# Polya class theory for Hermite-Biehler functions of finite order

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**Abstract**

A partition of the class of all Hermite-Biehler of finite order into subclasses $\mathcal{P}_\kappa$ is introduced. The belonging of a given function $E(z)$ to $\mathcal{P}_\kappa$ is characterized by $-z^{-1}\log E(z) \in \mathcal{N}_\kappa$. Hereby, the class $\mathcal{N}_\kappa$ is a well studied family of meromorphic functions on the upper half plane, which originates from operator theoretic problems. We also prove that the introduced subclasses are stable under bounded type perturbation.

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## 1 Introduction

In [dB] the *Polya class* $\mathcal{P}$ is defined as the set of all entire functions $E$ such that $E$ has no zeros in the open upper half plane $\mathbb{C}^+$, satisfies $|E(\overline{z})| \leq |E(z)|$ for all $z \in \mathbb{C}^+$ and possesses the property that $|E(x+iy)|$ is, for each fixed $x \in \mathbb{R}$, a nondecreasing function of $y > 0$. This class of entire functions has in fact been studied rather closely (cf. e.g. [B] or [L]) and origins from some investigations of G.Polya (cf. [Po]). It is a well known fact (cf. e.g. [dB, Theorem 7]) that a function $E$ belongs to $\mathcal{P}$ if and only if (we denote by $(z_n)_{n=1}^{N(E)}$, $N(E) \in \mathbb{N}_0 \cup \{\infty\}$, the sequence of nonzero zeros of $E$ counted according to their multiplicities)

$$\sum_{n=1}^{N(E)} \frac{1 - \operatorname{Im} z_n}{|z_n|^2} < \infty,$$

and $E$ allows the product representation

$$E(z) = cz^r e^{p(z)-ihz} \prod_{n=1}^{N(E)} \big(1 - \frac{z}{z_n}\big) e^{z \operatorname{Re} \frac{1}{z_n}}, \tag{1.1}$$

where $p(z) = p_2 z^2 + p_1 z$ is a real polynomial with $p_2 \leq 0$, $h \geq 0$, $c \in \mathbb{C}$ and $r \in \mathbb{N}_0$.

Let us recall (see e.g. [L]) that the *Hermite-Biehler class* $\mathcal{H}B$ is defined to be the set of all entire functions $F$ which are zerofree in $\mathbb{C}^+$ and satisfy

$$|F(\overline{z})| \leq |F(z)|, \; z \in \mathbb{C}^+.$$

Note that, a forteriori, $\mathcal{P} \subseteq \mathcal{H}B$.

Denote by $\mathcal{N}_0$ the *Nevanlinna class* of all functions $f$ analytic in $\mathbb{C}^+$ which have the property that $f(\mathbb{C}^+) \subseteq \mathbb{C}^+ \cup \mathbb{R}$. In [dB, Theorem 14] the following connection between the classes $\mathcal{P}$ and $\mathcal{N}_0$ was established: Let $E \in \mathcal{H}B$, $E(0) = 1$, be given and define $\log E(z)$ analytically on $\mathbb{C}^+ \cup \{0\}$ such that $\log E(0) = 0$. Then $E \in \mathcal{P}$ if and only if

$$-\frac{\log E(z)}{z} \in \mathcal{N}_0.$$

It is the aim of this note to provide an analogous result in a more general framework. Starting from the product representation (1.1) we define a generalization of Polya class:

**1.1 Definition.** Let $\kappa \in \mathbb{N}_0$. An entire function $E$ is said to belong to the class $\mathcal{P}_{\leq\kappa}$ if it belongs to $\mathcal{H}B$, satisfies

$$\sum_{n=1}^{N(E)} \frac{1}{|z_n|^{2\kappa+2}} < \infty, \tag{1.2}$$

and admits the product representation

$$E(z) = cz^r e^{p(z)-ihz} \prod_{n=1}^{N(E)} \left(1 - \frac{z}{z_n}\right) e^{z \operatorname{Re} \frac{1}{z_n} + \ldots + z^{2\kappa+1} \operatorname{Re} \frac{1}{(2\kappa+1) z_n^{2\kappa+1}}}, \tag{1.3}$$

where $p(z) = p_{2\kappa+2} z^{2\kappa+2} + \ldots + p_1 z$ is a real polynomial with $p_{2\kappa+2} \leq 0$, $h \geq 0$, $c \in \mathbb{C}$ and $r \in \mathbb{N}_0$.

For $\kappa \geq 1$ we put $\mathcal{P}_\kappa := \mathcal{P}_{\leq\kappa} \setminus \mathcal{P}_{\leq\kappa-1}$ and refer to $\mathcal{P}_\kappa$ as the *generalized Polya class of index* $\kappa$. Instead of $\mathcal{P}_{\leq 0}$ we shall also write $\mathcal{P}_0$.

This definition is consistent with the previously defined notion of the Polya class $\mathcal{P}$: We have $\mathcal{P} = \mathcal{P}_0$.

Let us recall the notion of generalized Nevanlinna class: Let $\kappa \in \mathbb{N}_0$ and let $f$ be meromorphic in $\mathbb{C}^+$. Then $f$ is said to belong to the class $\mathcal{N}_{\leq\kappa}$ if for any choice of $n \in \mathbb{N}$ and points $z_1, \ldots, z_n$ in the domain of holomorphy of $f$ the quadratic form

$$\sum_{i,j=1}^{n} \frac{f(z_i) - \overline{f(z_j)}}{z_i - \overline{z_j}} \xi_i \overline{\xi_j} \tag{1.4}$$

has at most $\kappa$ negative squares. It is well known (cf. e.g. [Pi]) that this notation is in the case $\kappa = 0$ consistent with the previously defined notion of $\mathcal{N}_0$: We have $\mathcal{N}_{\leq 0} = \mathcal{N}_0$. Again, for $\kappa \geq 1$ we set $\mathcal{N}_\kappa := \mathcal{N}_{\leq\kappa} \setminus \mathcal{N}_{\leq\kappa-1}$ and refer to $\mathcal{N}_\kappa$ as the *generalized Nevanlinna class with negativ index* $\kappa$.

We are now in position to formulate the main results of this note. First an exact analogue of [dB, Theorem 14]:

**1.2 Theorem.** *Let $E \in \mathcal{H}B$, $E(0) = 1$, and let $\log E(z)$ be defined analytically on $\mathbb{C}^+ \cup \{0\}$ such that $\log E(0) = 0$. Then $E \in \mathcal{P}_\kappa$ if and only if*

$$-\frac{\log E(z)}{z} \in \mathcal{N}_\kappa.$$

Secondly, a theorem which generalizes [dB, Problem 34], and turned out to be a most useful tool in some investigations concerned with Hermite-Biehler functions of finite order (cf. [KW3]).

Denote by $\mathcal{N}$ the set of all functions $f$ analytic on $\mathbb{C}^+$ which are of *bounded type* in this half plane, i.e. can be represented as a quotient of two functions analytic and bounded on $\mathbb{C}^+$.

**1.3 Theorem.** *Let $E, F \in \mathcal{H}B$, $E(0) = F(0) = 1$, be given and assume that $\frac{E}{F} \in \mathcal{N}$. Moreover, let $\kappa \in \mathbb{N}_0$. Then $E \in \mathcal{P}_\kappa$ if and only if $F \in \mathcal{P}_\kappa$.*

This result states that the generalized Polya class is stable with respect to bounded type perturbations. It gains significance particularly in view of the following statement:

**1.4 Proposition.** *We have*

$$\Big\{E \in \mathcal{HB} : E \text{ is of finite order}\Big\} = \bigcup_{\kappa\in\mathbb{N}_0} \mathcal{P}_\kappa. \tag{1.5}$$

*More exactly: If $E \in \mathcal{P}_\kappa$, then $E \in \mathcal{HB}$ and the order $\rho$ of $E$ satisfies $\rho \in [2\kappa, 2\kappa+2]$. Conversely, assume that $E$ belongs to $\mathcal{HB}$ and that the order $\rho$ of $E$ is finite. If $\rho$ is not an even integer, then $E \in \mathcal{P}_\kappa$ where $\kappa$ is the unique integer with $\rho \in (2\kappa, 2\kappa+2)$. In case $\rho \in 2\mathbb{Z}$, we have $E \in \mathcal{P}_{\frac{\rho}{2}-1}$ if $E$ is of convergence class and the coefficient of the power $z^\rho$ in the polynomial $q$ in the Hadamard product*

$$E(z) = cz^r e^{q(z)} \prod_{n=1}^{N(E)} \big(1 - \frac{z}{z_n}\big) \exp\left[\frac{z}{z_n} + \ldots + \frac{1}{\rho-1}\Big(\frac{z}{z_n}\Big)^{\rho-1}\right]$$

*is nonpositive. Otherwise $E \in \mathcal{P}_{\frac{\rho}{2}}$.*

Hence the set of all Hermite-Biehler functions of finite order is stable with respect to bounded type perturbations.

Let us outline the contents of the following sections. In §2 we give the proof of Proposition 1.4 which can be viewed also as a product representation for Hermite-Biehler functions of finite order. This result is independent from the other considerations and uses only the Hadamard factorization theorem and some elementary facts concerning complex sequences. In the third section we collect some facts about functions belonging to some generalized Nevanlinna class. In fact, we provide appropriate forms of integral representation and factorization theorems for functions of the class $\mathcal{N}_\kappa$ which are, in their original formulations, due to [KL1] and [DLLS]. These are important ingredients for the proof of our main results. The main task of §4 is to prove a couple of lemmata which deal with functions of the form $-z^{-1}\log F(z)$. These lemmata will play, besides the results of §3, a major role in establishing Theorem 1.2. The fifth, and last, section is devoted to the proof of our main results Theorem 1.2 and Theorem 1.3.

# 2 Product representation of Hermite-Biehler functions

We start with the following elementary, but essential, observation:

Let $(z_j)_{j\in\mathbb{N}}$ be a sequence of nonzero complex numbers which does not accumulate at 0 and satisfies

$$\sum_{j=1}^{\infty} \Big|\operatorname{Im}\frac{1}{z_j}\Big| < \infty. \tag{2.1}$$

Then for each $k \in \mathbb{N}$

$$\sum_{j=1}^{\infty} \Big|\operatorname{Im}\frac{1}{z_j^k}\Big| < \infty. \tag{2.2}$$

In order to see this note that

$$2i\,\mathrm{Im}\,\frac{1}{z_j^k}=\Big(\frac{1}{z_j}\Big)^k-\Big(\frac{1}{\overline{z_j}}\Big)^k=\Big[\Big(\frac{1}{z_j}\Big)-\Big(\frac{1}{\overline{z_j}}\Big)\Big]\cdot\sum_{l=0}^{k-1}\Big(\frac{1}{z_j}\Big)^l\Big(\frac{1}{\overline{z_j}}\Big)^{k-1-l}. \tag{2.3}$$

Since $|z_j|$ is bounded away from 0, the second factor is for each fixed $k\in\mathbb{N}$ bounded from above. The convergence of (2.2) now follows from the convergence of (2.1).

*Proof. (of Proposition 1.4)* First assume that $E\in\mathcal{P}_{\leq\kappa}$. Then by definition $E$ belongs to $\mathcal{H}B$. The product representation (1.3) can, by virtue of (2.2), be rewritten as

$$E(z)=cz^re^{q(z)}\prod_{n=1}^{N(E)}\big(1-\frac{z}{z_n}\big)\exp\Big[\frac{z}{z_n}+\ldots+\frac{1}{2\kappa+1}\Big(\frac{z}{z_n}\Big)^{2\kappa+1}\Big],$$

where

$$q(z)=p(z)-ihz-i\sum_{l=1}^{2\kappa+1}\left(\frac{z^l}{l}\sum_{n=1}^{N(E)}\mathrm{Im}\,\frac{1}{z_n^l}\right).$$

From this and (1.2) we conclude that the order of $E$ is at most $2\kappa+2$.

Conversely assume that $E$ belongs to $\mathcal{H}B$ and is of finite order $\rho$. Let $\kappa$ be the smallest integer with $2\kappa+2>\rho$. The convergent exponent $\rho_1$ of the zeros of $E$ is at most equal to $\rho$, and hence (1.2) follows. By the Hadamard factorization of $E$ we have, again appealing to (2.2),

$$\begin{aligned}E(z)&=cz^re^{q(z)}\prod_{n=1}^{N(E)}\big(1-\frac{z}{z_n}\big)\exp\Big[\frac{z}{z_n}+\ldots+\frac{1}{2\kappa+1}\Big(\frac{z}{z_n}\Big)^{2\kappa+1}\Big]=\\&=cz^re^{p(z)+ir(z)}\prod_{n=1}^{N(E)}\big(1-\frac{z}{z_n}\big)\exp\Big[z\,\mathrm{Re}\,\frac{1}{z_n}+\ldots+\frac{z^{2\kappa+1}}{2\kappa+1}\,\mathrm{Re}\,\frac{1}{z_n^{2\kappa+1}}\Big]\end{aligned}$$

with real polynomials $p$ and $r$ of degree less than $2\kappa+2$. We obtain

$$\frac{E^{\#}(z)}{E(z)}=e^{-2ir(z)}B(z) \tag{2.4}$$

where $B$ denotes the Blaschke product associated with the sequence $(\overline{z_n})_{n\in\mathbb{N}}$. Since $E\in\mathcal{H}B$, the quotient $E^{-1}E^{\#}$, and hence the function $\exp[-2ir(z)]$ is inner. In particular, $\exp[-2ir(z)]$ is of bounded type and nonpositive mean type in $\mathbb{C}^+$. Hence $r$ must be a linear polynomial and the coefficient of $z$ must be nonpositive. We conclude that $E\in\mathcal{P}_{\leq\kappa}$.

The relation (1.5) readily follows. Also we conclude that, if $E\in\mathcal{H}B$ is of finite order $\rho\notin 2\mathbb{Z}$, then $E\in\mathcal{P}_\kappa$ where $\kappa$ is such that $2\kappa<\rho<2\kappa+2$. Moreover, in the case $\rho\in 2\mathbb{Z}$ we either have $E\in\mathcal{P}_{\frac{\varrho}{2}-1}$ or $E\in\mathcal{P}_{\frac{\varrho}{2}}$. The desired assertion follows now on comparing the Hadamard product for $E$ with the product (1.3).

□

# 3 Some properties of $\mathcal{N}_\kappa$ functions

In this section we recall some results about $\mathcal{N}_\kappa$ functions. Let us start with the case $\kappa = 0$. Every $\mathcal{N}_0$ function $Q(z)$ can be written in a unique way as

$$Q(z) = a + bz + \int_{\mathbb{R}} \left( \frac{1}{t-z} - \frac{t}{t^2+1} \right) (t^2+1) d\sigma(t), \tag{3.1}$$

where $a, b \in \mathbb{R},\ b \geq 0$ and $\sigma$ is a positive and finite Borel measure on $\mathbb{R}$. Conversely, given $a, b, \sigma$, then the function represented by (3.1) is an $\mathcal{N}_0$ function. The quantities in (3.1) can be obtained from $Q(z)$ explicitly:

$$\lim_{y \nearrow \infty} \frac{Q(iy)}{iy} = b,\ \ \lim_{z \widehat{\to} \beta} (z-\beta)Q(z) = -(\beta^2+1)\sigma(\{\beta\}). \tag{3.2}$$

Here $z \widehat{\to} \beta$ denotes the nontangential limit from the upper half-plane. Moreover, the so-called Stieltjes Inversion formula holds (see e. g. [RR]):

$$\pi(\int_{(\alpha,\beta)} (x^2+1) d\sigma(x) + \frac{\alpha^2+1}{2}\sigma\{\alpha\} + \frac{\beta^2+1}{2}\sigma\{\beta\}) = \tag{3.3}$$

$$\lim_{y \searrow 0} \int_\alpha^\beta \operatorname{Im} Q(x+iy) dx.$$

As will be mentioned in Lemma 3.6 also $\mathcal{N}_\kappa$ functions for $\kappa \geq 0$ admit an integral representation (see [KL1], Satz 3.1).

Now we bring an elementary result concerning $\mathcal{N}_\kappa$ functions which will turn out to be useful in the sequel.

**3.1 Lemma.** *Let $Q(z) \in \mathcal{N}_\kappa,\ \kappa \in \mathbb{N} \cup \{0\}$. If $\alpha \in \mathbb{C}$, then $(z-\alpha)(z-\bar{\alpha})Q(z) \in \mathcal{N}_{\kappa'}$, where $\kappa' \in \{\kappa-1, \kappa, \kappa+1\}$. The same is true for $(z-\alpha)^{-1}(z-\bar{\alpha})^{-1}Q(z)$. Moreover, if in addition $\tilde{Q}(z) \in \mathcal{N}_{\tilde{\kappa}},\ \kappa \in \mathbb{N} \cup \{0\}$, then $Q(z) + \tilde{Q}(z) \in \mathcal{N}_{\leq \kappa + \tilde{\kappa}}$.*

*Proof.* For simplicity we assume $\alpha = 0$. Consider the following relation

$$\frac{z^2 Q(z) - \bar{w}^2 \bar{Q}(w)}{z - \bar{w}} = zQ(z) + \bar{w}\bar{Q}(w) + z\bar{w}\frac{Q(z) - \bar{Q}(w)}{z - \bar{w}}.$$

Thus the kernel on the left hand side is the sum of a kernel with one positive and one negative square and a kernel with $\kappa$ negative squares. Hence $z^2 Q(z) \in \mathcal{N}_{\kappa'}$ with $\kappa' \in \{\kappa-1, \kappa, \kappa+1\}$. The second assertion can be treated similarly.

If in addition $\tilde{Q}(z) \in \mathcal{N}_{\tilde{\kappa}}$, then the quadratic form (1.4) with $f(z)$ replaced by $Q(z) + \tilde{Q}(z)$ is the sum of the corresponding quadratic forms for $Q(z)$ and $\tilde{Q}(z)$, and hence, by standard Linear Algebra arguments it has at most $\kappa + \tilde{\kappa}$ negative squares.

□

To get a better understanding for the structure of an $\mathcal{N}_\kappa$ function we introduce the following notation. Given $Q(z) \in \mathcal{N}_\kappa$ a point $\beta$ in $\mathbb{R}$ is called a generalized zero of nonpositive type of multiplicity $d$, if

1. $\lim_{z \widehat{\to} \beta} \frac{Q(z)}{(z-\beta)^{2d-1}}$ is finite and nonpositive,

2. $\lim_{z \widehat{\to} \beta} \frac{Q(z)}{(z-\beta)^{2d+1}}$ is finite and positve, or is $\infty$.

The point $\infty$ is called a generalized zero of nonpositive type of multiplicity $d$, if $0$ has this property for the function $Q(-z^{-1})$. It is elementary to see that this function again belongs to $\mathcal{N}_\kappa$. A number $\beta$ in the upper half-plane is called a generalized zero of nonpositive type of multiplicity $d$, if it is a zero of order $d$. If $\beta$ is no generalized zero of nonpositive type, we also say that $\beta$ is a generalized zero of nonpositive type of multiplicity zero.

A number $\beta \in \mathbb{R} \cup \{\infty\} \cup \mathbb{C}^+$ is called a generalized pole of nonpositive type of multiplicity $d$, if this number is a generalized zero of nonpositive type of multiplicity $d$ for the $\mathcal{N}_\kappa$ function $-Q(z)^{-1}$. Note that a number $\beta$ cannot be generalized zero and pole of nonpositive type for the same function. See [DLLS] for further details.

For an $\mathcal{N}_\kappa$ function the sum of the multiplicities of all its generalized zeros of nonpositive type is exactly $\kappa$. The same is true for its generalized poles of nonpositive type. The most striking result concerning this concept is the following (see [DLLS]).

**3.2 Theorem.** *Let the function $Q(z) \in \mathcal{N}_\kappa,\ \kappa \in \mathbb{N} \cup \{0\}$, be given and let $\kappa_1, \kappa_2 \in \{0, \dots, \kappa\}$ be such that $\kappa - \kappa_1$ $(\kappa - \kappa_2)$ is the multiplicity of $\infty$ as a generalized pole (zero) of nonpositive type. Then at least one of the numbers $\kappa_1, \kappa_2$ coincides with $\kappa$.*

*Moreover, if $\alpha_1, \dots, \alpha_{\kappa_1}$ are the generalized poles of nonpositive type of $Q(z)$ in $\mathbb{R} \cup \mathbb{C}^+$ and if $\beta_1, \dots, \beta_{\kappa_2}$ are the generalized zeros of nonpositive type of $Q(z)$ in $\mathbb{R} \cup \mathbb{C}^+$, all counted according to their multiplicities, then the function*

$$Q(z)\frac{\prod_{j=1,\dots,\kappa_1}(z-\alpha_j)(z-\bar{\alpha}_j)}{\prod_{j=1,\dots,\kappa_2}(z-\beta_j)(z-\bar{\beta}_j)}$$

*belongs to the class $\mathcal{N}_0$.*

As an immediate corollary we formulate

**3.3 Corollary.** *Let $Q(z)$ belong to $\mathcal{N}_\kappa,\ \kappa \in \mathbb{N} \cup \{0\}$, with $\infty$ as its only generalized pole of nonpositive type. Then $\kappa$ is the multiplicity of $\infty$ as a generalized pole of nonpositive type.*

*If $\beta_1, \dots, \beta_\kappa$ denote the generalized zeros of nonpositive type of $Q(z)$, then the function*

$$\frac{Q(z)}{\prod_{j=1,\dots,\kappa}(z-\beta_j)(z-\bar{\beta}_j)},$$

*belongs to $\mathcal{N}_0$.*

**3.4 Corollary.** *If $p(z)$ is a real polynomial of degree $d$, then $p(z) \in \mathcal{N}_\kappa$, where $\kappa = \frac{d}{2}$ for even $p$. For odd $d$ we have $\kappa = \frac{d - \operatorname{sgn} a_d}{2}$, where $a_d$ is the coefficient of $z^d$ in $p(z)$.*

*Proof.* Since a constant real valued function belongs to $\mathcal{N}_0$ and since $bz \in \mathcal{N}_0$ for $b \geq 0$ and $bz \in \mathcal{N}_1$ for $b < 0$, we obtain from Lemma 3.1 that $bz^n$ belongs to $\mathcal{N}_{\leq n}$. By the second half of that lemma $p(z) \in \mathcal{N}_\nu$ for some finite $\nu \in \mathbb{N} \cup \{0\}$.

On the other hand it is easy to check that $p(z)$ has its only generalized pole of nonpositive type at $\infty$ and this pole has multiplicity $\kappa$, where $\kappa$ is as stated in the corollary. Now we are done, because the sum of the multiplicities of the poles of nonpositive type agrees with $\nu$.

$\square$

**3.5 Corollary.** *Let $Q(z)$ be an $\mathcal{N}_\kappa,\ \kappa \in \mathbb{N}$, having $\infty$ as generalize pole of nonpositive type of multiplicity $d \in \mathbb{N}$. If $l \in \mathbb{N},\ l \geq d$, and $p(z)$ is any polynomial of degree at most $2l$, then $Q(z) + p(z) \in \mathcal{N}_{\leq(\kappa-d+l)}$.*

*Proof.* By Corollary 3.4 and Lemma 3.1 $Q(z) + p(z) \in \mathcal{N}_{\leq\kappa+d}$. To obtain a sharper estimation for $\nu$ such that $Q(z) + p(z) \in \mathcal{N}_\nu$, note that the generalized poles of nonpositive type different from $\infty$ and their respective multiplicity are the same for $Q(z)$ and $Q(z) + p(z)$. Moreover, an elementary limit calculation shows that the multiplicity of the generalized pole of nonpositive type $\infty$ for $Q(z)+p(z)$ is at most $l$. Since the sum of the multiplicities of the generalized poles of nonpositive type is $\nu$, we obtain $\nu \leq \kappa-d+l$ □

We will extensively make use of the following lemma in the proof of our main results.

**3.6 Lemma.** *Let $Q(z)$ belong to $\mathcal{N}_\kappa,\ \kappa \in \mathbb{N}\cup\{0\}$ with $\infty$ as its only generalized pole of nonpositive type. Let $Q(z)$ be represented in the form (cf. [KL1], Satz 3.1)*

$$Q(z) = \int_{\mathbb{R}} \left( \frac{1}{t-z} - (t+z) \sum_{j=1}^{\kappa+1} \frac{(1+z^2)^{j-1}}{(t^2+1)^j} \right) (t^2+1) d\mu(t) + R(z), \qquad (3.4)$$

*where $R(z)$ is a real polynomial of degree at most $2\kappa+1$ and where*

$$\int_{\mathbb{R}} (1+t^2)^{-\kappa} d\mu(t) < \infty.$$

*If $d \in \mathbb{N},\ d \geq \kappa$, then we have*

$$\frac{Q(z)}{(z^2+1)^d} = b_0 z + R_0(\frac{1}{z-i}) + R_0^{\#}(\frac{1}{z+i}) + Q_0(z), \qquad (3.5)$$

*where $b_0 \in \mathbb{R},\ b_0 \geq 0$ and $b_0 = 0$ in the case $d > \kappa$. $R_0(z)$ is a polynomial of degree at most $d$ and the function $Q_0(z)$ belongs to $\mathcal{N}_0$, such that in its integral representation the constants $a$ and $b$ vanish and the measure $\sigma$ is given by*

$$d\sigma = \frac{d\mu}{(t^2+1)^d}.$$

*Proof.* First assume that $d = \kappa$ and consider the relation

$$\frac{1}{(z^2+1)^\kappa} \left( \frac{1}{t-z} - (t+z) \sum_{j=1}^{\kappa+1} \frac{(1+z^2)^{j-1}}{(t^2+1)^j} \right) = \frac{1}{(t^2+1)^\kappa} \left( \frac{1}{t-z} - \frac{t+z}{t^2+1} \right).$$

Using (3.1) we obtain the relation (3.5) with some real constant $b_0$. To show that $b_0 \geq 0$ note that for a an $\mathcal{N}_0$ function $\tilde{Q}(z)$ as in (3.1) with the corresponding notation one has

$$\lim_{y \nearrow \infty} \frac{\tilde{Q}(iy)}{iy} = \tilde{b} \geq 0.$$

Thus by Corollary 3.3 and a little bit of limit calculation we see that

$$\lim_{y \nearrow \infty} \frac{Q(iy)}{iy((iy)^2+1)^\kappa} = b_0 \geq 0.$$

In the case $d > \kappa$ we have

$$\frac{1}{(z^2+1)^d}\left(\frac{1}{t-z} - (t+z)\sum_{j=1}^{\kappa+1}\frac{(1+z^2)^{j-1}}{(t^2+1)^j}\right) =$$

$$= \frac{1}{(t^2+1)^d}\left(\frac{1}{t-z} + \frac{t+z}{t^2+1}\sum_{j=1}^{d-\kappa-1}\left(\frac{t^2+1}{z^2+1}\right)^j\right),$$

and we see from (3.4) that (3.5) holds with $b_0 = 0$.

□

# 4 Preparatory results

We turn to the investigation of the function

$$L(w;z) := -\frac{\log(1-\frac{z}{w})}{z},$$

where $w \in \mathbb{C}^- \cup \mathbb{R} \setminus \{0\}$ and the branch of the logarithm is chosen such that $\log 1 = 0$. Note that $L(w;z)$ is analytic on $\mathbb{C}$ with exception of a path connecting $w$ and $\infty$. We shall always choose a path which entirely belongs to the lower half plane, so that $L(w;z)$ can be considered as an analytic function on $\mathbb{C}^+ \cup \mathbb{R} \setminus \{w\}$. Of course one can similarly define $L(w;z)$ for $w \in \mathbb{C}^+ \cup \mathbb{R} \setminus \{0\}$, in which case $L(w;z)$ will be considered as analytic function on $\mathbb{C}^- \cup \mathbb{R} \setminus \{w\}$. Note that with these definitions $L(w;z) = L(-w;-z)$, $z \in \mathbb{C}^+, \mathbb{C}^- \cup \mathbb{R} \setminus \{0\}$.

**4.1 Lemma.** *Let $w \in \mathbb{C}^- \cup \mathbb{R} \setminus \{0\}$, then $L(w;z) \in \mathcal{N}_0$. In particular $\operatorname{Im} L(w;t) \geq 0$ for $t \in \mathbb{R} \setminus \{w\}$. In the integral representation* (3.1) *one has $b = 0$ and*

$$\pi\, d\sigma(t) = \frac{\operatorname{Im} L(w;t)}{t^2+1}\, dt.$$

*A similar integral representation (for the lower half plane) holds in the case that $w \in \mathbb{C}^+ \cup \mathbb{R} \setminus \{0\}$.*

*Proof.* The fact that the function under consideration belongs to $\mathcal{N}_0$ is proved in [dB, Theorem 14]. Hence it admits a representation of the form (3.1). From (3.2) and (3.3) we obtain $b = 0$ and the desired formula for $d\sigma$.

□

**4.2 Lemma.** *Let $w \in \mathbb{C}$, $|w| \geq 2$, and let $p \in \mathbb{N}$, $p > 2$. Then the following estimates are valid:*

$$\Big|\operatorname{Re}\frac{1}{w^{p-1}}\arg\frac{1}{1-w}\Big| \leq (p-1)\int_1^\infty \frac{\operatorname{Im} L(w;t)}{|t|^{p-1}}\, dt + p^2\Big|\operatorname{Im}\frac{1}{w}\Big|, \tag{4.1}$$

$$\Big|\operatorname{Re}\frac{1}{w^{p-1}}\arg\frac{1}{1+w}\Big| \leq (p-1)\int_{-\infty}^{-1} \frac{\operatorname{Im} L(w;t)}{|t|^{p-1}}\, dt + p^2\Big|\operatorname{Im}\frac{1}{w}\Big|, \tag{4.2}$$

*Proof.* Since (4.2) follows from (4.1) by the obvious substitution of variables ($w \mapsto -w$, $t \mapsto -t$), it suffices to establish the first estimate. The case $w \in \mathbb{R}$ can be settled in a couple of lines and hence will be postponed. For the core of the proof we will assume that $w \notin \mathbb{R}$.

Fix $w \in \mathbb{C} \setminus \mathbb{R}$, $|w| \geq 2$, and $p \in \mathbb{N}$, $p > 2$. Note that for $t > 0$ we have

$$\frac{\operatorname{Im} L(w;t)}{|t|^{p-1}} = -\operatorname{Im} \frac{\log(1-\frac{t}{w})}{t^p}.$$

By elementary calculus we obtain (differentiate both sides to establish equality)

$$\begin{aligned} &(p-1)\int \frac{\log(1-\frac{t}{w})}{t^p}\,dt = \\ &= -\frac{1}{t^{p-1}}\log\Big(1-\frac{t}{w}\Big) + \frac{1}{w^{p-1}}\log\Big(\frac{1}{t}-\frac{1}{w}\Big) + \sum_{j=1}^{p-2}\frac{1}{jw^{p-1-j}}\frac{1}{t^j}. \end{aligned} \tag{4.3}$$

Since $w \notin \mathbb{R}$ the function $\log(1-w^{-1}t)$ can be defined continuously on $\mathbb{R}$. We choose $\log z$ analytic on $\mathbb{C} \setminus (-\infty, 0]$ such that $\operatorname{Im}\log z = \arg z \in (-\pi, \pi)$. This corresponds to defining $\log(1-w^{-1}z)$ analytically on $\mathbb{C} \setminus \{uw : u \geq 1\}$. Note that with this choice of $\arg z$ we always have $\log(z^{-1}) = -\log z$. Moreover, since $z$ and $1+z^{-1}$ cannot both belong to the same halfplane $\mathbb{C}^+$ or $\mathbb{C}^-$, respectively, we have $\log z + \log(1+z^{-1}) = \log(z+1)$. Evaluation of the integral (4.3) now yields

$$\begin{aligned} &I^+(w) := \int_1^\infty \frac{\log(1-\frac{t}{w})}{t^p}\,dt = \\ &= \frac{\log(1-\frac{1}{w})}{p-1} + \frac{1}{(p-1)w^{p-1}}\log\frac{1}{1-w} - \frac{1}{p-1}\sum_{j=1}^{p-2}\frac{1}{jw^{p-1-j}}. \end{aligned}$$

Taking the imaginary part of this relation it follows that

$$\begin{aligned} \operatorname{Re}\frac{1}{w^{p-1}}\arg\frac{1}{1-w} &= (p-1)\operatorname{Im} I^+(w) - \operatorname{Im}\frac{1}{w^{p-1}}\ln\Big|\frac{1}{1-w}\Big| - \\ &- \arg\Big(1-\frac{1}{w}\Big) + \sum_{j=1}^{p-2}\frac{1}{j}\operatorname{Im}\frac{1}{w^{p-1-j}}. \end{aligned} \tag{4.4}$$

The second term on the right hand side of (4.4) can be estimated from above by making use of (2.3). Note that, since $|w| \geq 2$ and $p > 2$, surely

$$\big|\ln|1-w|\big| \leq \ln\big(1+|w|\big) \leq |w|^{p-2}.$$

Hence we get

$$\begin{aligned} &\Big|\operatorname{Im}\frac{1}{w^{p-1}}\ln\Big|\frac{1}{1-w}\Big|\Big| \leq \Big|\operatorname{Im}\frac{1}{w}\Big| \cdot \Big|\sum_{k=0}^{p-2}\frac{1}{w^k\bar{w}^{p-2-k}}\Big| \cdot \ln|1-w| \leq \\ &\leq \Big|\operatorname{Im}\frac{1}{w}\Big| \cdot \frac{p-1}{|w|^{p-2}} \cdot \ln(1+|w|) \leq (p-1)\Big|\operatorname{Im}\frac{1}{w}\Big|. \end{aligned}$$

Concerning the third term on the right hand side of (4.4) we have ($|w| \geq 2$ implies $1 - \operatorname{Re}\frac{1}{w} \geq \frac{1}{2}$)

$$\big|\arg\Big(1-\frac{1}{w}\Big)\big| = \arctan\frac{|\operatorname{Im}\frac{1}{w}|}{1-\operatorname{Re}\frac{1}{w}} \leq \frac{|\operatorname{Im}\frac{1}{w}|}{1-\operatorname{Re}\frac{1}{w}} \leq 2\,\big|\operatorname{Im}\frac{1}{w}\big|.$$

Moreover, for $j = 1, \dots, p-2$ and $|w| \geq 2$,

$$\big|\operatorname{Im}\frac{1}{w^{p-1-j}}\big| = \big|\operatorname{Im}\frac{1}{w}\big| \cdot \Big|\sum_{k=0}^{p-2-j}\frac{1}{w^k\bar{w}^{p-2-j-k}}\Big| \leq (p-1)\big|\operatorname{Im}\frac{1}{w}\big|.$$

Alltogether we obtain the following estimate:

$$\Big|\operatorname{Re}\frac{1}{w^{p-1}}\arg\frac{1}{1-w}\Big| \leq$$

$$\leq (p-1)\big|\operatorname{Im} I^+(w)\big| + (p-1)\big|\operatorname{Im}\frac{1}{w}\big| + 2\big|\operatorname{Im}\frac{1}{w}\big| + (p-2)\cdot(p-1)\big|\operatorname{Im}\frac{1}{w}\big| \leq$$

$$\leq (p-1)\big|\operatorname{Im} I^+(w)\big| + p^2\big|\operatorname{Im}\frac{1}{w}\big|,$$

which gives the desired conclusion.

It remains to consider the case $w \in \mathbb{R}$. For $w \geq 1$ we have

$$\operatorname{Im} I^+(w) = \int_1^\infty \operatorname{Im}\frac{\log(1-\frac{t}{w})}{t^p}\,dt = \int_w^\infty \pi t^{-p}\,dt = -\frac{\pi}{p-1}\frac{1}{w^{p-1}},$$

whereas $I^+(w) = 0$ for $w < 0$. This shows that for $w \in \mathbb{R}$, $w \notin [0,1)$,

$$\Big|\operatorname{Re}\frac{1}{w^{p-1}}\arg\frac{1}{1-w}\Big| \leq (p-1)\big|\operatorname{Im} I^+(w)\big|.$$

$\square$

**4.3 Corollary.** *Let $(z_j)_{j\in\mathbb{N}}$ be a sequence of nonzero complex numbers belonging to the closed lower half plane which has no finite accumulation point and satisfies $\sum_{j=1}^\infty \operatorname{Im}\frac{1}{z_j} < \infty$. Moreover, let $\kappa \in \mathbb{N}\cup\{0\}$ be fixed. If*

$$\sum_{j=1}^\infty \int_{\mathbb{R}} \frac{\operatorname{Im} L(z_j;t)}{(1+t^2)^{1+\kappa}}\,dt < \infty,$$

*then also*

$$\sum_{j=1}^\infty \Big|\operatorname{Re}\frac{1}{z_j^{2+2\kappa}}\Big| < \infty \tag{4.5}$$

*Proof.* As the sequence under consideration has no accumulation point in the complex plane, it is no loss of generality to assume that $|z_j| \geq 2$ for all $j \in \mathbb{N}$.

If $w$ is a point of the left halfplane which lies outside the disk $\{z \in \mathbb{C} : |z| < 2\}$, then $w+1$ and hence also $(w+1)^{-1}$ lie outside the angle $\{z \in \mathbb{C} : |\arg z| < \gamma\}$ with $\gamma = \arctan 2$. Hence we have $|\arg(1+z_j)^{-1}| \geq \gamma$ for $\arg z_j \in [-\pi, -\frac{\pi}{2}]$ and $|\arg(1-z_j)^{-1}| \geq \gamma$ for $\arg z_j \in [-\frac{\pi}{2}, 0]$.

From Lemma 4.2 and the fact that $2t^2 \geq t^2+1$ for $t \in \mathbb{R} \setminus (-1,1)$ we find that

$$\Big|\operatorname{Re}\frac{1}{z_j^{2+2\kappa}}\Big| \leq \left\{ \begin{array}{ll} \frac{1}{\gamma}\big|\operatorname{Re}\frac{1}{z_j^{2+2\kappa}}\arg\frac{1}{1+z_j}\big|, & \arg z_j \in [-\pi,-\frac{\pi}{2}] \\ \frac{1}{\gamma}\big|\operatorname{Re}\frac{1}{z_j^{2+2\kappa}}\arg\frac{1}{1-z_j}\big|, & \arg z_j \in [-\frac{\pi}{2},0] \end{array} \right\} \leq$$

$$\leq \frac{2+2\kappa}{\gamma} \int\limits_{\mathbb{R}\setminus(-1,1)} \frac{\operatorname{Im} L(z_j;t)}{|t|^{2+2\kappa}}\,dt + (3+2\kappa)^2\Big|\operatorname{Im}\frac{1}{z_j}\Big| \leq$$

$$\leq \frac{(2+2\kappa)2^{1+\kappa}}{\gamma} \int\limits_{\mathbb{R}\setminus(-1,1)} \frac{\operatorname{Im} L(z_j;t)}{(t^2+1)^{1+\kappa}}\,dt + (3+2\kappa)^2\Big|\operatorname{Im}\frac{1}{z_j}\Big|$$

We have obtained a convergent majorant for the series (4.5).

□

If $E \in \mathcal{HB}$, $E(0)=1$, we can similarly consider the function

$$L(E;z) := -\frac{\log E(z)}{z},\ z \in \mathbb{C}^+.$$

Obviously $L(E;z)$ is analytic on $\mathbb{C}^+ \cup \mathbb{R}$ with exception of the real zeros of $E$.

It is important to note that, although we can of course in general not expect $L(E;z)$ to belong to $\mathcal{N}_0$, positivity on the real axis is retained: Let us show that for $E \in \mathcal{HB}$, $E(0)=1$,

$$\operatorname{Im} L(E;t) \geq 0,\ t \in \mathbb{R}. \tag{4.6}$$

For any $x_0 \in \mathbb{R}$ with $E(x_0) \neq 0$ we have

$$0 \leq \Big(\frac{\partial}{\partial y}\operatorname{Re}\log E\Big)(x_0) = \Big(\frac{\partial}{\partial x}\operatorname{Im}\big(-\log E\big)\Big)(x_0),$$

whereas at a point $x_0 \in \mathbb{R}$ which is a zero of $E$ of multiplicity $r$

$$\operatorname{Im}\Big(-\log E(x+)\Big) - \operatorname{Im}\Big(-\log E(x-)\Big) = r\pi.$$

Hence $\operatorname{Im}(-\log E(x))$ is nondecreasing on $\mathbb{R}$. Our assumption $E(0)=1$ ensures that $\operatorname{Im}(-x^{-1}\log E(x))$ is nonnegative for all $x \in \mathbb{R}$.

Moreover, an elementary consideration shows that for any sequence $(z_n)_{n\in\mathbb{N}}$, $z_n \in \mathbb{C}^+$, with $\lim_{n\to\infty} z_n = x_0 \in \mathbb{R}$, we have

$$\limsup_{n\to\infty}\operatorname{Im}\Big(-\log E(z_n)\Big) \in \Big[\operatorname{Im}\Big(-\log E(x-)\Big), \operatorname{Im}\Big(-\log E(x+)\Big)\Big]. \tag{4.7}$$

In particular the function $\operatorname{Im} L(E;z)$ is bounded on every bounded subset of $\mathbb{C}^+ \cup \{x \in \mathbb{R} : E(x) \neq 0\}$.

Under the hypotheses that $L(E;z)$ belongs to some generalized Nevanlinna class, the analyticity of $E$ along the real axis implies a sharp result on the integral representation of $L(E;z)$.

**4.4 Lemma.** *Let $E \in \mathcal{HB}$, $E(0)=1$, be given and assume that $L(E;z) \in \mathcal{N}_{\leq\kappa}$. Then $L(E;z)$ has no pole of nonpositive type other than $\infty$. In the integral representation* (3.5) *which reads as ($d \geq \kappa$)*

$$\frac{L(E;z)}{(z^2+1)^d} = b_0 z + R_0\Big(\frac{1}{z-i}\Big) + R_0^{\#}\Big(\frac{1}{z+i}\Big) + \int\limits_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{t}{t^2+1}\Big)(t^2+1)\,d\sigma(t)$$

*the measure $d\sigma(t)$ is given as*

$$d\sigma(t) = \frac{\operatorname{Im} L(E;t)}{\pi(t^2+1)^d}\,dt. \tag{4.8}$$

*Proof.* Since $E$ is analytic on $\mathbb{R}$, for any $x_0 \in \mathbb{R}$ we can write $E(z) = (z - x_0)^r E_1(z)$ and a direct calculation using the definition of a generalized pole of nonpositive type verifies that $x_0$ cannot be such a generalized pole. Clearly, $L(E;z)$ also has no generalized pole of nonpositive type in $\mathbb{C}^+$.

In order to obtain the formula (4.8) we apply the Stieltjes Inversion formula (3.3) to the representation (3.5) of $L(E;z)$ and obtain with the notation of Lemma 3.6 for $a < b$

$$\int\limits_{(a,b)} (x^2+1)d\sigma(x) + \frac{a^2+1}{2}\sigma\{a\} + \frac{b^2+1}{2}\sigma\{b\} = \frac{1}{\pi}\lim_{y\searrow 0}\int\limits_a^b \operatorname{Im} Q_0(x+iy)\,dx.$$

As all addends but $Q_0(z)$ on the right hand side of (3.5) are real we obtain by Lebesgue's Bounded Convergence Theorem

$$\lim_{y\searrow 0}\int\limits_a^b \operatorname{Im} Q_0(x+iy)\,dx = \lim_{y\searrow 0}\int\limits_a^b \frac{\operatorname{Im} L(E;z)}{\big((x+iy)^2+1\big)^\kappa}\,dx = \int\limits_a^b \frac{\operatorname{Im} L(E;x)}{(x^2+1)^\kappa}\,dx.$$

□

# 5 Proof of the main results

We start with the proof of Theorem 1.2. Let $E \in \mathcal{H}B$, $E(0) = 1$, be fixed. We have to show that $E \in \mathcal{P}_\kappa$ if and only if $L(E;z) = -z^{-1}\log E(z) \in \mathcal{N}_\kappa$. Note that the case $\kappa = 0$ is nothing else but [dB, Theorem 14], hence we may assume throughout that $\kappa > 0$.

*Proof. (of Theorem 1.2, necessity)* Let $E \in \mathcal{P}_\kappa$ and write as in the definition of $\mathcal{P}_\kappa$

$$E(z) = e^{p(z)-ihz}\prod_{n=1}^{N(E)}\big(1 - \frac{z}{z_n}\big)e^{z\operatorname{Re}\frac{1}{z_n}+\ldots+z^{2\kappa+1}\operatorname{Re}\frac{1}{(2\kappa+1)z_n^{2\kappa+1}}}.$$

Then, with $r_n(z) = -\big(\operatorname{Re}\frac{1}{z_n} + \ldots + z^{2\kappa}\operatorname{Re}\frac{1}{(2\kappa+1)z_n^{2\kappa+1}}\big)$,

$$L(E;z) = -\frac{p(z)}{z} + ih + \sum_{n=1}^{N(E)}\Big(L(z_n;z) + r_n(z)\Big) =$$

$$= \lim_{N\to\infty}\left[ih + \Big(-\frac{p(z)}{z} + \sum_{n=1}^{\min\{N,N(E)\}} r_n(z)\Big) + \sum_{n=1}^{\min\{N,N(E)\}} L(z_n;z)\right].$$

Since $r_n$ is a real polynomial of degree at most $2\kappa$ and $-z^{-1}p(z)$ has degree at most $2\kappa+1$ where the coefficient of $z^{2\kappa+1}$ is nonnegative, by Corollary 3.4 the polynomial in the rounded brackets belongs to $\mathcal{N}_{\leq\kappa}$. Since $L(z_n;z)$ and the constant function $ih$ belong to $\mathcal{N}_0$, Lemma 3.1 shows that the function in the square brackets belongs to $\mathcal{N}_{\leq\kappa}$. As a limit of functions belonging to $\mathcal{N}_{\leq\kappa}$ the function $L(E;z)$ also belongs to $\mathcal{N}_{\leq\kappa}$. This can be seen by considering the respective quadratic forms (1.4) and the fact that the limit of quadratic forms of at most $\kappa$ negative squares has the same property.

$\square$

*Proof. (of Theorem 1.2, sufficiency)* Let $E \in \mathcal{HB}$, $E(0) = 1$, with $L(E;z) \in \mathcal{N}_{\leq\kappa}$ be given. For the proof that $E \in \mathcal{P}_{\leq\kappa}$ we will first settle the particular case that $E$ is zerofree. Then the general case is reduced to this particular one.

*Step 1:* Assume that $E$ has no zeros. Then $E^{-1}E^{\#}$ is an inner function on $\mathbb{C}^+$ which has no zeros and is analytic on $\mathbb{C}\cup\mathbb{R}$. Thus this function equals $\exp(2ihz)$ for some $h \geq 0$ (see e.g. Theorem 9 in [dB]). We conclude that $\exp(ihz)E(z)$ is a real function without zeros and, therefore, that

$$\operatorname{Im} L(E;x) = h,\ x \in \mathbb{R}.$$

Applying Lemma 4.4 to $L(E;z)(\in \mathcal{N}_{\leq\kappa})$ as well as to the function $ih(\in \mathcal{N}_0)$ we obtain

$$\begin{gathered}\frac{L(E;z)}{(z^2+1)^\kappa} = b_0 z + R_1\Big(\frac{1}{z-i}\Big) + R_1^{\#}\Big(\frac{1}{z+i}\Big) + \int_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{t}{t^2+1}\Big)\frac{h\,dt}{(t^2+1)^\kappa},\\ \frac{ih}{(z^2+1)^\kappa} = R_2\Big(\frac{1}{z-i}\Big) + R_2^{\#}\Big(\frac{1}{z+i}\Big) + \int_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{t}{t^2+1}\Big)\frac{h\,dt}{(t^2+1)^\kappa}.\end{gathered} \tag{5.1}$$

where $b_0 \geq 0$ and $R_1, R_2$ are polynomials of degree at most $\kappa$. Thus $-z(L(E;z)-ih) = p(z)$ for some real polynomial $p(z)$ of degree at most $2\kappa+2$ with $p(0) = 0$ and with coefficient $-b_0(\leq 0)$ for the power $z^{2\kappa+2}$. We conclude that $E \in \mathcal{P}_{\leq\kappa}$.

*Step 2:* Given arbitrary $E \in \mathcal{HB}$, $E(0) = 1$, with $L(E;z) \in \mathcal{N}_{\leq\kappa}$, we are going to factor out finitely many zeros without changing the assumptions on $E$. More exactly we shall prove the following: Let $z_1, \ldots, z_N$ be zeros of $E$ and put

$$E_N(z) := \frac{E(z)}{\prod_{j=1}^N (1 - \frac{z}{z_j})}.$$

Then $E_N \in \mathcal{HB}$, $E_N(0) = 1$ and $L(E_N;z) \in \mathcal{N}_{\leq\kappa}$.

Since $E \in \mathcal{HB}$, the function $E^{-1}E^{\#}$ is inner. Hence also, by taking out a part of the Blaschke product,

$$\frac{E_N^{\#}(z)}{E_N(z)} = \prod_{j=1}^N \frac{1-\frac{z}{z_j}}{1-\frac{z}{\overline{z_j}}} \cdot \frac{E^{\#}(z)}{E(z)}$$

is inner and thus $E_N \in \mathcal{HB}$. Obviously $E_N(0) = 1$, and we have

$$L(E_N;z) = L(E;z) - L\Big(\prod_{j=1}^N \big(1 - \frac{z}{z_j}\big); z\Big) = L(E;z) - \sum_{j=1}^N L(z_j;z).$$

By assumption $L(E;z) \in \mathcal{N}_{\leq\kappa}$ and by Lemma 4.1 $\sum_{j=1}^N L(z_j;z) \in \mathcal{N}_0$. Recall now that we assumed throughout the present proof that $\kappa > 0$. Therefore, Lemma 4.4 implies

$$\frac{L(E_N;z)}{(z^2+1)^\kappa} = b_0 z + R\Big(\frac{1}{z-i}\Big) + R^{\#}\Big(\frac{1}{z+i}\Big) + \int\limits_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{t}{t^2+1}\Big)(t^2+1)\,d\mu(t) -$$

$$- \int\limits_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{t}{t^2+1}\Big)(t^2+1)\,d\sigma_N(t) =$$

$$= b_0 z + R\Big(\frac{1}{z-i}\Big) + R^{\#}\Big(\frac{1}{z+i}\Big) + \int\limits_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{t}{t^2+1}\Big)(t^2+1)\,d\nu(t)$$

where

$$d\mu(t) = \frac{\operatorname{Im} L(E;t)}{(t^2+1)^\kappa}\,dt,\ \ d\sigma_N(t) = \frac{\sum_{j=1}^N \operatorname{Im} L(z_j;z)}{(t^2+1)^\kappa}\,dt,$$

and

$$d\nu(t) = d\mu(t) - d\sigma_N(t) = \frac{\operatorname{Im} L(E_N;z)}{(t^2+1)^\kappa}\,dt. \tag{5.2}$$

By (4.6) we know that $d\nu$ is a positive measure. Hence Lemma 3.1 and Corollary 3.5 yield $L(E_N;z) \in \mathcal{N}_{\leq\kappa}$.

Combining the result of the current step with the knowledge provided in *Step 1*, we see that the case that $E$ has only finitely many zeros is readily settled.

*Step 3:* In the last step we shall employ a limiting argument in order to prove the following: Let $(z_j)_{j\in\mathbb{N}}$ be the sequence of zeros of $E$. Then

$$\sum_{j=1}^\infty \frac{1}{|z_j|^{2\kappa+2}} < \infty. \tag{5.3}$$

The product

$$P(z) := \prod_{j=1}^\infty \big(1 - \frac{z}{z_j}\big) e^{z \operatorname{Re}\frac{1}{z_j} + \ldots + z^{1+2\kappa} \operatorname{Re}\frac{1}{(1+2\kappa) z_j^{1+2\kappa}}} \tag{5.4}$$

converges locally uniformly on $\mathbb{C}$ and

$$L\Big(\frac{E}{P};z\Big) \in \mathcal{N}_{\leq\kappa}.$$

Once this claim is established the proof of sufficiency is completed by an application of *Step 1* to the function $P^{-1}E$.

Since the measure $d\nu$ in (5.2) is positive, we have for all $N \in \mathbb{N}$

$$\sum_{j=1}^N \int\limits_{\mathbb{R}} \frac{\operatorname{Im} L(z_j;t)}{(t^2+1)^{\kappa+1}}\,dt = \int\limits_{\mathbb{R}} d\sigma_N(t) \leq \int\limits_{\mathbb{R}} d\mu(t) < \infty$$

An application of Corollary 4.3 yields

$$\sum_{j=1}^\infty \big|\operatorname{Re}\frac{1}{z_j^{2\kappa+2}}\big| < \infty.$$

Since $E \in \mathcal{H}B$ and therefore satisfies (2.2), we find that in fact (5.3) holds. Again appealing to (2.2) we conclude that with the Weierstraß product for $(z_j)_{j\in\mathbb{N}}$ also the product (5.4) converges.

Consider the function

$$G_N(z) := E(z) \cdot \left[ \prod_{j=1}^{N} \left(1 - \frac{z}{z_j}\right) e^{z \operatorname{Re} \frac{1}{z_j} + \ldots + z^{1+2\kappa} \operatorname{Re} \frac{1}{(1+2\kappa) z_j^{1+2\kappa}}} \right]^{-1}$$

Then $L(G_N; z)$ and $L(E_N; z)$ differ only by a polynomial of degree at most $2\kappa$ and Corollary 3.5 implies that with $L(E_N; z)$ also $L(G_N; z)$ belongs to $\mathcal{N}_{\leq\kappa}$. Hence

$$L\Big(\frac{E}{P}; z\Big) = \lim_{N\to\infty} L(G_N; z) \in \mathcal{N}_{\leq\kappa}.$$

□

We turn to the proof of Theorem 1.3. It will employ the Phragmen-Lindelöf principle in the following version (which is a slight extension of [dB, Theorem 1] and is proved exactly the same).

**5.1 Theorem** (Phragmen-Lindelöf principle)**.** *Let $f$ be analytic in $\mathbb{C}^+$ and assume that $|f(z)|$ has a continuous extension to a set $\Omega$ which contains $\mathbb{C}^+ \cup \mathbb{R}$ with possible exception of a discrete subset of $\mathbb{R}$. Moreover, assume that*

$$\liminf_{r\to\infty} \frac{1}{r} \int_0^\pi \log^+ |f(re^{i\theta})| \sin\theta \, d\theta = 0$$

*and that for all $x \in \mathbb{R}$*

$$\limsup_{z\to x, z\in\Omega} |f(z)| \leq 1.$$

*Then $|f(z)| \leq 1$ for all $z \in \mathbb{C}^+$.*

In order to apply Theorem 5.1 we need the following

**5.2 Lemma.** *If $Q(z) \in \mathcal{N}_0$ and in its integral representation* (3.1) *we have $b = 0$, then*

$$\lim_{r\to+\infty} \frac{1}{r} \int_0^\pi |Q(re^{i\theta})| \sin\theta \, d\theta = 0.$$

*Proof.* With the notation of (3.1) we can assume that $a = 0$ and calculate

$$0 \leq \frac{1}{r} \int_0^\pi |Q(re^{i\theta})| \sin\theta d\theta \leq \int_0^\pi \int_{\mathbb{R}} \frac{\sin\theta}{r} \left| \frac{1 + tre^{i\theta}}{t - re^{i\theta}} \right| d\sigma(t) \, d\theta. \tag{5.5}$$

The integrand converges pointwise (for all $t, \theta$) to zero as $r$ converges to $+\infty$. Moreover, as

$$\frac{\sin\theta}{r} \frac{1 + tre^{i\theta}}{t - re^{i\theta}} = \sin\theta e^{i\theta} + \frac{\frac{1}{r} + e^{2i\theta}}{\frac{t - r\cos\theta}{\sin\theta} - ir},$$

the integrand is bounded by $(r > 1)$

$$1 + \frac{\frac{1}{r} + 1}{r} \leq 3.$$

The Bounded Convergence Theorem yields that (5.5) converges to zero as $r$ tends to $+\infty$.

□

*Proof. (of Theorem 1.3)* Assume that $E \in \mathcal{P}_\kappa$ and $E^{-1}F \in \mathcal{N}$. By Theorem 1.2 and since the roles of $E$ and $F$ may be exchanged, it suffices to prove that $L(F;z) \in \mathcal{N}_{\leq\kappa}$.

Denote by $R((z-i)^{-1})$ the principal part of the Laurent expansion of $(z^2+1)^{-\kappa}L(F;z)$ at $i$. Hence $R$ is a polynomial of degree at most $\kappa$ and the function

$$\varphi(z) := \frac{L(F;z)}{(z^2+1)^\kappa} - R\Big(\frac{1}{z-i}\Big) - R^\#\Big(\frac{1}{z+i}\Big)$$

is analytic in $\mathbb{C}^+$. We shall show by an application of the Phragmen-Lindelöf principle that $\exp(i\varphi(z))$ is bounded by 1 in $\mathbb{C}^+$ or, equivalently, that $\varphi \in \mathcal{N}_0$.

Let us proceed checking the assumptions of Theorem 5.1 one after another. The function $\exp(i\varphi(z))$ is analytic in $\mathbb{C}^+$ and continuous on the real line with exception of the zeros of $F$ which of course form a discrete set. Moreover, by (4.6) and (4.7), we have

$$\operatorname{Im}\varphi(t) = \operatorname{Im}\frac{L(F;t)}{(t^2+1)^\kappa} \geq 0,\ \ t \in \mathbb{R}, F(t) \neq 0,$$

and

$$\liminf_{z\to x,\ z\in\mathbb{C}^+} \operatorname{Im}\frac{L(F;z)}{(z^2+1)^\kappa} \geq 0. \tag{5.6}$$

Hence the boundedness condition along the real line is satisfied.

It remains to show that

$$\liminf_{r\to+\infty} \frac{1}{r}\int_0^\pi \log^+\big|\exp(i\varphi(re^{i\theta}))\big| \sin\theta\, d\theta = 0. \tag{5.7}$$

Note that $|\exp(i\varphi(z))| = \exp(-\operatorname{Im}\varphi(z))$. The function $\varphi(z)$ can be written as

$$\varphi(z) = \frac{L\big(\frac{F}{E};z\big)}{(z^2+1)^\kappa} + \frac{L(E;z)}{(z^2+1)^\kappa} - R\Big(\frac{1}{z-i}\Big) - R^\#\Big(\frac{1}{z+i}\Big) \tag{5.8}$$

Since we assume that $E^{-1}F \in \mathcal{N}$ and obviously this function has no zeros in the upper half plane, [dB, Theorem 9] yields that

$$\log\frac{F(z)}{E(z)} = cz + Q_1(z) - Q_2(z),$$

with $c \in \mathbb{R}$ and $Q_1, Q_2 \in \mathcal{N}_0$ such that in the respective representations (3.1) of $Q_1$ and $Q_2$ the constant $b$ equals zero. Moreover, by assumption $L(E;z) \in \mathcal{N}_\kappa$. Hence, according to Lemma 3.6 we may write

$$\frac{L(E;z)}{(z^2+1)^\kappa} = R_0\Big(\frac{1}{z-i}\Big) + R_0^\#\Big(\frac{1}{z+i}\Big) + b_0 z + Q_0(z). \tag{5.9}$$

We collected the rational terms in (5.8) and (5.9)

$$S(z) := R\Big(\frac{1}{z-i}\Big) + R^{\#}\Big(\frac{1}{z+i}\Big) - R_0\Big(\frac{1}{z-i}\Big) - R_0^{\#}\Big(\frac{1}{z+i}\Big)$$

and obtain (say for $|z| \geq 2$)

$$\begin{aligned} -\operatorname{Im}\varphi(z) = \operatorname{Im}\frac{cz+Q_1(z)-Q_2(z)}{(z^2+1)^\kappa} + \operatorname{Im} S(z) - \operatorname{Im}\big(b_0 z + Q_0(z)\big) \leq \\ \leq \operatorname{Im}\frac{cz+Q_1(z)-Q_2(z)}{(z^2+1)^\kappa} + \operatorname{Im} S(z) \leq \\ \leq \left|\frac{c}{(z^2+1)^\kappa}\right| + |Q_1(z)| + |Q_2(z)| + |S(z)| \end{aligned} \tag{5.10}$$

The first as well as the last summand in (5.10) is bounded and hence does not contribute to the limes inferior in (5.7). Lemma 5.2 now implies (5.7).

We have proved that $\varphi(z) \in \mathcal{N}_0$. By Lemma 3.1,

$$(z^2+1)^\kappa \varphi(z) \in \mathcal{N}_{\leq\kappa}.$$

It is easy to see that this function has no finite generalized poles of nonpositive type. Since $(z^2+1)^\kappa\varphi(z)$ and $L(F;z)$ differ only by a polynomial of degree at most $2\kappa$ we may apply Corollary 3.5 to obtain $L(F;z) \in \mathcal{N}_{\leq\kappa}$. □

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