -\alpha_{n+1} - \sum_{j=0}^{n} \alpha_j \mathfrak{w}_{n-j+1}(s_-)_2\,,$$

$$[A^{-n}\binom{1}{0}, p_0] =$$
$$= \sum_{j=0}^{n} \Big[\hat{\alpha}_j d_{n-j} + \hat{\beta}_j \mathfrak{w}_{n-j+1}(s_+)_2\Big] - \hat{\gamma}_{n,0} - \mathfrak{w}_{n+1}(s_+)_2 = -\hat{\beta}_{n+1}\,,$$

$$[A^{-n}\binom{1}{0}, \binom{1}{0}] =$$
$$= \sum_{j=0}^{n} \Big[\hat{\alpha}_j\big(\mathfrak{w}_{n-j+1}(s_+)_2 - \mathfrak{w}_{n-j+1}(s_-)_2\big) + \hat{\beta}_j\big(B^{n-j+1}\chi_+\binom{1}{0}\big)(s_+)_2\Big] -$$
$$- \mathfrak{w}_{n+1}(s_-)_2 = -\hat{\alpha}_{n+1} - \sum_{j=1}^{n} \hat{\alpha}_j \mathfrak{w}_{n-j+1}(s_-)_2 - \mathfrak{w}_{n+1}(s_-)_2\,.$$

Putting these formulas together, yields the desired assertion. ❑

For the treatment of inverse problems it is interesting to see how the equations (4.2) can be solved for $(\alpha_n)_{n\in\mathbb{N}_0}$, $(\beta_n)_{n\in\mathbb{N}_0}$, $(\hat\alpha_n)_{n\in\mathbb{N}_0}$ and $(\hat\beta_n)_{n\in\mathbb{N}_0}$. It will follow from a short argument that, for given Δ, $\ddot{o}$, $\mathfrak{w}_k(s_\pm)_2$ and $\big(B^k\chi_+\binom{1}{0}\big)(s_+)_2$, the equations (4.1) can be solved most easily for X_n, $n=1,\dots,2\Delta$, and Y_n, $n=1,\dots,\ddot{o}$.

It is a bit more tricky to recover the number $\ddot{o}$ from the knowledge of Δ, $\mathfrak{w}_k(s_\pm)_2$ and $\big(B^k\chi_+\binom{1}{0}\big)(s_+)_2$. An explicitly computable result is obtained only if an a priori bound for $\ddot{o}$ is known. However, such an a priori bound is always present, e.g. certainly $\ddot{o}\leq 2\,\mathrm{ind}_-\,\omega(\mathfrak{B}(\mathfrak{h}))+2$.

4.3 Proposition. *Let $(\alpha_n)_{n\in\mathbb{N}}$, $(\beta_n)_{n\in\mathbb{N}}$, and $(\gamma_{n,l})_{n\in\mathbb{N}}$, $l\in\mathbb{N}_0$, be defined by the recurrance (4.1) using some parameters $\Delta,\ddot{o},U_k,U_k^-,V_k,X_k,Y_k$ with $Y_1\neq 0$ in case $\ddot{o}>0$, the parameter $\epsilon=0$ and the initial values*

$$\alpha_0:=1,\ \beta_0:=0,\quad \gamma_{0,l}=0,\ l\in\mathbb{N}_0\,.$$

Assume that the values Δ, U_k,V_k for all $k\in\mathbb{N}_0$, X_n for $n>2\Delta$, and the sequences $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$, are known.

Then the remaining parameters $\ddot{o}$, X_n, $n=1,\dots,2\Delta$, Y_n, $n=1,\dots,\ddot{o}$, and the sequence $\gamma_{n,0}$, $n\in\mathbb{N}_0$, can be computed as follows:

$$\gamma_{n,0}=0,\ n=1,\dots 2\Delta-1\,, \tag{4.4}$$

$$\begin{aligned} X_n&=-\beta_n-\sum_{j=1}^{n-1}\big[\alpha_jX_{n-j}+\beta_jU_{n-j}\big],\ n=1,\dots,2\Delta\,,\\ \gamma_{n,0}&=\beta_{n+1}+\sum_{j=0}^{n}\big[\alpha_jX_{n+1-j}+\beta_jU_{n+1-j}\big],\ n\geq 2\Delta\,. \end{aligned} \tag{4.5}$$

Define a sequence $(\hat Y_n)_{n\geq 2\Delta}$ recursively by $\hat Y_n=\gamma_{n,0}-\sum_{j=1}^{n-2\Delta}\alpha_j\hat Y_{n-j}$, $n\geq 2\Delta$. Then

$$\ddot{o}=\begin{cases}0 & ,\ \hat Y_n=0 \text{ for all } n\\ \max\{n:\hat Y_n\neq 0\}+1-2\Delta & ,\ \hat Y_n\neq 0 \text{ for some } n\end{cases}$$

and

$$Y_n=\hat Y_{\ddot{o}-n+2\Delta},\ n=1,\dots,\ddot{o}\,. \tag{4.6}$$

Proof. First we show inductively that

$$\gamma_{n,l}=\sum_{j=0}^{n-\Delta}\alpha_jY_{j+\ddot{o}-(n-\Delta)-(l-\Delta)},\ n,l\in\mathbb{N}_0\,.$$

For $n=0$ this is obvious since $\Delta\geq 1$ and $\gamma_{0,l}=0$, $l\geq 0$. By the recursive definition (4.1) we have for $n\geq 1$, $l\in\mathbb{N}_0$,

$$\begin{aligned}\gamma_{n,l}=&\gamma_{n-1,l+1}+\alpha_{n-\Delta}Y_{\Delta+\ddot{o}-l}=\\ =&\sum_{j=0}^{n-1-\Delta}\alpha_jY_{j+\ddot{o}-(n-1-\Delta)-(l+1-\Delta)}+\alpha_{n-\Delta}Y_{\Delta+\ddot{o}-l}=\\ =&\sum_{j=0}^{n-\Delta}\alpha_jY_{j+\ddot{o}-(n-\Delta)-(l-\Delta)}\,.\end{aligned}$$

For the proof of the present assertion we will repeatedly use the special case $l = 0$:

$$\gamma_{n,0} = \sum_{j=0}^{n-\Delta} \alpha_j Y_{j+\ddot{o}-n+2\Delta},\ n \in \mathbb{N}_0 \,.$$

If $n < 2\Delta$, then $\ddot{o} + (2\Delta - n) > \ddot{o}$, and hence $Y_{j+\ddot{o}+2\Delta-n} = 0$, $j = 0, \dots, n - \Delta$. This yields (4.4).

The relations (4.5) are now obvious from the second relation in (4.1), since $\alpha_0 = 1$ and $\beta_0 = 0$.

For $n = 2\Delta$ we have $\hat{Y}_{2\Delta} = \gamma_{2\Delta,0} = Y_{\ddot{o}}$. Let $n > 2\Delta$, then inductively

$$\hat{Y}_n = \gamma_{n,0} - \sum_{j=1}^{n-2\Delta} \alpha_j \underbrace{\hat{Y}_{n-j}}_{=Y_{\ddot{o}-(n-j)+2\Delta}} = Y_{\ddot{o}-n+2\Delta} \,.$$

This shows that the relation (4.6) holds actually for all $n \leq \ddot{o}$. Since $Y_1 \neq 0$ in case $\ddot{o} > 0$ and $Y_n = 0$ for all $n < 1$, the asserted formula for $\ddot{o}$ follows. ❑

4.4 Corollary. *Let $\mathfrak{h} = (H, \mathfrak{c}, \mathfrak{d})$ and $\mathfrak{h}' = (H', \mathfrak{c}', \mathfrak{d}')$ be elementary indefinite Hamiltonians of kind (A), and assume that $b_{\ddot{o}+1} = b'_{\ddot{o}'+1} = 0$. If $\omega_{\mathfrak{h}} = \omega_{\mathfrak{h}'}$, then $\mathfrak{h} = \mathfrak{h}'$.*

Proof. The domains of the chains $\omega_{\mathfrak{h}}$ and $\omega_{\mathfrak{h}'}$ coincide, and thus $I = I'$. Since $\omega_{\mathfrak{h}}$ and $\omega_{\mathfrak{h}'}$ are solutions of the respective differential equations (V.5.5), it follows that $H = H'$. Thus also

$$\Delta = \Delta', \quad \mathfrak{w}_k = \mathfrak{w}'_k, \quad B^k \chi_+ \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (B')^k \chi_+ \begin{pmatrix} 1 \\ 0 \end{pmatrix} .$$

Since $\omega(\mathfrak{B}(\mathfrak{h})) = \omega(\mathfrak{B}(\mathfrak{h}'))$, Proposition 4.1 combined with Proposition 4.3 yields that $\mathfrak{c} = \mathfrak{c}'$ and $\mathfrak{d} = \mathfrak{d}'$, i.e. $\mathfrak{h} = \mathfrak{h}'$. ❑

b. Bijectivity modulo reparameterization.

It is only a small technical effort to lift Corollary 4.4 and prove that general Hamiltonians $\mathfrak{h}, \mathfrak{h}'$ with $\omega_{\mathfrak{h}} \leftrightsquigarrow \omega_{\mathfrak{h}'}$ must be reparameterizations of each other. First we give a condition which ensures actual equality.

4.5 Lemma. *Let $\mathfrak{h}$ and $\mathfrak{h}'$ be general Hamiltonians such that $\omega_{\mathfrak{h}} = \omega_{\mathfrak{h}'}$, $E = E'$, $d_{i1} = d'_{i1} = 0$ if H_{i-1} and H_i both end with an indivisible interval towards σ_i, and $b_{\ddot{o}_i+1} = b_{\ddot{o}'_i+1} = 0$ for all other i. Then $\mathfrak{h} = \mathfrak{h}'$.*

Proof. First note that equality of associated chains immediately implies $n = n'$ and $\sigma_i = \sigma'_i$, $i = 0, \dots, n+1$. Hence the conditions $d_{i1} = d'_{i1} = 0$ and $b_{\ddot{o}_i+1} = b_{\ddot{o}'_i+1} = 0$, respectively, are meaningful. Also, since $\omega_{\mathfrak{h}}$ and $\omega_{\mathfrak{h}'}$ are solutions of the respective differential equations (V.5.5), it immediately follows that $H = H'$.

Write $E = E' = \{s_0, \dots, s_{N+1}\}$ with $s_0 < \dots < s_{N+1}$. Then, by Lemma V.5.4, we have

$$\omega_{\mathfrak{h}_{s_i \leftrightarrow s_{i+1}}}(t) = \omega_{\mathfrak{h}}(s_i)^{-1} \omega_{\mathfrak{h}}(t) = \omega_{\mathfrak{h}'}(s_i)^{-1} \omega_{\mathfrak{h}'}(t) = \omega_{\mathfrak{h}'_{s_i \leftrightarrow s_{i+1}}}(t),\ t \in [s_i, s_{i+1}] \,.$$

If $\mathfrak{h}_{s_i \leftrightarrow s_{i+1}}$ is positive definite, so is $\mathfrak{h}'_{s_i \leftrightarrow s_{i+1}}$, and we have

$$\mathfrak{h}_{s_i \leftrightarrow s_{i+1}} = H|_{(s_i, s_{i+1})} = H'|_{(s_i, s_{i+1})} = \mathfrak{h}'_{s_i \leftrightarrow s_{i+1}} \,.$$

Consider next the situation that $\mathfrak{h}_{s_i\leftrightarrow s_{i+1}}$, and hence also $\mathfrak{h}'_{s_i\leftrightarrow s_{i+1}}$, is indefinite, and let σ_j be the singularity contained in (s_i, s_{i+1}). Since $H = H'$, we have $\phi_j = \phi'_j$. Then $\circlearrowleft_{\phi_i} \mathfrak{h}_{s_i\leftrightarrow s_{i+1}}$ and $\circlearrowleft_{\phi_i} \mathfrak{h}'_{s_i\leftrightarrow s_{i+1}}$ are elementary indefinite, and are of the same kind. If they are of kind (B) or (C), we see from Proposition V.4.31 that $\ddot{o}_j = \ddot{o}'_j$, $b_{jl} = b'_{jl}$, $l = 1, \dots, \ddot{o}_j + 1$, and $d_{j0} = d'_{j0}$, i.e. $\circlearrowleft_{\phi_i} \mathfrak{h}_{s_i\leftrightarrow s_{i+1}} =\circlearrowleft_{\phi_i} \mathfrak{h}'_{s_i\leftrightarrow s_{i+1}}$. If they are of kind (A), we refer to Corollary 4.4 to obtain

$$\circlearrowleft_{\phi_i} \mathfrak{h}_{s_i\leftrightarrow s_{i+1}} =\circlearrowleft_{\phi_i} \mathfrak{h}'_{s_i\leftrightarrow s_{i+1}} .$$

Putting together all of these cases yields $\mathfrak{h} = \mathfrak{h}'$. ❑

It is now easy to finish the proof of Theorem 1.6.

Proof (of Theorem 1.6). Let general Hamiltonians $\mathfrak{h} = (H, \mathfrak{c}, \mathfrak{d})$ and $\mathfrak{h}' = (H', \mathfrak{c}', \mathfrak{d}')$ be given.

Assume first that $\mathfrak{h} \leftrightsquigarrow \mathfrak{h}'$, and let $\alpha : I' \to I$ be the increasing bijection such that α and α^{-1} are absolutely continuous and

$$H'(t') = (H \circ \alpha)(t') \cdot \frac{d\alpha}{dt}(t'), \quad \text{for a.e.} \quad t' \in I' . \tag{4.7}$$

We see that $\alpha(I'_{\text{reg}}) = I_{\text{reg}}$. Let $x' \in I'$ and set $x := \alpha(x')$. Inspecting the definition of a reparameterization, it follows that $\mathfrak{h} \leftrightsquigarrow \mathfrak{h}'$ implies $\mathfrak{h}_{\Lsh x} \leftrightsquigarrow \mathfrak{h}'_{\Lsh x'}$. Together with Remark V.3.39 and Proposition V.4.7 we get

$$\omega_{\mathfrak{h}'}(x') = \omega(\mathfrak{B}(\mathfrak{h}'_{\Lsh x'})) = \omega(\mathfrak{B}(\mathfrak{h}_{\Lsh x})) = \omega_{\mathfrak{h}}(x), \, , \tag{4.8}$$

for all $x' \in (I'_{\text{reg}} \cup \{\sigma'_0\}) \setminus I'_{\text{sing}}$, cf. Definition V.5.3. If $\mathfrak{h}$, and hence $\mathfrak{h}'$, is regular, then this equality also holds for $x' = \sigma'_{n+1}$. Since $\omega_{\mathfrak{h}}(t)$ satisfies the differential equation (V.5.5) with H, it follows from (4.7) that $(\omega_{\mathfrak{h}} \circ \alpha)(t)$ satisfies (V.5.5) with H'. The function $\omega_{\mathfrak{h}'}(t)$ however satsifies the same differential equation. Since the set of all x' which satisfy (4.8) intersects every component of $I' \cup \{\sigma'_0, \sigma'_{n+1}\}$, and $\omega_{\mathfrak{h}'}$ coincides with $\omega_{\mathfrak{h}} \circ \alpha$ on this set, it follows that $\omega_{\mathfrak{h}'} = \omega_{\mathfrak{h}} \circ \alpha$ on all of I'.

Conversely, assume that $\omega_{\mathfrak{h}} \leftrightsquigarrow \omega_{\mathfrak{h}'}$, and let $\alpha : I' \to I$ be the increasing bijection with $\omega_{\mathfrak{h}'} = \omega_{\mathfrak{h}} \circ \alpha$. Then $\mathfrak{t}(\omega_{\mathfrak{h}'}) = \mathfrak{t}(\omega_{\mathfrak{h}}) \circ \alpha$, and we obtain with the help of Corollary V.5.8 in particular that α and α^{-1} are both absolutely continuous. Let $\mathfrak{h}_1$ be the general Hamiltonian defined by the requirement that $\mathfrak{h} \sim_1 \mathfrak{h}_1$ by means of the map $\alpha : I' \to I$, cf. Remark V.3.38. Next, let $\mathfrak{h}_2$ be the general Hamiltonian defined by the requirement that $\mathfrak{h}_1 \sim_3 \mathfrak{h}_2$ and $E_2 = E'$. Finally, let $\mathfrak{h}_3$ be the general Hamiltonian such that $\mathfrak{h}_2 \sim_2 \mathfrak{h}_3$ and such that $d_{i1}^{(3)} = 0$ if σ_i is of polynomial type, and $b^{(3)}_{i,\ddot{o}_i+1} = 0$ otherwise. By what we have already proved in the previous step, $\omega_{\mathfrak{h}_3} = \omega_{\mathfrak{h}} \circ \alpha$. We see that $\mathfrak{h}_3$ and $\mathfrak{h}'$ satisfy the hypothesis of Lemma 4.5, and hence $\mathfrak{h}_3 = \mathfrak{h}'$. In particular, $\mathfrak{h} \leftrightsquigarrow \mathfrak{h}'$. ❑

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