Majorization in de Branges spaces III. Division by Blaschke products

Anton Baranov, Harald Woracek

Dedicated to Victor Petrovich Havin on the occasion of his 75th birthday.

Abstract

This paper is a part of a series dealing with subspaces of de Branges spaces of entire functions generated by majorization on subsets of the closed upper half-plane. In the present, third, part we continue our study of a certain Banach space generated by an admissible majorant. The main theme is "invariance of the unit ball with respect to division by Blaschke products". In connection with this topic representability by means of special types of majorants plays an important role. We obtain some (positive and negative) results on invariance when dividing by Blaschke factors, and characterize those unit balls representable by log-superharmonic majorants.

AMS Classification Numbers: 46E15, 46E22, 30D50
Keywords: de Branges subspace, majorant, subharmonic function, Blaschke product

1 Introduction

A de Branges space $\mathcal{H}$ is a Hilbert space whose elements are entire functions, and which satisfies the following axioms:

(dB1) For each $w \in \mathbb{C}$ the point evaluation functional $F \mapsto F(w)$ is continuous in the norm of $\mathcal{H}$.

(dB2) If $F \in \mathcal{H}$, also the function $F^{\#}(z) := \overline{F(\bar{z})}$ belongs to $\mathcal{H}$ and $\|F^{\#}\|_{\mathcal{H}} = \|F\|_{\mathcal{H}}$.

(dB3) If $w \in \mathbb{C} \setminus \mathbb{R}$ and $F \in \mathcal{H}$, $F(w) = 0$, then

$$\frac{z-\bar{w}}{z-w}F(z) \in \mathcal{H} \quad \text{and} \quad \Big\|\frac{z-\bar{w}}{z-w}F(z)\Big\|_{\mathcal{H}} = \|F\|_{\mathcal{H}}.$$

In [dB1]–[dB5] L. de Branges developed a deep and rich structure theory of such spaces. The key role thereby is played by the de Branges subspaces of a given space $\mathcal{H}$, i.e. those subspaces of $\mathcal{H}$ which become themselves de Branges spaces if endowed with the inner product inherited from $\mathcal{H}$.

Originating from the Beurling–Malliavin Multiplier Theorem, and some more recent generalizations to shift-coinvariant subspaces of the Hardy space, cf. [BM], [HM1], [HM2], a general concept of generating de Branges subspaces by majorization has evolved: If $\mathfrak{m}$ is a function defined on some subset D of the closed upper half-plane $\mathbb{C}^+ \cup \mathbb{R}$, one may define

$$\mathcal{R}_{\mathfrak{m}}(\mathcal{H}) := \operatorname{clos}_{\mathcal{H}} \left\{ F \in \mathcal{H} : \ \exists C > 0 : |F(z)|, |F(\overline{z})| \leq C\mathfrak{m}(z), z \in D \right\}.$$

Here $\operatorname{clos}_{\mathcal{H}}$ stands for the closure taken in the norm of $\mathcal{H}$. Provided that $\mathcal{R}_{\mathfrak{m}}(\mathcal{H}) \neq \{0\}$ and $\mathfrak{m}$ satisfies some mild regularity condition, $\mathcal{R}_{\mathfrak{m}}(\mathcal{H})$ is a de Branges subspace of $\mathcal{H}$. We have studied this concept in our previous papers [BW1]–[BW4]. For example, we have shown that every de Branges subspace of $\mathcal{H}$ can be obtained in this way, and have investigated which majorants defined on which subsets produce (or may produce) a prescribed (family of) de Branges subspace. Also, we have considered the linear space

$$R_{\mathfrak{m}}(\mathcal{H}) := \big\{ F \in \mathcal{H} : \ \exists C > 0 : |F(z)|, |F(\overline{z})| \leq C\mathfrak{m}(z), z \in D \big\},$$

which becomes a Banach space if endowed with the stronger norm

$$\|F\|_{\mathfrak{m}} := \max\big\{ \|F\|_{\mathcal{H}} \,, \min\{C \geq 0 : \ |F(z)|, |F^{\#}(z)| \leq C\mathfrak{m}(z), z \in D\} \big\}, \qquad F \in R_{\mathfrak{m}}(\mathcal{H}) \,.$$

Apparently, the space $R_{\mathfrak{m}}(\mathcal{H})$ is exactly that part of $\mathcal{R}_{\mathfrak{m}}(\mathcal{H})$ about whose elements one has explicit information. Hence, this Banach space is an object of interest. The geometric structure of $R_{\mathfrak{m}}(\mathcal{H})$ is in general complicated, but an intriguing topic. For example, in most cases $R_{\mathfrak{m}}(\mathcal{H})$ will not be reflexive, cf. [BW4].

The present paper is devoted to a further study of $R_{\mathfrak{m}}(\mathcal{H})$. It is more oriented towards complex analysis topics than towards Banach space questions; the main theme here is "dividing out zeros" and "subharmonicity".

We now give a short overview of the main results of the paper, sometimes in a simplified and less general form (in brackets we refer to the corresponding statements in the text).

It is well known that, if $\mathcal{H}$ is a de Branges space and $\mathcal{L}$ is one of its de Branges subspaces, then the unit ball

$$B(\mathcal{L}) := \big\{F \in \mathcal{L} : \ \|F\|_{\mathcal{H}} \leq 1\big\}$$

of $\mathcal{L}$ is invariant with respect to dividing out zeros in the following sense: *If $F \in B(\mathcal{L})$ and P is a Blaschke product for $\mathbb{C}^+$ such that $P^{-1}F$ is entire, then $P^{-1}F \in B(\mathcal{L})$.* We pose the question whether the same holds true for the unit ball

$$B_{\mathfrak{m}}(\mathcal{H}) := \big\{F \in R_{\mathfrak{m}}(\mathcal{H}) : \ \|F\|_{\mathfrak{m}} \leq 1\big\}$$

of the Banach space $R_{\mathfrak{m}}(\mathcal{H})$. In general the answer will be negative; obstacles are naturally appearing when majorization is required off the real axis (see, e.g. Example 4.2). However, for a particular class of majorants $\mathfrak{m}$ defined on open subsets of $\mathbb{C}^+$, positive answers can be given. The crucial property of $\mathfrak{m}$ in this respect is that the function $(-\log \mathfrak{m})$ is subharmonic; we will call such majorants log-*superharmonic*. This property allows us to employ the Phragmén–Lindelöf Principle.

1.1 Theorem. *(Theorem 4.4 and Corollary 4.5) Let D be an open subset of $\mathbb{C}^+$, and let $\mathfrak{m} \in \operatorname{Adm}_D \mathcal{H}$ be* log*-superharmonic. Moreover, let $F \in B_{\mathfrak{m}}(\mathcal{H}) \setminus \{0\}$ and let P be a Blaschke product for $\mathbb{C}^+$ with $\mathfrak{d}_P|_{\mathbb{C}^+} \leq \mathfrak{d}_F|_{\mathbb{C}^+}$. If, for some $\beta > 0$,*

$$\Big|\frac{F(\zeta)}{P(\zeta)}\Big| \leq \beta \liminf_{\substack{z\to\zeta \\ z\in D}} \mathfrak{m}(z), \quad \zeta \in \partial D \setminus \mathbb{R} \,,$$

then we have $P^{-1}F \in R_{\mathfrak{m}}(\mathcal{H})$, *and*

$$\Big|\frac{F(z)}{P(z)}\Big| \leq \max\{\beta, \|F\|_{\mathfrak{m}}\}\, \mathfrak{m}(z), \quad z \in D\,.$$

In particular, if $\mathfrak{m} \in \operatorname{Adm}_{\mathbb{C}^+}\mathcal{H}$ *is* log-*superharmonic,* $F \in B_{\mathfrak{m}}(\mathcal{H})$ *and* P *is a Blaschke product for* $\mathbb{C}^+$ *with* $\mathfrak{d}_P|_{\mathbb{C}^+} \leq \mathfrak{d}_F|_{\mathbb{C}^+}$*, then* $P^{-1}F \in B_{\mathfrak{m}}(\mathcal{H})$.

Of course, being log-superharmonic is a quite strong property. However, one should notice that still every de Branges subspace of $\mathcal{H}$ can be realized as $\mathcal{R}_{\mathfrak{m}}(\mathcal{H})$ with a majorant $\mathfrak{m}$ of this kind, cf. [BW3]. In connection with the division problem, the following question appears: *Which unit balls* $B_{\mathfrak{m}_0}(\mathcal{H})$ *(or, more generally, which subsets* B *of* $\mathcal{H}$*) are equal to a unit ball* $B_{\mathfrak{m}}(\mathcal{H})$ *with some* log-*superharmonic majorant* $\mathfrak{m}$ *defined on all of* $\mathbb{C}^+$*?*

We also consider two more special classes of majorants, namely, Smirnov class $\mathcal{N}_+$ and Hardy class H^p majorants. A function $\mathfrak{m} \in \operatorname{Adm}_{\mathbb{C}^+}\mathcal{H}$ is called an $\mathcal{N}_+$*-majorant* (a H^p*-majorant*, $p > 0$) for $\mathcal{H}$, if it is of the form

$$\mathfrak{m}(z) = \big|e^{-iaz} f(z) E(z)\big|, \quad z \in \mathbb{C}^+\,,$$

with some $a \leq 0$ and f being outer (respectively, f being an outer function in H^p).

The next theorem gives a description of those subsets B which can be realized as unit balls with some log-superharmonic majorant in $\mathbb{C}^+$. Interestingly, it turns out that these balls can already be realized by majorization along the real axis in conjunction with a restriction of exponential growth; also a majorant may always be chosen to be an $\mathcal{N}_+$-majorant. In the statement we use the following upper envelope majorant $\mathfrak{m}_B(w) = \sup_{F\in B}|F(w)|$.

1.2 Theorem. *(Theorem 5.3) Let* $\mathcal{H}$ *be a de Branges space, and let* $E \in \mathcal{H}B$ *be such that* $\mathcal{H} = \mathcal{H}(E)$*. Moreover, let* $B \subseteq B(\mathcal{H})$*,* $B \neq \emptyset, \{0\}$*. Then the following are equivalent:*

- (i) *There exists a* log-*superharmonic majorant* $\mathfrak{m} \in \operatorname{Adm}_{\mathbb{C}^+}\mathcal{H}$*, such that* $B = B_{\mathfrak{m}}(\mathcal{H})$.
- (ii) *There exists an* $\mathcal{N}_+$*-majorant* $\mathfrak{m} \in \operatorname{Adm}_{\mathbb{C}^+}\mathcal{H}$*, such that* $B = B_{\mathfrak{m}}(\mathcal{H})$.
- (iii) *We have* $B = B_{\mathfrak{m}_B|_{\mathbb{R}}}(\mathcal{H}_{(a)})$ *(majorization on the real axis) and*

$$\int_D \Big(\log^+ \frac{\mathfrak{m}_B(t)}{|E(t)|}\Big)\frac{dt}{1+t^2} < \infty\,. \tag{1.1}$$

Here $a = \operatorname{mt}_{\mathcal{H}} B$ *is the so-called mean type of the set* B*, and*

$$\mathcal{H}_{(a)} := \big\{F \in \mathcal{H} :\ \operatorname{mt}_{\mathcal{H}} F, \operatorname{mt}_{\mathcal{H}} F^{\#} \leq a\big\}$$

is the subspace defined by exponential growth.

The convergence of the logarithmic integral in (1.1) is a mild restriction, it is satisfied, e.g., whenever the function E is of finite order (see Proposition 5.10).

An analogous description may be given for sets B representable as unit balls $B_{\mathfrak{m}}(\mathcal{H})$ with some H^p-majorant $\mathfrak{m}$.

1.3 Proposition. *(Proposition 5.14) Let $\mathcal{H}$ be a de Branges space, let B be a nonempty subset of its unit ball, and let $p \in (0, \infty)$. Then the following are equivalent:*

(i) There exists an H^p-majorant $\mathfrak{m} \in \operatorname{Adm}_{\mathbb{C}^+} \mathcal{H}$, such that $B = B_{\mathfrak{m}}(\mathcal{H})$.

(ii) We have $B = B_{\mathfrak{m}_B|_{\mathbb{R}}}(\mathcal{H}_{(a)})$, $a = \operatorname{mt}_{\mathcal{H}} B$, and $\int_{\mathbb{R}} \big(\frac{\mathfrak{m}_B(t)}{|E(t)|}\big)^p \, dt < \infty$.

In the last part of the paper we study the question if a unit ball $B_{\mathfrak{m}_0}(\mathcal{H})$ generated by an arbitrary majorant is contained or does contain unit balls $B_{\mathfrak{m}}(\mathcal{H})$ generated by log-superharmonic majorants.

This question is also of interest for the following reason: In the first case, we obtain supersets B such that division of a function F in $B_{\mathfrak{m}_0}(\mathcal{H})$ by a Blaschke product cannot lead further out than B and, in the second case, we obtain subsets B which are invariant with respect to division by Blaschke products.

It is a noteworthy fact that the desription of those balls which are larger than $B_{\mathfrak{m}_0}(\mathcal{H})$ is fairly simple, whereas it is quite hard to get hands on the set of those which are contained in $B_{\mathfrak{m}_0}(\mathcal{H})$. The following theorem provides an answer to the first question. Here we denote by $\mathfrak{m}_0^\flat$ the upper envelope, $\mathfrak{m}_0^\flat(w) = \sup_{F \in B_{\mathfrak{m}_0}(\mathcal{H})} |F(w)|$, $w \in \mathbb{C}^+$.

1.4 Theorem. *(Theorem 6.3) Let $\mathcal{H} = \mathcal{H}(E)$ be a de Branges space, and let $\mathfrak{m}_0 \in \operatorname{Adm} \mathcal{H}$. Then a* log-*superharmonic majorant $\mathfrak{m}$ with $B_{\mathfrak{m}_0}(\mathcal{H}) \subset B_{\mathfrak{m}}(\mathcal{H})$ exists if and only if*

$$\int_{\mathbb{R}} \Big(\log^+ \frac{\mathfrak{m}_0^\flat(t)}{|E(t)|} \Big) \frac{dt}{1+t^2} < \infty.$$

The problem of existence of smaller balls generated by log-superharmonic majorants is more delicate, and is related to a completely different topic, namely the existence of real zerofree elements in $B_{\mathfrak{m}_0}(\mathcal{H})$.

1.5 Theorem. *(Theorem 6.5) Let $\mathcal{H}$ be a de Branges space, and let $\mathfrak{m}_0 \in \operatorname{Adm} \mathcal{H}$. Then the following are equivalent:*

(i) There exists an H^2-majorant $\mathfrak{m}$ with $B_{\mathfrak{m}}(\mathcal{H}) \subset B_{\mathfrak{m}_0}(\mathcal{H})$.

(ii) There exists an element $F \in B_{\mathfrak{m}_0}(\mathcal{H})$ which satisfies $F^\# = F$ and which has no zeros in $\mathbb{C} \setminus \mathbb{R}$.

The paper is organized as follows. Sections 2 and 3 are of preliminary character. In Section 2 we set up our notation, recall some basic facts on de Branges spaces, and provide some preparatory results, among them a variant of the Monotone Convergence Theorem for nondecreasing nets (rather than sequences) of functions. In Section 3, we introduce the majorants $\mathfrak{m}_B$ and $\mathfrak{m}_B^\perp$ associated to a subset $B \subseteq B(\mathcal{H})$. These are essential tools, and will be extensively used throughout the paper.

Sections 4 and 5 contain the main results of the paper. In Section 4 we study the problem of division by Blaschke products and prove Theorem 4.4 on dividing out zeros for log-superharmonic majorants, In Section 5 we characterizes those subsets B of a de Branges space $\mathcal{H}$ which are equal to a unit ball $B_{\mathfrak{m}}(\mathcal{H})$ with some log-superharmonic majorant defined on all of $\mathbb{C}^+$. Finally, in Section 6, we study the question if a unit ball $B_{\mathfrak{m}_0}(\mathcal{H})$ generated by an arbitrary majorant is contained or does contain unit balls $B_{\mathfrak{m}}(\mathcal{H})$ generated by log-superharmonic majorants.

2 Preliminaries

a. Functions of bounded type.

To start with, let us recall a few notations from bounded type theory. Our standard reference in this respect will be [RR].

We will denote by $\mathcal{N}(\mathbb{C}^+)$ the class of function which are analytic and of bounded type in $\mathbb{C}^+$, and by $H^p(\mathbb{C}^+)$, $p \in (0,\infty)$, the respective Hardy space. Moreover, $\mathcal{N}_+(\mathbb{C}^+)$ will denote the Smirnov class, i.e. those functions f of bounded type whose inner-outer factorization is of the form BSF with a Blaschke product B, a singular inner function S, and an outer function F. Since we will use these notations excusively for the open upper half-plane, we will drop the argument $\mathbb{C}^+$.

For later use, let us recall the definition of an outer function. If $k : \mathbb{R} \to [0,\infty]$ is measurable, and $\log k \in L^1(\frac{dt}{1+t^2})$, then an analytic function $\mathfrak{f}_k$ is well-defined on $\mathbb{C}^+$ by

$$\mathfrak{f}_k(z) := \exp\Big(\frac{1}{i\pi}\int_{\mathbb{R}}\Big(\frac{1}{t-z} - \frac{t}{1+t^2}\Big)\log k(t)\,dt\Big)\,.$$

The boundary values of $|\mathfrak{f}_k|$ along $\mathbb{R}$ are equal almost everywhere to k. Actually, the function $\log|\mathfrak{f}_k|$ is just the Poisson integral of $\log k$. Let us note explicitly that always $\mathfrak{f}_{k_1}\cdot\mathfrak{f}_{k_2} = \mathfrak{f}_{k_1\cdot k_2}$.

A function f is called outer, if it is of the form $\gamma\mathfrak{f}_k$ with some $\gamma\in\mathbb{C}$, $|\gamma|=1$, and some $k : \mathbb{R} \to [0,\infty]$, $\log k \in L^1(\frac{dt}{1+t^2})$. Sometimes, one speaks more specifically of an outer function for $\mathcal{N}$. If, additionally, the function k belongs to $L^p(dt)$, then f belongs to H^p, and one says that f is outer for H^p.

The following statement will be used later on. It is seen by a simple and standard argument and, apparently, is well known. Therefore we omit the proof.

2.1 Lemma. *Let $k : \mathbb{R} \to [0,\infty]$ be such that $\log k \in L^1(\frac{dt}{1+t^2})$. Assume that*

(i) k is continuous.

(ii) the set $S := k^{-1}(\{0,\infty\})$ is discrete.

(iii) for each $x_0 \in S$ there exists a number $n(x_0)\in\mathbb{Z}$, such that

$$\lim_{t\to x_0}\frac{k(t)}{|t-x_0|^{n(x_0)}} \in (0,\infty)\,. \tag{2.1}$$

Let F be a function which is defined and meromorphic on some domain $G \supseteq \mathbb{C}^+\cup\mathbb{R}$, and which satisfies

$$\lim_{z\to x_0}\frac{|F(z)|}{|z-x_0|^{n(x_0)}} \in (0,\infty),\ x_0\in S, \qquad F(z)\neq 0,\infty,\ z\in G\setminus S\,.$$

Then the function $|F^{-1}\mathfrak{f}_k|$ has a continuous and positive extension to $\mathbb{C}^+\cup\mathbb{R}$, namely by

$$\begin{cases} \frac{k(x)}{|F(x)|}, & x\notin S,\\ \lim_{t\to x_0}\frac{k(t)}{|F(t)|}, & x\in S. \end{cases}$$

b. Zero-divisors, mean type, ordering.

First, we deal with zero-divisors associated with a function. Let $D \subseteq \mathbb{C}$ be a nonempty open set, and let $f : D \to \mathbb{C}$ be analytic and not identically zero. Then the zero-divisor $\mathfrak{d}_f$ associated with f is the map which assigns to each point $w \in D$ the multiplicity of w as a zero of f. In other words, $\mathfrak{d}_f(w)$ is the unique nonnegative integer, such that

$$\lim_{z \to w} \frac{|f(z)|}{|z-w|^{\mathfrak{d}_f(w)}} \in (0, \infty) . \tag{2.2}$$

We will use a slightly different formulation of this definition, in order to be able to apply the notation $\mathfrak{d}_f$ to arbitrary functions.

2.2 Definition. Let $D \subseteq \mathbb{C}$ and $f : D \to \mathbb{C}$ be an arbitrary function. Then the zero-divisor $\mathfrak{d}_f$ of f is the function $\mathfrak{d}_f : \mathbb{C} \to \mathbb{N}_0 \cup \{+\infty\}$, defined as

$$\mathfrak{d}_f(w) := \inf \left\{ n \in \mathbb{N}_0 : \begin{array}{l} \exists \text{ neighborhood } U \text{ of } w, \text{ s.t.} \\ \inf_{\substack{z \in U \cap D \\ |z-w|^n \neq 0}} |z-w|^{-n} |f(z)| > 0 \end{array} \right\}, \quad w \in \mathbb{C} .$$

Here the infimum of the empty set is understood as $+\infty$.

For analytic functions f and points $w \in D$, this definition of $\mathfrak{d}_f(w)$ clearly coincides with the above stated usual definition via (2.2). Of course, for an arbitrary function f we will not have a limit relation like (2.2). But at least the speed of decrease of f towards w will be bounded by $|z-w|^{\mathfrak{d}_f(w)}$. Let us moreover note that the above definition of $\mathfrak{d}_f$ is made in such a way that $\mathfrak{d}_f(w) = 0$, $w \notin \overline{D}$.

The notation $\mathfrak{d}_{\mathsf{index}}$ will also be applied when ‘index is not a single function, but a set of functions: If B is a set of functions, we will denote

$$\mathfrak{d}_B(w) := \inf\{\mathfrak{d}_f(w) : \ f \in B\} .$$

Note that, provided B is nonempty, this infimum is attained.

Next, we discuss the notion of mean type. If f is analytic and of bounded type in $\mathbb{C}^+$, then there exists a number $c \in \mathbb{R}$ such that

$$\limsup_{\substack{r \to \infty \\ r \in M}} \frac{1}{r} \log |f(a + re^{i\theta})| = c \cdot \sin\theta , \tag{2.3}$$

whenever $\theta \in (0, \pi)$, $a \in \mathbb{R}$, and $M \subseteq \mathbb{R}^+$ is a subset of infinite logarithmic length. The number c is usually referred to as the mean type of f. Again, we wish to apply the notion of mean type to a broader class of functions.

2.3 Definition. Let $D \subseteq \mathbb{C}^+$ and $f : D \to \mathbb{C}$ be an arbitrary function. Then the mean type of f is defined as the number

$$\operatorname{mt} f := \inf \left\{ \frac{1}{\sin\theta} \limsup_{\substack{r \to \infty \\ r \in M}} \frac{1}{r} \log |f(a + re^{i\theta})| \right\} \in [-\infty, +\infty] ,$$

where the infimum is taken over all $\theta \in (0, \pi)$, $a \in \mathbb{R}$, and subsets $M \subseteq \mathbb{R}^+$ of infinite logarithmic length, such that $\{a + re^{i\theta} : r \in M\} \subseteq D$. If B is a set of functions, we set $\operatorname{mt} B := \sup\{\operatorname{mt} f : \ f \in B\}$.

Apparently, this notion coincides with the usual one if $f \in \mathcal{N}$. Of course, for an arbitrary function f, we cannot expect a regular growth behaviour like (2.3). But at least the asymptotic growth of f on some not too small set is controlled by $\operatorname{mt} f$.

Finally, let us define a partial order on functions taking nonnegative values.

2.4 Definition. Let $D_i \subseteq \mathbb{C}$ and $f_i : D_i \to [0, \infty)$, $i = 1, 2$. Then we will write

$$f_1 \preccurlyeq f_2 \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad D_1 \supseteq D_2 \text{ and } f_1|_{D_2} \leq f_2 .$$

In the context of majorization, the use of this notation is sometimes practical. Let us explain this fact. Denote by $\operatorname{Adm} \mathcal{H}$ the set of all functions $\mathfrak{m} : D \to [0, \infty]$, where D is some nonempty subset of $\mathbb{C}^+ \cup \mathbb{R}$, such that

(Adm1) $\operatorname{supp} \mathfrak{d}_{\mathfrak{m}} \subseteq \mathbb{R}$.

(Adm2) $R_{\mathfrak{m}}(\mathcal{H})$ contains a function which does not vanish identically.

If we wish to be specific about the domain of $\mathfrak{m}$, we will write $\mathfrak{m} \in \operatorname{Adm}_D \mathcal{H}$ when $\mathfrak{m}$ is defined on D. As we saw in [BW3, Theorem 3.1], these functions are exactly those for which $\mathcal{R}_{\mathfrak{m}}(\mathcal{H})$ becomes a de Branges subspace of $\mathcal{H}$.

Further, denote by β the assignment

$$\beta : \begin{cases} \operatorname{Adm} \mathcal{H} & \to \quad \{ B \subseteq \mathcal{H} : B \neq \emptyset, \{0\} \} \\ \qquad \mathfrak{m} & \mapsto \quad B_{\mathfrak{m}}(\mathcal{H}) \end{cases} \tag{2.4}$$

The relevance of the relation '$\preccurlyeq$' is seen from the following simple observation: *The assignment β is order preserving, i.e.* $\mathfrak{m}_1 \preccurlyeq \mathfrak{m}_2$ *implies that* $\beta(\mathfrak{m}_1) \subseteq \beta(\mathfrak{m}_2)$.

c. De Branges spaces.

Most of the facts collected in this subsection can be found in [dB6]. To start with, note that by (dB1) the space $\mathcal{H}$ is a reproducing kernel Hilbert space. Denote its reproducing kernel by $K(w, z)$, i.e. let $K(w, .)$ be the unique element of $\mathcal{H}$ such that

$$F(w) := \big(F, K(w, .)\big)_{\mathcal{H}}, \quad F \in \mathcal{H},\ w \in \mathbb{C} .$$

Moreover, set

$$\nabla_{\mathcal{H}}(w) := \|K(w, .)\|_{\mathcal{H}} = K(w, w)^{\frac{1}{2}}, \quad w \in \mathbb{C} .$$

The function $w \mapsto K(w, .)$ is a (weakly- and hence norm-) analytic Banach space valued function. This implies in particular that the function $\nabla_{\mathcal{H}}$ is continuous. Moreover, let us note that, by (dB2), we have $\nabla_{\mathcal{H}}(\overline{z}) = \nabla_{\mathcal{H}}(z)$, $z \in \mathbb{C}$. By (dB3), the function $\nabla_{\mathcal{H}}$ does not vanish at any point of $\in \mathbb{C} \setminus \mathbb{R}$.

In the very beginning of our exposition we gave an axiomatic definition of de Branges spaces, namely via (dB1)–(dB3). For many purposes it is essential that de Branges spaces can be constructed also in a more concrete way.

2.5 Definition. An entire function E is said to be of Hermite–Biehler class, if it satisfies

(HB) $\quad |E(\overline{z})| < |E(z)|, \quad z \in \mathbb{C}^+.$

The set of all Hermite–Biehler functions is denoted by $\mathcal{H}B$.

For each function $E \in \mathcal{H}B$, there exists a continuous and increasing function $\varphi_E : \mathbb{R} \to \mathbb{R}$, such that

$$E(x) = |E(x)| \exp(-i\varphi_E(x)), \quad x \in \mathbb{R}.$$

A function φ_E with these properties is called a phase function of E. Clearly, each two phase functions of E differ only by an additive constant.

Sometimes, it is also useful to relate the above notions to the zeros of the function E. Let us mention some facts in this direction:

2.6 Remark. Let $E \in \mathcal{H}B$, and denote by $(z_n)_n$ the (finite or infinite) sequence of nonreal zeros of E.

(i) The sequence $(z_n)_n$ satisfies the Blaschke condition

$$\sum_n \operatorname{Im} \frac{1}{z_n} < \infty,$$

and we have

$$\frac{E^\#(z)}{E(z)} = \gamma e^{-iaz} \prod_n \frac{1 - z/\overline{z_n}}{1 - z/z_n},$$

where $|\gamma| = 1$ and $a = \operatorname{mt}(E^{-1}E^\#) \leq 0$.

(ii) We have

$$\varphi_E'(x) = -\frac{a}{2} + \sum_n \frac{|\operatorname{Im} z_n|}{|x - z_n|^2}, \quad x \in \mathbb{R}.$$

With a function $E \in \mathcal{H}B$, a space of entire functions can be associated.

2.7 Definition. For $E \in \mathcal{H}B$ denote by $\mathcal{H}(E)$ and $\|.\|_E$ the linear space and norm

$$\mathcal{H}(E) := \Big\{ F \text{ entire} : \frac{F}{E}, \frac{F^\#}{E} \in H^2 \Big\},$$

$$\|F\|_E := \Big(\int_{\mathbb{R}} \Big| \frac{F(t)}{E(t)} \Big|^2 dt \Big)^{\frac{1}{2}}, \quad F \in \mathcal{H}(E).$$

2.8. *De Branges spaces via Hermite–Biehler functions:* *The following statements hold true:*

(i) *If $E \in \mathcal{H}B$, then $\mathcal{H}(E)$ endowed with the norm $\|.\|_E$ is a de Branges space.*

(ii) *If $\mathcal{H}$ is a de Branges space, then there exists a function $E \in \mathcal{H}B$ such that $\mathcal{H} = \mathcal{H}(E)$ and $\|.\|_{\mathcal{H}} = \|.\|_E$.*

(iii) *Let $E_1, E_2 \in \mathcal{H}B$. Then $\mathcal{H}(E_1) = \mathcal{H}(E_2)$ and $\|.\|_{E_1} = \|.\|_{E_2}$, if and only if there exists a matrix $U \in \mathbb{R}^{2\times 2}$ with $\det U = 1$ such that*

$$(A_2, B_2) = (A_1, B_1)U,$$

where $A := \frac{1}{2}(E + E^\#)$, $B := \frac{i}{2}(E - E^\#)$.

In the following we will always understand equality of de Branges spaces as including equality of norms, i.e. writing $\mathcal{H}(E_1) = \mathcal{H}(E_2)$ implicitly includes that $\|.\|_{E_1} = \|.\|_{E_2}$.

Since a de Branges space $\mathcal{H} = \mathcal{H}(E)$ is fully determined by the function E, all its properties must correspond to properties of E. Let us mention a couple of relations of this kind.

2.9. *Relations between* $\mathcal{H}(E)$ *and* E: *Let $\mathcal{H}$ be a de Branges space, and let $E \in \mathcal{H}B$ be such that $\mathcal{H} = \mathcal{H}(E)$. Then the following hold:*

(*i*) *We have $\mathfrak{d}_{\mathcal{H}}(x) = \mathfrak{d}_E(x)$, $x \in \mathbb{R}$.*

(*ii*) *The reproducing kernel $K(w, .)$ of $\mathcal{H}$ is given as*

$$K(w,z) = \frac{E(z)E^{\#}(\bar{w}) - E(\bar{w})E^{\#}(z)}{2\pi i(\bar{w} - z)}, \quad z, w \in \mathbb{C},\ z \neq \overline{w},$$

and

$$K(\overline{z}, z) := \frac{i}{2\pi}\Big(\frac{\partial E}{\partial z}(z)E^{\#}(z) - E(z)\frac{\partial E^{\#}}{\partial z}(z)\Big), \quad z \in \mathbb{C}.$$

(*iii*) *We have*

$$\nabla_{\mathcal{H}}(z) = \Big(\frac{|E(z)|^2 - |E(\overline{z})|^2}{4\pi \operatorname{Im} z}\Big)^{1/2}, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

and

$$\nabla_{\mathcal{H}}(x) = \pi^{-1/2}|E(x)|(\varphi_E'(x))^{1/2}, \quad x \in \mathbb{R}.$$

The following statement is very easy to see, but still often useful.

2.10 Lemma. *Let $\mathcal{H}$ be a de Branges space, and let $E \in \mathcal{H}B$ be such that $\mathcal{H} = \mathcal{H}(E)$. Then*

(*i*) *The function $|E(z)|^{-1}\nabla_{\mathcal{H}}(z)$, $z \in \mathbb{C}^+$, has a continuous and positive extension to $\mathbb{C}^+ \cup \mathbb{R}$.*

(*ii*) *We have $\operatorname{mt}(|E|^{-1}\nabla_{\mathcal{H}}) = 0$. More precisely,*

$$\lim_{r\to\infty} \frac{1}{r} \log \frac{\nabla_{\mathcal{H}}(re^{i\vartheta})}{|E(re^{i\vartheta})|} = 0, \quad \vartheta \in (0, \pi).$$

Proof. If E has no real zeros, the assertion in (i) is clear. The general case can be easily reduced to this case by dividing out the real zeros of E.

The assertion (ii) follows from the inequalities ($w_0 \in \mathbb{C}^+$ fixed)

$$\frac{|E(w_0)|(1 - |\frac{E(\overline{w_0})}{E(w_0)}|)}{2\pi\nabla_{\mathcal{H}}(w_0)} \frac{1}{|z - \overline{w_0}|} \leq \frac{\nabla_{\mathcal{H}}(z)}{|E(z)|} \leq \frac{1}{2\sqrt{\pi}} \frac{1}{\sqrt{\operatorname{Im} z}}, \quad z \in \mathbb{C}^+,$$

see e.g. [BW3, (2.6)]. ❑

Sometimes a property of a de Branges space $\mathcal{H} = \mathcal{H}(E)$ will be defined in terms of the function E. Proceeding in this way makes it of course necessary to show that the notion under consideration is well-defined. For this purpose, the following facts are useful: Let $E_1, E_2 \in \mathcal{H}B$ and assume that $\mathcal{H}(E_1) = \mathcal{H}(E_2)$. Then for some positive constants $c, C > 0$, we have

$$c \leq \Big|\frac{E_1(z)}{E_2(z)}\Big| \leq C, \quad z \in \mathbb{C}^+ \cup \mathbb{R}\,. \tag{2.5}$$

The function $E_2^{-1}E_1$ is of bounded type and has an analytic extension to some domain containing the closed half-plane $\mathbb{C}^+ \cup \mathbb{R}$. This extension does not vanish at any point of $\mathbb{R} \cup \mathbb{C}^+$, and satisfies $\operatorname{mt}(E_2^{-1}E_1) = 0$. Thus, as it is seen e.g. from the Theorem of Szegö-Solomentsev [RR, Theorem 3.13], it is outer.

It turns out practical to use a notation of mean type relative to a given de Branges space.

2.11 Definition. Let $\mathcal{H}$ be a de Branges space, let $D \subseteq \mathbb{C}$ and $f : D \to \mathbb{C}$. Then we denote

$$\operatorname{mt}_{\mathcal{H}} f := \operatorname{mt} \frac{f}{\nabla_{\mathcal{H}}}\,.$$

If B is a set of functions, we again set $\operatorname{mt}_{\mathcal{H}} B := \sup_{f \in B} \operatorname{mt}_{\mathcal{H}} f$.

Finally, let us spend a couple of lines on the notion of de Branges subspaces.

2.12 Definition. Let $\mathcal{H}$ be a de Branges space. A closed subspace $\mathcal{L}$ of $\mathcal{H}$ is called a de Branges subspace of $\mathcal{H}$, if it is itself, with the norm inherited from $\mathcal{H}$, a de Branges space.

A closed subspace $\mathcal{L}$ of $\mathcal{H}$ is a de Branges subspace of $\mathcal{H}$ if and only if

$$F \in \mathcal{L} \Rightarrow F^\# \in \mathcal{L}, \qquad F \in \mathcal{L}, w \in \mathbb{C}^+, F(w) = 0 \Rightarrow \frac{F(z)}{z-w} \in \mathcal{L}\,.$$

Certain de Branges subspace are defined by restriction of exponential growth: If $a \leq 0$, let us denote

$$\mathcal{H}_{(a)} := \big\{F \in \mathcal{H} :\ \operatorname{mt}_{\mathcal{H}} F, \operatorname{mt}_{\mathcal{H}} F^\# \leq a\big\}\,.$$

The fact that $\mathcal{H}_{(a)}$ is a de Branges subspace of $\mathcal{H}$, provided it contains a function which does not vanish identically, has been proved e.g. in [KW, §5]. Moreover, we have:

(i) If $\mathcal{L}$ is a de Branges subspace of $\mathcal{H}$, then $\operatorname{mt}_{\mathcal{H}} \mathcal{L} = \operatorname{mt}_{\mathcal{H}} \nabla_{\mathcal{L}}$.

(ii) Provided $\mathcal{H}_{(a)} \neq \{0\}$, we have $\operatorname{mt}_{\mathcal{H}} \mathcal{H}_{(a)} = a$.

d. Monotone Convergence Theorem for nets of functions.

We will need the following variant of the Lebesgue Monotone Convergence Theorem, which deals with nondecreasing nets of nonnegative functions, rather than nondecreasing sequences. We will thereby require some additional properties of the functions under consideration, namely lower semicontinuity. Other variants of the Monotone Convergence Theorem rather put some hypothesis on the order structure of the index set, see e.g. [H-J].

2.13 Proposition. *Let X be a locally compact and σ-compact Hausdorff space, and let λ be a positive Borel measure which is complete, regular and satisfies $\lambda(K) < \infty$ for all compact sets $K \subseteq X$.*

Let $(I, \leq)$ be a directed set, and let $f_i : X \to [0, \infty]$, $i \in I$, be a family of lower semicontinuous functions which is nondecreasing, i.e. $f_i(x) \leq f_j(x)$, $x \in X$, whenever $i \leq j$. Set $f(x) := \sup_{i \in I} f_i(x)$, $x \in X$. Then

$$\int_X f\, d\lambda = \sup_{i \in I} \int_X f_i\, d\lambda\,. \tag{2.6}$$

Proof. The inequality '$\geq$' is trivial. For the proof of the converse inequality, choose a function g_0 which is lower semicontinuous, everywhere positive, and satisfies $\int_X g_0\, d\lambda = 1$, e.g. an appropriate step-function with open level sets, cf. [R].

Let s be a nonnegative and bounded upper semicontinuous function with $s \leq f$. Moreover, let $\epsilon > 0$ be given. By semicontinuity, the set

$$E_i := \big\{ x \in X :\, f_i(x) - s(x) + \epsilon g_0(x) > 0 \big\}$$

is open. Let $x \in X$ be given. Assume that $f(x) < \infty$, then there exists $i_0 \in I$ such that $f_{i_0}(x) > f(x) - \epsilon g_0(x) \geq s(x) - \epsilon g_0(x)$, i.e. $x \in E_{i_0}$. If $f(x) = \infty$, since $s(x) < \infty$, again there exists $i_0 \in I$ with $f_{i_0}(x) > s(x) - \epsilon g_0(x)$. We conclude that $X = \bigcup_{i \in I} E_i$. By σ-compactness there exists a countable subcover $\{E_{i_1}, E_{i_2}, \ldots\}$.

For each $n \in \mathbb{N}$ choose an index $j_n \in I$ with $j_n \geq i_1, \ldots, i_n$. Since each f_i in nonnegative and the family $(f_i)_{i \in I}$ is monotone, we obtain

$$\sup_{i \in I} \int_X f_i\, d\lambda \geq \int_X f_{j_n}\, d\lambda \geq \int_{\bigcup_{k=1}^n E_{i_k}} f_{j_n}\, d\lambda \geq \int_{\bigcup_{k=1}^n E_{i_k}} s\, d\lambda - \epsilon \int_{\bigcup_{k=1}^n E_{i_k}} g_0\, d\lambda\,.$$

Letting $n \in \mathbb{N}$ tend to infinity, it follows that $\sup_{i \in I} \int_X f_i\, d\lambda \geq \int_X s\, d\lambda - \epsilon$. Since $\epsilon > 0$ was arbitrary, this yields $\sup_{i \in I} \int_X f_i\, d\lambda \geq \int_X s\, d\lambda$. Finally, since the measure λ is regular and s was arbitrary, it follows that $\sup_{i \in I} \int_X f_i\, d\lambda \geq \int_X f\, d\lambda$, cf. [R, Theorem 2.25] which applies to each nonnegative step-function not exceeding f. ❑

2.14 Remark.

(*i*) Since $(f_i)_{i \in I}$ is nondecreasing, both suprema appearing in Proposition 2.13 are actually limits. Hence, by taking linear combinations, we obtain that

$$\int_X \lim_{i \in I} f_i\, d\lambda = \lim_{i \in I} \int_X f_i\, d\lambda$$

whenever $(f_i)_{i \in I}$ satisfies the stated hypothesis, and λ is a complex (complete, regular) Borel measure.

(*ii*) Let us explicitly note that, in order to have (2.6), some hypothesis on the net $(f_i)_{i \in (I, \leq)}$ is needed. For example, let I be the set of all finite subsets of $[0, 1]$ ordered by set-theoretic inclusion. Moreover, let f_i, $i \in I$, be the indicator function of i. Then $f(x) := \sup_{i \in I} f_i(x) = 1$, $x \in [0, 1]$. Hence, integrating with respect to the Lebesgue measure,

$$\int_{[0,1]} f\, dx = 1 \quad \text{but} \quad \int_{[0,1]} f_i\, dx = 0, \quad i \in I\,.$$

This example also explains why we cannot employ a compactness argument in Proposition 2.13. Since $f_i = 0$ in $L^1(dx)$, $i \in I$, we trivially have $\lim_{i\in I}(f_i\,dx) = 0$ in $C([0,1])'$. The point here is that for a nondecreasing net of functions, unlike for a nondecreasing sequence, the L^1-limit and pointwise limit $(f_i)_{i\in I}$ need not coincide almost everywhere.

As a consequence of Proposition 2.13 we obtain the following two statements, and these are what will be needed later on in §6. A function f is called superharmonic, if $(-f)$ is subharmonic.

2.15 Corollary. *Let $D \subseteq \mathbb{C}^+$ be open, let $(I, \leq)$ be a directed set, and let $f_i : D \to (-\infty, \infty]$, $i \in I$, be a family of superharmonic functions. Assume that $(f_i)_{i\in I}$ is nondecreasing and that, for some index $i_0 \in I$, there exists a harmonic function $g : D \to \mathbb{R}$ with $g \leq f_{i_0}$. Set $f := \sup_{i\in I} f_i$. Then f is superharmonic.*

Proof. First note that f is, as supremum of a family of lower semicontinuous functions, itself lower semicontinuous. Moreover, since the family $(f_i)_{i\in I}$ is nondecreasing, we have $f = \sup_{i\geq i_0} f_i$.

Let U be a closed disk which is entirely contained in D, let a be its center and r its radius. We have

$$f_i(a) \geq \frac{1}{2\pi}\int_0^{2\pi} f_i(a + re^{it})\,dt,\ i \in I, \qquad g(t) = \frac{1}{2\pi}\int_0^{2\pi} g(a + re^{it})\,dt\,,$$

and hence also

$$(f_i - g)(a) \geq \frac{1}{2\pi}\int_0^{2\pi} (f_i - g)(a + re^{it})\,dt, \quad i \in I\,.$$

Taking the supremum over all $i \geq i_0$ yields

$$f(a) - g(a) \geq \sup_{i\geq i_0} \frac{1}{2\pi}\int_0^{2\pi} (f_i - g)(a + re^{it})\,dt\,.$$

The family $(f_i - g)_{i\geq i_0}$ is a nondecreasing family of nonnegative lower semicontinuous functions. Hence, by Proposition 2.13,

$$\sup_{i\geq i_0} \frac{1}{2\pi}\int_0^{2\pi} (f_i - g)(a + re^{it})\,dt = \frac{1}{2\pi}\int_0^{2\pi} (f - g)(a + re^{it})\,dt\,.$$

Using once again the mean value property of g, we conclude that

$$f(a) \geq \frac{1}{2\pi}\int_0^{2\pi} f(a + re^{it})\,dt\,.$$

❑

2.16 Corollary. *Let $(I, \leq)$ be a directed set, and let $k_i : \mathbb{R} \to [0, \infty]$, $i \in I$, be a nondecreasing family of continuous functions. Assume that*

$$\log k_i \in L^1\big(\tfrac{dt}{1+t^2}\big),\ i \in I, \qquad \sup_{i\in I}\int_{\mathbb{R}} \log^+ k_i(t)\frac{dt}{1+t^2} < \infty\,. \tag{2.7}$$

Denote $k(x) := \sup_{i\in I} k_i(x) = \lim_{i\in I} k_i(x)$, $x \in \mathbb{R}$. Then $\log k \in L^1(\frac{dt}{1+t^2})$ and

$$\lim_{i\in I} \mathfrak{f}_{k_i}(z) = \mathfrak{f}_k(z), \quad z \in \mathbb{C}^+\,.$$

Proof. Clearly, we have

$$\log^- k = -\min\{\log k, 0\} \leq \log^- k_i \leq |\log k_i|, \quad i \in I\,.$$

In particular, $\log^- k \in L^1(\frac{dt}{1+t^2})$.

The functions $\log^+ k_i$, $i \in I$, are continuous, nonnegative, and form a nondecreasing family. Moreover, $\log^+ k = \sup_{i\in I} \log^+ k_i$. Thus, by Proposition 2.13 and the present assumption,

$$\int\limits_{\mathbb{R}} \log^+ k(t) \frac{dt}{1+t^2} = \int\limits_{\mathbb{R}} \big[\sup_{i\in I} \log^+ k_i(t)\big] \frac{dt}{1+t^2} = \sup_{i\in I} \int\limits_{\mathbb{R}} \log^+ k_i(t) \frac{dt}{1+t^2} < \infty\,.$$

It follows that $\log k \in L^1(\frac{dt}{1+t^2})$.

Let $z \in \mathbb{C}^+$ be fixed. Then, as mentioned in Remark 2.14, (i), we may apply Proposition 2.13 with the complex measure

$$d\lambda(t) := \Big(\frac{1}{t-z} - \frac{t}{1+t^2}\Big)\, dt\,.$$

It follows that

$$\lim_{i\in I} \int_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{t}{1+t^2}\Big) \log^+ k_i(t)\, dt = \lim_{i\in I} \int_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{t}{1+t^2}\Big) \log^+ k(t)\, dt\,.$$

Fix $i_0 \in I$. The functions $\log^- k_{i_0} - \log^- k_i$, $i \in I$, $i \geq i_0$, are continuous, nonnegative, and form a nondecreasing family. Thus

$$\lim_{i\in I} \int_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{t}{1+t^2}\Big) \log^- k_i(t)\, dt =$$

$$= -\lim_{i\in I} \int_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{t}{1+t^2}\Big) \big[\log^- k_{i_0}(t) - \log^- k_i(t)\big]\, dt +$$

$$+ \int_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{t}{1+t^2}\Big) \log^- k_{i_0}(t)\, dt = \int_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{t}{1+t^2}\Big) \log^- k(t)\, dt\,.$$

❑

3 The functions $\mathfrak{m}_B$ and $\mathfrak{m}_B^\perp$

If $\mathcal{H}$ is a de Branges space, denote its unit ball by $B(\mathcal{H})$, i.e.

$$B(\mathcal{H}) := \big\{F \in \mathcal{H} :\ \|F\|_{\mathcal{H}} \leq 1\big\}\,.$$

3.1 Definition. Let $\mathcal{H}$ be a de Branges space, and let B be a subset of its unit ball which contains a function that does not vanish identically. Denote by $\mathfrak{m}_B$ the function

$$\mathfrak{m}_B : \left\{ \begin{array}{rcl} \mathbb{C}^+ \cup \mathbb{R} & \to & [0,\infty) \\ w & \mapsto & \sup_{F\in B} |F(w)| \end{array} \right. .$$

3.2 Remark. Let us explicitly note the following facts:

(i) The supremum in the definition of $\mathfrak{m}_B$ is finite, since $B \subseteq B(\mathcal{H})$ implies that $|F(z)| \leq \nabla_{\mathcal{H}}(z)$, $F \in B$.

(ii) We have $\mathfrak{m}_{B(\mathcal{H})}(w) = \nabla_{\mathcal{H}}(w)$, $w \in \mathbb{C}^+ \cup \mathbb{R}$.

(iii) If $B_1 \subseteq B_2$, then $\mathfrak{m}_{B_1} \leq \mathfrak{m}_{B_2}$.

For particular sets B, namely for unit balls $B_{\mathfrak{m}}(\mathcal{H})$ generated by majorization, we had already used this function in our previous work, cf. [BW4, §4]. There, we had defined for a majorant $\mathfrak{m}$ the function

$$\mathfrak{m}^\flat(w) := \sup_{F \in B_{\mathfrak{m}}(\mathcal{H})} |F(w)| \qquad \big(= \mathfrak{m}_{B_{\mathfrak{m}}(\mathcal{H})} \big) .$$

We will keep this notation also in the present paper.

It will be important to know that the function $\mathfrak{m}_B$ is fairly smooth, and reflects properties of B in many respects. Let us collect some statements of this kind. Similar as in [BW4, Proposition 4.6], the proof is based on a normal family argument.

3.3 Lemma. *Let $\mathcal{H}$ be a de Branges space, and let $B \subseteq B(\mathcal{H})$, $B \neq \emptyset, \{0\}$. Then the following hold:*

(i) *The function $\nabla_{\mathcal{H}}^{-1}\mathfrak{m}_B$ is continuous on $\mathbb{C}^+ \cup \mathbb{R}$, and we have*

$$\lim_{z \to z_0, z \in \mathbb{C}^+} \frac{1}{|z - z_0|^{\mathfrak{d}_B(z_0) - \mathfrak{d}_{\mathcal{H}}(z_0)}} \frac{\mathfrak{m}_B(z)}{\nabla_{\mathcal{H}}(z)} \in (0, \infty), \quad z_0 \in \mathbb{C}^+ \cup \mathbb{R} .$$

(ii) *We have $\mathfrak{d}_{\mathfrak{m}_B} = \mathfrak{d}_B$; moreover, $\lim_{z \to w} |z - z_0|^{-\mathfrak{d}_B(w)} \mathfrak{m}_B(z) \in (0, \infty)$, $w \in \mathbb{C}^+ \cup \mathbb{R}$.*

(iii) *The function $\log \mathfrak{m}_B$ is subharmonic in $\mathbb{C}^+$.*

(iv) *We have $\operatorname{mt}_{\mathcal{H}} \mathfrak{m}_B = \operatorname{mt}_{\mathcal{H}} B$.*

Proof. Let $E \in \mathcal{H}B$, and let G be a domain which contains the closed half-plane $\mathbb{C}^+ \cup \mathbb{R}$, such that E has no nonreal zeros in G. Then the family $\mathcal{F}_{\mathcal{H}(E)} := \left\{ \frac{F}{E}\Big|_G : F \in B(\mathcal{H}) \right\}$ is normal. Note that all elements of $\mathcal{F}_{\mathcal{H}(E)}$ are analytic, since $F \in \mathcal{H}(E)$ implies that $\mathfrak{d}_F|_{\mathbb{R}} \geq \mathfrak{d}_E|_{\mathbb{R}}$. If E has no real zeros, this fact is clear. Otherwise, we can immediately reduce to this case.

The proof of (i) can now be given. Let $w \in \mathbb{C}^+ \cup \mathbb{R}$, and set $n := \mathfrak{d}_B(w) - \mathfrak{d}_{\mathcal{H}}(w) \geq 0$. Since $B \subseteq B(\mathcal{H})$, the family $\{E^{-1}F : F \in B\}$ is normal. Thus also the family

$$\mathcal{F} := \left\{ \frac{F(z)}{(z - w)^n E(z)} : F \in B \right\}$$

is normal, and hence equicontinuous. Using Lemma 2.10, it follows that

$$\frac{\mathfrak{m}_B(z)}{|z - w|^n \nabla_{\mathcal{H}}(z)} = \frac{|E(z)|}{\nabla_{\mathcal{H}}(z)} \cdot \frac{\mathfrak{m}_B(z)}{|z - w|^n |E(z)|} = \frac{|E(z)|}{\nabla_{\mathcal{H}}(z)} \cdot \sup_{F \in \mathcal{F}} \left| \frac{F(z)}{(z - w)^n E(z)} \right|$$

is continuous. Moreover, since there exists a function $F \in B$ with $\mathfrak{d}_F(w) = \mathfrak{d}_B(w)$, the supremum on the right side is positive. This shows (i).

In view of Lemma 2.10 and 2.9, (i), the assertion (ii) is an immediate consequence of (i).

We turn to the proof of (iii). By (i), the function $\log \mathfrak{m}_B$ is a continuous function of $\mathbb{C}^+$ into $[-\infty, \infty)$. Moreover, it is the supremum of the subharmonic functions $\log |F(z)|$, $F \in B$. Thus it is itself subharmonic.

Finally, we come to the proof of (iv). For each $F \in B$ we have $|F(z)| \leq \mathfrak{m}_B(z)$, $z \in \mathbb{C}^+$. Thus

$$\operatorname{mt}_{\mathcal{H}} B = \sup_{F \in B} \operatorname{mt}_{\mathcal{H}} F \leq \operatorname{mt}_{\mathcal{H}} \mathfrak{m}_B .$$

Conversely, we have $B \subseteq \mathcal{H}_{(a)}$ where $a := \operatorname{mt}_{\mathcal{H}} B$. Thus $\mathfrak{m}_B \leq \nabla_{\mathcal{H}_{(a)}}$, and hence

$$\operatorname{mt}_{\mathcal{H}} \mathfrak{m}_B \leq \operatorname{mt}_{\mathcal{H}} \nabla_{\mathcal{H}_{(a)}} = a .$$

❏

3.4 Lemma. *Let $\mathfrak{m}_0 \in \operatorname{Adm} \mathcal{H}$, and denote the domain of $\mathfrak{m}_0$ by $D_0 \subseteq \mathbb{C}^+ \cup \mathbb{R}$. If $D \subseteq \mathbb{C}^+ \cup \mathbb{R}$ is such that $D_0 \subseteq \overline{D}$, and $a \in [\operatorname{mt}_{\mathcal{H}} B_{\mathfrak{m}_0}(\mathcal{H}), 0]$, then*

$$B_{\mathfrak{m}_0}(\mathcal{H}) = B_{\mathfrak{m}_0^\flat|_D}(\mathcal{H}_{(a)}) .$$

Proof. If $F \in B_{\mathfrak{m}_0}(\mathcal{H})$, then $F \in \mathcal{H}_{(a)}$ and $|F(z)| \leq \mathfrak{m}_0^\flat(z)$, $z \in \mathbb{C}^+ \cup \mathbb{R}$. In particular, $F \in B_{\mathfrak{m}_0^\flat|_D}(\mathcal{H}_{(a)})$. Conversely, assume that F belongs to this set. Then $F \in \mathcal{H}$ and $|F(z)| \leq \mathfrak{m}_0^\flat(z)$, $z \in D$. By continuity, it follows that $|F(z)| \leq \mathfrak{m}_0^\flat(z)$, $z \in \overline{D}$. Since $D_0 \subseteq \overline{D}$, we obtain

$$|F(z)| \leq \mathfrak{m}_0^\flat(z) \leq \mathfrak{m}_0(z), \quad z \in D_0 .$$

❏

If B is a subset of the unit ball of $\mathcal{H}$, define

$$M_{\geq}(B) := \big\{ \mathfrak{m} \in \operatorname{Adm} \mathcal{H} :\ B_{\mathfrak{m}}(\mathcal{H}) \supseteq B \big\} ,$$

and set $B_{\geq}(B) := \beta(M_{\geq}(B))$ where β is the map $\mathfrak{m} \mapsto B_{\mathfrak{m}}(\mathcal{H})$, cf. (2.4).

3.5 Lemma. *Let $\mathcal{H}$ be a de Branges space, and let $B \subseteq B(\mathcal{H})$, $B \neq \emptyset, \{0\}$.*

- (i) *We have $\mathfrak{m} \in M_{\geq}(B)$ if and only if $\mathfrak{m} \in \operatorname{Adm} \mathcal{H}$ and $\mathfrak{m} \succcurlyeq \mathfrak{m}_B$.*
- (ii) *Assume that for each $w \in \mathbb{C}^+$ there exists an element $F \in B$ with $F(w) \neq 0$. Then $\mathfrak{m}_B \in \operatorname{Adm} \mathcal{H}$, and is the smallest element of $M_{\geq}(B)$. The ball $B_{\mathfrak{m}_B}(\mathcal{H})$ is the smallest element of $B_{\geq}(B)$.*

Proof. Let $D \subseteq \mathbb{C}^+ \cup \mathbb{R}$ be the domain of $\mathfrak{m}$. Assume that $B_{\mathfrak{m}}(\mathcal{H}) \supseteq B$, then

$$\mathfrak{m}_B(z) = \sup_{F \in B} |F(z)| \leq \sup_{F \in B_{\mathfrak{m}}(\mathcal{H})} |F(z)| \leq \mathfrak{m}(z), \quad z \in D .$$

Conversely, if $\mathfrak{m} \geq \mathfrak{m}_B|_D$ and $F \in B$, then

$$|F(z)| \leq \mathfrak{m}_B(z) \leq \mathfrak{m}(z), \quad z \in D ,$$

and hence $F \in B_{\mathfrak{m}}(\mathcal{H})$. This shows (i).

If there exists $F \in B$ with $F(w) \neq 0$, then $\mathfrak{m}_B(w) > 0$. Hence, the assumption in (ii) together with continuity of $\mathfrak{m}_B$ implies that $\mathfrak{d}_{\mathfrak{m}_B}(w) = 0$, $w \in \mathbb{C}^+$. Since B contains a function which does not vanish identically, we obtain $\mathfrak{m}_B \in \operatorname{Adm}\mathcal{H}$.

By (i), clearly, $\mathfrak{m}_B$ is the smallest element of $M_{\geq}(B)$. Since the map $\beta : \mathfrak{m} \mapsto B_{\mathfrak{m}}(\mathcal{H})$ preserves the respective orders and maps $M_{\geq}(B)$ surjectively onto $B_{\geq}(B)$, this implies that $\beta(\mathfrak{m}_B)$ is the smallest element of $B_{\geq}(B)$. ❑

3.6 Lemma. *Let $\mathcal{H}$ be a de Branges space, and let $B \subseteq B(\mathcal{H})$, $B \neq \emptyset, \{0\}$. Assume that B is invariant with respect to division by Blaschke products, i.e. assume that for each function $F \in B$ and Blaschke product P for $\mathbb{C}^+$ with $\mathfrak{d}_P|_{\mathbb{C}^+} \leq \mathfrak{d}_F|_{\mathbb{C}^+}$, we have $P^{-1}F \in B$. Then*

$$\mathfrak{m}_B(w) = \sup\left\{|F(w)| : F \in B, F \text{ zerofree in } \mathbb{C}^+\right\}.$$

Proof. The inequality '$\geq$' holds by the definition of $\mathfrak{m}_B$. Conversely, denote the supremum on the right side by M, and let $F \in B$ be given. Let P be the Blaschke product with $\mathfrak{d}_P|_{\mathbb{C}^+} = \mathfrak{d}_F|_{\mathbb{C}^+}$. Then $P^{-1}F \in B$, and hence

$$|F(z)| \leq \left|\frac{F(z)}{P(z)}\right| \leq M, \quad z \in \mathbb{C}^+.$$

The inequality '$\leq$' follows. ❑

With a subset B of the unit ball $B(\mathcal{H})$, which is in some sense not too big, we can associate another function closely related to B.

3.7 Definition. Let $\mathcal{H}$ be a de Branges space, and choose $E \in \mathcal{H}B$ with $\mathcal{H} = \mathcal{H}(E)$. Moreover, let $B \subseteq B(\mathcal{H})$, $B \neq \emptyset, \{0\}$, and assume that

$$\int_{\mathbb{R}} \Big(\log^+ \frac{\mathfrak{m}_B(t)}{|E(t)|}\Big) \frac{dt}{1+t^2} < \infty. \tag{3.1}$$

Then we define a function $\mathfrak{m}_B^{\perp} : \mathbb{C}^+ \to [0, \infty)$ as

$$\mathfrak{m}_B^{\perp}(z) := \left|e^{-i(\operatorname{mt}_{\mathcal{H}} B)\cdot z} \mathfrak{f}_{|E|^{-1}\mathfrak{m}_B}(z) E(z)\right|, \quad z \in \mathbb{C}^+.$$

First of all, we have to show that $\mathfrak{m}_B^{\perp}$ is well-defined.

3.8 Lemma. *Let B be a subset of $B(\mathcal{H})$, $B \neq \emptyset, \{0\}$, which satisfies* (3.1). *Then the following hold:*

(i) *We have $\int_{\mathbb{R}} |\log(|E|^{-1}\mathfrak{m}_B)|(1+t^2)^{-1}dt < \infty$. Hence the outer function $\mathfrak{f}_{|E|^{-1}\mathfrak{m}_B}$ in Definition 3.7 exists.*

(ii) *Neither the validity of* (3.1), *nor the function $\mathfrak{m}_B^{\perp}$ itself depends on the particular choice of E in Definition 3.7.*

Proof. Choose $F \in B \setminus \{0\}$, then $E^{-1}F \in H^2$ and does not vanish identically. Hence ($\log^- x := -\min\{\log x, 0\}$)

$$\int_{\mathbb{R}} \Big(\log^- \frac{\mathfrak{m}_B(t)}{|E(t)|}\Big) \frac{dt}{1+t^2} \leq \int_{\mathbb{R}} \Big(\log^- \frac{|F(t)|}{|E(t)|}\Big) \frac{dt}{1+t^2} < \infty.$$

Together with (3.1) this gives $\log(|E|^{-1}\mathfrak{m}_B) \in L^1(\frac{dt}{1+t^2})$.

The fact that the validity of (3.1) does not depend on the choice of E, follows from (2.5). Moreover, for each two functions $E_1, E_2 \in \mathcal{H}B$ with $\mathcal{H}(E_1) = \mathcal{H}(E_2)$, we have $E_1^{-1}E_2 = \gamma \mathfrak{f}_{|E_1|^{-1}|E_2|}$ with some $|\gamma| = 1$. Hence,

$$\left|\mathfrak{f}_{|E_1|^{-1}\mathfrak{m}_B}(z)E_1(z)\right| = \left|\mathfrak{f}_{|E_2|^{-1}\mathfrak{m}_B}(z)E_2(z)\right|, \quad z \in \mathbb{C}^+ .$$

❑

Let us show some properties of $\mathfrak{m}_B^\perp$ which will be needed later on.

3.9 Lemma. *Let $B \subseteq B(\mathcal{H})$, $B \neq \emptyset, \{0\}$, and assume that* (3.1) *holds. Then*

- (i) *The function $\mathfrak{m}_B^\perp$ has a continuous extension to the closed half-plane $\mathbb{C}^+ \cup \mathbb{R}$, namely by setting $\mathfrak{m}_B^\perp(x) := \mathfrak{m}_B(x)$, $x \in \mathbb{R}$.*
- (ii) *We have $\mathfrak{m}_B \preccurlyeq \mathfrak{m}_B^\perp$.*

Proof. In order to prove (i), we wish to apply Lemma 2.1 with the function $k(t) := |E(t)|^{-1}\mathfrak{m}_B(t)$. The required hypothesis for this application have been established in Lemma 3.3, (i). Actually, the limit relation (2.1) holds for all $x_0 \in \mathbb{R}$ when we set $n(x_0) := \mathfrak{d}_B(x_0) - \mathfrak{d}_E(x_0)$. Remember here that, concerning continuity and zeros, by Lemma 2.10, (i), it does not matter whether we consider the quotient $\nabla_{\mathcal{H}}^{-1}\mathfrak{m}_B$ or $|E|^{-1}\mathfrak{m}_B$.

If the function F is chosen as in Lemma 2.1, we can write

$$\mathfrak{m}_B^\perp(z) = \left|e^{-i(\operatorname{mt}_{\mathcal{H}} B)\cdot z}\right| \cdot |E(z)F(z)| \cdot \left|\frac{\mathfrak{f}_{|E|^{-1}\mathfrak{m}_B}(z)}{F(z)}\right|, \quad z \in \mathbb{C}^+ .$$

From this formula, it is apparent that $\mathfrak{m}_B^\perp$ has a continuous extension $\tilde{\mathfrak{m}}_B^\perp$ to $\mathbb{C}^+ \cup \mathbb{R}$. Moreover, we see that for this extension

$$\lim_{\substack{z \to x \\ z \in \mathbb{C}^+ \cup \mathbb{R}}} \frac{\tilde{\mathfrak{m}}_B^\perp(z)}{|z-x|^{\mathfrak{d}_B(x)}} \in (0,\infty), \quad x \in \mathbb{R} .$$

Again by Lemma 2.1, the boundary values of $\mathfrak{m}_B^\perp$ along $\mathbb{R}$ are given by $\mathfrak{m}_B$.

For the proof of (ii), let $F \in B$ be given. Then $E^{-1}F \in H^2$ and $\operatorname{mt}(E^{-1}F) \leq \operatorname{mt}_{\mathcal{H}} B$. Hence, the function

$$g(z) := e^{i(\operatorname{mt}_{\mathcal{H}} B)\cdot z}\frac{F(z)}{E(z)}, \quad z \in \mathbb{C}^+ ,$$

belongs to the class $\mathcal{N}_+$. The function g is actually continuous on $\mathbb{C}^+ \cup \mathbb{R}$, and along the real axis we have

$$|g(x)| = \left|\frac{F(x)}{E(x)}\right| \leq \frac{\mathfrak{m}_B(x)}{|E(x)|}, \quad x \in \mathbb{R},\ E(x) \neq 0 .$$

By the Smirnov Maximum Principle, it follows that

$$|g(z)| \leq \left|\mathfrak{f}_{|E|^{-1}\mathfrak{m}_B}(z)\right|, \quad z \in \mathbb{C}^+ .$$

This, however, just says that $|F| \leq \mathfrak{m}_B^\perp$ throughout the half-plane $\mathbb{C}^+$. Thus

$$\mathfrak{m}_B(z) = \sup_{F \in B} |F(z)| \leq \mathfrak{m}_B^\perp(z), \quad z \in \mathbb{C}^+ .$$

❑

4 Division by Blaschke products

If majorization takes place along the real line, the situation is clear and simple.

4.1 Lemma. *Let $\mathcal{H}$ be a de Branges space, let $D \subseteq \mathbb{R}$, and let $\mathfrak{m} \in \operatorname{Adm}_D \mathcal{L}$. Moreover, let $\mathcal{L}$ be a de Branges subspace of $\mathcal{H}$. Then $B_{\mathfrak{m}}(\mathcal{H}) \cap \mathcal{L}$ $(= B_{\mathfrak{m}}(\mathcal{L}))$ is invariant with respect to division by Blaschke products.*

Proof. It is well-known that each de Branges space is invariant with respect to division by Blaschke products. Since the norm in a de Branges space can be computed as a weighted L^2-integral along $\mathbb{R}$, even its unit ball has this property. For the same reason, majorization along any subset of $\mathbb{R}$ will be preserved when dividing by a Blaschke product. ❑

Things change when majorization is required off the real axis.

4.2 Example. Consider the function

$$E(z) := \Big(1 + \frac{z}{i}\Big) \prod_{n\geq 2} \Big(1 + \frac{z}{in^2}\Big)^2 .$$

Then E is of order $\frac{1}{2}$, belongs to $\mathcal{H}B$, and has no real zeros. By [B, Theorem 1], the space $\mathcal{H} := \mathcal{H}(E)$ contains the set $\mathbb{C}[z]$ of all polynomials as a dense linear subspace. This yields that the chain of all de Branges subspaces $\mathcal{L}$ of $\mathcal{H}$ with $\mathfrak{d}_{\mathcal{L}} = 0$ is equal to

$$\{\mathbb{C}[z]_n : n \in \mathbb{N}_0\} \cup \{\mathcal{H}\} .$$

Here $\mathbb{C}[z]_n$ denotes the set of all polynomials of degree at most n. Moreover, it follows from [KW, Theorem 3.4] that every element of $\mathcal{H}(E)$ has order at most $\frac{1}{2}$.

Denote by P_0 the Blaschke product

$$P_0(z) := \prod_{n\geq 2} \frac{1 - z/in^2}{1 + z/in^2} .$$

Then the function

$$F_0(z) := P_0(z) \frac{E(z)}{1 + z/i}$$

satisfies $F_0^{\#} = F_0$ and belongs to $\mathcal{H}$. The zeros of F_0 are all simple and located at the points $\pm in^2$, $n \geq 2$.

Let c_n, $n \geq 2$, be a sequence of positive real numbers with $c_n \to 0$. Choose radii r_n, $n \geq 2$, with $r_n \leq 1$, such that

$$|F_0(z)| \leq c_n, \quad |z - in^2| \leq r_n .$$

Let $D \subseteq \mathbb{C}^+$ be any subset which contains the points in^2, $n \geq 2$, and define $\mathfrak{m} : D \to [0, \infty)$ by

$$\mathfrak{m}(z) := \begin{cases} c_n, & |z - in^2| \leq r_n, \\ |F_0(z)|, & \text{otherwise}, \end{cases} \qquad z \in D .$$

Then $\mathfrak{d}_{\mathfrak{m}} = 0$ and $F_0 \in R_{\mathfrak{m}}(\mathcal{H})$, hence $\mathfrak{m} \in \operatorname{Adm}_D \mathcal{H}$.

Assume that $F \in R_{\mathfrak{m}}(\mathcal{H})$ and F does not vanish in $\mathbb{C}^+$. Then, since F has order $\leq \frac{1}{2}$, it has the property that $|F(iy)|$ is a nondecreasing function of $y \geq 0$, see e.g. [Bo]. Since

$$\liminf_{\substack{y\to+\infty\\ iy\in D}} \mathfrak{m}(iy) \leq \lim_{n\to+\infty} c_n = 0\,,$$

and $|F(iy)| \leq C\mathfrak{m}(iy)$, $iy \in D$, for some $C > 0$, we obtain that also $\liminf_{y\to+\infty} |F(iy)| = 0$. Together with monotonicity this implies that $F(iy) = 0$, $y > 0$, and we have reached a contradiction.

It already follows that $R_{\mathfrak{m}}(\mathcal{H})$ cannot be invariant with respect to division by Blaschke products. We obtain even more information: Since $R_{\mathfrak{m}}(\mathcal{H})$ is always invariant with respect to division by polynomials whose zeros lie off the real axis, every nonzero element of $R_{\mathfrak{m}}(\mathcal{H})$ has infinitely many zeros in $\mathbb{C}^+$. In particular, $R_{\mathfrak{m}}(\mathcal{H}) \cap \mathbb{C}[z] = \{0\}$, and hence $\mathcal{R}_{\mathfrak{m}}(\mathcal{H}) = \mathcal{H}$.

From this example we see that in general the situation may be complicated. Still, under an additional hypothesis on the majorant $\mathfrak{m}$ positive results can be obtained. The crucial notion in this context is the following:

4.3 Definition. Let $D \subseteq \mathbb{C}^+$ be open, and let $\mathfrak{m} : D \to [0,\infty)$. We say that $\mathfrak{m}$ is log-*superharmonic*, if the function $(-\log \mathfrak{m})$ is subharmonic in D.

For log-superharmonic majorants the Phragmén-Lindelöf Principle, as stated in [RR, Theorem 6.2], can be applied. This gives the following theorem, which in turn leads to results on dividing out zeros of elements of $R_{\mathfrak{m}}(\mathcal{H})$.

4.4 Theorem. *Let D be an open subset of $\mathbb{C}^+$, and let $\mathfrak{m} \in \operatorname{Adm}_D \mathcal{H}$ be* log-*superharmonic. Moreover, let $F \in B_{\mathfrak{m}}(\mathcal{H}) \setminus \{0\}$ and let P be a Blaschke product for $\mathbb{C}^+$ with $\mathfrak{d}_P|_{\mathbb{C}^+} \leq \mathfrak{d}_F|_{\mathbb{C}^+}$. Denote by $\alpha \in [0,\infty]$ the supremum of all nonnegative real numbers α', such that*

$$\alpha' \Big|\frac{F(\zeta)}{P(\zeta)}\Big| \leq \liminf_{\substack{z\to\zeta\\ z\in D}} \mathfrak{m}(z), \quad \zeta \in \partial D \setminus \mathbb{R}\,. \tag{4.1}$$

Then we have

$$\Big|\frac{F(z)}{P(z)}\Big| \leq \max\{\alpha^{-1}, \|F\|_{\mathfrak{m}}\}\, \mathfrak{m}(z), \quad z \in D\,.$$

We always have

$$\alpha \geq \inf_{\zeta\in\partial D\setminus\mathbb{R}} |P(\zeta)|\,.$$

Proof.
Step 1: Let $F \in B_{\mathfrak{m}}(\mathcal{H})$, let P be a finite Blaschke product with $\mathfrak{d}_P|_{\mathbb{C}^+} \leq \mathfrak{d}_F|_{\mathbb{C}^+}$, and let $\alpha' \in (0,1]$ be such that (4.1) holds.

Consider the function

$$u(z) = -\log \mathfrak{m}(z) + \log \Big|\alpha' \frac{F(z)}{P(z)}\Big|, \quad z \in D\,,$$

where $u(z)$ is interpreted as $-\infty$ if $\mathfrak{m}(z) = \infty$ or $P(z)^{-1}F(z) = 0$. Note here that always $\mathfrak{m}(z) > 0$ since $z \in D \subseteq \mathbb{C}^+$. Then u is subharmonic in D. Note that, if $P(z) \neq 0$ for some $z \in D$, then we can also write

$$u(z) = -\log \mathfrak{m}(z) + \log|F(z)| + \log\alpha' - \log|P(z)|\,. \tag{4.2}$$

We write the boundary $\partial_\infty D$ of D in the extended complex plane $\mathbb{C}\cup\{\infty\}$ as the disjoint union of the two sets

$$R := \partial D \setminus \mathbb{R}, \qquad S := (\partial D\cap\mathbb{R})\cup\{\infty\}\,.$$

By our hypothesis (4.1), for $\zeta\in R$,

$$\lim_{\substack{z\to\zeta\\ z\in D}} \alpha'\Big|\frac{F(z)}{P(z)}\Big| = \alpha'\Big|\frac{F(\zeta)}{P(\zeta)}\Big| \leq \liminf_{\substack{z\to\zeta\\ z\in D}} \mathfrak{m}(z)\,.$$

and therefore

$$\limsup_{\substack{z\to\zeta\\ z\in D}} \big(-\log\mathfrak{m}(z)\big) \leq -\lim_{\substack{z\to\zeta\\ z\in D}} \log\Big(\alpha'\Big|\frac{F(z)}{P(z)}\Big|\Big)\,.$$

Since $\mathfrak{d}_\mathfrak{m}(\zeta)=0$ and $P^{-1}F$ is analytic at ζ, neither of these limits can be equal to $+\infty$. It follows that

$$\limsup_{\substack{z\to\zeta\\ z\in D}} u(z) = \limsup_{\substack{z\to\zeta\\ z\in D}} \big(-\log\mathfrak{m}(z)\big) + \lim_{\substack{z\to\zeta\\ z\in D}} \log\Big(\alpha'\Big|\frac{F(z)}{P(z)}\Big|\Big) \leq 0\,.$$

Next, let $\zeta\in S$, $\zeta\neq\infty$, and let $\epsilon>0$ be given. Since $\zeta\in\mathbb{R}$ there exists a neighbourhood U of ζ, such that

$$-\log|P(z)|<\epsilon,\quad z\in U\,.$$

Since $F\in B_\mathfrak{m}(\mathcal{H})$ and $\alpha'\leq 1$, we have

$$-\log\mathfrak{m}(z)+\log|F(z)|+\log\alpha'\leq 0,\quad z\in U\cap D\,.$$

Thus, by (4.2), $u(z)<\epsilon$ for $z\in U\cap D$.

Finally, consider $\zeta=\infty$ and let $\epsilon>0$ be given. Since P is a finite Blaschke product, there exists a neighbourhood U of ζ such that $-\log|P(z)|<\epsilon$, $z\in U\setminus\{\infty\}$. In the same way as above we now obtain that $u(z)<\epsilon$, $z\in U\cap D$.

We can apply the Phragmén-Lindelöf Principle, cf. [RR, Theorem 6.2], and conclude that $u(z)\leq 0$ throughout D, i.e. that

$$\alpha'\Big|\frac{F(z)}{P(z)}\Big|\leq\mathfrak{m}(z),\quad z\in D\,.$$

Since $\alpha'\leq 1$ and $\|F\|_\mathcal{H}\leq 1$, we also have $\|\alpha' P^{-1}F\|_\mathcal{H}\leq 1$. Thus $\alpha' P^{-1}F\in B_\mathfrak{m}(\mathcal{H})$.

Step 2: Let $F\in B_\mathfrak{m}(\mathcal{H})$, let P be an arbitrary Blaschke product with $\mathfrak{d}_P|_{\mathbb{C}^+}\leq\mathfrak{d}_F|_{\mathbb{C}^+}$, and let $\alpha'\in(0,1]$ be such that (4.1) holds.

For $N\in\mathbb{N}$ denote by P_N the N-th partial product of P. Since $|P_N(z)|\geq|P(z)|$, $z\in\mathbb{C}^+$, the number α' satisfies (4.1) also with P_N instead of P. Therefore, by Step 1, $\alpha' P_N^{-1}F\in B_\mathfrak{m}(\mathcal{H})$. Since $B_\mathfrak{m}(\mathcal{H})$ is weakly compact, there is a sequence $(N_k)_{k\in\mathbb{N}}$ and a function $G\in B_\mathfrak{m}(\mathcal{H})$, such that $\lim^w_{k\to\infty}\alpha' P_N^{-1}F=G$. Since weak convergence implies pointwise convergence, it follows that $G=\alpha' P^{-1}F$.

Step 3: Set

$$\alpha := \sup\left\{\alpha' \geq 0 : \text{(4.1) holds}\right\},$$

$$\beta := \max\left\{\beta' \geq 0 : \beta'\frac{F}{P} \in B_{\mathfrak{m}}(\mathcal{H})\right\}.$$

Then we have

$$1 \geq \left\|\beta\frac{F}{P}\right\|_{\mathfrak{m}} \geq \beta\|F\|_{\mathfrak{m}},$$

i.e. $\beta \leq \|F\|_{\mathfrak{m}}^{-1}$. Also, $\beta|P(z)^{-1}F(z)| \leq \mathfrak{m}(z)$, $z \in D$, and hence (4.1) holds with $\alpha' := \beta$, i.e. $\beta \leq \alpha$.

On the other hand, let $0 < \beta' < \min\{\alpha, \|F\|_{\mathfrak{m}}^{-1}\}$. Then $G := \|F\|_{\mathfrak{m}}^{-1}F \in B_{\mathfrak{m}}(\mathcal{H})$ and $\alpha' := \beta'\|F\|_{\mathfrak{m}}$ satisfies (4.1) for G. Moreover, $\alpha' < 1$, and hence

$$\beta'\frac{F}{P} = \alpha'\frac{G}{P} \in B_{\mathfrak{m}}(\mathcal{H}).$$

This shows that $\beta' \leq \beta$. We conclude that $\beta = \min\{\alpha, \|F\|_{\mathfrak{m}}^{-1}\}$.

Step 4: The estimate $\alpha \geq \inf_{\zeta\in\partial D}|P(\zeta)|$ follows since $F \in B_{\mathfrak{m}}(\mathcal{H})$ implies that $|F(\zeta)| \leq \liminf_{\substack{z\to\zeta\\ z\in D}} \mathfrak{m}(z)$, $\zeta \in \partial D$. ❑

Let us demonstrate in two particular situations how Theorem 4.4 can be applied.

4.5 Corollary. *Let $\mathfrak{m} \in \operatorname{Adm}_{\mathbb{C}^+}\mathcal{H}$ be* log-*superharmonic. If $F \in B_{\mathfrak{m}}(\mathcal{H})$ and P is a Blaschke product for $\mathbb{C}^+$ with $\mathfrak{d}_P|_{\mathbb{C}^+} \leq \mathfrak{d}_F|_{\mathbb{C}^+}$, then $P^{-1}F \in B_{\mathfrak{m}}(\mathcal{H})$.*

Proof. This follows from Theorem 4.4, since $\partial\mathbb{C}^+ = \mathbb{R}$ and thus (4.1) is trivially satisfied with $\alpha' = 1$. ❑

Combining this statement with Example 4.2 leads to the following observation.

4.6 Remark. Not for every majorant $\mathfrak{m}_0$ there exists a log-superharmonic majorant $\mathfrak{m} \in \operatorname{Adm}_{\mathbb{C}^+}\mathcal{H}$ which generates the same unit ball.

Note that this contrasts the situation which prevails for log-subharmonic majorants, i.e. such majorants $\mathfrak{m} \in \operatorname{Adm}_{\mathbb{C}^+}\mathcal{H}$ for which the function $\log\mathfrak{m}$ is subharmonic. Remember that for each majorant $\mathfrak{m}_0 \in \operatorname{Adm}\mathcal{H}$ the function $\mathfrak{m}_0^\flat|_{\mathbb{C}^+}$ is log-subharmonic and $B_{\mathfrak{m}_0}(\mathcal{H}) = B_{\mathfrak{m}_0^\flat|_{\mathbb{C}^+}}(\mathcal{H})$, cf. Lemma 3.3 and Lemma 3.4.

4.7 Corollary. *Let $\delta \in (0, \frac{\pi}{2})$, let D be the Stolz angle $D := \{z \in \mathbb{C} : \delta < \arg z < \pi - \delta\}$, and let $\mathfrak{m} \in \operatorname{Adm}_D\mathcal{H}$. If $F \in B_{\mathfrak{m}}(\mathcal{H})$ and P is a Blaschke product for $\mathbb{C}^+$ with $\mathfrak{d}_P|_{\mathbb{C}^+} \leq \mathfrak{d}_F|_{\mathbb{C}^+}$ and*

$$\sum_{w\in\mathbb{C}^+} \mathfrak{d}_P(w)\frac{\operatorname{Im} w}{|w|} < \infty, \tag{4.3}$$

then $P^{-1}F \in R_{\mathfrak{m}}(\mathcal{H})$.

Proof. We write

$$P(z) = \prod_{k=1}^{\infty}\frac{1 - z/z_k}{1 - z/\overline{z_k}}$$

with the zeros repeated according to their multiplicities. We have

$$2\log|P(z)| = \sum_{w\in\mathbb{C}^+} \mathfrak{d}_P(w)\log\left(1-\frac{4\operatorname{Im} z\operatorname{Im} w}{|z-\overline{w}|^2}\right) < \infty. \tag{4.4}$$

For each $\delta\in(0,\frac{\pi}{2})$ there exists a constant $C=C(\delta)>0$, such that $|z-\overline{w}|\geq C(|z|+|w|)$ for any $z\in D$, $w\in\mathbb{C}^+$. Now it follows from (4.3) and (4.4) that there exist $N=N(\delta)\in\mathbb{N}$ such that

$$P_N(z) := \prod_{k=N}^{\infty}\frac{1-z/z_k}{1-z/\overline{z_k}}.$$

satisfies $|B_N(z)|\geq 1/2$ for any $z\in D$. Hence $(P_N)^{-1}F\in R_{\mathfrak{m}}(\mathcal{H})$. ❑

5 Unit balls of log-superharmonic majorants

Our aim in the present section is to investigate which subsets B of a de Branges space $\mathcal{H}$ can be realized as $B_{\mathfrak{m}}(\mathcal{H})$ with some log-superharmonic majorant $\mathfrak{m}\in\operatorname{Adm}_{\mathbb{C}^+}\mathcal{H}$. It is an interesting result, that such balls can also be realized by majorization along $\mathbb{R}$ in conjunction with a restriction of exponential growth towards infinity, cf. Theorem 5.3. We will also discuss representability with more particular kinds of majorants which appear naturally in this context, namely $\mathcal{N}_+$-majorants and H^p-majorants, cf. Definition 5.1, Definition 5.13.

a. Representabilty by log-superharmonic majorants.

In the study of log-superharmonic majorants defined on all of $\mathbb{C}^+$ a smaller class of majorants plays a key role.

5.1 Definition. Let $\mathcal{H}$ be a de Branges space and choose $E\in\mathcal{H}B$ with $\mathcal{H}=\mathcal{H}(E)$. A function $\mathfrak{m}\in\operatorname{Adm}_{\mathbb{C}^+}\mathcal{H}$ is called an $\mathcal{N}_+$-majorant for $\mathcal{H}$, if it is of the form

$$\mathfrak{m}(z)=\left|e^{-iaz}f(z)E(z)\right|,\quad z\in\mathbb{C}^+,$$

with some $a\leq 0$ and f being outer for $\mathcal{N}$.

The fact whether or not a given function $\mathfrak{m}$ is an $\mathcal{N}_+$-majorant for $\mathcal{H}$ does not depend on the particular choice of the function E in Definition 5.1. This follows since, for two functions $E_1,E_2\in\mathcal{H}B$ with $\mathcal{H}(E_1)=\mathcal{H}(E_2)$, the quotient $E_1^{-1}E_2$ is outer.

5.2 Example. Let $\mathcal{H}=\mathcal{H}(E)$ be a de Branges space, and let $B\subseteq B(\mathcal{H})$, $B\neq\emptyset,\{0\}$. Assume that (3.1) holds, i.e. that $\int_{\mathbb{R}}[\log^+(|E|^{-1}\mathfrak{m}_B)](1+t^2)^{-1}\,dt<\infty$. Then the function $\mathfrak{m}_B^{\perp}$ is an $\mathcal{N}_+$-majorant. We have

$$B\subseteq B_{\mathfrak{m}_B^{\perp}}(\mathcal{H}).$$

These assertions are immediate from the respective definitions, and the fact that $\mathfrak{m}_B^{\perp}\succcurlyeq\mathfrak{m}_B$, cf. Lemma 3.9.

For each $\mathcal{N}_+$-majorant $\mathfrak{m}$, the function $\log\mathfrak{m}$ is harmonic in $\mathbb{C}^+$. Hence, in general, the set of all $\mathcal{N}_+$-majorants will be a fairly small subset of the set of all log-superharmonic majorants.

5.3 Theorem. *Let $\mathcal{H}$ be a de Branges space, and let $E \in \mathcal{H}B$ be such that $\mathcal{H} = \mathcal{H}(E)$. Moreover, let $B \subseteq B(\mathcal{H})$, $B \neq \emptyset, \{0\}$. Then the following are equivalent:*

(*i*) *There exists a* log*-superharmonic majorant* $\mathfrak{m} \in \operatorname{Adm}_{\mathbb{C}^+} \mathcal{H}$*, such that* $B = B_{\mathfrak{m}}(\mathcal{H})$.

(*ii*) *There exists an* $\mathcal{N}_+$*-majorant* $\mathfrak{m} \in \operatorname{Adm}_{\mathbb{C}^+} \mathcal{H}$*, such that* $B = B_{\mathfrak{m}}(\mathcal{H})$.

(*iii*) *There exists a subset* $D \subseteq \mathbb{R}$ *such that* $\mathbb{R} \setminus D$ *has measure zero, a number* $a \leq 0$*, and a measurable majorant* $\mathfrak{m} \in \operatorname{Adm}_D \mathcal{H}_{(a)}$ *satisfying*

$$\int_D \Big(\log^+ \frac{\mathfrak{m}(t)}{|E(t)|}\Big) \frac{dt}{1+t^2} < \infty \,,$$

such that $B = B_{\mathfrak{m}}(\mathcal{H}_{(a)})$.

(*iv*) *We have* $B = B_{\mathfrak{m}_B|_{\mathbb{R}}}(\mathcal{H}_{(\operatorname{mt}_{\mathcal{H}} B)})$ *and*

$$\int_D \Big(\log^+ \frac{\mathfrak{m}_B(t)}{|E(t)|}\Big) \frac{dt}{1+t^2} < \infty \,. \tag{5.1}$$

Proof. First of all note that the implications (*ii*) $\Rightarrow$ (*i*) and (*iv*) $\Rightarrow$ (*iii*) are trivial. We will carry out the proof by showing the implications

$$(i) \Rightarrow (ii) \Rightarrow (iv), \quad (iii) \Rightarrow (ii) \,.$$

Step 1: Assume that $\mathfrak{m} \in \operatorname{Adm}_{\mathbb{C}^+} \mathcal{H}$ is log-superharmonic. In this step we will show that then the subharmonic function $\log(|E(z)|^{-1}\mathfrak{m}^\flat(z))$, $z \in \mathbb{C}^+$, has a nonnegative harmonic majorant.

Denote $\mathcal{F} := \{F \in B_{\mathfrak{m}}(\mathcal{H}) : F \text{ zerofree in } \mathbb{C}^+\}$. We have, for any $F \in \mathcal{F}$,

$$\log \Big|\frac{F}{E}\Big| \leq \log \frac{\mathfrak{m}^\flat}{|E|} \leq \log \frac{\mathfrak{m}}{|E|}$$

throughout the half-plane $\mathbb{C}^+$. Since, by Corollary 4.5, $\mathcal{F}$ is nonempty, this tells us in particular that the superharmonic function $\log(|E|^{-1}\mathfrak{m})$ has a harmonic minorant. Thus it has the largest harmonic minorant, which we denote by h. For each element $F \in \mathcal{F}$, the function $\log(|E|^{-1}|F|)$ is a harmonic minorant of $\log(|E|^{-1}\mathfrak{m})$, and hence does not exceed h. Lemma 3.6 implies that also $\log(|E|^{-1}\mathfrak{m}^\flat) \leq h$. This tells us that the subharmonic function $\log(|E|^{-1}\mathfrak{m}^\flat)$ has a harmonic majorant. Thus it has the least harmonic majorant, which we denote by $h^\flat$. Clearly, $h^\flat \leq h$, and altogether

$$\log \Big|\frac{F}{E}\Big| \leq \log \frac{\mathfrak{m}^\flat}{|E|} \leq h^\flat \leq h \leq \log \frac{\mathfrak{m}}{|E|}, \quad F \in \mathcal{F} \,. \tag{5.2}$$

Fix $F_0 \in \mathcal{F}$. Since $E^{-1}F_0 \in H^2 \subseteq \mathcal{N}$, the subharmonic function $\log^+ |E^{-1}F_0|$ has a harmonic majorant, let us denote one such function by k. Define

$$k^\flat := h^\flat + k - \log \Big|\frac{F_0}{E}\Big| \,.$$

Then $k^\flat$ is harmonic, and

$$k^\flat = h^\flat + \underbrace{\Big(k - \log\big|\frac{F_0}{E}\big|\Big)}_{\geq 0} \geq \log\frac{\mathfrak{m}^\flat}{|E|}, \quad k^\flat = \underbrace{\Big(h^\flat - \log\big|\frac{F_0}{E}\big|\Big)}_{\geq 0} + k \geq 0\,.$$

We see that $k^\flat$ is a nonnegative harmonic majorant of $\log(|E|^{-1}\mathfrak{m}^\flat)$.

Step 2: Assume that $\mathfrak{m} \in \operatorname{Adm}_{\mathbb{C}^+}\mathcal{H}$ is log-superharmonic. In this step we determine the least harmonic majorant $h^\flat$ of $\log(|E|^{-1}\mathfrak{m}^\flat)$ explicitly.

In Step 1 we showed that $\log(|E|^{-1}\mathfrak{m}^\flat)$ has a nonnegative harmonic majorant. Hence, the least harmonic majorant $h^\flat$ of this function can be represented as a Poisson integral:

$$h^\flat(z) = ay + \frac{y}{\pi}\int_{\mathbb{R}} \frac{d\mu(t)}{(t-x)^2+y^2}, \quad z = x+iy \in \mathbb{C}^+\,,$$

where $a \in \mathbb{R}$ and μ is a real (signed) Borel measure on $\mathbb{R}$ which satisfies $\int_{\mathbb{R}}(1+t^2)^{-1}d|\mu|(t) < \infty$. We employ the Theorem of Szegö–Solomentsev to determine the data a, μ, see e.g. [RR, Theorem 3.13] where the version for the unit disk is stated.

Since $|E|^{-1}\mathfrak{m}^\flat$ has a continuous extension to the closed half-plane $\mathbb{C}^+\cup\mathbb{R}$, the Radon–Nikodym derivative of μ with respect to the Lebesgue measure is equal to $\log(|E|^{-1}\mathfrak{m}^\flat)$, see [RR, Theorem 3.13, (iv)]. Since $|E|^{-1}\mathfrak{m}^\flat$ does not vanish on $\mathbb{R}$ with possible exception of a discrete set where it tends to zero polynomially, the measure μ is absolutely continuous with respect to the Lebesgue measure. To see this, use [RR, Theorem 3.13, (ii)] e.g. with the function $\varphi(t) := t^2$.

Set $B := B_{\mathfrak{m}}(\mathcal{H})$. We just showed that $d\mu = \log(|E|^{-1}\mathfrak{m}^\flat)\,dt$, in particular the integral $\int_{\mathbb{R}}|\log(|E|^{-1}\mathfrak{m}^\flat)|(1+t^2)^{-1}dt$ converges. Hence, the function $\mathfrak{m}_B^\perp$ is well-defined. Moreover, $\log(|E|^{-1}\mathfrak{m}_B^\perp)$ is harmonic and greater or equal to $\log(|E|^{-1}\mathfrak{m}^\flat)$, cf. Lemma 3.9, (ii). Since $h^\flat$ is the least harmonic majorant of $\log(|E|^{-1}\mathfrak{m}^\flat)$, this implies $\log(|E|^{-1}\mathfrak{m}_B^\perp) \geq h^\flat$. Using Lemma 3.3, (iv), and the definition of $\mathfrak{m}_B^\perp$, it follows that

$$\operatorname{mt}_{\mathcal{H}} B = \operatorname{mt}\frac{\mathfrak{m}^\flat}{|E|} \leq \underbrace{\operatorname{mt}\big[\exp(h^\flat)\big]}_{=a} \leq \operatorname{mt}\frac{\mathfrak{m}_B^\perp}{|E|} = \operatorname{mt}_{\mathcal{H}} B\,,$$

i.e. $a = \operatorname{mt}_{\mathcal{H}} B$. Altogether, we obtain that

$$h^\flat = \log\frac{\mathfrak{m}_B^\perp}{|E|}\,. \tag{5.3}$$

Step 3, $(i) \Rightarrow (ii)$: It is now easy to establish this implication. Let B be given, and assume that $\mathfrak{m} \in \operatorname{Adm}_{\mathbb{C}^+}\mathcal{H}$ is log-superharmonic and satisfies $B = B_{\mathfrak{m}}(\mathcal{H})$. By (5.2) and (5.3), we have

$$\mathfrak{m}^\flat \leq |E|\cdot\exp(h^\flat) = \mathfrak{m}_B^\perp \leq \mathfrak{m}\,, \tag{5.4}$$

and it follows that $B_{\mathfrak{m}}(\mathcal{H}) = B_{\mathfrak{m}_B^\perp}(\mathcal{H})$.

Step 4: Let $\mathfrak{m} : \mathbb{C}^+ \to [0,\infty)$ be a function of the form

$$\mathfrak{m}(z) = \big|e^{-iaz}f(z)E(z)\big|, \quad z \in \mathbb{C}^+\,,$$

where f is outer, and $a = \operatorname{mt}_{\mathcal{H}} \mathfrak{m} \leq 0$. Denote by $D \subseteq \mathbb{R}$ the set of all points $x \in \mathbb{R}$ such that the nontangential limit $f^*(x) := \lim_{z \hat{\to} x} f(z)$ exists and that $E(x) \neq 0$, and set

$$\mathfrak{m}^*(x) := \lim_{z \hat{\to} x} \mathfrak{m}(z) = |f^*(x)E(x)|, \quad x \in D. \tag{5.5}$$

Our aim in this step is to show that

$$B_{\mathfrak{m}}(\mathcal{H}) = B_{\mathfrak{m}^\flat|_{\mathbb{R}}}(\mathcal{H}_{(b)}) = B_{\mathfrak{m}^*}(\mathcal{H}_{(b)}), \quad b \in \big[\operatorname{mt}_{\mathcal{H}} B_{\mathfrak{m}}(\mathcal{H}), \operatorname{mt}_{\mathcal{H}} \mathfrak{m}\big]. \tag{5.6}$$

Thereby, the inclusion '$\subseteq$' in the first asserted equality holds since $b \geq \operatorname{mt}_{\mathcal{H}} B_{\mathfrak{m}}(\mathcal{H})$. The inclusion '$\subseteq$' in the second equality follows from

$$\mathfrak{m}^\flat(x) = \lim_{z \hat{\to} x} \mathfrak{m}^\flat(z) \leq \lim_{z \hat{\to} x} \mathfrak{m}(z) = \mathfrak{m}^*(x), \quad x \in D, \tag{5.7}$$

i.e. $\mathfrak{m}^\flat|_{\mathbb{R}} \preccurlyeq \mathfrak{m}^*$. Let $F \in B_{\mathfrak{m}^*}(\mathcal{H}_{(b)})$ be given, and consider the function

$$g(z) := e^{iaz} \frac{F(z)}{E(z)}, \quad z \in \mathbb{C}^+ \cup \mathbb{R}.$$

Note that g is continuous on $\mathbb{C}^+ \cup \mathbb{R}$. Since $F \in \mathcal{H}_{(b)} \subseteq \mathcal{H}$, we have $E^{-1}F \in H^2$ and $\operatorname{mt}(E^{-1}F) \leq b \leq a$. It follows that $g \in \mathcal{N}_+$. Along the real axis we have

$$|g(x)| \leq \frac{\mathfrak{m}^*(x)}{|E(x)|} = |f^*(x)|, \quad x \in D.$$

Since f is outer, the Smirnov Maximum Principle implies that the inequality $|g(z)| \leq |f(z)|$ prevails throughout the half-plane $\mathbb{C}^+$. This just says that

$$|F(z)| \leq |e^{-iaz}E(z)| \cdot |f(z)| = \mathfrak{m}(z), \quad z \in \mathbb{C}^+,$$

and we obtain that $F \in B_{\mathfrak{m}}(\mathcal{H})$. The proof of (5.6) is complete.

Step 5, (ii) $\Rightarrow$ (iv): Again, this implication is now easy to see. Let B be given, and assume that $\mathfrak{m} \in \operatorname{Adm}_{\mathbb{C}^+} \mathcal{H}$ is an $\mathcal{N}_+$-majorant with $B = B_{\mathfrak{m}}(\mathcal{H})$. What we have shown in Step 4 applies to $\mathfrak{m}$. The first equality in (5.6) with $b = \operatorname{mt}_{\mathcal{H}} B_{\mathfrak{m}}(\mathcal{H})$, however, is just the first condition in (*iv*). Moreover, we see from (5.7) and the fact that $\mathbb{R} \setminus D$ has measure zero, that

$$\int_{\mathbb{R}} \Big(\log^+ \frac{\mathfrak{m}^\flat(t)}{|E(t)|} \Big) \frac{dt}{1+t^2} \leq \int_D \Big(\log^+ \frac{\mathfrak{m}^*(t)}{|E(t)|} \Big) \frac{dt}{1+t^2} = \int_D \big[\log^+ f^*(t) \big] \frac{dt}{1+t^2} < \infty.$$

Step 6, (iii) $\Rightarrow$ (ii): Assume that $B = B_{\mathfrak{m}}(\mathcal{H}_{(a)})$ with data $\mathfrak{m}, a$ as stated in (*iii*). Since $\mathfrak{m}^\flat(x) \leq \mathfrak{m}(x)$, $x \in D$, and $\mathbb{R} \setminus D$ has measure zero, we have

$$\int_{\mathbb{R}} \Big(\log^+ \frac{\mathfrak{m}^\flat(t)}{|E(t)|} \Big) \frac{dt}{1+t^2} \leq \int_D \Big(\log^+ \frac{\mathfrak{m}(t)}{|E(t)|} \Big) \frac{dt}{1+t^2} < \infty.$$

Hence, the function $\mathfrak{m}_B^\perp$ is well-defined and an $\mathcal{N}_+$-majorant. What we have shown in Step 4 thus applies to $\mathfrak{m}_B^\perp$. The relation (5.6) with $b = \operatorname{mt}_{\mathcal{H}} \mathfrak{m}_B^\perp = \operatorname{mt}_{\mathcal{H}} B$ gives

$$B_{\mathfrak{m}_B^\perp}(\mathcal{H}) = B_{(\mathfrak{m}_B^\perp)^*}(\mathcal{H}_{(\operatorname{mt}_{\mathcal{H}} B)}).$$

However, by Lemma 3.9, (i), we have $(\mathfrak{m}_B^{\perp})^* = \mathfrak{m}^\flat|_{\mathbb{R}}$. Moreover, since the domain of $\mathfrak{m}$ is contained in $\mathbb{R}$, it follows from [BW1, Theorem 3.4] that $\operatorname{mt}_{\mathcal{H}_{(a)}} B = 0$. Hence,

$$\operatorname{mt}_{\mathcal{H}} B = \operatorname{mt}_{\mathcal{H}} \mathcal{H}_{(a)} = a\,, \tag{5.8}$$

and we conclude that

$$B = B_{\mathfrak{m}}(\mathcal{H}_{(a)}) = B_{\mathfrak{m}^\flat|_{\mathbb{R}}}(\mathcal{H}_{(a)}) = B_{\mathfrak{m}_B^{\perp}}(\mathcal{H})\,.$$

Here, for the second equality, we have used Lemma 3.4. ❑

Let us explicitly point out the following facts, which have been established in the course of the proof of Theorem 5.3. For a subset B of the unit ball of $\mathcal{H}$ denote

$$M^{\log}(B) := \big\{\mathfrak{m} \in \operatorname{Adm}_{\mathbb{C}^+} \mathcal{H} :\ \mathfrak{m} \text{ log-superharmonic, } B_{\mathfrak{m}}(\mathcal{H}) = B\big\}\,.$$

5.4 Corollary. *Let $\mathcal{H}$ be a de Branges space, let $B \subseteq B(\mathcal{H})$, $B \neq \emptyset, \{0\}$, and assume that $M^{\log}(B) \neq \emptyset$. Then $\mathfrak{m}_B^{\perp}$ is well-defined, belongs to $M^{\log}(B)$, and is the smallest element of this set. In particular, $M^{\log}(B)$ possesses a smallest element, and this element is an $\mathcal{N}_+$-majorant.*

Proof. The fact that $\mathfrak{m}_B^{\perp}$ is well-defined and belongs to $M^{\log}(B)$ was established in Steps 2 and 3 of the proof of Theorem 5.3. The relation (5.4) gives that $\mathfrak{m}_B^{\perp} \leq \mathfrak{m}$ whenever $\mathfrak{m} \in M^{\log}(B)$, hence $\mathfrak{m}_B^{\perp}$ is the smallest element of this set. ❑

5.5 Corollary. *Let $\mathcal{H}$ be a de Branges space, and let $E \in \mathcal{H}B$ be such that $\mathcal{H} = \mathcal{H}(E)$. Then*

$$\big\{B_{\mathfrak{m}}(\mathcal{H}) :\ \mathfrak{m} \in \operatorname{Adm}_{\mathbb{C}^+} \mathcal{H},\ \mathfrak{m} \text{ log-}\textit{superharmonic}\big\} = \\ = \left\{ B_{\mathfrak{m}}(\mathcal{H}_{(a)}) :\ \begin{array}{l} a \leq 0,\ \mathfrak{m} \in \operatorname{Adm}_{\mathbb{R}} \mathcal{H}_{(a)},\ \mathfrak{m} \textit{ continuous,} \\ \int_{\mathbb{R}} (\log^+ \frac{\mathfrak{m}(t)}{|E(t)|}) \frac{dt}{1+t^2} < \infty \end{array} \right\}$$

Proof. The inclusion '$\subseteq$' holds because of Theorem 5.3, $(i) \Rightarrow (iv)$. The converse inclusion follows from the implication $(iii) \Rightarrow (i)$. ❑

5.6 Corollary. *Let $\mathcal{H}$ be a de Branges space, and let $\mathfrak{m} \in \operatorname{Adm}_{\mathbb{C}^+} \mathcal{H}$ be an $\mathcal{N}_+$-majorant. Then*

$$B_{\mathfrak{m}}(\mathcal{H}) = B_{\mathfrak{m}^*}(\mathcal{H}_{(\operatorname{mt}_{\mathcal{H}} \mathfrak{m})}) \quad \textit{and} \quad \operatorname{mt}_{\mathcal{H}} B_{\mathfrak{m}}(\mathcal{H}) = \operatorname{mt}_{\mathcal{H}} \mathfrak{m}\,.$$

Proof. These assertions follow immediately from (5.6) and (5.8). ❑

In order to gain in logical clarity, we wish to make the following point explicit.

5.7 Remark. In Lemma 4.1 and Corollary 4.5 we had obtained examples for unit balls $B_{\mathfrak{m}}(\mathcal{H})$ which are invariant with respect to division by Blaschke products. Theorem 5.3 implies that the set of unit balls described in (the almost trivial) Lemma 4.1 already includes the set of unit balls described by (the nontrivial) Corollary 4.5. However, this does not make Corollary 4.5 superfluous; it was essentially needed for the proof of Theorem 5.3.

In this context, let us note that the class given by Lemma 4.1 is generically even strictly larger than the class of balls generated by log-superharmonic majorants.

5.8 Example. Let $E \in \mathcal{H}B$ have no real zeros, and assume that there exists a de Branges subspace $\mathcal{L}_0$ of $\mathcal{H}(E)$ which is not of the form $\mathcal{H}_{(a)}$. Consider the majorant

$$\mathfrak{m}(x) := \nabla_{\mathcal{H}(E)}(x), \quad x \in \mathbb{R}.$$

Then, for each subspace $\mathcal{L}$ of $\mathcal{H}(E)$, we have $B_{\mathfrak{m}}(\mathcal{L}) = B(\mathcal{L})$. In particular, the ball $B_{\mathfrak{m}}(\mathcal{L}_0)$ is not of the form $B_{\mathfrak{m}}(\mathcal{H}_{(a)})$ with some $a \leq 0$.

b. The condition (5.1)**.**

The first condition in Theorem 5.3, *(iv)*, can be viewed as a geometric requirement, whereas the second one, i.e. (5.1), is an analytic condition. We will in the present subsection show that, generically, (5.1) plays only a minor role. Actually, it is satisfied in "most" cases. First let us show when the integral condition in Corollary 5.5 is superfluous.

5.9 Lemma. *Let $\mathcal{H}$ be a de Branges space, choose $E \in \mathcal{H}B$ with $\mathcal{H} = \mathcal{H}(E)$, and denote by φ_E a phase function of E. Then*

$$\big\{B_{\mathfrak{m}}(\mathcal{H}) : \ \mathfrak{m} \in \operatorname{Adm}_{\mathbb{C}^+} \mathcal{H},\ \mathfrak{m} \text{ log-}\textit{superharmonic}\big\} = \\ = \big\{B_{\mathfrak{m}}(\mathcal{H}_{(a)}) : \ a \leq 0,\ \mathfrak{m} \in \operatorname{Adm}_{\mathbb{R}} \mathcal{H}_{(a)},\ \mathfrak{m} \textit{ continuous}\big\}$$

if and only if

$$\int_{\mathbb{R}} \big[\log^+ \varphi_E'(t)\big] \frac{dt}{1+t^2} < \infty. \tag{5.9}$$

Proof. By the formula for $\nabla_{\mathcal{H}}$ given in 2.9, we have $|E|^{-1}\nabla_{\mathcal{H}} = (\sqrt{\pi})^{-1}\sqrt{\varphi_E'}$ along the real axis. Hence,

$$\frac{1}{2}\log^+ \varphi_E'(t) - \log\sqrt{\pi} \leq \log^+ \frac{\nabla_{\mathcal{H}}(t)}{|E(t)|} \leq \frac{1}{2}\log^+ \varphi_E'(t), \quad t \in \mathbb{R},$$

and we conclude that convergence of the integral (5.9) is equivalent to

$$\int_{\mathbb{R}} \Big(\log^+ \frac{\nabla_{\mathcal{H}}(t)}{|E(t)|}\Big) \frac{dt}{1+t^2} < \infty. \tag{5.10}$$

Assume that the stated equality of sets holds. The unit ball of $\mathcal{H}$ can be written as $B(\mathcal{H}) = B_{\nabla_{\mathcal{H}}|_{\mathbb{R}}}(\mathcal{H})$. Since $\mathfrak{m}_{B(\mathcal{H})} = \nabla_{\mathcal{H}}$, the convergence of the integral in (5.10) follows from Theorem 5.3.

Conversely, assume that (5.10) holds, and let a continuous majorant $\mathfrak{m} \in \operatorname{Adm}_{\mathbb{R}} \mathcal{H}_{(a)}$ be given. Since $B_{\mathfrak{m}}(\mathcal{H}_{(a)}) = B_{\mathfrak{m}^\flat|_{\mathbb{R}}}(\mathcal{H}_{(a)})$, and $\mathfrak{m}^\flat \leq \nabla_{\mathcal{H}}$, we obtain from Theorem 5.3 that $B_{\mathfrak{m}}(\mathcal{H}_{(a)})$ can be represented with some log-superharmonic majorant. This shows the inclusion '$\supseteq$' in the stated equality. The inclusion '$\subseteq$' holds in any case. ❑

In view of this fact, it is interesting to see that convergence of the integral in (5.9) follows from a mild condition on the density of zeros of E.

5.10 Proposition. *Let $E \in \mathcal{H}B$ and denote by $n(t)$ the counting function for the nonreal zeros of E, i.e. let $n(t)$ be the number of nonreal zeros of E located in the disk $\{z \in \mathbb{C} : |z| \leq t\}$ counted according to their multiplicities. If*

$$\int_1^\infty \frac{\log n(t)}{t^2}\, dt < \infty, \tag{5.11}$$

then (5.9) *holds.*

Proof. Denote by $z_n = x_n - iy_n$, $n = 1, 2, \ldots$, the (finite or infinite) sequence of nonreal zeros of E, and set $c := -2^{-1}\,\mathrm{mt}(E^{-1}E^{\#})$. Moreover, let φ_E be a phase function of E. Then

$$\sum_n \operatorname{Im}\frac{1}{z_n} = \sum_n \frac{y_n}{|z_n|^2} < \infty \quad\text{and}\quad \varphi_E'(t) = c + \sum_n \frac{y_n}{|t-z_n|^2}\,.$$

Let $k \in \mathbb{N}$ be fixed. Then, for $t \in [k-1,k]$ and $|z| > 2k$, we have

$$\frac{|\operatorname{Im} z|}{|t-z|^2} = \frac{|\operatorname{Im} z|}{|z|^2}\cdot\frac{1}{|1-\frac{t}{z}|^2} \leq 2\frac{|\operatorname{Im} z|}{|z|^2}\,.$$

Thus

$$\sum_{|z_n|>2k}\frac{y_n}{|t-z_n|^2} \leq 2\sum_{|z_n|>2k}\frac{y_n}{|z_n|^2} \leq 2\sum_n\frac{y_n}{|z_n|^2}\,.$$

Set $A := 1 + c + 2\sum_n |z_n|^{-2}y_n$, then for $t \in [k-1,k]$ we have

$$\log^+\varphi_E'(t) \leq \log\big(1+\varphi_E'(t)\big) \leq \log\Big(1+c+\sum_{|z_n|>2k}\frac{y_n}{|t-z_n|^2}+\sum_{|z_n|\leq 2k}\frac{y_n}{|t-z_n|^2}\Big) \leq$$

$$\leq \log\Big(A+\sum_{|z_n|\leq 2k}\frac{y_n}{|t-z_n|^2}\Big)\,.$$

By the Jensen inequality, we have

$$\int_{k-1}^{k}\log^+\varphi_E'(t)\,dt \leq \int_{k-1}^{k}\log\Big(A+\sum_{|z_n|\leq 2k}\frac{y_n}{|t-z_n|^2}\Big)\,dt \leq$$

$$\leq \log\int_{k-1}^{k}\Big(A+\sum_{|z_n|\leq 2k}\frac{y_n}{|t-z_n|^2}\Big)\,dt \leq \log\big(A+\pi n(2k)\big)\,.$$

Here we have used the fact that

$$\int_{k-1}^{k}\frac{y_n}{|t-z_n|^2}\,dt \leq \int_{\mathbb{R}}\frac{y_n}{|t-z_n|^2}\,dt = \pi\,.$$

Now we can estimate

$$\int_0^\infty\big[\log^+\varphi_E'(t)\big]\frac{dt}{1+t^2} = \sum_{k=1}^{\infty}\int_{k-1}^{k}\big[\log^+\varphi_E'(t)\big]\frac{dt}{1+t^2} \leq$$

$$\leq \sum_{k=1}^{\infty}\frac{1}{1+(k-1)^2}\int_{k-1}^{k}\log^+\varphi_E'(t)\,dt \leq \sum_{k=1}^{\infty}\frac{\log(A+\pi n(2k))}{1+(k-1)^2}\,.$$

Our hypothesis (5.11) implies that the last series converges. The integral $\int_{-\infty}^{0}[\log^+\varphi_E'(t)](1+t^2)^{-1}dt$ can be estimated in the same way, and we obtain that (5.9) holds. ❑

5.11 Corollary. *If $E \in \mathcal{H}B$ is of finite order, then the condition* (5.11) *holds true. Hence, for de Branges spaces $\mathcal{H} = \mathcal{H}(E)$ where E is of finite order, the analytic condition in* (5.1) *is always satisfied.* ❑

5.12 Example. Although the condition (5.11) is not necessary for (5.9) to hold, the following example shows that it is pretty sharp. Let

$$z_n = \ln|n| + \frac{i}{|n|\ln^2|n|}, \quad n \in \mathbb{Z},\ n \neq 0\,.$$

Then $c_1 e^t \leq n(t) \leq c_2 e^t$ with some constants $c_1, c_2 > 0$, and thus the integral in (5.11) "just" diverges. Let $t \in [\ln k, \ln k + 1]$, $k > 1$. Then

$$\varphi_E'(t) > \sum_{k/2 \leq n \leq k-1} \frac{1}{n \ln^2 n \Big[(\ln n - \ln k)^2 + n^{-2}\ln^{-4} n\Big]}\,.$$

Note that $\ln k - \ln n \asymp \frac{k-n}{n} > \frac{1}{n\ln^2 n}$. Then

$$\varphi_E'(t) > C \sum_{k/2 \leq n \leq k-1} \frac{n}{(k-n)^2 \ln^2 n} \geq C_1 \frac{k}{\ln^2 k} \geq C_2 t^{-2} e^t\,,$$

and the log-integral diverges.

c. Hardy class majorants.

Let $\mathcal{H} = \mathcal{H}(E)$ be a de Branges space, and let $\mathcal{L} = \mathcal{H}(E_1)$ be one of its de Branges subspaces. A standard majorant, which was already used in [BW1] and investigated in greater detail in [BW3], is the function

$$\mathfrak{m}_{E_1}(z) := |z+i|^{-1}|E_1(z)|, \quad z \in \mathbb{C}^+ \cup \mathbb{R}\,.$$

This majorant reproduces the space $\mathcal{L}$ in the sense that $\mathcal{R}_{\mathfrak{m}_{E_1}}(\mathcal{H}) = \mathcal{L}$. Since we can write

$$\mathfrak{m}_{E_1}(z) = \Big| e^{-i\,\mathrm{mt}[(E^{-1}E_1)]z} \cdot e^{i[\mathrm{mt}(E^{-1}E_1)]z} \frac{E_1(z)}{(z+i)E(z)} \cdot E(z) \Big|, \quad z \in \mathbb{C}^+\,, \tag{5.12}$$

this function is an $\mathcal{N}_+$-majorant. Note here that $E^{-1}E_1$ is outer for $\mathcal{N}$, since it has a continuous extension to $\mathbb{C}^+ \cup \mathbb{R}$ which vanishes only on a discrete set and tends to zero at most polynomially, see once more [RR, Theorem 3.13]. However, the function

$$f(z) := e^{i[\mathrm{mt}(E^{-1}E_1)]z} \frac{E_1(z)}{(z+i)E(z)}$$

is not only outer for $\mathcal{N}$, but actually outer for H^2.

5.13 Definition. Let $\mathcal{H}$ be a de Branges space and choose $E \in \mathcal{H}B$ with $\mathcal{H} = \mathcal{H}(E)$. Moreover, let $p \in (0,\infty)$. A function $\mathfrak{m} \in \mathrm{Adm}_{\mathbb{C}^+}\mathcal{H}$ is called an H^p-majorant for $\mathcal{H}$, if it is of the form

$$\mathfrak{m}(z) = \big| e^{-iaz} f(z) E(z) \big|, \quad z \in \mathbb{C}^+\,,$$

with some $a \leq 0$ and f being outer for H^p.

Since $\mathcal{H}(E_1) = \mathcal{H}(E_2)$ implies not only that $E_1^{-1}E_2$ is outer, but also that this quotient is bounded above and bounded away from zero throughout the half-plane $\mathbb{C}^+$, this notion does not depend on the particular choice of E in Definition 5.13.

For a subset B of the unit ball of $\mathcal{H}$ denote

$$M^p(B) := \big\{\mathfrak{m} \in \operatorname{Adm}\mathcal{H} : \mathfrak{m} \text{ is a } H^p\text{-majorant, } B_{\mathfrak{m}}(\mathcal{H}) = B\big\}\,.$$

Clearly, $M^p(B) \subseteq M^{\log}(B)$. The analogue of Theorem 5.3, including Corollary 5.4, corresponding to the set $M^p(B)$ now reads as follows.

5.14 Proposition. *Let $\mathcal{H}$ be a de Branges space, let B be a nonempty subset of its unit ball, and let $p \in (0,\infty)$. Then the following are equivalent:*

- (i) *$M^p(B) \neq \emptyset$, i.e. there exists an H^p-majorant $\mathfrak{m} \in \operatorname{Adm}_{\mathbb{C}^+}\mathcal{H}$, such that $B = B_{\mathfrak{m}}(\mathcal{H})$.*
- (ii) *There exists a subset $D \subseteq \mathbb{R}$ such that $\mathbb{R}\setminus D$ has measure zero, a number $a \leq 0$, and a measurable majorant $\mathfrak{m} \in \operatorname{Adm}_D \mathcal{H}_{(a)}$ satisfying*
$$\int_D \Big(\frac{\mathfrak{m}(t)}{|E(t)|}\Big)^p\,dt < \infty\,, \tag{5.13}$$
such that $B = B_{\mathfrak{m}}(\mathcal{H}_{(a)})$.
- (iii) *We have $B = B_{\mathfrak{m}_B|_{\mathbb{R}}}(\mathcal{H}_{(\operatorname{mt}_{\mathcal{H}} B)})$ and $\int_{\mathbb{R}}\big(\frac{\mathfrak{m}_B(t)}{|E(t)|}\big)^p\,dt < \infty$.*

In this case, the set $M^p(B)$ contains a smallest element. Actually, the majorant $\mathfrak{m}_B^{\perp}$ is an H^p-majorant.

Proof. Assume that (i) holds, and let $\mathfrak{m} \in \operatorname{Adm}_{\mathbb{C}^+}\mathcal{H}$ be an H^p-majorant with $B = B_{\mathfrak{m}}(\mathcal{H})$. Since $\mathfrak{m}$ is in particular log-superharmonic, Theorem 5.3 yields that $B = B_{\mathfrak{m}_B|_{\mathbb{R}}}(\mathcal{H}_{(\operatorname{mt}_{\mathcal{H}} B)})$. Let $\mathfrak{m}^*$ be the boundary function of $\mathfrak{m}$, cf. (5.5). Then $|E|^{-1}\mathfrak{m}^* \in L^p(dt)$, and we conclude from $\mathfrak{m}_B \preccurlyeq \mathfrak{m}^*$ that also $|E|^{-1}\mathfrak{m}_B \in L^p(dt)$.

The implication $(iii) \Rightarrow (ii)$ is trivial. It remains to show that (ii) implies (i). To this end assume that $B = B_{\mathfrak{m}}(\mathcal{H}_{(a)})$ with data $\mathfrak{m}, a$ as in (ii). Then, by Theorem 5.3, $M^{\log}(B) \neq \emptyset$, and we obtain from Corollary 5.4 that $B = B_{\mathfrak{m}_B^{\perp}}(\mathcal{H})$. Since $\mathfrak{m}_B \preccurlyeq \mathfrak{m}$, convergence of the integral (5.13) implies that $(|E|^{-1}\mathfrak{m}_B)|_{\mathbb{R}} \in L^p(dt)$. This, however, says that the outer function $\mathfrak{f}_{|E|^{-1}\mathfrak{m}_B}$ used in the definition of $\mathfrak{m}_B^{\perp}$ is outer for H^p, i.e. $\mathfrak{m}_B^{\perp}$ is an H^p-majorant. ❑

Let $\mathfrak{m}_0 \in \operatorname{Adm}\mathcal{H}$, choose $E_1 \in \mathcal{H}B$ with $\mathcal{R}_{\mathfrak{m}_0}(\mathcal{H}) = \mathcal{H}(E_1)$, and let $\mathfrak{m}_{E_1}$ be the H^2-majorant defined in (5.12). Then $\mathcal{R}_{\mathfrak{m}_0}(\mathcal{H}) = \mathcal{R}_{\mathfrak{m}_{E_1}}(\mathcal{H})$. But, of course, the balls $B_{\mathfrak{m}_0}(\mathcal{H})$ and $B_{\mathfrak{m}_{E_1}}(\mathcal{H})$ will in general by no means be comparable. The next statement shows that, at least as far as majorization along $\mathbb{R}$ is concerned, an H^2-majorant realizing $\mathcal{R}_{\mathfrak{m}_0}(\mathcal{H})$ can be choosen to be small.

5.15 Proposition. *Let $a \leq 0$ and $\mathfrak{m}_0 \in \operatorname{Adm}_{\mathbb{R}}\mathcal{H}_{(a)}$ be given. Then there exists an H^2-majorant $\mathfrak{m} \in \operatorname{Adm}\mathcal{H}$, such that*

$$\mathcal{R}_{\mathfrak{m}}(\mathcal{H}) = \mathcal{R}_{\mathfrak{m}_0}(\mathcal{H}_{(a)}) \quad\textit{and}\quad B_{\mathfrak{m}}(\mathcal{H}) \subseteq B_{\mathfrak{m}_0}(\mathcal{H}_{(a)}).$$

Proof. Since $\mathcal{H}$ is a separable Hilbert space, we can choose a countable set $\{F_n : n \in \mathbb{N}\} \subseteq B_{\mathfrak{m}_0}(\mathcal{H})$ which is dense in $B_{\mathfrak{m}_0}(\mathcal{H})$ in the norm of $\mathcal{H}$. Set $G_n := 2^{-n} F_n$, $n \in \mathbb{N}$, and define a continuous function $\mathfrak{m}_1 : \mathbb{R} \to [0, \infty)$ as

$$\mathfrak{m}_1(x) := \sup_{n \in \mathbb{N}} |G_n(x)|, \quad x \in \mathbb{R}\,.$$

The continuity follows since the functions G_n, $n \in \mathbb{N}$, belong to $B(\mathcal{H})$, and hence form an equicontinuous family, cf. Step 1 of the proof of Lemma 3.3.

We have $\mathfrak{m}_1 \leq \mathfrak{m}_0$, and hence

$$B_{\mathfrak{m}_1}(\mathcal{H}_{(a)}) \subseteq B_{\mathfrak{m}_0}(\mathcal{H}_{(a)})\,. \tag{5.14}$$

Since $G_n \in B_{\mathfrak{m}_1}(\mathcal{H}_{(a)})$ for each $n \in \mathbb{N}$, we see that $\mathfrak{m}_1 \in \operatorname{Adm}_{\mathbb{R}} \mathcal{H}_{(a)}$, and actually $R_{\mathfrak{m}_1}(\mathcal{H}_{(a)}) \supseteq \operatorname{span}\{F_n : n \in \mathbb{N}\}$. This implies that $\mathcal{R}_{\mathfrak{m}_1}(\mathcal{H}_{(a)}) \supseteq \mathcal{R}_{\mathfrak{m}_0}(\mathcal{H}_{(a)})$. However, by (5.14), the converse inclusion also holds, and it follows that

$$\mathcal{R}_{\mathfrak{m}_1}(\mathcal{H}_{(a)}) = \mathcal{R}_{\mathfrak{m}_0}(\mathcal{H}_{(a)})\,.$$

Choose $E \in \mathcal{H}B$ with $\mathcal{H} = \mathcal{H}(E)$. Then

$$\Big(\int_{\mathbb{R}} \Big| \frac{G_n(t)}{E(t)} \Big|^2 dt \Big)^{\frac{1}{2}} = \|G_n\|_{\mathcal{H}} = \frac{1}{2^n} \|F_n\|_{\mathcal{H}} \leq \frac{1}{2^n}\,,$$

and therefore

$$\int_{\mathbb{R}} \Big(\frac{\mathfrak{m}_1(t)}{|E(t)|} \Big)^2 dt = \int_{\mathbb{R}} \Big[\sup_{n \in \mathbb{N}} \Big| \frac{G_n(t)}{E(t)} \Big| \Big]^2 dt \leq \int_{\mathbb{R}} \sum_{n=1}^{\infty} \Big| \frac{G_n(t)}{E(t)} \Big|^2 dt =$$

$$= \sum_{n=1}^{\infty} \int_{\mathbb{R}} \Big| \frac{G_n(t)}{E(t)} \Big|^2 dt \leq \sum_{n=1}^{\infty} \frac{1}{4^n} < \infty\,.$$

Proposition 5.14 furnishes us with an H^2-majorant $\mathfrak{m} \in \operatorname{Adm}_{\mathbb{C}^+} \mathcal{H}$ such that $B_{\mathfrak{m}}(\mathcal{H}) = B_{\mathfrak{m}_1}(\mathcal{H}_{(a)})$. Using that $\mathcal{H}_{(a)}$ is a closed subspace of $\mathcal{H}$, we obtain that also $\mathcal{R}_{\mathfrak{m}}(\mathcal{H}) = \mathcal{R}_{\mathfrak{m}_1}(\mathcal{H}_{(a)})$. ❑

6 Non-representable unit balls

Let $\mathfrak{m}_0 \in \operatorname{Adm} \mathcal{H}$, and consider the unit ball $B_{\mathfrak{m}_0}(\mathcal{H})$. Then, in general, this ball need not be representable as $B_{\mathfrak{m}}(\mathcal{H})$ with some log-superharmonic majorant $\mathfrak{m}$, i.e. $M^{\log}(B_{\mathfrak{m}_0}(\mathcal{H}))$ may be empty. However, we may ask the question if the ball $B_{\mathfrak{m}_0}(\mathcal{H})$ is contained or does contain some balls generated by such special kinds of majorants.

This question is also of interest for the following reason: In the first case, we obtain supersets B such that division of a function F in $B_{\mathfrak{m}_0}(\mathcal{H})$ by a Blaschke product cannot lead further out than B and, in the second case, we obtain subsets B which are invariant with respect to division by Blaschke products.

6.1 Definition. Let $\mathcal{H}$ be a de Branges space, and let $\mathfrak{m}_0 \in \operatorname{Adm} \mathcal{H}$. Then we denote

$$M^{\log}_{\geq}(\mathfrak{m}_0) := \big\{ \mathfrak{m} \in \operatorname{Adm}_{\mathbb{C}^+} \mathcal{H} : \ \mathfrak{m} \text{ log-superharmonic}, \ \mathfrak{m} \succcurlyeq \mathfrak{m}_0^\flat \big\}$$

$$M^{\log}_{\leq}(\mathfrak{m}_0) := \big\{\mathfrak{m} \in \operatorname{Adm}_{\mathbb{C}^+}\mathcal{H} :\ \mathfrak{m} \text{ log-superharmonic, } \mathfrak{m}^\flat \preccurlyeq \mathfrak{m}_0\big\}$$

and

$$B^{\log}_{\geq}(\mathfrak{m}_0) := \beta\big(M^{\log}_{\geq}(\mathfrak{m}_0)\big), \quad B^{\log}_{\leq}(\mathfrak{m}_0) := \beta\big(M^{\log}_{\leq}(\mathfrak{m}_0)\big)\,,$$

where β is the map $\mathfrak{m} \mapsto B_{\mathfrak{m}}(\mathcal{H})$, cf. (2.4).

Note here that

$$\mathfrak{m} \succcurlyeq \mathfrak{m}_0^\flat \iff B_{\mathfrak{m}}(\mathcal{H}) \supseteq B_{\mathfrak{m}_0}(\mathcal{H}) \iff \mathfrak{m}^\flat \geq \mathfrak{m}_0^\flat$$

$$\mathfrak{m}^\flat \preccurlyeq \mathfrak{m}_0 \iff B_{\mathfrak{m}}(\mathcal{H}) \subseteq B_{\mathfrak{m}_0}(\mathcal{H}) \iff \mathfrak{m}^\flat \leq \mathfrak{m}_0^\flat$$

It will follow from the results of the previous section that the structure of the sets $M^{\log}_{\geq}(\mathfrak{m}_0)$ and $B^{\log}_{\geq}(\mathfrak{m}_0)$ is fairly simple. Contrasting this, we cannot say much about $M^{\log}_{\leq}(\mathfrak{m}_0)$ and $B^{\log}_{\leq}(\mathfrak{m}_0)$; the obvious obstacle being that in the definition of $M^{\log}_{\leq}(\mathfrak{m}_0)$ not the function $\mathfrak{m}$ itself but only $\mathfrak{m}^\flat$ appears.

a. The set $M^{\log}_{\geq}(\mathfrak{m}_0)$.

Whether or not a log-superharmonic majorant belongs to $M^{\log}_{\geq}(B)$ is decided by behaviour of boundary values along $\mathbb{R}$ and exponential growth.

6.2 Proposition. *Let $\mathcal{H}$ be a de Branges space, and choose $E \in \mathcal{H}B$ with $\mathcal{H} = \mathcal{H}(E)$. If $\mathfrak{m}_0 \in \operatorname{Adm}\mathcal{H}$, then the following hold:*

(i) We have

$$M^{\log}_{\geq}(\mathfrak{m}_0) = \left\{\mathfrak{m} \in \operatorname{Adm}_{\mathbb{C}^+}\mathcal{H} :\ \begin{array}{l}\mathfrak{m} \text{ log-}\textit{superharmonic, and}\\ \mathfrak{m}^\flat|_{\mathbb{R}} \geq \mathfrak{m}_0^\flat|_{\mathbb{R}},\ \operatorname{mt}_{\mathcal{H}}\mathfrak{m}^\flat \geq \operatorname{mt}_{\mathcal{H}}\mathfrak{m}_0^\flat\end{array}\right\} =$$

$$= \left\{\begin{array}{l}\mathfrak{m} \in \operatorname{Adm}_{\mathbb{C}^+}\mathcal{H} :\ \mathfrak{m} \text{ log-}\textit{superharmonic, and}\\ \displaystyle\liminf_{z\to x, z\in\mathbb{C}^+}\mathfrak{m}(z) \geq \mathfrak{m}_0^\flat(x),\ x\in\mathbb{R},\quad \limsup_{|z|\to\infty, z\in\mathbb{C}^+}\frac{1}{\operatorname{Im} z}\log\frac{\mathfrak{m}_0^\flat(z)}{\mathfrak{m}(z)} \leq 0\end{array}\right\},$$

$$B^{\log}_{\geq}(\mathfrak{m}_0) =$$

$$= \left\{B_{\mathfrak{m}}(\mathcal{H}_{(a)}) :\ \begin{array}{l}\mathfrak{m} \in \operatorname{Adm}_{\mathbb{R}}\mathcal{H},\ \mathfrak{m} \textit{ continuous, and}\\ \int_{\mathbb{R}}(\log^+\frac{\mathfrak{m}(t)}{|E(t)|})\frac{dt}{1+t^2} < \infty,\ \mathfrak{m} \geq \mathfrak{m}_0^\flat|_{\mathbb{R}},\ a \geq \operatorname{mt}_{\mathcal{H}}\mathfrak{m}_0^\flat\end{array}\right\}.$$

(ii) Let $\mathfrak{m} \in \operatorname{Adm}_{\mathbb{C}^+}\mathcal{H}$ be an $\mathcal{N}_+$-majorant, and denote by $\mathfrak{m}^*$ its boundary function, cf. (5.5)*. Then*

$$\mathfrak{m} \in M^{\log}_{\geq}(\mathfrak{m}_0) \iff \mathfrak{m}^* \succcurlyeq \mathfrak{m}_0^\flat,\ \operatorname{mt}_{\mathcal{H}}\mathfrak{m} \geq \operatorname{mt}_{\mathcal{H}}\mathfrak{m}_0^\flat$$

Proof. Assume first that $\mathfrak{m} \in M^{\log}_{\geq}(\mathfrak{m}_0)$, then $\mathfrak{m}^\flat \geq \mathfrak{m}_0^\flat$, and hence in particular

$$\mathfrak{m}^\flat|_{\mathbb{R}} \geq \mathfrak{m}_0^\flat|_{\mathbb{R}}, \qquad \operatorname{mt}_{\mathcal{H}}\mathfrak{m}^\flat \geq \operatorname{mt}_{\mathcal{H}}\mathfrak{m}_0^\flat\,.$$

Assume next that $\mathfrak{m}$ is log-superharmonic and that these two inequalities hold. Then $\mathfrak{m}^\perp_{B_\mathfrak{m}(\mathcal{H})} \geq \mathfrak{m}^\perp_{B_{\mathfrak{m}_0}(\mathcal{H})}$, and hence $\mathfrak{m} \geq \mathfrak{m}^\perp_{B_\mathfrak{m}(\mathcal{H})} \geq \mathfrak{m}^\perp_{B_{\mathfrak{m}_0}(\mathcal{H})} \succcurlyeq \mathfrak{m}_0^\flat$. In particular,

$$\liminf_{z\to x, z\in\mathbb{C}^+} \mathfrak{m}(z) \geq \mathfrak{m}_0^\flat(x),\ x\in\mathbb{R}, \qquad \limsup_{|z|\to\infty, z\in\mathbb{C}^+} \frac{1}{\operatorname{Im} z} \log \frac{\mathfrak{m}_0^\flat(z)}{\mathfrak{m}(z)} \leq 0\,. \tag{6.1}$$

We see that both inclusions '$\subseteq$' in the first asserted line of equalities hold.

Assume that $\mathfrak{m}$ is log-superharmonic and satisfies (6.1). An application of the Phragmén–Lindelöf Principle [RR, Theorem 6.2] with the functions and sets

$$u(z) := \log \mathfrak{m}_0^\flat - \log \mathfrak{m},\ h(z) := \operatorname{Im} z, \qquad R := \mathbb{R},\ S := \{\infty\}\,,$$

yields that $\mathfrak{m}_0^\flat(z) \leq \mathfrak{m}(z)$ for all $z\in\mathbb{C}^+$. Thus $\mathfrak{m}\in M_\geq^{\log}(\mathcal{H})$.

In order to see the stated form of $B_\geq^{\log}(\mathfrak{m}_0)$, it is in view of Corollary 5.5 enough to note that

$$B_\mathfrak{m}(\mathcal{H}_{(a)}) \supseteq B_{\mathfrak{m}_0}(\mathcal{H}) \iff a \geq \operatorname{mt}_\mathcal{H} B_{\mathfrak{m}_0}(\mathcal{H}) = \operatorname{mt}_\mathcal{H} \mathfrak{m}_0^\flat,\ \mathfrak{m} \succcurlyeq \mathfrak{m}_0^\flat$$

Finally, assume that $\mathfrak{m}$ is an $\mathcal{N}_+$-majorant. Since $\mathfrak{m} \succcurlyeq \mathfrak{m}^\flat$ implies $\mathfrak{m}^* \succcurlyeq \mathfrak{m}^\flat$, the implication '$\Rightarrow$' in the asserted equivalence is immediate from what we already know about $M_\geq^{\log}(\mathfrak{m}_0)$. Hence assume that $\mathfrak{m}$ satisfies the conditions stated on the right hand side. Then, by the definition of $\mathfrak{m}^\perp_{B_{\mathfrak{m}_0}(\mathcal{H})}$ and Lemma 3.9, (*ii*), we have $\mathfrak{m} \geq \mathfrak{m}^\perp_{B_{\mathfrak{m}_0}(\mathcal{H})} \succcurlyeq \mathfrak{m}_0^\flat$. ❑

It is now easy to state the analogoue of Theorem 5.3 and Corollary 5.4 corresponding to the set $M_\geq^{\log}(\mathfrak{m}_0)$.

6.3 Theorem. *Let $\mathcal{H}$ be a de Branges space, and choose $E\in\mathcal{H}B$ with $\mathcal{H} = \mathcal{H}(E)$. Moreover, let $\mathfrak{m}_0 \in \operatorname{Adm}\mathcal{H}$. Then*

$$M_\geq^{\log}(\mathfrak{m}_0) \neq \emptyset \iff \int_\mathbb{R} \Big(\log^+ \frac{\mathfrak{m}_0^\flat(t)}{|E(t)|}\Big) \frac{dt}{1+t^2} < \infty.$$

In this case the set $M_\geq^{\log}(B)$ contains a smallest element, namely the $\mathcal{N}_+$-majorant $\mathfrak{m}^\perp_{B_{\mathfrak{m}_0}(\mathcal{H})}$. Moreover, $\beta(\mathfrak{m}^\perp_{B_{\mathfrak{m}_0}(\mathcal{H})})$ is the smallest element of $B_\geq^{\log}(\mathfrak{m}_0)$.

Proof. The implication '$\Rightarrow$' is obvious. Assume that the integral on the right side of the asserted equivalence converges. Then the function $\mathfrak{m}^\perp_{B_{\mathfrak{m}_0}(\mathcal{H})}$ is well-defined, is an $\mathcal{N}_+$-majorant, and satisfies

$$(\mathfrak{m}^\perp_{B_{\mathfrak{m}_0}(\mathcal{H})})^* = \mathfrak{m}_0^\flat|_\mathbb{R}, \quad \operatorname{mt}_\mathcal{H} \mathfrak{m}^\perp_{B_{\mathfrak{m}_0}(\mathcal{H})} = \operatorname{mt}_\mathcal{H} \mathfrak{m}_0^\flat.$$

Thus $\mathfrak{m}^\perp_{B_{\mathfrak{m}_0}(\mathcal{H})} \in M_\geq^{\log}(\mathfrak{m}_0)$. Let $\mathfrak{m}\in M_\geq^{\log}(\mathfrak{m}_0)$ be given. Then the $\mathcal{N}_+$-majorant $\mathfrak{m}^\perp_{B_\mathfrak{m}(\mathcal{H})}$ generates the same unit ball as $\mathfrak{m}$ does, and satisfies $\mathfrak{m}^\perp_{B_\mathfrak{m}(\mathcal{H})} \leq \mathfrak{m}$, cf. Corollary 5.4. Hence $\mathfrak{m}^\perp_{B_\mathfrak{m}(\mathcal{H})} \in M_\geq^{\log}(\mathfrak{m}_0)$, and it follows that

$$(\mathfrak{m}^\perp_{B_\mathfrak{m}(\mathcal{H})})^* \geq \mathfrak{m}_0^\flat|_\mathbb{R} = (\mathfrak{m}^\perp_{B_{\mathfrak{m}_0}(\mathcal{H})})^*\,,$$

$$\operatorname{mt}_\mathcal{H} \mathfrak{m}^\perp_{B_\mathfrak{m}(\mathcal{H})} \geq \operatorname{mt}_\mathcal{H} \mathfrak{m}_0^\flat = \operatorname{mt}_\mathcal{H} \mathfrak{m}^\perp_{B_{\mathfrak{m}_0}(\mathcal{H})}\,.$$

This implies that $\mathfrak{m} \geq \mathfrak{m}^{\perp}_{B_{\mathfrak{m}}(\mathcal{H})} \geq \mathfrak{m}^{\perp}_{B_{\mathfrak{m}_0}(\mathcal{H})}$.

Since β is order-preserving and maps $M^{\log}_{\geq}(\mathfrak{m}_0)$ onto $B^{\log}_{\geq}(\mathfrak{m}_0)$, the image of the smallest element of $M^{\log}_{\geq}(\mathfrak{m}_0)$ is the smallest element in $B^{\log}_{\geq}(\mathfrak{m}_0)$. ❑

6.4 Remark. The above statements concerning $M^{\log}_{\geq}(\mathfrak{m}_0)$ and $B^{\log}_{\geq}(\mathfrak{m}_0)$ have obvious analogues for arbitrary nonempty subsets B of $B(\mathcal{H})$ instead of $B_{\mathfrak{m}_0}(\mathcal{H})$. One only should everywhere replace $\mathfrak{m}_0^{\flat}$ by $\mathfrak{m}_B$, and copy the above proofs word by word. We have decided to stick to the case $B = B_{\mathfrak{m}_0}(\mathcal{H})$, in order to stress to contrast between approximation from above and from below; compare Theorem 6.3 with Theorem 6.5 and Proposition 6.9 below.

b. The set $M^{\log}_{\leq}(\mathfrak{m}_0)$.

Interestingly, the structure of $M^{\log}_{\leq}(\mathfrak{m}_0)$ is much more delicate, and is related to a completely different topic, namely the existence of zerofree elements in $B_{\mathfrak{m}_0}(\mathcal{H})$.

6.5 Theorem. *Let $\mathcal{H}$ be a de Branges space, and let $\mathfrak{m}_0 \in \operatorname{Adm}\mathcal{H}$. Then the following are equivalent:*

- (i) $M^{\log}_{\leq}(\mathfrak{m}_0) \neq \emptyset$.
- (ii) *There exists an H^2-majorant $\mathfrak{m}$ with $\mathfrak{m} \leq \mathfrak{m}_0^{\flat}|_{\mathbb{C}^+}$.*
- (iii) *There exists an element $F \in B_{\mathfrak{m}_0}(\mathcal{H})$ which satisfies $F^{\#} = F$ and which has no zeros in $\mathbb{C} \setminus \mathbb{R}$.*

Let us formulate the crucial argument of the proof of this result separately.

6.6 Lemma. *Let $\mathfrak{m} \in \operatorname{Adm}\mathcal{H}$ and assume that $R_{\mathfrak{m}}(\mathcal{H})$ is invariant with respect to division by Blaschke products. Then $B_{\mathfrak{m}}(\mathcal{H})$ contains a function F with $F = F^{\#}$ which has no zeros off the real axis. In fact, whenever $G \in R_{\mathfrak{m}}(\mathcal{H}) \setminus \{0\}$, the choice of F can be made such that $\mathfrak{d}_F|_{\mathbb{R}} \geq \mathfrak{d}_G|_{\mathbb{R}}$.*

Proof. Let $G \in R_{\mathfrak{m}}(\mathcal{H}) \setminus \{0\}$ and set $F_1 := G + G^{\#}$ (in case $G = -G^{\#}$, use $F_1 := i(G - G^{\#})$ instead). Then $F_1 \in R_{\mathfrak{m}}(\mathcal{H}) \setminus \{0\}$, $F_1 = F_1^{\#}$, and $\mathfrak{d}_{F_1}|_{\mathbb{R}} \geq \mathfrak{d}_G|_{\mathbb{R}}$.

If F_1 has no zeros in $\mathbb{C}^+$, then $F := \|F_1\|_{\mathfrak{m}}^{-1} F_1$ has all the desired properties, and we are done. Otherwise, let P be the Blaschke product for $\mathbb{C}^+$ built with the zeros of F_1, and define

$$F_2 := \frac{F_1}{P} + \left(\frac{F_1}{P}\right)^{\#} = F_1\left(\frac{1}{P} + P\right).$$

Clearly, $F_2 = F_2^{\#}$ and $\mathfrak{d}_{F_2}|_{\mathbb{R}} \geq \mathfrak{d}_{F_1}|_{\mathbb{R}} \geq \mathfrak{d}_G|_{\mathbb{R}}$.

Let $w \in \mathbb{C}^+$ be given. If $F_1(w) \neq 0$, then $0 < |P(w)| < 1$. Hence also $P(w)^{-1} + P(w) \neq 0$, and we obtain $F_2(w) \neq 0$. In case $F_1(w) = 0$, we have $F_1(w)P^{-1}(w) \neq 0$ and $(F_1P)(w) = 0$, and again it follows that $F_2(w) \neq 0$. Setting $F := \|F_2\|_{\mathfrak{m}}^{-1} F_2$, we obtain a function with all the required properties. ❑

Proof (of Theorem 6.5). We first establish the implication $(i) \Rightarrow (iii)$. Assume that $\mathfrak{m} \in M^{\log}_{\leq}(\mathfrak{m}_0)$. Then by Corollary 4.5 the unit ball $B_{\mathfrak{m}}(\mathcal{H})$ is invariant with respect to division by Blaschke products, and hence also $R_{\mathfrak{m}}(\mathcal{H})$ has this property. An application of Lemma 6.6 yields a function F as required in (iii).

Next we show that $(iii) \Rightarrow (ii)$. Let F be as in (iii), and set $\mathfrak{m}(z) := |F(z)|$, $z \in \mathbb{C}^+$. Since F is zerofree in $\mathbb{C}^+$, we have $\mathfrak{d}_{\mathfrak{m}}(w) = 0$, $w \in \mathbb{C}^+$. Since F is real, we have $F \in B_{\mathfrak{m}}(\mathcal{H})$. Moreover, $\mathfrak{m}(z) = |(E^{-1}F) \cdot E|$, and hence $\mathfrak{m}$ is an H^2-majorant. Finally, since $F \in B_{\mathfrak{m}_0}(\mathcal{H})$, we have $\mathfrak{m} \leq \mathfrak{m}_0^\flat|_{\mathbb{C}^+}$. Thus $\mathfrak{m} \in M^{\log}_{\leq}(\mathfrak{m}_0)$.

The implication $(ii) \Rightarrow (i)$ is trivial. ❑

Similar as we have asked for minimal elements of $M^{\log}_{\geq}(\mathfrak{m}_0)$ and $B^{\log}_{\geq}(\mathfrak{m}_0)$, it is natural to seek for maximal elements $M^{\log}_{\leq}(\mathfrak{m}_0)$ or $B^{\log}_{\leq}(\mathfrak{m}_0)$. It will turn out, cf. item (iii) of the below Proposition 6.9, that in general such will not exist. However, a positive result can be obtained, when restricting considerations to a specific subclass of $M^{\log}_{\leq}(\mathfrak{m}_0)$.

6.7 Definition. Let $\mathcal{H}$ be a de Branges space, let $\mathfrak{m}_0 \in \operatorname{Adm}\mathcal{H}$, and denote by $D \subseteq \mathbb{C}^+ \cup \mathbb{R}$ the domain of $\mathfrak{m}_0$. Then we define

$$\widetilde{M}^{\log}_{\leq}(\mathfrak{m}_0) := \big\{\mathfrak{m} \in M^{\log}_{\leq}(\mathfrak{m}_0) : \ \mathfrak{m}(z) \leq \mathfrak{m}_0(z), z \in D \cap \mathbb{C}^+\big\},$$

and set $\widetilde{B}^{\log}_{\leq}(\mathfrak{m}_0) := \beta(\widetilde{M}^{\log}_{\leq}(\mathfrak{m}_0))$.

Of course, the additional requirement in the definition of $\widetilde{M}^{\log}_{\leq}(\mathfrak{m}_0)$ is a restriction only if $D \cap \mathbb{C}^+ \neq \emptyset$. In case $D \subseteq \mathbb{R}$, we have $\widetilde{M}^{\log}_{\leq}(\mathfrak{m}_0) = M^{\log}_{\leq}(\mathfrak{m}_0)$, and hence also $\widetilde{B}^{\log}_{\leq}(\mathfrak{m}_0) = B^{\log}_{\leq}(\mathfrak{m}_0)$.

Note that, by the equivalence of (i) and (ii) in Theorem 6.5, we have in particular

$$\widetilde{M}^{\log}_{\leq}(\mathfrak{m}_0) \neq \emptyset \iff M^{\log}_{\leq}(\mathfrak{m}_0) \neq \emptyset$$

Let us explicitly state the following observation.

6.8 Remark. Let B be a nonempty subset of the unit ball of $\mathcal{H}$. Then $B \in \widetilde{B}^{\log}_{\leq}(\mathfrak{m}_0)$ if and only if the function $\mathfrak{m}_B^{\perp}$ is well-defined, satisfies $\mathfrak{m}_B^{\perp}(z) \leq \mathfrak{m}_0(z)$, $z \in D \cap \mathbb{C}^+$, and $B = \beta(\mathfrak{m}_B^{\perp})$.

6.9 Proposition. *Let $\mathcal{H}$ be a de Branges space, let $\mathfrak{m}_0 \in \operatorname{Adm}\mathcal{H}$, and denote by D the domain of $\mathfrak{m}_0$. Assume that $\widetilde{M}^{\log}_{\leq}(\mathfrak{m}_0) \neq \emptyset$. Then the following hold:*

- (i) *If $D \subseteq \mathbb{C}^+$, then for each element $\mathfrak{m} \in \widetilde{M}^{\log}_{\leq}(\mathfrak{m}_0)$ there exists a maximal element $\hat{\mathfrak{m}}$ of $\widetilde{M}^{\log}_{\leq}(\mathfrak{m}_0)$, with $\mathfrak{m} \leq \hat{\mathfrak{m}}$.*
- (ii) *If $D \cap \mathbb{C}^+ \neq \emptyset$, then for each element $B \in \widetilde{B}^{\log}_{\leq}(\mathfrak{m}_0)$ there exists a maximal element $\hat{B}$ of $\widetilde{B}^{\log}_{\leq}(\mathfrak{m}_0)$, with $B \subseteq \hat{B}$.*
- (iii) *If $D \subseteq \mathbb{R}$ and $\int_{\mathbb{R}} [\log^+(|E|^{-1}\mathfrak{m}_0^\flat)](1+t^2)^{-1} dt < \infty$, then $B^{\log}_{\leq}(\mathfrak{m}_0)$ contains the largest element, namely $B_{\mathfrak{m}_0}(\mathcal{H})$ itself.*
- (iv) *If $D \subseteq \mathbb{R}$ and $\int_{\mathbb{R}} [\log^+(|E|^{-1}\mathfrak{m}_0^\flat)](1+t^2)^{-1} dt = \infty$, then no element of $B^{\log}_{\leq}(\mathfrak{m}_0)$ is maximal in this set.*

Proof.
Step 1, Case $D\subseteq\mathbb{C}^+$: We establish the hypothesis of Zorn's Lemma for the set $\widetilde{M}^{\log}_{\leq}(\mathfrak{m}_0)$. Let an ascending chain $\mathfrak{M}$ of elements of $\widetilde{M}^{\log}_{\leq}(\mathfrak{m}_0)$ be given, and set

$$\tilde{\mathfrak{m}}:=\sup_{\mathfrak{m}\in\mathfrak{M}}\mathfrak{m}\,.$$

Since $\mathfrak{m}|_D\leq\mathfrak{m}_0$, $\mathfrak{m}\in\mathfrak{M}$, we also have $\tilde{\mathfrak{m}}|_D\leq\mathfrak{m}_0$. Note that the domain of $\tilde{\mathfrak{m}}$, i.e. $\mathbb{C}^+$, entirely contains the domain of $\mathfrak{m}_0$, i.e. D. We conclude that $B_{\tilde{\mathfrak{m}}}(\mathcal{H})\subseteq B_{\mathfrak{m}_0}(\mathcal{H})$.

The family $(\log\mathfrak{m})_{\mathfrak{m}\in\mathfrak{M}}$ is a nondecreasing net of superharmonic functions. Pick an element $\mathfrak{m}_1\in\mathfrak{M}$. Then there exists a function $F\in B_{\mathfrak{m}_1}(\mathcal{H})$ which has no zeros in $\mathbb{C}^+$. Hence $\log|F|$ is harmonic in $\mathbb{C}^+$ and satisfies $\log|F|\leq\log\mathfrak{m}_1$. Corollary 2.15 implies that $\log\tilde{\mathfrak{m}}$ is superharmonic. We conclude that $\tilde{\mathfrak{m}}\in\widetilde{M}^{\log}_{\leq}(\mathfrak{m}_0)$ and have found an upper bound of $\mathfrak{M}$.

Step 2, Case $D\cap\mathbb{C}^+\neq\emptyset$: Let $\mathfrak{B}$ be an ascending chain in $B^{\log}_{\leq}(\mathfrak{m}_0)$. For each $B\in\mathfrak{B}$, the majorant $\mathfrak{m}_B^{\perp}$ is well-defined, belongs to $M^{\log}_{\leq}(\mathfrak{m}_0)$, and we have $B=\beta(\mathfrak{m}_B^{\perp})$. Set

$$\tilde{\mathfrak{m}}:=\sup_{B\in\mathfrak{B}}\mathfrak{m}_B^{\perp}\,,$$

and

$$k_B:=\frac{\mathfrak{m}_B}{|E|}\Big|_{\mathbb{R}},\quad k:=\sup_{B\in\mathfrak{B}}k_B,\quad a:=\sup_{B\in\mathfrak{B}}\operatorname{mt}_{\mathcal{H}}B\,.$$

First we are going to show that

$$\tilde{\mathfrak{m}}(z)=\big|e^{-iaz}\mathfrak{f}_k(z)E(z)\big|,\qquad z\in\mathbb{C}^+\,. \tag{6.2}$$

To this end we verify the hypothesis of Corollary 2.16 for the family $(k_B)_{B\in\mathfrak{B}}$. By Lemma 3.3, the function k_B is continuous. The fact that $B_1\subseteq B_2$ implies $k_{B_1}\leq k_{B_2}$ is clear, and the condition $\log k_B\in L^1(\frac{dt}{1+t^2})$ is just the fact that $\mathfrak{m}_B^{\perp}$ is well-defined.

It remains to establish the second condition in (2.7). Here we employ our assumption that $D\cap\mathbb{C}^+\neq\emptyset$. Choose $z_0=x_0+iy_0\in D\cap\mathbb{C}^+$, and let $c>0$ be such that

$$c\frac{1}{1+t^2}\leq\frac{y_0}{(t-x_0)^2+y_0^2},\quad t\in\mathbb{R}\,.$$

Moreover, fix $B_0\in\mathfrak{B}$. Then, for each $B\in\mathfrak{B}$ with $B\supseteq B_0$,

$$c\Big(\int_{\mathbb{R}}\big[\log k_B(t)\big]\frac{dt}{1+t^2}-\int_{\mathbb{R}}\big[\log k_{B_0}(t)\big]\frac{dt}{1+t^2}\Big)\leq$$
$$\leq\int_{\mathbb{R}}\big[\log k_B(t)-\log k_{B_0}(t)\big]\frac{y_0}{(t-x_0)^2+y_0^2}\,dt=\pi\log\Big|\frac{\mathfrak{f}_{k_B}(z_0)}{\mathfrak{f}_{k_{B_0}}(z_0)}\Big|,$$

$$|\mathfrak{f}_{k_B}(z_0)|=e^{-y_0\operatorname{mt}_{\mathcal{H}}B}\frac{\mathfrak{m}_B^{\perp}(z_0)}{|E(z_0)|}\leq e^{-y_0\operatorname{mt}_{\mathcal{H}}B_0}\frac{\mathfrak{m}_0(z_0)}{|E(z_0)|}\,,$$

$$\int_{\mathbb{R}}\big[\log^-k_B(t)\big]\frac{dt}{1+t^2}\leq\int_{\mathbb{R}}\big[\log^-k_{B_0}(t)\big]\frac{dt}{1+t^2}\,.$$

Putting together these estimates yields

$$\int_{\mathbb{R}}\big[\log^+k_B(t)\big]\frac{dt}{1+t^2}=\int_{\mathbb{R}}\big[\log k_B(t)\big]\frac{dt}{1+t^2}+\int_{\mathbb{R}}\big[\log^-k_B(t)\big]\frac{dt}{1+t^2}\leq$$

$$\leq \frac{\pi}{c}\Big(\log \mathfrak{m}_0(z_0) - y_0 \operatorname{mt}_{\mathcal{H}} B_0 - \log|E(z_0)| - \log|\mathfrak{f}_{k_{B_0}}(z_0)|\Big) +$$

$$+ 2\int_{\mathbb{R}} \big|\log k_B(t)\big| \frac{dt}{1+t^2} =: C < \infty\,.$$

Hence, by monotonicity,

$$\sup_{B\in\mathfrak{B}} \int_{\mathbb{R}} \big[\log^+ k_B(t)\big]\frac{dt}{1+t^2} = \sup_{\substack{B\in\mathfrak{B}\\ B\supseteq B_0}} \int_{\mathbb{R}} \big[\log^+ k_B(t)\big]\frac{dt}{1+t^2} \leq C\,.$$

An application of Corollary 2.16 gives $\mathfrak{f}_k(z) = \lim_{B\in\mathfrak{B}} \mathfrak{f}_{k_B}(z)$, $z\in\mathbb{C}^+$, and hence (6.2) follows.

Next we are going to show that $\tilde{\mathfrak{m}} \in \widetilde{M}^{\log}_{\leq}(\mathfrak{m}_0)$. Clearly, $\tilde{\mathfrak{m}}$ is an $\mathcal{N}_+$-majorant, and satisfies

$$\tilde{\mathfrak{m}}(z) \leq \mathfrak{m}_0(z), \quad z\in D\cap\mathbb{C}^+\,. \tag{6.3}$$

Since, for each $B\in\mathfrak{B}$, we have $B\subseteq B_{\mathfrak{m}_0}(\mathcal{H})$, it follows that

$$\exp\big(k_B(x)\big)|E(x)| = \mathfrak{m}_B(x) \leq \mathfrak{m}_0^\flat(x), \quad x\in\mathbb{R}\,.$$

Thus also $\exp(k(x))|E(x)| \leq \mathfrak{m}_0^\flat(x)$, $x\in\mathbb{R}$. Let $F\in B_{\tilde{\mathfrak{m}}}(\mathcal{H})$ be given. Then, for each Lebesgue point of the function $\log k$, we have

$$|F(x)| = \lim_{z\hat{\to} x} |F(z)| \leq \lim_{z\hat{\to} x} \tilde{\mathfrak{m}}(z) = \exp(k(x))|E(x)| \leq \mathfrak{m}_0^\flat(x)\,.$$

By continuity, and the fact that the set of Lebesgue points of $\log k$ is dense in $\mathbb{R}$, it follows that

$$|F(x)| \leq \mathfrak{m}_0^\flat(x), \quad x\in\mathbb{R}\,.$$

In particular, $|F(x)|\leq \mathfrak{m}_0(x)$, $x\in D\cap\mathbb{R}$. Together with (6.3) this shows that $F\in B_{\mathfrak{m}_0}(\mathcal{H})$. We conclude that indeed $\tilde{\mathfrak{m}}\in\widetilde{M}^{\log}_{\leq}(\mathcal{H})$. Moreover, clearly, $\beta(\tilde{\mathfrak{m}})$ is an upper bound of $\mathfrak{B}$.

Step 3, Case $D\subseteq\mathbb{R}$: If the integral $\int_{\mathbb{R}}[\log(|E|^{-1}\mathfrak{m}_0^\flat)](1+t^2)^{-1}dt$ converges, then by Theorem 5.3 we have $B_{\mathfrak{m}_0}(\mathcal{H})\in B^{\log}_{\leq}(\mathfrak{m}_0)$. Trivially, it is the largest element of this set.

Assume that the above logarithmic integral diverges. If $B\in B^{\log}_{\leq}(\mathfrak{m}_0)$, then $\mathfrak{m}_B\leq\mathfrak{m}_0^\flat$ and $\int_{\mathbb{R}}[\log(|E|^{-1}\mathfrak{m}_B)](1+t^2)^{-1}dt<\infty$. Hence, we cannot have $\mathfrak{m}_B=\mathfrak{m}_0^\flat$. Choose $x_0\in\mathbb{R}$ such that $\mathfrak{m}_B(x_0)<\mathfrak{m}_0^\flat(x_0)$, and let $F\in B_{\mathfrak{m}_0}(\mathcal{H})$ be such that $\mathfrak{m}_B(x_0)<|F(x_0)|$. Set

$$\mathfrak{m}_1(x) := \max\big\{|F(x)|, \mathfrak{m}_B(x)\big\}, \quad x\in\mathbb{R}\,,$$

then $\mathfrak{m}_B\preccurlyeq\mathfrak{m}_1\preccurlyeq\mathfrak{m}_0$. However, $F\in B_{\mathfrak{m}_1}(\mathcal{H})\setminus B$, and hence

$$B\subsetneq B_{\mathfrak{m}_1}(\mathcal{H})\subseteq B_{\mathfrak{m}_0}(\mathcal{H})\,.$$

Moreover, since $E^{-1}F\in H^2$, it follows that

$$\int_{\mathbb{R}}\Big(\log^+\frac{\mathfrak{m}_1(t)}{|E(t)|}\Big)\frac{dt}{1+t^2} \leq$$

$$\leq \int_{\mathbb{R}}\Big(\log^+\Big|\frac{F(t)}{E(t)}\Big|\Big)\frac{dt}{1+t^2} + \int_{\mathbb{R}}\Big(\log^+\frac{\mathfrak{m}_B(t)}{|E(t)|}\Big)\frac{dt}{1+t^2} < \infty\,.$$

We obtain from Theorem 5.3 that $B_{\mathfrak{m}_1}(\mathcal{H})\in B^{\log}_{\leq}(\mathfrak{m}_0)$. ❑

References

[B] Baranov A. D., *Polynomials in the de Branges spaces of entire functions*, Ark. Mat. **44** (2006), 16–38.

[BW1] Baranov A., Woracek H., *Subspaces of de Branges spaces generated by majorants*, Canad. Math. J. **61** (2009), 3, 503–517.

[BW2] Baranov A., Woracek H., *Finite-dimensional de Branges subspaces generated by majorants*, Operator Theory Adv. Appl. **188** (2009), 37–48.

[BW3] Baranov A., Woracek H., *Majorization in de Branges spaces I. Representability of subspaces*, arXiv:0906.2939v1 [math.CV].

[BW4] Baranov A., Woracek H., *Majorization in de Branges spaces II. Banach spaces generated by majorants*, arXiv:0906.2943v1 [math.CV].

[BM] Beurling A., Malliavin P., *On Fourier transforms of measures with compact support*, Acta Math. **107** (1962), 291–309.

[Bo] Boas R., *Entire functions*, Academic Press, New York, 1954.

[dB1] De Branges L., *Some Hilbert spaces of entire functions*, Proc. Amer. Math. Soc. **10** (1959), 840–846.

[dB2] De Branges L., *Some Hilbert spaces of entire functions*, Trans. Amer. Math. Soc. **96** (1960), 259–295.

[dB3] De Branges L., *Some Hilbert spaces of entire functions II*, Trans. Amer. Math. Soc. **99** (1961), 118–152.

[dB4] De Branges L., *Some Hilbert spaces of entire functions III*, Trans. Amer. Math. Soc. **100** (1961), 73–115.

[dB5] De Branges L., *Some Hilbert spaces of entire functions IV*, Trans. Amer. Math. Soc. **105** (1962), 43–83.

[dB6] De Branges L., *Hilbert spaces of entire functions*, Prentice-Hall, London, 1968.

[HM1] Havin V. P., Mashreghi J., *Admissible majorants for model subspaces of H^2. Part I: slow winding of the generating inner function*, Can. J. Math. **55** (2003), 6, 1231–1263.

[HM2] Havin V. P., Mashreghi J., *Admissible majorants for model subspaces of H^2. Part II: fast winding of the generating inner function*, Can. J. Math. **55** (2003), 6, 1264–1301.

[H-J] Hoffmann-Jørgensen J., *Stochastic processes on Polish spaces*, Åarhus Universitet, Matematisk Institut 39, Aarhus, 1991.

[KW] Kaltenbäck M., Woracek H., *De Branges spaces of exponential type: General theory of growth*, Acta Sci. Math (Szeged) **71** (2005), 1/2, 231–284.

[RR] Rosenblum M., Rovnyak J., *Topics in Hardy classes and univalent functions*, Birkhäuser Verlag, Basel, 1994.

[R] Rudin W., *Real and complex analysis*, McGraw-Hill, New York, 1987.

H. Woracek
Institut für Analysis und Scientific Computing
Technische Universität Wien
Wiedner Hauptstr. 8–10/101
A–1040 Wien
AUSTRIA
email: harald.woracek@tuwien.ac.at

A. Baranov
Department of Mathematics and Mechanics
Saint Petersburg State University
28, Universitetski pr.
198504 Petrodvorets
RUSSIA
email: a.baranov@ev13934.spb.edu