PONTRYAGIN SPACES OF ENTIRE FUNCTIONS I

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We give a generalization of L.de Branges theory of Hilbert spaces of entire functions to the Pontryagin space setting. The aim of this - first - part is to provide some basic results and to investigate subspaces of Pontryagin spaces of entire functions. Our method makes strong use of L.de Branges's results and of the extension theory of symmetric operators as developed by M.G.Krein.

1 Introduction

In [dB1]-[dB6] L. de Branges developed a theory of Hilbert spaces $\mathfrak{P}$, which consist of entire functions and satisfy certain additional axioms:

(i) $\mathfrak{P}$ is a reproducing kernel space.

(ii) The mapping $F(z) \mapsto \overline{F(\overline{z})}$ is an (antilinear) isometry of $\mathfrak{P}$ onto itself.

(iii) If $F \in \mathfrak{P}$, $w \in \mathbb{C} \setminus \mathbb{R}$, and $F(w) = 0$, then $F(z)\frac{z-\overline{w}}{z-w} \in \mathfrak{P}$ and has the same norm as $F(z)$.

If $\mathfrak{P}$ satisfies these axioms we shall speak of a dB-Hilbert space. A main subject of the theory of such spaces is to determine those subspaces of a given dB-Hilbert space, or more general those subspaces of a given space $L^2(\mu)$, which are again dB-Hilbert spaces. L. de Branges proved that these subspaces (satisfying a certain additional condition) form a chain with respect to inclusion. The spaces belonging to this chain are related by an integral equation, or equivalently by a so-called canonical system of differential equations.

An example for such a situation can be obtained by using the theorem of Payley and Wiener describing the Fourier transforms of $L^2(\mathbb{R})$ functions with compact support. The Payley-Wiener space $\mathfrak{P}_a$, $0 < a < \infty$, is the space of all entire functions F of exponential type at most a such that $F|_{\mathbb{R}} \in L^2(\mathbb{R})$, endowed with the $L^2(\mathbb{R})$-norm. The chain $(\mathfrak{P}_a)_{a\in(0,\infty)}$ of subspaces of $L^2(\mathbb{R})$ is exactly the chain of subspaces in the above described sense.

The mentioned results find various applications, for example in the solution of inverse spectral problems and related questions (see e.g. [DK], [KL4], [KL5], [W]).

Independently, M.G.Krein developed a theory of entire operators (see e.g. [K2], [K3], [K6], [GG]) and studied their selfadjoint extensions. Such operators play, besides their theoretical interest, an important role in various classical problems (compare e.g. [K1], [K4]). For example the continuation problem of a positive definite function on a finite interval gives rise to an entire operator.

It turns out that there exists an intimate connection between these two theories. In fact, an entire operator can be represented as the operator of multiplication by the independent variable in a conveniently chosen dB-Hilbert space. Under a certain additional condition also the converse is true. Moreover, so called transfer matrices, which play a vital role in the theory of dB-subspaces, have been identified as resolvent matrices in the sense of [KL3] or [KW].

The present paper is concerned with a generalization of L.de Branges's theory to indefinite inner product spaces. More precisely, we consider inner product spaces $\mathfrak{P}$, consisting of entire functions, such that the isotropic part $\mathfrak{P}^\circ$ of $\mathfrak{P}$ has finite dimension, $\mathfrak{P}/\mathfrak{P}^\circ$ is a Pontryagin space and which satisfy additional axioms similar to *(i)-(iii)*. Some basic results concerning such dB-spaces are given and subspaces of a given dB-space which are themselves dB-spaces are studied. In particular, we obtain that the dB-subspaces form a chain and are connected by transfer matrices.

Although our presentation is based on [dB1]-[dB6] and [dB7], the proofs rely in many cases on the above mentioned connection with M.G.Krein's theory and on results given in an operator theoretic context concerning selfadjoint extensions of symmetric operators.

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The preliminary Section 2 contains some results concerning Nevanlinna functions which are derived from their well known integral representation (compare [KL2]). The rest of the paper splits into three parts.

In Sections 3-7 we develop a basic theory of dB-spaces, and study the operator $\mathcal{S}$ of multiplication by the independent variable in such spaces. Section 3 contains the proper definition of a dB-space, and it is shown that for each dB-space $\mathfrak{P}$ there exists a positive definite inner product on the same set of entire functions which turns $\mathfrak{P}$ into a dB-Hilbert space. This allows us to use the results given in [dB1]-[dB6]. In Section 5 we show that, similar as in the Hilbert space case, a nondegenerated dB-space is determined by a single entire function $E(z)$ with specific properties. Also the converse result holds. Sections 4 and 6 deal with the operator $\mathcal{S}$ and its extensions, in particular with its selfadjoint extensions. The extensions of $\mathcal{S}$ correspond to certain entire functions, the so-called associated functions. We determine those associated functions which correspond to selfadjoint extensions of $\mathcal{S}$ and compute the respective Q-functions. Finally, in Section 7, we discuss decompositions of a (nondegenerated) dB-space, which relate to the spectral decomposition of such a Q-function. This result gives an analogue to the notion of a phase function in [dB7].

The Sections 8 and 9 deal with matrix functions which satisfy a certain kernel relation. In these sections results concerning resolvent matrices come into play. Via the so-called Potapov-Ginzburg transformation, spectral properties of matrix Nevanlinna functions are used to obtain results on the structure of the reproducing kernel space generated by the above mentioned matrix kernel. In Section 10 these results are applied to obtain a characterization of associated functions. In this context there occurs the notion of generalized elements or triplet spaces in the sense of [KW].

Sections 11-13 are devoted to the study of subspaces of a given dB-space. First we collect some basic results which follow from [dB7]. The main subject of Section 11 is to show that only finitely many members of the (unique) chain of subspaces of a given dB-space can be degenerated. In Sections 12 and 13 transfer matrices of nondegenerated subspaces of a dB-space are investigated. The existence and uniqueness of such matrices and their factorizations is essential for the construction of the canonical system (of differential equations) connected with the chain of subspaces.

Concerning the theory of inner product spaces we use the notation and results of [IKL] (or [B]), concerning symmetric and selfadjoint operators (or relations) we use [DS1] and [L]. Our standard reference for the theory of Hilbert spaces of entire functions is [dB7]. The results on resolvent matrices and generalized elements are taken from [KW] (compare also [KL3]). We also use the notion of reproducing kernel Pontryagin spaces and some results which can be found in [ADSR1] (see also [dB8], [ADSR2], [ADSR3]).

It will be the subject of a forthcoming note to study the subspaces of $L^2(\mu)$ (understood in a certain distributional sense), the canonical system of differential equations associated with a chain of subspaces and to give some applications.

2 Some results on Nevanlinna functions

In this preliminary section we give some results concerning Nevanlinna functions, which will be useful later. First let us recall the notion of a Nevanlinna function.

Definition 2.1. An $n\times n$-matrix valued function Q, which is analytic in an open set $O\neq\emptyset$, is said to be element of the Nevanlinna class $\mathcal{N}_\kappa^{n\times n}$, if $Q(\overline{z})=Q(z)^*$, whenever $z,\overline{z}\in O$, and if the kernel

$$N_Q(z,w) = \frac{Q(z) - Q(w)^*}{z - \overline{w}},\ z, w \in O,$$

has κ negative squares.

It is well known that Q can be continued analytically to its maximal domain of holomorphy $\rho(Q)$ which contains $\mathbb{C} \setminus \mathbb{R}$ with a possible exception of a finite number of points.

For abbreviation we write $\mathcal{N}_\kappa$ instead of $\mathcal{N}_\kappa^{1\times 1}$. The reproducing kernel Pontryagin space generated by the kernel N_Q for a Nevanlinna function Q (compare [ABDS1]) is denoted by $\mathfrak{K}(Q)$.

It has been proved in [KL2] that a function Q, meromorphic in $\mathbb{C}^+$, which belongs to $\mathcal{N}_\kappa$ allows an integral representation:

$$Q(z) = \int_{-\infty}^{\infty} \left(\frac{1}{t-z} - \sum_{j=0}^{r} \chi_j(t) S_j(t,z) \right) \varphi(t)(1+t^2) d\sigma(t) + R_0(z) +$$

$$+ \sum_{j=1}^{r} R_j\left(\frac{1}{\alpha_j - z}\right) + \sum_{k=1}^{s} \left(T_k\left(\frac{1}{z-\beta_k}\right) + T_k\left(\frac{1}{z-\beta_k}\right)^{\#} \right). \tag{2.1}$$

Here $r \geq 0$, $s \geq 0$, $\alpha_1, \ldots, \alpha_r \in \mathbb{R}$ are the so called critical points of Q and $\beta_1, \ldots, \beta_s \in \mathbb{C}^+$. The function σ is a nondecreasing bounded function on $\mathbb{R}$, and is continuous from the left. The function $\varphi(t)$ is given by

$$\varphi(t) = (1+t^2)^{\rho_0} \prod_{j=1}^{r} \frac{(1+t^2)^{\rho_j}}{(t-\alpha_j)^{2\rho_j}},$$

where ρ_j, $j = 1, \ldots, r$, is the order of the critical point α_j, and ρ_0 is the order of the point ∞ as a critical point of Q. Moreover, R_0 and R_j, $j = 1, \ldots, r$, are polynomials with real coefficients, of degree $\leq 2\rho_0 + 1$ and $\leq 2\rho_j$, respectively, and $R_j(0) = 0$, $j = 1, \ldots, r$. T_k are polynomials with $T_k(0) = 0$. The expressions $S_j(t,z)$ are given by

$$S_j(t,z) = -\sum_{i=1}^{2\rho_j} \frac{(t-\alpha_j)^{i-1}}{(z-\alpha_j)^i},\ j = 1, \ldots, r,$$

$$S_0(t,z) = (t+z) \sum_{i=1}^{\rho_0+1} \frac{(1+z^2)^{i-1}}{(1+t^2)^i},$$

and χ_j is the characteristic function of U_j, where U_0, U_j, $j = 1, \ldots, r$, are pairwise disjoint neighbourghoods of ∞ and α_j, respectively. If $\deg T_k = \tau_k$, it is proved in [KL2] that

$$\sum_{j=0}^{r} \rho_j + \sum_{k=1}^{s} \tau_k = \kappa. \tag{2.2}$$

We may assume that, if $\rho_j > 0$ or $\rho_0 > 0$,

$$\int_{-\infty}^{\infty} \frac{d\sigma(t)}{(t-\alpha_j)^2} = \infty,\ \int_{-\infty}^{\infty} t^2 d\sigma(t) = \infty.$$

We shall use this integral representation to prove the following results.

Proposition 2.2. *Assume that Q_1 and Q_2 are meromorphic functions in $\mathbb{C}^+$ which allow a continuous extension to $\mathbb{R}$, and that $Q_1 \in \mathcal{N}_{\kappa_1}$, $Q_2 \in \mathcal{N}_{\kappa_2}$. If $\operatorname{Im} Q_1(x) = \operatorname{Im} Q_2(x)$ for $x \in \mathbb{R}$, we have*

$$Q_1(z) - Q_2(z) = \sum_{k=1}^{s_1} \left(T_{1,k}(\frac{1}{z-\beta_{1,k}}) + T_{1,k}(\frac{1}{z-\beta_{1,k}})^{\#} \right) -$$

$$- \sum_{k=1}^{s_2} \left(T_{2,k}(\frac{1}{z-\beta_{2,k}}) + T_{2,k}(\frac{1}{z-\beta_{2,k}})^{\#} \right) + p(z),$$

where $p(z)$ is a polynomial with real coefficients, and

$$\deg p(z) \leq 2 \max \left\{ \kappa_1 - \sum_{k=1}^{s_1} \deg T_{1,k}(z), \kappa_2 - \sum_{k=1}^{s_2} \deg T_{2,k}(z) \right\} + 1. \tag{2.3}$$

Proof : Consider the integral representation (2.1) of Q_1 (Q_2). The terms involving $R_{1,0}$ ($R_{2,0}$) and $T_{1,k}$ ($T_{2,k}$) are analytic on $\mathbb{R}$ and assume there real values. Hence we only need to consider the integral terms and the terms involving $R_{1,j}$ ($R_{2,j}$), $j \geq 1$.

Let $\alpha_1, \dots, \alpha_r \in \mathbb{R}$ be those points which are critical points either for Q_2 or for Q_2, and denote by $\rho_{1,j}, \rho_{2,j}$ the respective orders. Note that if α_1 is critical, say for Q_1 but not for Q_2, we put $\rho_{2,j} = 0$. Choose pairwise disjoint right open intervals U_j, $j = 1, \dots, r$, such that U_j contains α_j in its interior. Moreover, let $U_0 := \mathbb{R} \setminus [-N, N)$ where N is sufficiently large in order that $U_j \cap U_0 = \emptyset$ for all j. Denote by χ_j, $j = 0, 1, \dots, r$, the characteristic function of the set U_j.

The, for our purposes, essential terms in the integral representation of Q_1 (Q_2) are

$$f_1(z) := \int_{-\infty}^{\infty} \left(\frac{1}{t-z} - \sum_{j=0}^{r} \chi_j(t) S_{1,j}(t,z) \right) \varphi_1(t)(1+t^2) d\sigma_1(t) +$$

$$+ \sum_{j=1}^{r} R_{1,j}(\frac{1}{\alpha_j - z}),$$

$$f_2(z) := \int_{-\infty}^{\infty} \left(\frac{1}{t-z} - \sum_{j=0}^{r} \chi_j(t) S_{2,j}(t,z) \right) \varphi_2(t)(1+t^2) d\sigma_2(t) +$$

$$+ \sum_{j=1}^{r} R_{2,j}(\frac{1}{\alpha_j - z}).$$

Consider a critical point α_j. We show that $\rho_{1,j} = \rho_{2,j}$. Without loss of generality assume that $\rho_{1,j} \geq \rho_{2,j}$ and that for sufficiently small $\delta > 0$

$$\int_{\alpha_j - \delta}^{\alpha_j + \delta} \frac{d\sigma_1(t)}{(t-\alpha_j)^2} = \infty.$$

If we put $F_1(z) := (z-\alpha_j)^{2\rho_{1,j}} f_1(z)$ and $F_2(z) := (z-\alpha_j)^{2\rho_{2,j}} f_2(z)$, we have by [KL2]

$$F_1(z) = \int_{-\infty}^{\infty} (\chi_j(t)\frac{1}{t-z} + (1-\chi_j(t))\psi_1(z,t))(t-\alpha_j)^{2\rho_{1,j}}\varphi_1(t)(1+t^2)d\sigma_1(t)+$$

$$+(z-\alpha_j)^{2\rho_{1,j}}\sum_{j=1}^{r} R_{1,j}(\frac{1}{\alpha_j - z}),$$

and

$$F_2(z) = \int_{-\infty}^{\infty} (\chi_j(t)\frac{1}{t-z} + (1-\chi_j(t))\psi_2(z,t))(t-\alpha_j)^{2\rho_{2,j}}\varphi_2(t)(1+t^2)d\sigma_2(t)+$$

$$+(z-\alpha_j)^{2\rho_{2,j}}\sum_{j=1}^{r} R_{2,j}(\frac{1}{\alpha_j - z}),$$

where ψ_1 (ψ_2) collect all expressions which occur for $t \notin U_j$. The last sum in the expressions F_1, F_2 is analytic at α_j. Applying twice the Stieltjes-Livsic inversion formula (see [GG]), we obtain for $a, b \in U_j$, $a < b$

$$\int_a^b (t-\alpha_j)^{2\rho_{1,j}}\varphi_1(t)(1+t^2)d\sigma_1(t) = \lim_{\varepsilon\searrow 0}\frac{1}{\pi}\int_a^b \operatorname{Im} F_1(u-i\varepsilon)du =$$

$$= \frac{1}{\pi}\int_a^b u^{2\rho_{1,j}} \operatorname{Im} f_1(u)du = \frac{1}{\pi}\int_a^b u^{2(\rho_{1,j}-\rho_{2,j})} \operatorname{Im} (u^{2\rho_{2,j}} f_2(u))du =$$

$$= \lim_{\varepsilon\searrow 0}\frac{1}{\pi}\int_a^b (u-i\varepsilon)^{2(\rho_{1,j}-\rho_{2,j})} \operatorname{Im} F_2(u-i\varepsilon)du =$$

$$= \int_a^b t^{2(\rho_{1,j}-\rho_{2,j})}(t-\alpha_j)^{2\rho_{2,j}}\varphi_2(t)(1+t^2)d\sigma_2(t).$$

If $\rho_{1,j} > \rho_{2,j}$, we obtain for sufficiently small $\delta > 0$ that $\int_{\alpha_j-\delta}^{\alpha_j+\delta} \frac{d\sigma_1(t)}{(t-\alpha_j)^2} < \infty$, a contradiction. Hence $\rho_{1,j} = \rho_{2,j}$.

A similar argument yields $\rho_{1,0} = \rho_{2,0}$. Hence the integrands in the representation of F_1 and F_2 are the same and we find, again appealing to the Stieltjes-Livsic inversion formula, that $d\sigma_1(t) = d\sigma_2(t)$.

The relation (2.3) follows from (2.2).

□

Corollary 2.3. *Let $Q \in \mathcal{N}_\kappa^{k\times k}$ for $k = 1$ or $k = 2$ be given. Assume that Q is analytic in $\mathbb{C}^+$, has a continuous extension to $\mathbb{R}$, and satisfies $\operatorname{Im} Q(x) = 0$ for $x \in \mathbb{R}$. Then Q is a polynomial of degree at most $2\kappa + 1$.*

Proof : For scalar functions the assertion follows immediately from Proposition 2.2. If $Q \in \mathcal{N}_\kappa^{2\times 2}$ consider the scalar functions $a^*Q(z)a$ with the vectors

$$a = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ i \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Since by the already proved each of the scalar functions $a^*Q(z)a$ is a polynomial, also $Q(z)$ itself must be a polynomial.

$\square$

Recall that a scalar function which is meromorphic in a certain region O, is said to be of bounded type (or bounded characteristic) if it can be written as quotient of two functions, which are analytic and bounded in O (compare e.g. [H]). It is well known (and follows immediately from some considerations in [dB7]) that for a function Q which is of bounded type in $\mathbb{C}^+$ and has only finitely many singularities, we have

$$-\infty < \operatorname{mt} Q := \limsup_{y\to\infty} \frac{\log|Q(iy)|}{y} < \infty.$$

We will refer to this number as the mean type of Q. Clearly a rational function is of bounded type in $\mathbb{C}^+$ and has mean type 0.

Proposition 2.4. *If $Q \in \mathcal{N}_\kappa$, then Q is of bounded type in $\mathbb{C}^+$ and* $\operatorname{mt} Q = 0$.
Proof : Consider the integral in (2.1), and split it into the summands where integrations runs through U_j, $j = 1, \dots, r$, U_0 and the remaining part of the real axis. The formula (2.1), together with [GG], imply that Q is of bounded type in $\mathbb{C}^+$ and that $\operatorname{mt} Q \leq 0$. Since with Q also $-\frac{1}{Q}$ belongs to $\mathcal{N}_\kappa$, we find $\operatorname{mt} Q = 0$.

$\square$

3 de Branges spaces of entire functions

In this section we introduce certain inner product spaces whose elements are entire functions. This generalizes the construction of [dB7].

Let $\langle \mathfrak{P}, (.,.)\rangle$ be a Hilbert space whose elements are entire functions, and assume that for each $w \in \mathbb{C}$ there exists a number $\gamma_w \geq 0$, such that

$$|F(w)| \leq \gamma_w \|F\|,\ F \in \mathfrak{P}, \tag{3.1}$$

i.e. assume that $\langle \mathfrak{P}, (.,.)\rangle$ is a reproducing kernel Hilbert space. Moreover, let $\mathcal{G}$ be a selfadjoint operator on $\mathfrak{P}$, such that for some $\varepsilon > 0$ the set $\sigma(\mathcal{G}) \cap (-\infty, \varepsilon)$ consists only of a finite number of eigenvalues of finite multiplicity. If we endow $\mathfrak{P}$ with the new inner product $[.,.] := (\mathcal{G}.,.)$, the space $\langle \mathfrak{P}, [.,.]\rangle$ has the following properties:

(i) The isotropic part $\mathfrak{P}^\circ$ of $\mathfrak{P}$ is finite dimensional.

(ii) The factor space $\mathfrak{P}/\mathfrak{P}^\circ$ is a Pontryagin space.

The condition (3.1) shows that, if $0 \notin \sigma(\mathcal{G})$, the space $\langle \mathfrak{P}, [.,.]\rangle$ is a reproducing kernel Pontryagin space, i.e. there exist elements $K(w,z) \in \mathfrak{P}$, $w \in \mathbb{C}$, such that

$$F(w) = [F(z), K(w,z)],\ F \in \mathfrak{P}.$$

In the following denote by $F^\#$ the function $F^\#(z) := \overline{F(\overline{z})}$. A function F with $F = F^\#$ is called real. Note that any function F can be decomposed into its "real-" and "imaginary-" part by

$$F = \left(\frac{F + F^\#}{2}\right) - i\left(i\frac{F - F^\#}{2}\right). \tag{3.2}$$

Definition 3.1. Consider the space $\langle\mathfrak{P}, [.,.]\rangle$, where $[.,.] = (\mathcal{G}.,.)$ is subject to the above conditions. Then $\langle\mathfrak{P}, [.,.]\rangle$ is called a de Branges inner product space (dB-space), if it satisfies the following axioms:

(i) If $F \in \mathfrak{P}$, then $F^\# \in \mathfrak{P}$. Moreover,

$$[F^\#, G^\#] = [G, F], \ F, G \in \mathfrak{P}. \tag{3.3}$$

(ii) If $w \in \mathbb{C} \setminus \mathbb{R}$ and $F \in \mathfrak{P}$, $F(w) = 0$, then the function $\frac{z-\overline{w}}{z-w}F(z)$ is also contained in $\mathfrak{P}$. Moreover, if $F, G \in \mathfrak{P}$, $F(w) = 0$, $G(\overline{w}) = 0$, then

$$[\frac{z-\overline{w}}{z-w}F(z), G(z)] = [F(z), \frac{z-w}{z-\overline{w}}G(z)]. \tag{3.4}$$

If $0 \notin \sigma(\mathcal{G})$ ($\mathcal{G} > 0$) we call $\langle\mathfrak{P}, [.,.]\rangle$ a dB-Pontryagin (dB-Hilbert) space.

Our first aim is to show that to each dB-space there corresponds in a natural way a dB-Hilbert space. This allows us, up to a certain extend, to use the theory developed in [dB7].

Lemma 3.2. *Let $\langle\mathfrak{P}, (.,.)\rangle$ be given, let $\mathcal{G}$ and $\mathcal{G}_1$ be subject to the above conditions and assume that $\langle\mathfrak{P}, (\mathcal{G}.,.)\rangle$ is a dB-space. Then $\langle\mathfrak{P}, (\mathcal{G}_1.,.)\rangle$ is a dB-space if and only if (3.3) and (3.4) hold with $\mathcal{G}$ replaced by $\mathcal{T} := \mathcal{G}_1 - \mathcal{G}$.*

Proof : If $F, G \in \mathfrak{P}$, we have

$$(\mathcal{G}_1 F, G) = (\mathcal{G}F, G) + (\mathcal{T}F, G).$$

Since $\mathcal{G}$ satisfies (3.3) and (3.4), the assertion follows.

$\square$

Theorem 3.3. *Let $\langle\mathfrak{P}, (\mathcal{G}.,.)\rangle$ be a dB-space. Then there exists a finite rank perturbation $\mathcal{G}_1$ of $\mathcal{G}$, such that $\langle\mathfrak{P}, (\mathcal{G}_1.,.)\rangle$ is a dB-Hilbert space.*

Proof : Denote by $K'(w, z) \in \mathfrak{P}$ the reproducing kernel of the Hilbert space $\langle\mathfrak{P}, (.,.)\rangle$. Let $\{t_1, t_2, \ldots\} \subseteq \mathbb{R}$ be a sequence which has a finite accumulation point. Since $\mathfrak{P}$ consists of entire functions, the linear subspace

$$\mathfrak{L} := \text{span}\,\{K'(t_i, z) | i = 1, 2, \ldots\}$$

is dense in $\mathfrak{P}$.

According to the spectral decomposition $(E_t)_{t\in\mathbb{R}}$ of $\mathcal{G}$ we write $\mathcal{G}=\mathcal{G}_+-\mathcal{G}_-$ where $\mathcal{G}_-:=-E_{(-\infty,0]}\mathcal{G}$ and $\mathcal{G}_+:=E_{(0,\infty)}\mathcal{G}$. It follows from our assumption on $\mathcal{G}$ that $\mathcal{G}_+|_{E_{(0,\infty)}\mathfrak{P}}>\!>0$ and that $\mathcal{G}_-$ is finite dimensional. Note that $\mathcal{G}_-\geq 0$, hence we may write ($n=\dim E_{(-\infty,0]}\mathfrak{P}$):

$$\mathcal{G}_-=\sum_{j=1}^n \lambda_j(.,e_j)e_j,$$

where $0\leq\lambda_1\leq\ldots\leq\lambda_n$, $(e_i,e_j)=0$ for $i\neq j$ and $(e_i,e_i)=1$. We set

$$\mathcal{G}^-:=\sum_{j=1}^n(1+\lambda_n)(.,e_j)e_j=(1+\lambda_n)\sum_{j=1}^n(.,e_j)e_j.$$

Then clearly $\mathcal{G}^-|_{E_{(-\infty,0]}\mathfrak{P}}\gg\mathcal{G}_-|_{E_{(-\infty,0]}\mathfrak{P}}\geq 0$ and hence $\mathcal{G}_++\mathcal{G}^-\gg 0$. Since $\mathfrak{L}$ is dense in $\mathfrak{P}$ there exist, for arbitrary $\varepsilon>0$, elements $f_1,\ldots,f_n\in\mathfrak{L}$, such that with

$$\mathcal{G}_\varepsilon:=(1+\lambda_n)\sum_{j=1}^n(.,f_j)f_j$$

we have $\|\mathcal{G}^--\mathcal{G}_\varepsilon\|<\epsilon$. For sufficiently small ε we obtain

$$0\ll\mathcal{G}_++\mathcal{G}^--2(\mathcal{G}^--\mathcal{G}_\varepsilon)=\mathcal{G}_+-\mathcal{G}^-+2\mathcal{G}_\varepsilon\leq$$

$$\leq\mathcal{G}_+-\mathcal{G}_-+2\mathcal{G}_\varepsilon=\mathcal{G}+2\mathcal{G}_\varepsilon.$$

This implies that the hermitian form $(.,.)_0:=((\mathcal{G}+2\mathcal{G}_\varepsilon).,.)$ is positive definite and topologically equivalent to $(.,.)$.

Now let $t_1,\ldots,t_m$ be such that $f_1,\ldots,f_n\in\operatorname{span}\{K'(t_1,z),\ldots,K'(t_m,z)\}$, then we can write

$$f_j=\sum_{k=1}^m\mu_k^jK'(t_k,z),j=1,\ldots,n.$$

Define a hermitian form $(.,.)_1$ on $\mathfrak{P}$ by

$$(F,G)_1:=[F,G]+C\sum_{k=1}^m F(t_k)\overline{G(t_k)},\ \ F,G\in\mathfrak{P}$$

with $C=2(1+\lambda_n)\sum_{j=1}^n\sum_{k=1}^m|\mu_k^j|^2$. We calculate for $F\in\mathfrak{P}$

$$(F,F)_0=[F,F]+2(1+\lambda_n)\sum_{j=1}^n\sum_{k,l=1}^m\mu_k^j\overline{\mu_l^j}(K'(t_k,z),F)(F,K'(t_l,z))=$$

$$=[F,F]+2(1+\lambda_n)\sum_{j=1}^n\sum_{k,l=1}^m\mu_k^j\overline{\mu_l^j}\overline{F(t_k)}F(t_l)\leq$$

$$\leq[F,F]+2(1+\lambda_n)\sum_{j=1}^n(\sum_{k=1}^m|\mu_k^j|^2)(\sum_{k=1}^m|F(t_k)|^2)=(F,F)_1,$$

Hence $(.,.)_1$ is a positive definite inner product on $\mathfrak{P}$ which is equivalent to $(.,.)$. Moreover, we find $(.,.)_1 = (\mathcal{G}_1.,.)$ with the finite rank perturbation

$$\mathcal{G}_1 := \mathcal{G} + C\sum_{k=1}^{m}(., K'(t_k,z))K'(t_k,z)$$

of $\mathcal{G}$.

Let $F, G \in \mathfrak{P}$, then $F^{\#}(t_i)\overline{G^{\#}(t_i)} = G(t_i)\overline{F(t_i)}$. Moreover, if $F(w) = 0$ and $G(\overline{w}) = 0$, then

$$\frac{t_i - \overline{w}}{t_i - w}F(t_i)\overline{G(t_i)} = F(t_i)\overline{\frac{t_i - w}{t_i - \overline{w}}G(t_i)}.$$

By Lemma 3.2 the space $\langle\mathfrak{P}, (.,.)_1\rangle$ is a dB-Hilbert space.

$\square$

4 The operator of multiplication by the independent variable

If a function F is analytic or has an isolated singularity at a point $w \in \mathbb{C}$, we denote by $\mathrm{Ord}_w F \in \mathbb{Z} \cup \{\pm\infty\}$ the order of w as a zero (minus the order of w as a pole) of F. More precisely, if F has the Laurent expansion

$$F(z) = \sum_{n\in\mathbb{Z}} a_n (z-w)^n$$

at w, then $\mathrm{Ord}_w F := \inf\{n \in \mathbb{Z} | a_n \neq 0\}$. We have $\mathrm{Ord}_w F = +\infty$ if and only if $F \equiv 0$, and $\mathrm{Ord}_w F = -\infty$ if and only if F has an essential singularity at w. Note that $\mathrm{Ord}_w \frac{1}{F} = -\mathrm{Ord}_w F$, and that for two functions F and G for which $\mathrm{Ord}_w F$ and $\mathrm{Ord}_w G$ is finite, we have

$$\mathrm{Ord}_w(F\cdot G) = \mathrm{Ord}_w F + \mathrm{Ord}_w G,\ \mathrm{Ord}_w(F+G) \geq \min(\mathrm{Ord}_w F, \mathrm{Ord}_w G),$$

where strict inequality can occur only if $\mathrm{Ord}_w F = \mathrm{Ord}_w G$.

We assume in the sequel that $\langle\mathfrak{P}, [.,.]\rangle$ is a dB-space. The operator of multiplication by the independent variable is defined as

$$(\mathcal{S}F)(z) := zF(z),\ F \in \operatorname{dom}\mathcal{S},$$

where

$$\operatorname{dom}\mathcal{S} := \{F \in \mathfrak{P} | zF(z) \in \mathfrak{P}\}.$$

Note that the definition of $\mathcal{S}$ does not depend on the choice of an inner product.

For a given dB-space $\mathfrak{P}$, the function $\mathfrak{d}(\mathfrak{P}) : \mathbb{C} \to \mathbb{N} \cup \{0\}$ defined by

$$\mathfrak{d}(\mathfrak{P})(w) := \min_{F\in\mathfrak{P}} \mathrm{Ord}_w F,\ w \in \mathbb{C}, \tag{4.1}$$

is called the divisor associated with $\mathfrak{P}$. Making use of Theorem 3.3, the results of [dB7] imply:

Lemma 4.1. *Let $\mathfrak{P}$ be a dB-space. Then $\mathfrak{d}(\mathfrak{P})(w) = 0$ for all $w \in \mathbb{C} \setminus \mathbb{R}$. The axiom (ii) of Definition 3.1 can be strengthened in the following sense: If $F \in \mathfrak{P}$, $w \in \mathbb{C}$ and $\mathrm{Ord}_w F > \mathfrak{d}(\mathfrak{P})(w)$, then the function $G(z) := \frac{F(z)}{z-w}$ is also contained in $\mathfrak{P}$.*

Proposition 4.2. *The operator $\mathcal{S}$ is closed and symmetric, the codimension of $\overline{\mathrm{dom}\,\mathcal{S}}$ in $\mathfrak{P}$ is either 0 or 1. The defect index of $\mathcal{S}$ is $(1,1)$, in fact*

$$\mathrm{ran}\,(\mathcal{S} - w) = \{F \in \mathfrak{P} | \mathrm{Ord}_w F > \mathfrak{d}(\mathfrak{P})(w)\}, w \in \mathbb{C}, \tag{4.2}$$

and $\mathrm{codim}\,\mathrm{ran}\,(\mathcal{S}-w) = 1$ *for all $w \in \mathbb{C}$. The operator $\mathcal{S}$ is real with respect to the involution $\#$.*

Proof : It is proved in [dB7] that, in case of a dB-Hilbert space, the assertions of Proposition 4.2 hold. The fact that the definition of $\mathcal{S}$ does not depend on the choice of an inner product shows with the aid of Theorem 3.3 that $\mathcal{S}$ is closed, satisfies (4.2), has defect index $(1,1)$, and that $\mathrm{codim}\,\overline{\mathrm{dom}\,\mathcal{S}}$ is either 0 or 1.

It follows from the axiom *(ii)* of Definition 3.1, that $\mathcal{S}$ is symmetric. In fact, for $w \in \mathbb{C} \setminus \mathbb{R}$, $\mathcal{S}$ is the Cayley transform (compare [DS1]) of the isometry

$$\mathcal{V}_w : F(z) \mapsto \frac{z - \overline{w}}{z - w} F(z),\ F \in \mathrm{ran}\,(\mathcal{S} - w). \tag{4.3}$$

The fact that $\mathcal{S}$ is real, i.e. satisfies $\mathcal{S}(F^{\#}) = (\mathcal{S}F)^{\#}$ for $F \in \mathrm{dom}\,\mathcal{S}$, is obvious from the definition of S.

□

Corollary 4.3. *Whenever $M \subseteq \mathbb{C}$ has a finite accumulation point, we have*

$$\bigcap_{w \in M} \mathrm{ran}\,(\mathcal{S} - w) = \{0\}. \tag{4.4}$$

If $\mathfrak{P}$ is a dB-Pontryagin space, $K(w,z)$ denotes the reproducing kernel of the space $\langle \mathfrak{P}, [.,.] \rangle$ and $w \in \mathbb{C}$ is such that $\mathfrak{d}(\mathfrak{P})(w) = 0$, then

$$\mathrm{ran}\,(\mathcal{S} - w)^{[\perp]} = \mathrm{span}\,\{K(w,z)\}. \tag{4.5}$$

Proof : Since $\mathfrak{P}$ consists of entire functions, the relations (4.4) and (4.5) are an immediate consequence of (4.2).

□

By (4.4) $\mathcal{S}$ has no eigenvalues. Note that, if $\mathfrak{P}$ is a dB-Pontryagin space and $\mathfrak{d}(\mathfrak{P})(M) = \{0\}$, the relation (4.4) is equivalent to

$$\overline{\mathrm{span}\,\{K(w,z) | w \in M\}} = \mathfrak{P},$$

hence we may speak of $\mathcal{S}$ as a simple operator in the sense of [KL3].

Definition 4.4. An entire function $S(z)$ is said to be associated to the dB-space $\mathfrak{P}$, if there exists a number $w \in \mathbb{C}$ and a function $F \in \mathfrak{P}$ with $F(w) \neq 0$, such that

$$\frac{F(z)S(w) - S(z)F(w)}{z - w} \in \mathfrak{P}. \tag{4.6}$$

The set of associated functions will be denoted by $\operatorname{Ass}\mathfrak{P}$.

Since the definition of an associated function does not depend on the inner product of $\mathfrak{P}$, we can use the results developed in [dB7]. Recall that $\mathfrak{P} \subseteq \operatorname{Ass}\mathfrak{P}$ and that, if $S \in \operatorname{Ass}\mathfrak{P}$, the relation (4.6) holds in fact for each $w \in \mathbb{C}$ and $F \in \operatorname{Ass}\mathfrak{P}$.

Let us recall that the set $\operatorname{Ass}\mathfrak{P}$ can easily be constructed from $\mathfrak{P}$ itself.

Lemma 4.5. *Let $S_0 \in \operatorname{Ass}\mathfrak{P}$ and let $z_0 \in \mathbb{C}$ be such that $S_0(z_0) \neq 0$. Then*

$$\operatorname{Ass}\mathfrak{P} = (z - z_0) \cdot \mathfrak{P} + \mathbb{C} \cdot S_0(z).$$

Proof : Assume that $S \in \operatorname{Ass}\mathfrak{P}$. Then

$$F(z) := \frac{S(z) - \frac{S_0(z)}{S_0(z_0)} S(z_0)}{z - z_0} \in \mathfrak{P},$$

and hence

$$S(z) = (z - z_0)F(z) + \frac{S(z_0)}{S_0(z_0)} S_0(z).$$

Conversely, let $F \in \mathfrak{P}$ and $\lambda \in \mathbb{C}$ be given. Then for any $G \in \mathfrak{P}$ we have

$$\frac{((z - z_0)F(z) + \lambda S_0(z))G(z_0) - G(z)\lambda S_0(z_0)}{z - z_0} = F(z)G(z_0) +$$

$$+\lambda \frac{S_0(z)G(z_0) - G(z)S_0(z_0)}{z - z_0} \in \mathfrak{P}.$$

$\square$

Note that, in particular, we can take in Lemma 4.5 for S_0 any nonzero element of $\mathfrak{P}$, together with a convenient number z_0.

Proposition 4.6. *The relations $\mathcal{A} \subseteq \mathfrak{P}^2$ which extend $\mathcal{S}$ and have nonempty resolvent set correspond bijectively to the functions $S \in \operatorname{Ass}\mathfrak{P}$. This correspondence is given by*

$$(\mathcal{A} - w)^{-1}F(z) = \frac{F(z) - \frac{S(z)}{S(w)} F(w)}{z - w}, \; w \in \rho(\mathcal{A}), F \in \mathfrak{P}, \tag{4.7}$$

and we have

$$\rho(\mathcal{A}) \cap \mathbb{C} = \{w \in \mathbb{C} | \operatorname{Ord}_w S = \mathfrak{d}(\mathfrak{P})(w)\}. \tag{4.8}$$

Moreover, $\mathcal{A}$ is a proper relation, i.e. $\mathcal{A}(0) \neq \{0\}$, if and only if $S \in \mathfrak{P}$. In this case $\mathcal{A}(0) = \operatorname{span}\{S(z)\}$.

Proof : Let $S \in \operatorname{Ass}\mathfrak{P}$. A computation shows that the operator valued function $(S(w) \neq 0)$

$$\mathcal{R}_S(w) : F(z) \mapsto \frac{F(z) - \frac{S(z)}{S(w)}F(w)}{z - w}, \quad F \in \mathfrak{P},$$

satisfies the resolvent identity

$$\mathcal{R}_S(w) - \mathcal{R}_S(w') = (w - w')\mathcal{R}_S(w)\mathcal{R}_S(w').$$

It is proved in [dB7] that $\mathcal{R}_S(w)$ is a bounded operator. By [DS1] there exists a relation $\mathcal{A} \subseteq \mathfrak{P}^2$, such that (4.7) holds. If $F \in \operatorname{ran}(\mathcal{S} - w)$ we have $F(w) = 0$, hence $\mathcal{R}_S(w) \supseteq (\mathcal{S} - w)^{-1}$, i.e. $\mathcal{A}$ extends $\mathcal{S}$.

Conversely let $\mathcal{A} \subseteq \mathfrak{P}^2$, $\rho(\mathcal{A}) \neq \emptyset$, $\mathcal{A} \supseteq \mathcal{S}$ be given. Choose $F \in \mathfrak{P}$ and $w_0 \in \rho(\mathcal{A})$, such that $F(w_0) \neq 0$. Consider the function

$$S(z) := \frac{1}{F(w_0)}(F(z) - (z - w_0)(\mathcal{A} - w_0)^{-1}F(z)) \in \operatorname{Ass}\mathfrak{P},$$

then $(\mathcal{A} - w_0)^{-1} = \mathcal{R}_S(w_0)$. By the resolvent identity and the analyticity of $\mathcal{R}_S(w)$ this relation holds for all $w \in \rho(\mathcal{A})$, $S(w) \neq 0$.

The relation (4.8) follows from Lemma 4.1 and the fact that $\rho(\mathcal{A})$ is the maximal domain of holomorphy of $\mathcal{R}_S(w)$. The last assertion follows since $\mathcal{A}$ being a proper relation is equivalent to the fact that $(\mathcal{A} - w)^{-1}$ has a nontrivial kernel for all $w \in \mathbb{C}$.

$\square$

Note that the resolvent $(\mathcal{A}-w)^{-1}$ can be extended to $\operatorname{Ass}\mathfrak{P}$ by (4.7). Then for any $F \in \operatorname{Ass}\mathfrak{P}$ the function $S(w)(\mathcal{A} - w)^{-1}F(z)$ is entire with respect to z and w (in the norm of $\mathfrak{P}$).

Recall that a point $w \in \mathbb{C}$ is said to be of regular type for $\mathcal{S}$, if there exists a number $\gamma_w > 0$, such that

$$\|F\| \leq \gamma_w \|(\mathcal{S} - w)F\|, \quad F \in \operatorname{dom}\mathcal{S}.$$

Corollary 4.7. *The set of points of regular type of $\mathcal{S}$ equals $\mathbb{C}$.*

Proof : By Lemma 4.1 and Proposition 4.6 there exists an extension $\mathcal{A}$ of $\mathcal{S}$ with $w \in \rho(\mathcal{A})$. Hence w is a point of regular type.

$\square$

5 Construction of dB-Pontryagin spaces by Hermite-Biehler functions

In this section we show that a dB-Pontryagin space is completely determined by a single entire function (in the case of a dB-Hilbert space compare besides [dB7] also [DK]).

Let us recall the notion of a Schur function and a Hermite-Biehler function (compare [DLS] and [Le]). For a meromorphic function Q we denote in the following by $\rho(Q)$ its domain of holomorphy.

Definition 5.1. Let $\kappa \in \mathbb{N} \cup \{0\}$. By $\mathcal{S}_\kappa$ we denote the set of all functions Q, meromorphic in $\mathbb{C}^+$, such that the kernel

$$S_Q(z,w) := i\frac{1 - Q(z)\overline{Q(w)}}{z - \overline{w}},\ z, w \in \rho(Q),$$

has κ negative squares. By $\mathcal{HB}_\kappa$ we denote the set of all entire functions E, such that $\frac{E^\#}{E} \in \mathcal{S}_\kappa$, E and $E^\#$ have no common nonreal zeros and $\frac{E^\#}{E}$ is not constant.

Note that, if we decompose E as in (3.2) as $E = A - iB$ with real functions A and B, the function $\frac{E^\#}{E}$ is constant if and only if A and B are linearly dependent. Moreover, the common zeros of E and $E^\#$ are exactly the common zeros of A and B. The following remark shows that the set $\mathcal{HB}_\kappa$ is nonempty.

Remark 5.2. Let $Q \in \mathcal{N}_\kappa$ be given, assume that Q is meromorphic in $\mathbb{C}$ and not constant. Then $Q = -\frac{A}{B}$ with real entire functions A, B which have no common zeros. The function $E := A - iB$ belongs to $\mathcal{HB}_\kappa$.

Proof : Let t_i, $i \in \mathbb{N}$, be the real poles of Q, $\tau_i := -\mathrm{Ord}_{t_i} Q$, and let $r_j, \overline{r_j}$, $j = 1, \dots, k$, be the nonreal poles of Q, $\rho_j := -\mathrm{Ord}_{r_j} Q$. Denote by $P(z)$ a Weierstraß product with zeros t_i of order τ_i, and define real entire functions B and A by

$$B(z) := P(z)\prod_{j=1}^{k}(z - r_j)^{\rho_j}(z - \overline{r_j})^{\rho_j},\ A(z) := -Q(z)B(z).$$

Clearly A and B are linearly independent and have no common zeros. Moreover, an elementary calculation shows that ($E := A - iB$)

$$i\frac{1 - \frac{E^\#(z)}{E(z)}\overline{(\frac{E^\#(w)}{E(w)})}}{z - \overline{w}} = \frac{B(z)}{E(z)}\frac{Q(z) - \overline{Q(w)}}{z - \overline{w}}\overline{(\frac{B(w)}{E(w)})},$$

and we conclude that $E \in \mathcal{HB}_\kappa$.

□

Theorem 5.3. *Let $\langle \mathfrak{P}, [.,.] \rangle$ be a dB-Pontryagin space, and denote by $K(w,z)$ its reproducing kernel. Then*

$$K(w,z) = \frac{B(z)A(\overline{w}) - A(z)B(\overline{w})}{z - \overline{w}} \tag{5.1}$$

for some real entire functions A and B. We have $E(z) := A(z) - iB(z) \in \mathcal{HB}_\kappa$ where $\kappa := \mathrm{Ind}\_\mathfrak{P}$.

Conversely, if $E \in \mathcal{HB}_\kappa$ is given, $E(z) = A(z) - iB(z)$ with real entire functions A and B, and $K(w,z)$ is defined by (5.1), then the reproducing kernel Pontryagin space $\mathfrak{P}(E)$ with kernel $K(w,z)$ is a dB-Pontryagin space.

Proof : First let a dB-Pontryagin space $\langle\mathfrak{P},[.,.]\rangle$ with reproducing kernel $K(w,z)$ be given. Since the function $K(w,z)$ depends analytically on z and $\overline{w}$ and does not vanish identically (for all z and w), there exists a nonreal number w_0, such that $K(w_0,w_0)\neq 0$. In fact the set of zeros of the function $K(z,z)$ contains no interior point, is closed, and lies symmetric with respect to the real axis.

As $^{\#}$ is an antiisometry, we have $K(w,z)^{\#}=K(\overline{w},z)$, in particular $K(w,w)=K(\overline{w},\overline{w})$. Choose $w_0\in\mathbb{C}\setminus\mathbb{R}$, such that $K(w_0,w_0)\operatorname{Im}w_0>0$, and define

$$E(z):=i\sqrt{\frac{\pi}{K(w_0,w_0)\operatorname{Im}w_0}}(\overline{w_0}-z)K(w_0,z),$$

$$\tilde{K}(w,z):=\frac{E(z)\overline{E(w)}-E^{\#}(z)\overline{E^{\#}(w)}}{-2\pi i(z-\overline{w})}.$$

By a straightforward calculation we obtain

$$\tilde{K}(w,z)=\frac{i}{2K(w_0,w_0)\operatorname{Im}w_0}\Big[\frac{|w_0|^2-\overline{w}(w_0+\overline{w_0})+z\overline{w}}{z-\overline{w}}(K(w_0,z)\overline{K(w_0,w)}-$$

$$-K(\overline{w_0},z)\overline{K(\overline{w_0},w)})-w_0K(w_0,z)\overline{K(w_0,w)}+\overline{w_0}K(\overline{w_0},z)\overline{K(\overline{w_0},w)}\Big].$$

Let $w\in\mathbb{C}$ be such that $\mathfrak{d}(\mathfrak{P})(w)=0$. If $F\in\mathfrak{P}$, $F(w)=0$, we find

$$-\frac{2K(w_0,w_0)\operatorname{Im}w_0}{i}[F(z),\tilde{K}(w,z)]=[F(z),(|w_0|^2-\overline{w}(w_0+\overline{w_0})+\overline{w}\mathcal{S})(\mathcal{S}-\overline{w})^{-1}$$

$$(K(w_0,z)\overline{K(w_0,w)}-K(\overline{w_0},z)\overline{K(\overline{w_0},w)})]-\overline{w_0}K(w_0,w)F(w_0)+w_0K(\overline{w_0},w)F(\overline{w_0})=$$

$$=\Big[\frac{|w_0|^2-w(w_0+\overline{w_0})+wz}{z-w}F(z),K(w_0,z)\overline{K(w_0,w)}-K(\overline{w_0},z)\overline{K(\overline{w_0},w)}\Big]-$$

$$-\overline{w_0}K(w_0,w)F(w_0)+w_0K(\overline{w_0},w)F(\overline{w_0})=0.$$

By Corollary 4.3, we have $\tilde{K}(w,z)=c(w)K(w,z)$. The function $c(w)$ is in fact a constant, since

$$c(w)K(w,w_0)=\tilde{K}(w,w_0)=\overline{\tilde{K}(w_0,w)}=\overline{c(w_0)}K(w,w_0).$$

Comparing $\tilde{K}(w_0,w_0)$ and $K(w_0,w_0)$, we find that $c(w)=1$.

If we write $E=A-iB$ with real entire functions A,B, we obtain (5.1). Moreover, we have

$$\frac{K(w,z)}{E(z)\overline{E(w)}}=\frac{1}{2\pi}i\frac{1-\frac{E^{\#}(z)}{E(z)}\Big(\overline{\frac{E^{\#}(w)}{E(w)}}\Big)}{z-\overline{w}},\tag{5.2}$$

hence $\frac{E^{\#}(z)}{E(z)}\in\mathcal{S}_\kappa$ with $\kappa=\operatorname{Ind}_-\mathfrak{P}$. Since $K(w,z)$ does not vanish identically, $\frac{E^{\#}(z)}{E(z)}$ is not constant. If $E(w)=0$ and $E^{\#}(w)=0$ for some number $w\in\mathbb{C}\setminus\mathbb{R}$, we have also $A(w)=0$ and $B(w)=0$, hence $K(\overline{w},z)=0$, $z\in\mathbb{C}$. This is not possible by Lemma 4.1. We conclude that $E\in\mathcal{HB}_\kappa$.

Now let conversely a function $E\in\mathcal{HB}_\kappa$ be given, and define an entire function $K(w,z)$ by (5.1). Clearly $K(w,z)=\overline{K(z,w)}$, hence $K(w,z)$ depends analytically on $\overline{w}$.

As the relation (5.2) holds, $K(w,z)$ is a kernel function with κ negative squares. Hence, the reproducing kernel Pontryagin space $\mathfrak{P} := \mathfrak{P}(E)$ consists of entire functions and has negative index κ (compare [ADSR1]). First we are concerned with the proof of axiom *(ii)* of Definition 3.1. Choose $w_0 \in \mathbb{C} \setminus \mathbb{R}$ such that $K(w_0, w_0) \neq 0$. A straightforward calculation shows that the relation

$$\left(K(w,z) - \frac{K(w_0,z)K(w,w_0)}{K(w_0,w_0)} \right) \frac{z-\overline{w_0}}{z-w_0} =$$

$$= \left(K(w,z) - \frac{K(\overline{w_0},z)K(w,\overline{w_0})}{K(\overline{w_0},\overline{w_0})} \right) \frac{\overline{w}-\overline{w_0}}{\overline{w}-w_0} \tag{5.3}$$

holds. Consider the mapping

$$\mathcal{V}_{w_0} : F(z) \mapsto \frac{z-\overline{w_0}}{z-w_0} F(z),$$

with domain

$$\operatorname{dom} \mathcal{V}_{w_0} := \{ F \in \mathfrak{P} | \frac{z-\overline{w_0}}{z-w_0} F(z) \in \mathfrak{P} \}.$$

By (5.3) we have

$$[\mathcal{V}_{w_0} \left(K(w,z) - \frac{K(w_0,z)K(w,w_0)}{K(w_0,w_0)} \right), \mathcal{V}_{w_0} \left(K(w',z) - \frac{K(w_0,z)K(w',w_0)}{K(w_0,w_0)} \right)] =$$

$$[\left(K(w,z) - \frac{K(\overline{w_0},z)K(w,\overline{w_0})}{K(\overline{w_0},\overline{w_0})} \right) \frac{\overline{w}-\overline{w_0}}{\overline{w}-w_0}, (K(w',z) -$$

$$- \frac{K(\overline{w_0},z)K(w',\overline{w_0})}{K(\overline{w_0},\overline{w_0})}) \frac{\overline{w'}-\overline{w_0}}{\overline{w'}-w_0}] =$$

$$= \frac{\overline{w}-\overline{w_0}}{\overline{w}-w_0} \frac{w'-w_0}{w'-\overline{w_0}} \left(K(w,w') - \frac{K(\overline{w_0},w')K(w,\overline{w_0})}{K(\overline{w_0},\overline{w_0})} \right) =$$

$$= \left(K(w,w') - \frac{K(w_0,w')K(w,w_0)}{K(w_0,w_0)} \right) =$$

$$= [\left(K(w,z) - \frac{K(w_0,z)K(w,w_0)}{K(w_0,w_0)} \right), \left(K(w',z) - \frac{K(w_0,z)K(w',w_0)}{K(w_0,w_0)} \right)],$$

i.e. $\mathcal{V}_{w_0}$ is an isometry of

$$\operatorname{dom} \mathcal{V}_{w_0} = \operatorname{span} \{ K(w,z) - \frac{K(w_0,z)K(w,w_0)}{K(w_0,w_0)} | w \in \mathbb{C} \}$$

onto

$$\operatorname{ran} \mathcal{V}_{w_0} = \operatorname{span} \{ K(w,z) - \frac{K(\overline{w_0},z)K(w,\overline{w_0})}{K(\overline{w_0},\overline{w_0})} | w \in \mathbb{C} \}.$$

Since $\overline{\operatorname{dom} \mathcal{V}_{w_0}} = \{ F \in \mathfrak{P} | F(w_0) = 0 \}$ and $\overline{\operatorname{ran} \mathcal{V}_{w_0}} = \{ F \in \mathfrak{P} | F(\overline{w_0}) = 0 \}$ are both regular subspaces of $\mathfrak{P}$, we may extend $\mathcal{V}_{w_0}$ to $\overline{\operatorname{dom} \mathcal{V}_{w_0}}$ by continuity, and obtain an isometry

$$\tilde{\mathcal{V}}_{w_0} : \{ F \in \mathfrak{P} | F(w_0) = 0 \} \to \{ F \in \mathfrak{P} | F(\overline{w_0}) = 0 \}.$$

In particular, we find that

$$\frac{F(z)}{z-w_0}=\frac{1}{w_0-\overline{w_0}}(\tilde{\mathcal{V}}_{w_0}F(z)-F(z))\in\mathfrak{P},$$

whenever $F\in\mathfrak{P}$, $F(w_0)=0$.

If $\mathcal{S}$ denotes the operator of multiplication by the independent variable in $\mathfrak{P}$, we may therefore write

$$\tilde{\mathcal{V}}_{w_0}F(z)=\frac{z-\overline{w_0}}{z-w_0}F(z)=(\mathcal{S}-\overline{w_0})(\mathcal{S}-w_0)^{-1}F(z),\ F\in\mathfrak{P},F(w_0)=0.$$

As we have seen above $\tilde{\mathcal{V}}_{w_0}$ is isometric, therefore its inverse Cayley transform $\mathcal{S}$ is symmetric. It follows that each Cayley transform

$$(\mathcal{S}-\overline{w})(\mathcal{S}-w)^{-1},\ w\in\mathbb{C}\setminus\mathbb{R},$$

is isometric. Since $\mathcal{S}$ is closed, has defect index $(1,1)$ and has no eigenvalues, we find (see [DS1]) that for each $w\in\mathbb{C}\setminus\mathbb{R}$

$$\dim\operatorname{ran}(\mathcal{S}-w)^{\perp}=1.$$

Since E and $E^{\#}$ have no common nonreal zeros, $K(w,z)$ does not vanish identically for any $w\in\mathbb{C}\setminus\mathbb{R}$. Hence

$$\operatorname{codim}\{F\in\mathfrak{P}|F(w)=0\}=1,\ w\in\mathbb{C}\setminus\mathbb{R},$$

which shows that

$$\operatorname{ran}(\mathcal{S}-w)=\{F\in\mathfrak{P}|F(w)=0\}.$$

Therefore the axiom *(ii)* of Definition 3.1 is fullfilled.

In order to prove axiom *(i)* of Definition 3.1 note that by the definition of $K(w,z)$

$$K(w,z)^{\#}=K(\overline{w},z),\ w\in\mathbb{C}.$$

We find

$$[K(w,z)^{\#},K(w',z)^{\#}]=K(\overline{w},\overline{w'})=\overline{K(w,w')}=K(w',w)=[K(w',z),K(w,z)].$$

Hence the (antilinear) involution $F\mapsto F^{\#}$ extends by continuity to an antiisometry on $\mathfrak{P}$.

□

The real zeros of the function E are connected with the divisor of the space $\mathfrak{P}(E)$.

Lemma 5.4. *Let $E\in\mathcal{HB}_\kappa$. For each $x\in\mathbb{R}$ we have*

$$\operatorname{Ord}_x E=\mathfrak{d}(\mathfrak{P})(x).$$

Proof : We first show that (as a function of z)

$$\frac{\partial^n}{(\partial\overline{w})^n}K(w,z)\in\mathfrak{P},\ n=0,1,2,\dots,$$

and that

$$[F(z), \frac{\partial^n}{(\partial \overline{w})^n} K(w,z)|_{w=w_0}] = F^{(n)}(w_0),\ F \in \mathfrak{P}, n = 0,1,2,\dots. \tag{5.4}$$

This follows inductively, since

$$\frac{F(w) - F(w_0)}{w - w_0} = [F(z), \frac{K(w,z) - K(w_0,z)}{\overline{w} - \overline{w_0}}],\ F \in \mathfrak{P}.$$

As every $F \in \mathfrak{P}$ is entire, the left hand side converges to $F'(w_0)$ if $w \to w_0$. Hence $\frac{K(w,z)-K(w_0,z)}{\overline{w}-\overline{w_0}}$ has a weak limit in $\mathfrak{P}$, which is (since weak convergence implies pointwise convergence) given by

$$\lim_{w \to w_0} \frac{K(w,z) - K(w_0,z)}{\overline{w} - \overline{w_0}} = \frac{\partial}{\partial \overline{w}} K(w,z)|_{w=w_0}.$$

Now proceed by induction to obtain the desired formula for higher derivatives.

The relation (5.4) shows that

$$\mathfrak{d}(\mathfrak{P})(x) = \max\{n \in \mathbb{N}_0 | \frac{\partial^k}{(\partial \overline{w})^k} K(w,z)|_{w=x} \equiv 0,\ k < n\}.$$

We have

$$(z - \overline{w}) \frac{\partial^n}{(\partial \overline{w})^n} K(w,z) - n \frac{\partial^{n-1}}{(\partial \overline{w})^{n-1}} K(w,z) = B(z) \frac{\partial^n}{(\partial \overline{w})^n} A(\overline{w}) - A(z) \frac{\partial^n}{(\partial \overline{w})^n} B(\overline{w}).$$

As A and B are linearly independent we have

$$\max\{n \in \mathbb{N}_0 | \frac{\partial^k}{(\partial \overline{w})^k} K(w,z)|_{w=x} \equiv 0,\ k < n\} =$$

$$= \max\{n \in \mathbb{N}_0 | \frac{\partial^k}{(\partial \overline{w})^k} A(\overline{w})|_{w=x} = \frac{\partial^k}{(\partial \overline{w})^k} B(\overline{w})|_{w=x} = 0,\ k < n\},$$

and the assertion follows.

□

Corollary 5.5. *Let $\mathfrak{P}$ be a dB-Pontryagin space, and let $\mathfrak{d}$ be a given divisor, such that $\mathfrak{d}(w) = 0$ with exception of an isolated subset of $\mathbb{R}$. Then there exists a dB-Pontryagin space $\mathfrak{Q}$ with $\mathfrak{d}(\mathfrak{Q}) = \mathfrak{d}$, which is isometrically isomorphic to $\mathfrak{P}$.*

Proof : Let $\mathfrak{P} = \mathfrak{P}(E)$ as in Theorem 5.3. Let $x_1, x_2, \dots$ be the real zeros of E taking into account their multiplicities. Denote by $U(z)$ a Weierstraß product with zeros $x_1, x_2, \dots$. Let $y_1, y_2, \dots$ be those real points where $\mathfrak{d}(y) \neq 0$, each as often as the value $\mathfrak{d}(y)$, and denote by $V(z)$ a Weierstraß product with zeros $y_1, y_2, \dots$.

Consider the function $E_1(z) := \frac{V(z)}{U(z)} E(z)$. By virtue of (5.2) and

$$\frac{E_1^{\#}(z)}{E_1(z)} = \frac{E^{\#}(z)}{E(z)}$$

we find $E_1 \in \mathcal{HB}_\kappa$. Again by (5.2) we find that

$$K_1(w,z) = \overline{\frac{V(w)}{U(w)}} K(w,z) \frac{V(z)}{U(z)},$$

hence $F(z) \mapsto \frac{V(z)}{U(z)} F(z)$ is an isometry of $\mathfrak{P}(E)$ onto $\mathfrak{P}(E_1)$.

$\square$

Remark 5.6. Note that, if $E = A - iB$, then

$$A, B, E \in \operatorname{Ass} \mathfrak{P}(E).$$

This follows from Lemma 4.5 and the fact that $K(w,z) \in \mathfrak{P}(E)$ for all w.

In the case of a dB-Hilbert space $\mathfrak{P}(E)$ the functions $F(z)$, $F \in \mathfrak{P}$ satisfy certain growth conditions. A similar result holds for dB-Pontryagin spaces. Recall the notion of bounded type and mean type as introduced in Section 2.

Proposition 5.7. *Let $\mathfrak{P} = \mathfrak{P}(E)$, $E = A - iB$, be a dB-Pontryagin space. The functions $\frac{F(z)}{A(z)}$, $F \in \operatorname{Ass} \mathfrak{P}$, are of bounded type in $\mathbb{C}^+$. We have*

$$\max_{F \in \mathfrak{P}} \operatorname{mt} \frac{F(z)}{A(z)} = \max_{F \in \operatorname{Ass} \mathfrak{P}} \operatorname{mt} \frac{F(z)}{A(z)} = 0.$$

If there exists a function $F_0 \in \operatorname{Ass} \mathfrak{P}$ which is of bounded type in $\mathbb{C}^+$, then all functions $F \in \operatorname{Ass} \mathfrak{P}$ are in fact of exponential type (compare e.g. [Bo] *or* [Le]*) and*

$$\max_{F \in \mathfrak{P}} \operatorname{et} F(z) = \max_{F \in \operatorname{Ass} \mathfrak{P}} \operatorname{et} F(z) = \operatorname{et} A(z).$$

The same assertion holds with A replaced by B or E.

Proof : Choose a positiv definite inner product on $\mathfrak{P}$ which turns $\mathfrak{P}$ into a dB-Hilbert space, $\mathfrak{P} = \mathfrak{P}(E_0)$, with $E_0 \in \mathcal{HB}_0$. Then $\frac{A}{E_0}$ is of bounded type in $\mathbb{C}^+$ and has nonpositive mean type.

Assume on the contrary that $\operatorname{mt} \frac{A}{E_0} = \rho < 0$. Then, by Proposition 2.4, also $\operatorname{mt} \frac{B}{E_0} = \rho$. Hence $\operatorname{mt} \frac{K(w,z)}{E_0} \leq \rho < 0$.

By [dB7] the mapping $F(z) \mapsto e^{iz\rho} F(z)$ is an isometry of the linear subspace $\operatorname{span} \{K(w,z) | w \in \mathbb{C}\}$ of $\mathfrak{P}(E_0)$ into $\mathfrak{P}(E_0)$. Hence it can be extended to an isometry of $\operatorname{cls} \{K(w,z) | w \in \mathbb{C}\} = \mathfrak{P}(E_0)$ into $\mathfrak{P}(E_0)$. This is a contradiction since, by the results of [dB7]

$$\max_{F \in \mathfrak{P}(E_0)} \operatorname{mt} \frac{F}{E_0} = 0, \tag{5.5}$$

and we conclude that $\rho = 0$. Now the assertion for A follows from (5.5).

Since $-\frac{A}{B} \in \mathcal{N}_\kappa$, we have $\operatorname{mt} \frac{A}{B} = 0$, hence the assertion with B instead of A holds.

Clearly $\operatorname{mt} \frac{E}{A} \leq 0$. The function $\frac{E^\#}{E}$ is contained in $\mathcal{S}_\kappa$. By [KL2] it can be written as a product of a rational function and a function contained in $\mathcal{S}_0$. Hence $\operatorname{mt} \frac{E^\#}{E} \leq 0$, and we find that $\operatorname{mt}\frac{A}{E} = \operatorname{mt}\frac{E+E^\#}{2E} \leq 0$. Now the assertion with A replaced by E follows.

□

6 Selfadjoint extensions of S

Let $\mathfrak{P}$ be a dB-Pontryagin space. As we have seen in Proposition 4.6 the extensions of $\mathcal{S}$ correspond to the functions $S \in \operatorname{Ass}\mathfrak{P}$. The following result determines those associated functions S which lead to selfadjoint extensions of $\mathcal{S}$.

As in Theorem 5.3 let $\mathfrak{P} = \mathfrak{P}(E)$ with a function $E \in \mathcal{HB}_\kappa$, and write $E(z) = A(z) - iB(z)$ with real entire functions A and B.

Proposition 6.1. *Let $S \in \operatorname{Ass}\mathfrak{P}$. The relation $\mathcal{A}$ corresponding to S via (4.7) is selfadjoint if and only if*

$$S(z) = uA(z) + vB(z),\ \ u, v \in \mathbb{C}, u\overline{v} \in \mathbb{R},$$

or, equivalently,

$$S(z) = \lambda(e^{i\alpha}E(z) - e^{-i\alpha}E^\#(z)),\ \ \lambda \in \mathbb{C}\setminus\{0\}, \alpha \in [0, \pi).$$

Proof : Note that $\mathcal{A}$ being selfadjoint is equivalent to the fact that the relation

$$[(\mathcal{A} - w)^{-1}K(a, z), K(b, z)] - [K(a, z), (\mathcal{A} - \overline{w})^{-1}K(b, z)] = 0 \tag{6.1}$$

holds for all $a, b \in \mathbb{C}$, $w \in \rho(\mathcal{A})$.

Assume first that $\mathcal{A}$ is selfadjoint. For $a = w \in \rho(\mathcal{A})$ the relation (6.1) can be written as

$$-S(w)\overline{S(\overline{w})}(B(b)A(\overline{w}) - A(b)B(\overline{w})) + \overline{S(\overline{w})}S(b)(B(w)A(\overline{w}) - A(b)B(\overline{w}))+$$
$$+S(w)S(\overline{w})(B(b)A(w) - A(b)B(w)) = 0. \tag{6.2}$$

Choose $w \in \mathbb{C}^+$ such that $B(w)A(\overline{w}) - A(w)B(\overline{w}) \neq 0$ and $S(w) \neq 0$. This is possible since $K(w, w)$ vanishes only on a set which contains no interior points. Then (6.2), considered as an identity among functions of b, states that S is a linear combination of A and B.

Let $S = uA + vB$ with $u, v \in \mathbb{C}$. Then the left hand side of (6.1) can be written as

$$\frac{K(\overline{b}, w)K(a, w)}{S(w)\overline{S(\overline{w})}}(u\overline{v} - \overline{u}v).$$

Hence $\mathcal{A}$ being selfadjoint is equivalent to $u\overline{v} \in \mathbb{R}$.

The fact that the functions of the form $uA + vB$ with $u\overline{v} \in \mathbb{R}$ are exactly those of the form $\lambda(e^{i\alpha}E(z) - e^{-i\alpha}E^\#(z))$ with $\lambda \in \mathbb{C}$ and $\alpha \in [0, \pi)$ follows by an elementary computation.

□

As a corollary of Proposition 6.1 we obtain a uniqueness result on the function E connected with a space $\mathfrak{P}$ via Theorem 5.3.

Corollary 6.2. *Let $E_1, E_2 \in \mathcal{HB}_\kappa$. The spaces $\mathfrak{P}(E_1)$ and $\mathfrak{P}(E_2)$ are identical, i.e. contain the same functions and have the same inner product, if and only if there exist four numbers $u_1, v_1, u_2, v_2 \in \mathbb{R}$, such that $u_1v_2 - u_2v_1 = 1$ and*

$$(A_1, B_1) = (A_2, B_2) \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}. \tag{6.3}$$

Proof : The spaces $\mathfrak{P}(E_1)$ and $\mathfrak{P}(E_2)$ are identical if and only they have the same kernel functions: $K_1(w, z) = K_2(w, z)$. Assume first that E_1 and E_2 satisfy (6.3) for some choice of u_1, v_1, u_2, v_2. Then

$$K_1(w, z) = (u_1v_2 - v_1u_2)K_2(w, z). \tag{6.4}$$

Since $u_1v_2 - v_1u_2 = 1$ by assumption we find that $\mathfrak{P}(E_1)$ and $\mathfrak{P}(E_2)$ are identical.

On the other hand assume that $\mathfrak{P}(E_1)$ and $\mathfrak{P}(E_2)$ are identical. Then both spaces have the same multiplication operator $\mathcal{S}$, the same associated functions, and allow the same selfadjoint extensions of $\mathcal{S}$. Since A_1 (B_1) yields a selfadjoint extension of $\mathcal{S}$ in $\mathfrak{P}(E_1)$ and hence in $\mathfrak{P}(E_2)$, there exist by Proposition 6.1 numbers $u_1, v_1, u_2, v_2 \in \mathbb{C}$ such that (6.3) holds. Since A_i and B_i $(i = 1, 2)$ are real and linearly independent, in fact $u_1, v_1, u_2, v_2 \in \mathbb{R}$. Now (6.4) shows that $u_1v_2 - u_2v_1 = 1$.

$\square$

Another corollary gives a characterization whether $\mathcal{S}$ is densely defined or not. Denote in the following for $\alpha \in [0, \pi)$ by S_α the function

$$S_\alpha(z) := -\frac{1}{2i}(e^{i\alpha}E(z) - e^{-i\alpha}E^\#(z)) = -\sin\alpha A(z) + \cos\alpha B(z).$$

As we have noted in Proposition 6.1 a function G can be written as $G(z) = uA(z) + vB(z)$ with $u, v \in \mathbb{C}$, $u\overline{v} \in \mathbb{R}$, if and only if $G(z) = \lambda S_\alpha(z)$ for some $\alpha \in [0, \pi)$ and $\lambda \in \mathbb{C}$.

Corollary 6.3. *We have $\overline{\operatorname{dom}\mathcal{S}} \neq \mathfrak{P}$ if and only if there exist $u, v \in \mathbb{C}$, not both zero, such that $u\overline{v} \in \mathbb{R}$ and $uA(z) + vB(z) \in \mathfrak{P}$ (or, equivalently, there exists $\alpha \in [0, \pi)$ such that $S_\alpha \in \mathfrak{P}$). In this case we have*

$$(\operatorname{dom}\mathcal{S})^\perp = \operatorname{span}\{uA + vB\} = \operatorname{span}\{S_\alpha\}.$$

Proof : The domain of $\mathcal{S}$ is not dense in $\mathfrak{P}$ if and only if $\mathcal{S}$ has a canonical selfadjoint extension which is a proper relation. By Proposition 4.6 and Proposition 6.1 this is the case if and only if there exist numbers $u, v \in \mathbb{C}$, $u\overline{v} \in \mathbb{R}$, such that $S := uA + vB \in \mathfrak{P}$. Since the codimension of $\overline{\operatorname{dom}\mathcal{S}}$ is at most one, and S itself is always in the relational part of the induced relation $\mathcal{A}$, we find that

$$(\operatorname{dom}\mathcal{S})^\perp = \operatorname{span}\{S(z)\}.$$

$\square$

The following lemma determines the Q-function (compare [KL1], [LT]) associated to a selfadjoint extension of $\mathcal{S}$.

Lemma 6.4. *Let $\mathfrak{P} = \mathfrak{P}(E)$ be given, $E = A - iB$. Consider the selfadjoint extension $\mathcal{A}_\alpha$ of $\mathcal{S}$ induced by S_α ($\alpha \in [0,\pi)$) via Proposition 6.1. A parametrization of the defect spaces of $\mathcal{S}$ associated with $\mathcal{A}_\alpha$ is given by ($w \in \rho(\mathcal{A}_\alpha)$)*

$$X(w,z) := \frac{1}{S_\alpha(w)} K(\overline{w}, z), \ \operatorname{ran}(\mathcal{S} - \overline{w})^\perp = \operatorname{span}\{X(w,z)\}. \tag{6.5}$$

A Q-function of $\mathcal{A}_\alpha$ and $\mathcal{S}$ is given by

$$Q_\alpha(z) := -\frac{\cos\alpha A(z) + \sin\alpha B(z)}{-\sin\alpha A(z) + \cos\alpha B(z)} = \frac{S_{\alpha+\frac{\pi}{2}}(z)}{S_\alpha(z)}. \tag{6.6}$$

Proof : We consider first the case $\alpha = 0$, i.e. $S_\alpha(z) = B(z)$. Define $X(w,z)$ by (6.5), then a straightforward calculation shows that

$$X(w,z) = (\mathcal{I} + (w - w_0)(\mathcal{A}_0 - w)^{-1}) X(w_0, z),$$

i.e. that $X(w,z)$ is an appropriate parametrization of the defect spaces of $\mathcal{S}$. Since

$$[X(w,z), X(w',z)] = \frac{1}{B(w)} K(\overline{w}, \overline{w'}) \frac{1}{\overline{B(w')}} = \frac{\left(-\frac{A(w)}{B(w)}\right) - \overline{\left(-\frac{A(w')}{B(w')}\right)}}{w - \overline{w'}},$$

the function Q_0 is a Q-function associated with $\mathcal{A}_0$ and $\mathcal{S}$.

By Corollary 6.2 we may substitute $E(z)$ by $e^{i\alpha}E(z)$ in the assumption of Lemma 6.4. Then $K(w,z)$ remains unchanged, whereas the functions A and B have to be substituted by $\cos\alpha A(z) + \sin\alpha B(z) = -S_{\alpha+\frac{\pi}{2}}(z)$ and $-\sin\alpha A(z) + \cos\alpha B(z) = S_\alpha(z)$, respectively. The first part of the proof yields (6.6).

$\square$

As a corollary of Proposition 6.1 and Lemma 6.4 we have:

Corollary 6.5. *Let $\kappa := \operatorname{Ind}_-\mathfrak{P}(E)$ and $\alpha \in [0,\pi)$. Then $Q_\alpha \in \mathcal{N}_\kappa$. The point ∞ is a critical point for Q_α for at most one value of α.*

Proof : Since the Q-function of an extension $\mathcal{A}$ of $\mathcal{S}$ can have ∞ as a critical point only if $\mathcal{A}$ is a proper relation, the assertion follows from Corollary 6.3.

$\square$

7 Orthogonal sets in a space $\mathfrak{P}$

Consider a dB-Pontryagin space $\mathfrak{P} = \mathfrak{P}(E)$, $E = A - iB$, with $\mathfrak{d}(\mathfrak{P}) = 0$. As in Lemma 6.4 denote by $\mathcal{A}_0$ the selfadjoint extension of $\mathcal{S}$ induced by $B(z)$. Moreover, if $B \in \mathfrak{P}$, let n be the supremum of all numbers such that $z^n B(z) \in \mathfrak{P}$. If $B \notin \mathfrak{P}$ we put for notational convenience $n := -1$.

Let $\{\gamma_1, \gamma_2, \ldots\} \subseteq \mathbb{R}$ be the set of all real simple zeros of B, $\alpha_1, \ldots, \alpha_r \in \mathbb{R}$ be the real multiple zeros and $\{\beta_1, \ldots, \beta_s; \overline{\beta_1}, \ldots, \overline{\beta_s}\}$ be the set of all nonreal zeros. Note that there exist only finitely many real multiple or nonreal zeros of B. This follows since $Q_0 = -\frac{A}{B} \in \mathcal{N}_\kappa$ and A and B have no common zeros. Hence the zeros of B are exactly the poles of Q_0.

Since Q_0 is a Q-function of $\mathcal{S}$ and $\mathcal{A}_0$, this observation leeds to a connection of the zeros of B and the spectrum of $\mathcal{A}_0$ (for the notation used in the following compare [L]). Note that by Proposition 4.6 the finite spectrum of $\mathcal{A}_0$ coincides with $\{w \in \mathbb{C} | B(w) = 0\}$ and that $\infty \in \sigma(\mathcal{A}_0)$ if and only if $n \geq 0$.

The assertion of the following lemma follows immediately from Proposition 4.6 and the definitions given in [L].

Lemma 7.1. *The set of finite critical points of $\mathcal{A}_0$ equals $\{\alpha_1, \dots, \alpha_r\}$. The spectral subspace of $\mathcal{A}_0$ corresponding to*

(i) γ_i, $i = 1, 2, \dots$, *is* $\operatorname{span}\{\frac{B(z)}{z-\gamma_i}\}$.

(ii) α_i, $i = 1, \dots, r$, *is*

$$\operatorname{span}\{\frac{B(z)}{z-\alpha_i}, \dots, \frac{B(z)}{(z-\alpha_i)^{r_i}}\}, \tag{7.1}$$

where $r_i = \operatorname{Ord}_{\alpha_i} B$.

(iii) $\{\beta_i, \overline{\beta_i}\}$, $i = 1, \dots, s$, *is*

$$\operatorname{span}\{\frac{B(z)}{z-\beta_i}, \dots, \frac{B(z)}{(z-\beta_i)^{\tau_i}}, \frac{B(z)}{z-\overline{\beta_i}}, \dots, \frac{B(z)}{(z-\overline{\beta_i})^{\tau_i}}\}, \tag{7.2}$$

where $\tau_i = \operatorname{Ord}_{\beta_i} B$.

(iv) ∞, *is*

$$\mathfrak{S}_\infty = \operatorname{span}\{B(z), \dots, z^n B(z)\},$$

in particular n is finite.

The point ∞ is not a critical point for $\mathcal{A}_0$ if and only if $n = -1$ or $n = 0$, $[B, B] > 0$ or $n = 0$, $[B, B] < 0$, $\dim \mathfrak{P} < \infty$. It is a singular critical point if and only if $[B, z^n B] = 0$.

Remark 7.2. The inner product with $\frac{B(z)}{z-\gamma_i}$ is given by

$$\left[F(z), \frac{B(z)}{z-\gamma_l}\right] = \frac{1}{A(\gamma_l)} F(\gamma_l), \; F \in \mathfrak{P}(E), l = 1, 2, \dots, \tag{7.3}$$

The Gram matrices of the spaces (7.1) and (7.2) are of the form

$$\mathcal{G}_{\alpha_j} = \begin{pmatrix} 0 & 0 & \cdots & c_1 \\ 0 & & & c_2 \\ \vdots & & & \vdots \\ c_1 & c_2 & \cdots & c_{r_j} \end{pmatrix}, \; c_1 \neq 0,$$

and

$$\mathcal{G}_{\beta_j} = \left(\begin{array}{cccc|cccc} 0 & \cdots & \cdots & 0 & 0 & 0 & \cdots & \overline{d_1} \\ \vdots & & & \vdots & 0 & & & \overline{d_2} \\ \vdots & & & \vdots & \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 & \overline{d_1} & \overline{d_2} & \cdots & \overline{d_{\tau_j}} \\ \hline 0 & 0 & \cdots & d_1 & 0 & \cdots & \cdots & 0 \\ 0 & & & d_2 & \vdots & & & \vdots \\ \vdots & & & \vdots & \vdots & & & \vdots \\ d_1 & d_2 & \cdots & d_{\tau_j} & 0 & \cdots & \cdots & 0 \end{array}\right), \ d_1 \neq 0.$$

The numbers c_i and d_i can be computed from the relations

$$\left[F(z), \sum_{i=0}^{k} \binom{k}{i} \frac{A^{(k-i)}(\overline{w})}{(i+1)!} \frac{B(z)}{(z-\overline{w})^{i+1}}\right] = F^{(k)}(w), F \in \mathfrak{P}(E), \tag{7.4}$$

for $w = \alpha_j$, $k = 0, \ldots, r_j - 1$ and $w = \beta_j, \overline{\beta_j}$, $k = 0, \ldots, \tau_j - 1$.

The relations (7.3) and (7.4) hold since $K(\gamma_l, z) = \frac{B(z)A(\gamma_l)}{z-\gamma_l}$, and since if $\mathrm{Ord}_{\overline{w}} B = l$, we have $(k \leq l-1)$

$$\frac{\partial^k}{(\partial\overline{w})^k} K(w,z) = \sum_{i=0}^{k} \binom{k}{i} \frac{A^{(k-1)}(\overline{w})}{(i+1)!} \frac{B(z)}{(z-\overline{w})^{i+1}} \in \mathfrak{P}(E),$$

and that (compare Lemma 5.4)

$$F^{(k)}(w) = [F(z), \frac{\partial^k}{(\partial\overline{w})^k} K(w,z)].$$

Theorem 7.3. *Assume that ∞ is not a critical point of $\mathcal{A}_0$. The span of the spaces*

$$\overline{\mathrm{span}\,\{\frac{B(z)}{z-\gamma_l} | l = 1, 2, \ldots\}}, \tag{7.5}$$

(7.1) and (7.2) is not dense in $\mathfrak{P}$ if and only if

$$\lim_{y\to+\infty} \frac{1}{y} \operatorname{Im} \frac{A(z)}{B(z)} \neq 0.$$

In this case the orthogonal complement of the above span equals span $\{B(z)\}$*, and we have*

$$[B(z), B(z)] = -\lim_{y\to+\infty} y(\operatorname{Im} \frac{A(z)}{B(z)})^{-1}. \tag{7.6}$$

The elements $K(w,z)$ are given by

$$K(w,z) = \sum_{l\in\mathbb{N}, \gamma_l\neq 0} \sigma_l \frac{B(\overline{w})}{\gamma_l - \overline{w}} \frac{B(z)}{\gamma_l - z} + \sigma_0 \frac{B(\overline{w})}{\overline{w}} \frac{B(z)}{z} + \mu_1 B(\overline{w}) B(z) +$$

$$+\sum_{i=1}^{r_j}\sum_{k=0}^{i-1}\lambda_{ji}\frac{B(\overline{w})}{(\alpha_j-\overline{w})^{1+k}}\frac{B(z)}{(\alpha_j-z)^{i-k}}+$$

$$+\sum_{i=1}^{\tau_j}\sum_{k=0}^{i-1}\left(\chi_{ji}\frac{B(\overline{w})}{(\beta_j-\overline{w})^{1+k}}\frac{B(z)}{(\beta_j-z)^{i-k}}+\overline{\chi_{ji}}\frac{B(\overline{w})}{(\overline{\beta_j}-\overline{w})^{1+k}}\frac{B(z)}{(\overline{\beta_j}-z)^{i-k}}\right), \tag{7.7}$$

where the series converges in the norm of $\mathfrak{P}$.

Proof : The function $Q_0(z)$ has only isolated singularities, namely the zeros of B. Moreover, by assumption, ∞ is not a critical point of Q_0. It is proved in [KL2] that in this case Q_0 has the representation (compare with the more general integral representation discussed in Section 2)

$$Q_0(z)=\sum_{l\in\mathbb{N},\gamma_l\neq 0}\frac{z}{\gamma_l(\gamma_l-z)}\sigma_l-\frac{\sigma_0}{z}+\mu_0+\mu_1 z+\sum_{j=1}^{r}R_j(\frac{1}{\alpha_j-z})+$$

$$+\sum_{j=1}^{s}\left(T_j(\frac{1}{\beta_j-z})+T_j^{\#}(\frac{1}{\overline{\beta_j}-z})\right), \tag{7.8}$$

where $\sigma_l>0$, $l\in\mathbb{N}$, $\sum_{l\in\mathbb{N}}\frac{\sigma_l}{\gamma_l^2}<\infty$, $\sigma_0\geq 0$, $\mu_0,\mu_1\in\mathbb{R}$, R_j are real polynomials with $R_j(0)=0$ and T_j are complex polynomials with $T_j(0)=0$. The term $\frac{\sigma_0}{z}$ occurs if and only if $B(0)=0$.

We compute the Nevanlinna kernel for each single summand of Q_0:

$$\frac{\sum_{l\in\mathbb{N},\gamma_l\neq 0}\frac{z}{\gamma_l(\gamma_l-z)}\sigma_l-\sum_{l\in\mathbb{N},\gamma_l\neq 0}\frac{\overline{w}}{\gamma_l(\gamma_l-\overline{w})}\sigma_l}{z-\overline{w}}=\sum_{l\in\mathbb{N},\gamma_l\neq 0}\frac{\sigma_l}{(\gamma_l-z)(\gamma_l-\overline{w})},$$

$$\frac{-\frac{\sigma_0}{z}+\frac{\sigma_0}{\overline{w}}}{z-\overline{w}}=\sigma_0\frac{1}{z\overline{w}},$$

$$\frac{\mu_1 z-\mu_1\overline{w}}{z-\overline{w}}=\mu_1.$$

If $R_j(z)=\sum_{i=1}^{r_j}\lambda_{ji}z^i$, we find

$$\frac{R_j(\frac{1}{\alpha_j-z})-R_j(\frac{1}{\alpha_j-\overline{w}})}{z-\overline{w}}=\sum_{i=1}^{r_j}\sum_{k=0}^{i-1}\lambda_{ji}\frac{1}{(\alpha_j-z)^{i-k}(\alpha_j-\overline{w})^{1+k}},$$

if $T_j(z)=\sum_{i=1}^{\tau_j}\chi_{ji}z^i$,

$$\frac{\left(T_j(\frac{1}{\beta_j-z})+T_j^{\#}(\frac{1}{\overline{\beta_j}-z})\right)-\left(T_j(\frac{1}{\beta_j-\overline{w}})+T_j^{\#}(\frac{1}{\overline{\beta_j}-\overline{w}})\right)}{z-\overline{w}}=$$

$$=\sum_{i=1}^{\tau_j}\sum_{k=0}^{i-1}\left(\chi_{ji}\frac{1}{(\beta_j-z)^{i-k}(\beta_j-\overline{w})^{1+k}}+\overline{\chi_{ji}}\frac{1}{(\overline{\beta_j}-z)^{i-k}(\overline{\beta_j}-\overline{w})^{1+k}}\right).$$

On the other hand the Nevanlinna kernel of $Q_0(z)$ is given by

$$\frac{Q_0(z)-Q_0(\overline{w})}{z-\overline{w}}=\frac{K(w,z)}{B(z)\overline{B(w)}},$$

hence we obtain (7.7).

Since the functions $\frac{B(z)}{\gamma_l - z}$ are an orthogonal sequence, only finitely many of these elements can be nonpositive. Moreover, in the relation (7.7) there are only finitely many summands added to the series

$$\sum_{l\in\mathbb{N},\gamma_l\neq 0}\sigma_l\frac{B(\overline{w})}{\gamma_l-\overline{w}}\frac{B(z)}{\gamma_l-z}. \tag{7.9}$$

Since by assumption ∞ is not a critical point the space $\mathfrak{S}_\infty$ is either $\{0\}$ or nondegenerated. Thus it can be proved as in [dB7], that the series (7.9) converges in fact in the norm of $\mathfrak{P}(E)$.

We obtain by the Lebesgue dominated convergence theorem that

$$\lim_{y\to+\infty}\frac{1}{y}Q_0(iy)=\mu_1.$$

Hence the term $B(z)$ occurs in (7.7) if and only if the above limit is not equal to 0. Clearly this is the case if and only if $B\in\mathfrak{P}(E)$. Taking the inner product of (7.7) with $B(z)$ yields (7.6)

□

Remark 7.4. By considering $e^{i\alpha}E(z)$ instead of $E(z)$, it is seen that the preceeding results remain valid, if only $A(z)$ and $B(z)$ is substituted by $-S_{\alpha+\frac{\pi}{2}}(z)$ and $S_\alpha(z)$, respectively. Since $\infty\in\sigma(\mathcal{A}_\alpha)$ if and only if $S_\alpha\in\mathfrak{P}$, the condition

$$\lim_{y\to+\infty}\frac{1}{y}\operatorname{Im}\frac{S_{\alpha+\frac{\pi}{2}}(z)}{S_\alpha(z)}=0$$

is violated for at most one value $\alpha_0\in[0,\pi)$. For all other values of α, the point ∞ is not critical for $\mathcal{A}_\alpha$, hence Theorem 7.3 can be applied for all $\alpha\in[0,\pi)$ with possible exception of one value α_0 and this exception can occur only if $S_{\alpha_0}\in\mathfrak{P}$. Note that, if $\mathfrak{P}$ is not finite dimensional, $S_{\alpha_0}\in\mathfrak{P}$ and ∞ is not a critical point for $\frac{S_{\alpha+\frac{\pi}{2}}(z)}{S_\alpha(z)}$, then (by the definition of a critical point in [L]) $[S_{\alpha_0},S_{\alpha_0}]>0$.

By applying Corollary 5.5 it is seen that analoguous results hold for spaces $\mathfrak{P}$ with $\mathfrak{d}(\mathfrak{P})\neq 0$.

8 Matrix functions of the class $\mathcal{M}_\kappa^S$

We start this section with some considerations concerning a certain kernel associated with a 2×2-matrix function. For a matrix M let M^* be its adjoint, and denote by J the matrix

$$J:=\begin{pmatrix}0&-1\\1&0\end{pmatrix}.$$

Definition 8.1. Let $M(z)$ be a 2×2-matrix valued function whose entries are meromorphic functions in $\mathbb{C}$ and let $\rho(M)$ be its domain of holomorphy. Moreover, let S be a scalar function meromorphic in $\mathbb{C}$. We write $M \in \mathcal{M}_\kappa^S$ if

$$M(z)JM(\overline{z})^* = S(z)J\overline{S(\overline{z})}, \tag{8.1}$$

whenever $z, \overline{z} \in \rho(M)$, and if the kernel

$$H_M(z,w) := \frac{M(z)JM(w)^* - S(z)J\overline{S(w)}}{z-\overline{w}},\ \ z,w \in \rho(M)$$

has κ negative squares.

The reproducing kernel Pontryagin space generated by the kernel H_M for a function $M \in \mathcal{M}_\kappa^S$ is denoted by $\mathfrak{K}(M)$. Although we have used the notation $\mathfrak{K}(.)$ already in Definition 2.1, this will not cause confusion. If the matrix $M(z)$ is given by

$$M(z) = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix},$$

the kernel $H_M(w,z)$ can be written as

$$H_M(w,z) = \begin{pmatrix} \frac{B(z)\overline{A(w)}-A(z)\overline{B(w)}}{z-\overline{w}} & \frac{B(z)\overline{C(w)}-A(z)\overline{D(w)}+S(z)\overline{S(w)}}{z-\overline{w}} \\ \frac{D(z)\overline{A(w)}-C(z)\overline{B(w)}-S(z)\overline{S(w)}}{z-\overline{w}} & \frac{D(z)\overline{C(w)}-C(z)\overline{D(w)}}{z-\overline{w}} \end{pmatrix} \tag{8.2}$$

Lemma 8.2. *Let $T(z)$ be a meromorphic function. The matrix $M(z)$ is an element of $\mathcal{M}_\kappa^S$ if and only if $T(z)\cdot M(z) \in \mathcal{M}_\kappa^{ST}$. In fact the mapping $\begin{pmatrix} F_+ \\ F_- \end{pmatrix} \mapsto T\begin{pmatrix} F_+ \\ F_- \end{pmatrix}$ is an isometry of $\mathfrak{K}(M)$ onto $\mathfrak{K}(TM)$, the respective kernels satisfy*

$$H_{TM}(w,z) = T(z)H_M(w,z)\overline{T(w)}. \tag{8.3}$$

If the entries of M are real, then (8.1) is equivalent to $\det M(z) = S(z)S^\#(z)$.

Proof : The relation (8.3) is obvious. The fact that $\begin{pmatrix} F_+ \\ F_- \end{pmatrix} \mapsto T\begin{pmatrix} F_+ \\ F_- \end{pmatrix}$ is an isometry of $\mathfrak{K}(M)$ onto $\mathfrak{K}(TM)$ follows from (8.3).

To prove the second assertion, consider the function $(z-\overline{w})H_M(w,z)$ given by (8.2). If the entries of M are real, the entries in the left upper and right lower corner of $(z-\overline{w})H_M(w,z)$ vanish for $w=\overline{z}$. The entry in the left lower corner equals

$$D(z)A(z) - C(z)B(z) - S(z)S^\#(z).$$

Since the right upper entry is given by a similar formula, we find that the condition

$$M(z)JM(\overline{z})^* - S(z)J\overline{S(\overline{z})} = 0$$

is equivalent to $\det M(z) = S(z)S^{\#}(z)$.

$\square$

In the following we denote by $\mathcal{R}_S(w)$ the difference quotient (compare with the notation of Proposition 4.6)

$$\mathcal{R}_S(w)X = \frac{X(z) - \frac{S(z)}{S(w)}X(w)}{z-w}.$$

Here X is allowed to be a scalar- or vector- valued function. From Lemma 8.2 above, Theorem 5.3 and Corollary 6.7 of [KW] we obtain:

Proposition 8.3. *If $M \in \mathcal{M}_\kappa^S$, then $\mathfrak{K}(M)$ is invariant under $\mathcal{R}_S(w)$. For $a, b \in \mathbb{C}$ and elements $\begin{pmatrix} F_+ \\ F_- \end{pmatrix}, \begin{pmatrix} G_+ \\ G_- \end{pmatrix} \in \mathfrak{K}(M)$, we have*

$$\frac{1}{S(a)\overline{S(b)}} \begin{pmatrix} G_+(b) \\ G_-(b) \end{pmatrix}^* J \begin{pmatrix} F_+(a) \\ F_-(a) \end{pmatrix} = [\begin{pmatrix} F_+ \\ F_- \end{pmatrix}, \mathcal{R}_S(b)\begin{pmatrix} G_+ \\ G_- \end{pmatrix}] -$$

$$-[\mathcal{R}_S(a)\begin{pmatrix} F_+ \\ F_- \end{pmatrix}, \begin{pmatrix} G_+ \\ G_- \end{pmatrix}] + (a-\overline{b})[\mathcal{R}_S(a)\begin{pmatrix} F_+ \\ F_- \end{pmatrix}, \mathcal{R}_S(b)\begin{pmatrix} G_+ \\ G_- \end{pmatrix}]. \tag{8.4}$$

Corollary 8.4. *Assume that $M \in \mathcal{M}_\kappa^1$ and that $\begin{pmatrix} u \\ v \end{pmatrix} \in \mathfrak{K}(M)$, $u, v \in \mathbb{C}$. Then $u\overline{v} \in \mathbb{R}$. If $\begin{pmatrix} u \\ v \end{pmatrix}$ and $\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$ both belong to $\mathfrak{K}(M)$, they are linearly dependent.*

Proof : Choosing in (8.4) the elements $\begin{pmatrix} F_+ \\ F_- \end{pmatrix} = \begin{pmatrix} G_+ \\ G_- \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$ we obtain $u\overline{v} - \overline{u}v = 0$. The remaining assertion follows by an elementary consideration.

$\square$

Let $M \in \mathcal{M}_\kappa^S$ be given. In order to study the structure of the reproducing kernel Pontryagin space $\mathfrak{K}(M)$, we introduce the component spaces

$$\mathfrak{K}_+(M) := \text{cls}\,\{H_M(w,z)\begin{pmatrix} 1 \\ 0 \end{pmatrix}\},\ \mathfrak{K}_-(M) := \text{cls}\,\{H_M(w,z)\begin{pmatrix} 0 \\ 1 \end{pmatrix}\}.$$

Obviously $\overline{\mathfrak{K}_+(M) + \mathfrak{K}_-(M)} = \mathfrak{K}(M)$. Denote in the following by π_+ and π_- the mappings

$$\pi_+ : \begin{pmatrix} F_+ \\ F_- \end{pmatrix} \mapsto F_+,\ \pi_- : \begin{pmatrix} F_+ \\ F_- \end{pmatrix} \mapsto F_-.$$

We also consider the scalar kernel functions

$$H_M^+(w,z) := (1,0)H_M(w,z)\begin{pmatrix} 1 \\ 0 \end{pmatrix},\ H_M^-(w,z) := (0,1)H_M(w,z)\begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and the respective reproducing kernel Pontryagin spaces $\mathfrak{K}(H_M^+)$ and $\mathfrak{K}(H_M^-)$. Clearly

$$\text{Ind}_-\mathfrak{K}(H_M^+) \le \text{Ind}_-\mathfrak{K}(M),\ \text{ and } \text{Ind}_-\mathfrak{K}(H_M^-) \le \text{Ind}_-\mathfrak{K}(M).$$

Remark 8.5. Assume that the matrix function $M(z) \in \mathcal{M}_\kappa^S$ has real and entire entries A, B, C, D. If the functions A and B (C and D) are linearly independent and have no common nonreal zeros, then $A - iB \in \mathcal{HB}_{\kappa'}$ ($D + iC \in \mathcal{HB}_{\kappa''}$) for some $\kappa' \leq \kappa$ ($\kappa'' \leq \kappa$). In this case we have $\mathfrak{K}(H_M^+) = \mathfrak{P}(A - iB)$ ($\mathfrak{K}(H_M^-) = \mathfrak{P}(D + iC)$).

Lemma 8.6. *We have*

$$\mathfrak{K}_+(M)^\perp = \ker \pi_+,$$

$$\dim\left(\pi_+\mathfrak{K}(M)/\pi_+\mathfrak{K}_+(M)\right) = \mathrm{Ind}_0\mathfrak{K}_+(M).$$

Moreover,

$$\mathfrak{K}_+(M)/\mathfrak{K}_+(M)^\circ \cong \mathfrak{K}(H_M^+),$$

and if $\mathrm{Ind}_0\mathfrak{K}_+(M) = 0$ *we have* $\mathfrak{K}(H_M^+) = \pi_+\mathfrak{K}(M)$ *as a set of functions, in fact* π_+ *is a partial isometry. If* $\mathfrak{K}_+(M) = \mathfrak{K}(M)$*, the mapping* π_+ *is an isometry of* $\mathfrak{K}(M)$ *onto* $\mathfrak{K}(H_M^+)$*. These assertions remain true if everywhere* $_+$ *is replaced by* $_-$*.*

Proof : The relation $\mathfrak{K}_+(M)^\perp = \ker \pi_+$ holds by definition. Decompose $\mathfrak{K}(M)$ as

$$\mathfrak{K}(M) = (\mathfrak{K}_+(M) \dot{+} \mathfrak{H}_1)[\dot{+}]\mathfrak{H}_2,$$

where $\mathfrak{H}_1$ is skewly linked to $\mathfrak{K}_+(M)^\circ$, then $\mathfrak{H}_2 \subseteq \ker \pi_+$ and $\mathfrak{H}_1 \cap \ker \pi_+ = \{0\}$. Hence the codimension of $\pi_+\mathfrak{K}_+(M)$ in $\pi_+\mathfrak{K}(M)$ equals $\dim \mathfrak{H}_1 = \mathrm{Ind}_0\mathfrak{K}_+(M)$.

The mapping π_+ maps $H_M(w, z)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ onto $H_M^+(w, z)$, hence extends to an isometry of $\mathfrak{K}_+(M)/\mathfrak{K}_+(M)^\circ$ onto $\mathfrak{K}(H_M^+)$.

$\square$

Corollary 8.7. *Let* $\mathfrak{L} \subseteq \mathfrak{K}(M)$ *be a Pontryagin space. If* $\mathfrak{L} \perp \ker \pi_+$*, then* $\pi_+|_{\mathfrak{L}}$ *is an isometry of* $\mathfrak{L}$ *into* $\mathfrak{K}(H_M^+)$*.*

Corollary 8.8. *Assume that* $\mathrm{Ind}_0\mathfrak{K}_+(M) = 0$*. Then* $\mathfrak{K}(H_M^+)$ *is invariant under application of* $\mathcal{R}_S(w)$*. If even* $\mathfrak{K}_+(M) = \mathfrak{K}(M)$*, then the mapping* $\psi := \pi_-(\pi_+)^{-1}$ *satisfies*

$$\frac{F(a)\overline{(\psi G)(b)} - (\psi F)(a)\overline{G(b)}}{S(a)\overline{S(b)}} = [F, \mathcal{R}_S(b)G] - [\mathcal{R}_S(a)F, G] +$$

$$+(a - \overline{b})[\mathcal{R}_S(a)F, \mathcal{R}_S(b)G], \tag{8.5}$$

for $F, G \in \mathfrak{K}(H_M^+)$ *and* $a, b \in \mathbb{C}$ *such that* $S(a), S(b) \neq 0$*. Any mapping* ψ' *which assigns to each function* $F \in \mathfrak{K}(H_M^+)$ *an entire function and which satisfies (8.5) is of the form*

$$\psi' = \psi + \lambda, \ \lambda \in \mathbb{R}.$$

Proof : Note first that $\mathrm{Ind}_0\mathfrak{K}_+(M) = 0$ implies by Lemma 8.6 that $\pi_+\mathfrak{K}_+(M) = \mathfrak{K}(H_M^+)$. Since $\mathcal{R}_S(w)$ commutes with π_+, it follows from $\pi_+\mathfrak{K}(M) = \mathfrak{K}(H_M^+)$ and Proposition 8.3 that $\mathfrak{K}(H_M^+)$ is invariant under $\mathcal{R}_S(w)$.

If $\mathfrak{K}_+(M) = \mathfrak{K}(M)$, the mapping $(\pi_+)^{-1}$ is an isometry of $\mathfrak{K}(H_M^+)$ onto $\mathfrak{K}(M)$. The relation (8.5) follows from (8.4).

Assume that ψ' is given. By (8.5) we have for any $F, G \in \mathfrak{K}(H_M^+)$ and $a, b \in \mathbb{C}$

$$F(a)\overline{((\psi - \psi')G)(b)} = \overline{G(b)}((\psi - \psi')F)(a).$$

By a convenient choice of G and b the assertion follows.

□

If a matrix function $M \in \mathcal{M}_\kappa^1$ is given,

$$M(z) =: \begin{pmatrix} m_{11}(z) & m_{12}(z) \\ m_{21}(z) & m_{22}(z) \end{pmatrix},$$

and the entry $m_{21}(z)$ does not vanish identically, then we will consider the so called Potapov-Ginzburg transform:

$$\Psi(M)(z) := \begin{pmatrix} \frac{m_{11}(z)}{m_{21}(z)} & \frac{m_{11}(z)m_{22}(z) - m_{21}(z)m_{12}(z)}{m_{21}(z)} \\ \frac{1}{m_{21}(z)} & \frac{m_{22}(z)}{m_{21}(z)} \end{pmatrix}, \ z \in \rho(\Psi(M)(z)).$$

Let us recall the following result from [Br] (compare also [KL3], [KW]).

Lemma 8.9. *Let $M(z)$ be a 2×2-matrix function. Then $M \in \mathcal{M}_\kappa^1$ if and only if $\Psi(M)(z) \in \mathcal{N}_\kappa^{2\times 2}$. In fact the kernel relation*

$$H_M(w, z) = \begin{pmatrix} -1 & m_{11}(z) \\ 0 & m_{21}(z) \end{pmatrix} \frac{\Psi(z) - \Psi(w)^*}{z - \bar{w}} \begin{pmatrix} -1 & m_{11}(w) \\ 0 & m_{21}(w) \end{pmatrix}^* \tag{8.6}$$

holds. The mapping

$$\begin{pmatrix} F_+ \\ F_- \end{pmatrix} \mapsto \begin{pmatrix} -1 & m_{11}(z) \\ 0 & m_{21}(z) \end{pmatrix} \begin{pmatrix} F_+ \\ F_- \end{pmatrix}$$

is an isometry of $\mathfrak{K}(\Psi(M))$ onto $\mathfrak{K}(M)$. Under application of this isometry the subspace $\ker \pi_-$ *of $\mathfrak{K}(\Psi(M))$ is mapped bijectively onto the subspace* $\ker \pi_-$ *of $\mathfrak{K}(M)$.*

Remark 8.10. Note that by Lemma 8.2 similar results hold for matrices of the class $\mathcal{M}_\kappa^S$. For the sake of simplicity we will consider only matrices of the class $\mathcal{M}_\kappa^1$. However, we could avoid the use of Lemma 8.2 if we define for $M \in \mathcal{M}_\kappa^S$ a Potapov-Ginzburg transform by

$$\Psi_S(M)(z) := \begin{pmatrix} \frac{m_{11}(z)}{m_{21}(z)} & \frac{1}{S(z)}\left(\frac{m_{11}(z)m_{22}(z)}{m_{21}(z)} - m_{12}(z)\right) \\ \frac{S(z)}{m_{21}(z)} & \frac{m_{22}(z)}{m_{21}(z)} \end{pmatrix}.$$

The kernel relation corresponding to (8.6) then is

$$H_M(w, z) = \begin{pmatrix} -S(z) & m_{11}(z) \\ 0 & m_{21}(z) \end{pmatrix} \frac{\Psi(z) - \Psi(w)^*}{z - \bar{w}} \begin{pmatrix} -S(w) & m_{11}(w) \\ 0 & m_{21}(w) \end{pmatrix}^*.$$

9 The structure of the reproducing kernel space $\mathfrak{K}(M)$

In this section we investigate the structure of the space $\mathfrak{K}(M)$ for functions $M \in \mathcal{M}_\kappa^1$ which satisfy an additional condition: if M is given by

$$M(z) = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \in \mathcal{M}_\kappa^1,$$

we assume throughout this section that $M(z)$ is meromorphic in $\mathbb{C}$, that

$$m_{21}(z), m_{22}(z) \not\equiv 0,\ \min(\mathrm{Ord}_w m_{21}, \mathrm{Ord}_w m_{22}) \leq 0,\ w \in \mathbb{C}, \tag{9.1}$$

and that

$$\min(\mathrm{Ord}_w m_{11}, \mathrm{Ord}_w m_{12}) \geq \min(\mathrm{Ord}_w m_{21}, \mathrm{Ord}_w m_{22}),\ w \in \mathbb{C}, \tag{9.2}$$

holds.

Remark 9.1. If M is real and entire, m_{21} and m_{22} are linearly independent and have no common zeros (i.e. satisfy (9.1)), then $m_{22} + im_{21} \in \mathcal{HB}_{\kappa'}$ ($\kappa' \leq \kappa$) and satisfies (9.2). The space $\mathfrak{K}(H_M^-)$ is a dB-Pontryagin space, $\mathfrak{K}(H_M^-) = \mathfrak{P}(m_{22} + im_{21})$.

Note that, if S is any meromorphic function, the condition (9.2) holds for a matrix M if and only if it holds for $S \cdot M$. Hence, by Lemma 8.2, the restriction to functions of the class $\mathcal{M}_\kappa^1$ instead of $\mathcal{M}_\kappa^S$ is in some respects not essential.

The following results show that the above conditions on M reflect in a very special structure of the Potapov-Ginzburg transform $\Psi(M)$.

Lemma 9.2. *Let $M(z) \in \mathcal{M}_\kappa^1$ be meromorphic in $\mathbb{C}$ and satisfy (9.1) and (9.2). Then the Potapov-Ginzburg transform*

$$\Psi(M)(z) =: \begin{pmatrix} n_{11}(z) & n_{12}(z) \\ n_{21}(z) & n_{22}(z) \end{pmatrix}$$

satisfies

$$\rho(\Psi(M)) = \rho(n_{22}) \tag{9.3}$$

and

$$\mathrm{Ord}_w n_{11}, \mathrm{Ord}_w n_{12}, \mathrm{Ord}_w n_{21}, \mathrm{Ord}_w \det \Psi(M) \geq \mathrm{Ord}_w n_{22},\ w \notin \rho(\Psi(M)). \tag{9.4}$$

Proof : In the following fix $w \notin \rho(\Psi(M))$. Assume that $w \notin \rho(n_{11})$, i.e. that $\mathrm{Ord}_w m_{11} < \mathrm{Ord}_w m_{21}$. With (9.2) we find

$$\mathrm{Ord}_w m_{11} \geq \min(\mathrm{Ord}_w m_{21}, \mathrm{Ord}_w m_{22}) = \mathrm{Ord}_w m_{22},$$

hence $\mathrm{Ord}_w n_{11} \geq \mathrm{Ord}_w n_{22}$, in particular $w \notin \rho(n_{22})$.

Assume that $w \notin \rho(n_{21})$, i.e. that $\mathrm{Ord}_w m_{21} > 0$. By (9.1) we have $\mathrm{Ord}_w m_{22} \leq 0$, hence

$$\mathrm{Ord}_w n_{22} = \mathrm{Ord}_w m_{22} - \mathrm{Ord}_w m_{21} \leq -\mathrm{Ord}_w m_{21} = \mathrm{Ord}_w n_{21},$$

and again $w \notin \rho(n_{22})$. Note that, since M satisfies (8.1), we have $\Psi(M)(z)^* = \Psi(M)(\overline{z})$. Hence the assumption concerning n_{12} follows by symmetry.

For a pole w of n_{22} we have $\min(\mathrm{Ord}_w m_{21}, \mathrm{Ord}_w m_{22}) = \mathrm{Ord}_w m_{22}$. By (9.2), $\mathrm{Ord}_w m_{12} \geq \mathrm{Ord}_w m_{22}$, and since $\det \Psi(M) = \frac{m_{12}}{m_{21}}$, we find

$$\mathrm{Ord}_w \det \Psi(M) \geq \mathrm{Ord}_w n_{22}.$$

□

In the sequel we investigate the structure of a meromorphic function

$$Q(z) = \begin{pmatrix} n_{11}(z) & n_{12}(z) \\ n_{21}(z) & n_{22}(z) \end{pmatrix} \in \mathcal{N}_\kappa^{2\times 2},$$

which satisfies (9.3) and (9.4) (with $\det Q$ instead of $\det \Psi(M)$). These considerations will in turn be applied to the Potapov-Ginzburg transform of a matrix M which satisfies (9.1) and (9.2).

It is well known (compare e.g. [KL1] or [HSW]) that there exists a model for Q, i.e. a Pontryagin space $\mathfrak{P}$, a selfadjoint relation $\mathcal{A}$ with $\rho(\mathcal{A}) \neq \emptyset$, and linear mappings $\Gamma_z : \mathbb{C}^2 \to \mathfrak{P},\ z \in \rho(\mathcal{A})$ with

$$\Gamma_z = (I + (z-w)(\mathcal{A} - z)^{-1})\Gamma_w,\ z, w \in \rho(\mathcal{A}), \tag{9.5}$$

such that $\rho(Q) = \rho(\mathcal{A})$ and

$$Q(z) = C - i \,\mathrm{Im}\, z_0 \Gamma_{z_0}^* \Gamma_{z_0} + (z - \overline{z_0})\Gamma_{z_0}^* \Gamma_z \tag{9.6}$$

with a constant selfadjoint 2×2-matrix C. Moreover, the space $\mathfrak{P}$ can be chosen such that

$$\mathrm{cls} \bigcup_{z \in \rho(\mathcal{A})} \Gamma_z \mathbb{C}^2 = \mathfrak{P}. \tag{9.7}$$

As a short argument shows this condition implies that $\mathcal{A}$ has eigenvalues (including ∞) of geometric multiplicity at most two. We will show that the properties (9.3) and (9.4) of Q ensure that the geometric multiplicity of the eigenvalues of $\mathcal{A}$ is in fact one.

Since Q is meromorphic, $\sigma(\mathcal{A}) \cap \mathbb{C}$ consists of isolated points only:

$$\sigma(\mathcal{A}) \cap \mathbb{C} = \{\lambda_j | j = 1, 2, \ldots\} \cup \{\beta_k, \overline{\beta_k} | k = 1, \ldots, m\},$$

where $\lambda_j \in \mathbb{R}$, $j = 1, 2, \ldots$, and $\beta_k \in \mathbb{C}^+$, $k = 1, \ldots, m$. There exist selfadjoint projectors $E_{\{\lambda_j\}}$ and $E_{\{\beta_k;\overline{\beta_k}\}}$ commuting with $\mathcal{A}$ such that (compare [DS2])

$$\sigma(\mathcal{A} \cap (E_{\{\lambda_j\}}\mathfrak{P})^2) = \{\lambda_j\},\ \lambda_j \in \rho(\mathcal{A} \cap (((I - E_{\{\lambda_j\}})\mathfrak{P})^2))$$

for $j = 1, 2, \ldots$, and

$$\sigma(\mathcal{A} \cap (E_{\{\beta_k;\overline{\beta_k}\}}\mathfrak{P})^2) = \{\beta_k, \overline{\beta_k}\},\ \beta_k, \overline{\beta_k} \in \rho(\mathcal{A} \cap (((I - E_{\{\beta_k;\overline{\beta_k}\}})\mathfrak{P})^2))$$

for $k = 1, \ldots, m$. In fact, $E_{\{\lambda_j\}}$ is the Riesz projection of $\mathfrak{P}$ onto the generalized eigenspace of $\mathcal{A}$ at λ_j, and $E_{\{\beta_k;\overline{\beta_k}\}}$ is the Riesz projection of $\mathfrak{P}$ onto the span of the generalized eigenspaces of β_k and $\overline{\beta_k}$. As $\mathfrak{P}$ is a Pontryagin space and satisfies (9.7), all these generalized eigenspaces are finite dimensional.

If $E := E_{\{\lambda_j\}}$ or $E := E_{\{\beta_k;\overline{\beta_k}\}}$ and we put $\Gamma_z^1 := (I - E)\Gamma_z$ and $\Gamma_z^2 := E\Gamma_z$, then Γ_z^1 and Γ_z^2 satisfy (9.5) and the function Q can be written as

$$Q(z) = Q_1(z) + Q_2(z), \tag{9.8}$$

with

$$Q_1(z) := D + C - i \operatorname{Im} z_0 (\Gamma_{z_0}^1)^* \Gamma_{z_0}^1 + (z - \overline{z_0})(\Gamma_{z_0}^1)^* \Gamma_z^1,$$
$$Q_2(z) := (1 - D) - i \operatorname{Im} z_0 (\Gamma_{z_0}^2)^* \Gamma_{z_0}^2 + (z - \overline{z_0})(\Gamma_{z_0}^2)^* \Gamma_z^2,$$

and an arbitrary constant selfadjoint 2×2-matrix D. Then $Q_1(z)$ is analytic at λ_j or $\beta_k, \overline{\beta_k}$, respectively, and $Q_2(z)$ has only one pole at λ_j (a pair of poles at $\beta_k, \overline{\beta_k}$, respectively). Moreover, since $E\mathfrak{P}$ is finite dimensional, it follows from [KL2] that we can choose D such that $Q_2(z) = \Gamma^*(\mathcal{A} - z)^{-1}\Gamma$ for a linear mapping $\Gamma : \mathbb{C}^2 \to E\mathfrak{P}$ with $\Gamma_z^2 = (\mathcal{A} - z)^{-1}\Gamma$, $z \in \rho(\mathcal{A})$. Note that

$$\operatorname{cls} \bigcup_{z \in O} (\mathcal{A} - z)^{-1}\Gamma\mathbb{C}^2 = E\mathfrak{P}$$

for any open set O contained in $\rho(\mathcal{A})$.

Lemma 9.3. *Let $\mathfrak{Q} \neq \{0\}$ be a finite dimensional Pontryagin space, let $\mathcal{L}$ be a selfadjoint operator in $\mathfrak{Q}$ with $\sigma(\mathcal{L}) = \{\lambda\}$, $\lambda \in \mathbb{R}$, and assume that the geometric multiplicity of λ is at most two. Moreover, let $\Gamma : \mathbb{C}^2 \to \mathfrak{Q}$ be a linear mapping such that*

$$\operatorname{span} \bigcup_{z \in O} (\mathcal{L} - z)^{-1}\Gamma\mathbb{C}^2 = \mathfrak{Q} \tag{9.9}$$

for some open set $O \subseteq \rho(\mathcal{L})$. Let

$$\Gamma^*(\mathcal{L} - z)^{-1}\Gamma =: \begin{pmatrix} n_{11}(z) & n_{12}(z) \\ n_{21}(z) & n_{22}(z) \end{pmatrix},$$

and assume that (9.3) and (9.4) hold. Then $\dim \mathfrak{Q} = -\mathrm{Ord}_\lambda n_{22}$,

$$\operatorname{span} \{(\mathcal{L} - z)^{-1}\Gamma \begin{pmatrix} 0 \\ 1 \end{pmatrix} | z \in O\} = \mathfrak{Q}, \tag{9.10}$$

and the geometric multiplicity of λ is one.

Proof : Without loss of generality we assume that $\lambda = 0$.

We set $x_0 := \Gamma \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $x_j := \mathcal{L}^j x_0$, $y_0 := \Gamma \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $y_j := \mathcal{L}^j y_0$. Note that (9.9) is equivalent to

$$\operatorname{span} \{x_0, x_1, \ldots, y_0, y_1, \ldots\} = \mathfrak{Q}, \tag{9.11}$$

whereas (9.10) means

$$\operatorname{span} \{x_0, x_1, \ldots\} = \mathfrak{Q}. \tag{9.12}$$

Note that $x_0 \neq 0$, since by (9.3) and (9.9) we have

$$0 \notin \rho(\Gamma^*(\mathcal{L}-z)^{-1}\Gamma) = \rho(n_{22}).$$

If $y_0 = 0$ the relation (9.12) holds. Otherwise let $l \in \mathbb{N}$ ($m \in \mathbb{N}$) be such that $x_l \neq 0$ and $x_{l+1} = 0$ ($y_m \neq 0$ and $y_{m+1} = 0$), and put $\mathfrak{L} = \operatorname{span}\{x_0, \dots, x_l\}$. Note that, $\{x_0, \dots, x_l\}$ is a basis of $\mathfrak{L}$, and that the Gram matrix with respect to this basis is a Hankel matrix.

The function $n_{22}(z)$ can be written as

$$n_{22}(z) = -\sum_{k=0}^{\infty} [\mathcal{L}^k x_0, x_0] \frac{1}{z^{k+1}},$$

and similar formulas hold for n_{12}, n_{21} and n_{11}.

The subspace $\mathfrak{L}$ is degenerated if and only if $[x_l, x_0] = 0$. In this case $\mathrm{Ord}_0 n_{22} > -(l+1)$. By (9.11) and since $\mathfrak{Q}$ is nondegenerated we must have $[x_l, y_0] \neq 0$, hence $\mathrm{Ord}_0 n_{12}(z) \leq -(l+1)$. This contradicts (9.4), and we arrive at the conclusion that $\mathfrak{L}$ is nondegenerated. Moreover, we find $\mathrm{Ord}_0 n_{22} = -(l+1)$.

Let $\mathcal{P}$ be the orthogonal projection of $\mathfrak{Q}$ onto $\mathfrak{L}$. Then $\mathcal{L}\mathcal{P} = \mathcal{P}\mathcal{L}$ because $\mathfrak{L}$ is invariant under $\mathcal{L}$. Hence

$$n_{12} = [(\mathcal{L}-z)^{-1}x_0, \mathcal{P}y_0], \; n_{21} = [(\mathcal{L}-z)^{-1}\mathcal{P}y_0, x_0],$$

and

$$n_{11} = [(\mathcal{L}-z)^{-1}\mathcal{P}y_0, \mathcal{P}y_0] + [(\mathcal{L}-z)^{-1}(I-\mathcal{P})y_0, (I-\mathcal{P})y_0].$$

Put $(I-\mathcal{P})y_0 =: \hat{y}_0$. If $\hat{y}_0 \neq 0$, let r be such that $\mathcal{L}^r \hat{y}_0 \neq 0$ and $\mathcal{L}^{r+1}\hat{y}_0 = 0$. Then $\{\hat{y}_0, \mathcal{L}\hat{y}_0, \dots, \mathcal{L}^r \hat{y}_0\}$ is a basis of $\mathfrak{L}^\perp$. Since $\mathfrak{L}^\perp$ is nondegenerated, we have $[\mathcal{L}^r \hat{y}_0, \hat{y}_0] \neq 0$, hence $\mathrm{Ord}_0[(\mathcal{L}-z)^{-1}\hat{y}_0, \hat{y}_0] = -(r+1) < 0$. Therefore

$$\mathrm{Ord}_0([(\mathcal{L}-z)^{-1}\hat{y}_0, \hat{y}_0] n_{22}) < -(l+1).$$

By Lemma 3.3 of [KL3] we have

$$\mathrm{Ord}_0([(\mathcal{L}-z)^{-1}\mathcal{P}y_0, \mathcal{P}y_0] n_{22} - n_{12}n_{21}) \geq -(l+1),$$

hence $\mathrm{Ord}_0 \det Q < -(l+1)$, which contradicts (9.4).

We conclude that $\hat{y}_0 = 0$, i.e. that (9.12) holds. In particular, $\dim \mathfrak{Q} = l+1 = -\mathrm{Ord}_0 n_{22}$ and the geometric multiplicity of 0 is one.

$\square$

Lemma 9.4. *Let $\mathfrak{Q} \neq \{0\}$ be a finite dimensional Pontryagin space, let $\mathcal{L}$ be a selfadjoint operator in $\mathfrak{Q}$ with $\sigma(\mathcal{L}) = \{\beta, \overline{\beta}\}$, and assume that the geometric multiplicity of β ($\overline{\beta}$) is at most two. Moreover, let $\Gamma : \mathbb{C}^2 \to \mathfrak{Q}$ be a linear mapping such that*

$$\operatorname{span} \bigcup_{z \in O} (\mathcal{L}-z)^{-1}\Gamma\mathbb{C}^2 = \mathfrak{Q} \tag{9.13}$$

for some open set $O \subseteq \rho(\mathcal{L})$. Let

$$\Gamma^*(\mathcal{L}-z)^{-1}\Gamma =: \begin{pmatrix} n_{11}(z) & n_{12}(z) \\ n_{21}(z) & n_{22}(z) \end{pmatrix},$$

and assume that (9.3) and (9.4) hold. Then $\dim \mathfrak{Q} = -2\mathrm{Ord}_\beta n_{22}$*,*

$$\operatorname{span}\left\{(\mathcal{L}-z)^{-1}\Gamma \begin{pmatrix} 0 \\ 1 \end{pmatrix} : z \in O\right\} = \mathfrak{Q},$$

and the geometric multiplicity of β ($\overline{\beta}$) is one.

Proof : Denote by E the Riesz projection of $\mathfrak{Q}$ onto the generalized eigenspace at β with the generalized eigenspace at $\overline{\beta}$ as its kernel, and let $E' := I - E$. Note that $E\mathfrak{Q}$ and $E'\mathfrak{Q}$ are neutral and skewly linked.

Set $x_0 := E\Gamma \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $x_j := (\mathcal{L}-\beta)^j x_0$, and let l be such that $x_l \neq 0$ and $x_{l+1} = 0$. If $x_0 = 0$ put formally $l = -1$. Set $x_0' := E'\Gamma \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $x_j' := (\mathcal{L}-\overline{\beta})^j x_0'$, and let l' be such that $x_{l'} \neq 0$ and $x_{l'+1} = 0$. If $x_0' = 0$ put $l' = -1$. In a similar manner define elements y_j, y_j' and numbers m, m'.

Again (9.13) implies that

$$\operatorname{span}\{x_0, \ldots, x_l, x_0', \ldots, x_{l'}', y_0, \ldots, y_m, y_0', \ldots, y_{m'}'\} = \mathfrak{Q}, \tag{9.14}$$

and we have to show that in fact

$$\operatorname{span}\{x_0, \ldots, x_l, x_0', \ldots, x_{l'}'\} = \mathfrak{Q}. \tag{9.15}$$

If $y_0 = y_0' = 0$, we are done. Otherwise put $\mathfrak{L} := \operatorname{span}\{x_0, \ldots, x_l, x_0', \ldots, x_{l'}'\}$, note that the elements $x_0, \ldots, x_l, x_0', \ldots, x_{l'}'$ are a basis of $\mathfrak{L}$, and that the Gram matrix with respect to this basis has 2×2-block form with zero blocks in the diagonal and Hankel matrices as off-diagonal blocks. Moreover, note that at least one of x_0 and x_0' is not equal to zero as is seen by the same reasoning as in the proof of Lemma 9.3.

If the subspace $\mathfrak{L}$ is degenerated, we have $[x_l, x_0'] = 0$ or $[x_{l'}', x_0] = 0$. In the first case we find $\mathrm{Ord}_\beta n_{22}(z) > -(l+1)$. By (9.14), and since $\mathfrak{Q}$ is nondegenerated this yields $0 \neq [x_l, y_0']$, hence $\mathrm{Ord}_\beta n_{12}(z) \leq -(l+1)$ which contradicts (9.4). In the second case the same argument yields a contradiction, and we conclude that $\mathfrak{L}$ is nondegenerated. In particular,

$$-(l+1) = \mathrm{Ord}_\beta n_{22} = \mathrm{Ord}_{\overline{\beta}} n_{22} = -(l'+1).$$

Let $\mathcal{P}$ be the orthogonal projection of $\mathfrak{Q}$ onto $\mathfrak{L}$. Then $\mathcal{L}\mathcal{P} = \mathcal{P}\mathcal{L}$ because $\mathfrak{L}$ is invariant under $\mathcal{L}$, and hence $E\mathcal{P} = \mathcal{P}E$. This gives

$$n_{12} = [(\mathcal{L}-z)^{-1}(x_0 + x_0'), \mathcal{P}(y_0 + y_0')],\ n_{21} = [(\mathcal{L}-z)^{-1}\mathcal{P}(y_0 + y_0'), (x_0 + x_0')],$$

and

$$n_{11} = [(\mathcal{L}-z)^{-1}\mathcal{P}(y_0 + y_0'), \mathcal{P}(y_0 + y_0')] + [(\mathcal{L}-z)^{-1}(I-\mathcal{P})(y_0 + y_0'), (I-\mathcal{P})(y_0 + y_0')].$$

Put $\hat{y}_0 := (I-\mathcal{P})y_0$ and $\hat{y}_0' := (I-\mathcal{P})y_0'$. The elements

$$\hat{y}_0, (\mathcal{L}-\beta)\hat{y}_0, \dots, (\mathcal{L}-\beta)^r\hat{y}_0, \hat{y}_0', (\mathcal{L}-\overline{\beta})\hat{y}_0', \dots, (\mathcal{L}-\overline{\beta})^{r'}\hat{y}_0',$$

span $\mathfrak{L}^\perp$. Here r is such that $(\mathcal{L}-\beta)^r\hat{y}_0 \neq 0$ and $(\mathcal{L}-\beta)^{r+1}\hat{y}_0 = 0$. If $\hat{y}_0 = 0$, we put $r = -1$. The number r' is defined similar.

If we assume that $\mathfrak{L}^\perp \neq \{0\}$, we have, since $\mathfrak{L}^\perp$ is nondegenerated, $r = r' \geq 0$ and $[(\mathcal{L}-\beta)^r\hat{y}_0, \hat{y}_0'] \neq 0$. Hence $\mathrm{Ord}_\beta[(\mathcal{L}-z)^{-1}(\hat{y}_0+\hat{y}_0'), \hat{y}_0+\hat{y}_0'] = -(r+1) < 0$. Therefore

$$\mathrm{Ord}_\beta([(\mathcal{L}-z)^{-1}(\hat{y}_0+\hat{y}_0'), \hat{y}_0+\hat{y}_0')]n_{22}) < -(l+1).$$

By Lemma 3.3 of [KL3] we have

$$\mathrm{Ord}_\beta([(\mathcal{L}-z)^{-1}\mathcal{P}y_0+\mathcal{P}y_0', \mathcal{P}y_0+\mathcal{P}y_0']n_{22}-n_{12}n_{21}) \geq -(l+1),$$

hence $\mathrm{Ord}_\beta \det Q < -(l+1)$, which contradicts (9.4).

We conclude that $\hat{y}_0 = \hat{y}_0' = 0$, i.e. that (9.15) holds. In particular, $\dim \mathfrak{Q} = 2(l+1) = -2\mathrm{Ord}_\beta n_{22}$, and the geometric multiplicity of β and $\overline{\beta}$ is one.

□

Proposition 9.5. *Let $Q \in \mathcal{N}_\kappa^{2\times 2}$ satisfy the conditions (9.3) and (9.4). Let $\mathfrak{P}$, $\mathcal{A}$ and Γ_z be a model for Q (as introduced above). Then each finite eigenvalue of $\mathcal{A}$ is of geometric multiplicity one. Moreover, we have*

$$(\mathrm{cls}\,\{\Gamma_z \begin{pmatrix} 0 \\ 1 \end{pmatrix} | z \in \rho(\mathcal{A})\})^\perp \subseteq S_\infty,$$

where S_∞ denotes the generalized eigenspace at ∞.

Proof : The first assertion follows from (9.8), Lemma 9.3 and Lemma 9.4. Moreover, these lemmata together with the fact that $\mathrm{cls}\,\{\Gamma_z \begin{pmatrix} 0 \\ 1 \end{pmatrix} | z \in \rho(\mathcal{A})\}$ is invariant under E_{λ_j} ($E_{\{\beta_k;\overline{\beta_k}\}}$), show that for

$$x \perp \mathrm{cls}\,\{\Gamma_z \begin{pmatrix} 0 \\ 1 \end{pmatrix} | z \in \rho(\mathcal{A})\},$$

we have $E_{\{\lambda_j\}}x = 0$ and $E_{\{\beta_k;\overline{\beta_k}\}}x = 0$. By [L] we obtain $x \in S_\infty$.

□

There exists a model for n_{22}, i.e. a Pontryagin space $\mathfrak{P}'$, a selfadjoint relation $\mathcal{A}'$, $\rho(\mathcal{A}') \neq \emptyset$, and elements $\gamma_z \in \mathfrak{P}'$, such that

$$\gamma_z = (I+(z-w)(\mathcal{A}'-z)^{-1})\gamma_w, \ z, w \in \rho(\mathcal{A}'),$$

$$\mathrm{cls}\,\{\gamma_z | z \in \rho(\mathcal{A})\} = \mathfrak{P}',$$

and (for some $c \in \mathbb{R}$)

$$n_{22}(z) = c - i\,\mathrm{Im}\, z_0[\gamma_{z_0}, \gamma_{z_0}] + (z-\overline{z_0})[\gamma_z, \gamma_{z_0}].$$

Lemma 9.6. *Let $Q(z)$ be as in Proposition 9.5. Assume that $\mathcal{A}'$ introduced above is an operator, i.e. that $\mathcal{A}'(0) = \{0\}$. Put*

$$\mathfrak{L} := \mathrm{cls}\,\{\Gamma_z \begin{pmatrix} 0 \\ 1 \end{pmatrix} |z \in \rho(\mathcal{A})\},$$

then

$$\mathfrak{L} = \mathfrak{L}^\circ + S_\infty^\perp,\ \mathfrak{L}^\perp = (\mathfrak{L}^\circ)^\perp \cap S_\infty, \tag{9.16}$$

and

$$\dim \mathfrak{L}^\perp \geq \mathrm{Ind}_0(S_\infty) + \max(\mathrm{Ind}_-(S_\infty), \mathrm{Ind}_+(S_\infty)). \tag{9.17}$$

Proof : By the construction of the model for Q the space $\mathfrak{L}$ is invariant under $(\mathcal{A}-z)^{-1}$, $z \in \rho(\mathcal{A})$. Hence also $\mathfrak{L}^\perp$ and $\mathfrak{L}^\circ$ are invariant under $(\mathcal{A}-z)^{-1}$, $z \in \rho(\mathcal{A})$. Proposition 9.5 implies that $\mathfrak{L}^\perp$, and hence $\mathfrak{L}^\circ$ is contained in S_∞.

An elementary consideration shows that the triplet

$$(\mathfrak{L}/\mathfrak{L}^\circ, (\mathcal{A} \cap \mathfrak{L}^2)/(\mathfrak{L}^\circ)^2, \Gamma_z \begin{pmatrix} 0 \\ 1 \end{pmatrix} /\mathfrak{L}^\circ)$$

is also a model for n_{22}, hence unitarily equivalent to $(\mathfrak{P}', \mathcal{A}', \gamma_z)$ (compare [HSW]). As $\mathcal{A}'$ is an operator we have $\overline{(\mathcal{A}' - z)^{-k}\mathfrak{P}'} = \mathfrak{P}'$ for all $k \in \mathbb{N}$. This means that $\overline{(\mathcal{A} - z)^{-k}\mathfrak{L}} + \mathfrak{L}^\circ = \mathfrak{L}$.

Denote by n the maximal length of a Jordan chain of $\mathcal{A}$ at infinity. Then $S_\infty^\perp = \overline{\mathrm{ran}\,(\mathcal{A} - z)^{-k}}$ for $k \geq n$. It follows that $\mathfrak{L} = S_\infty^\perp + \mathfrak{L}^\circ$.

Since $\mathfrak{L}^\circ/S_\infty^\circ$ is a neutral subspace of S_∞/S_∞°, its dimension cannot exceed

$$\min(\mathrm{Ind}_- S_\infty, \mathrm{Ind}_+ S_\infty).$$

Hence the dimension of its orthogonal companion is at least

$$\dim S_\infty/S_\infty^\circ - \min(\mathrm{Ind}_- S_\infty, \mathrm{Ind}_+ S_\infty) = \max(\mathrm{Ind}_- S_\infty, \mathrm{Ind}_+ S_\infty).$$

The relation (9.17) follows.

□

Corollary 9.7. *Let $M(z) \in \mathcal{M}_\kappa^1$ satisfy (9.1) and (9.2). Then the subspace* $\ker \pi_-$ *of* $\mathfrak{K}(M)$ *is finite dimensional. If $F(z) \in \pi_+ \ker \pi_-$, then F is a polynomial of degree at most 2κ.*

Proof : The reproducing kernel space $\mathfrak{K}(M)$ remains the same if we replace $M(z)$ by $M(z)U$, with an (iJ)-unitary matrix U (compare [KW]). By an elementary consideration it is seen that the demand that $\mathcal{A}'$ (for the matrix $Q(z) := \Psi(M(z)U)$ as introduced above) is an operator, can be achieved by an appropriate choice of U. Hence we may assume without loss of generality that $\mathcal{A}'(0) = \{0\}$.

It is well known (compare [ABDS1]) that as a model space $\mathfrak{P}$ for the function Q we may choose $\mathfrak{K}(Q)$ (recall Definition 2.1). Then

$$(\mathcal{A} - w)^{-1}F(z) = \frac{F(z) - F(w)}{z - w},\ F \in \mathfrak{K}(Q), w \in \rho(\mathcal{A}),$$

and

$$\Gamma_w(z) = N_Q(z, \overline{w}),\ w \in \rho(\mathcal{A}).$$

Hence, $\mathcal{A}(0)$ contains only constants, and S_∞ consists of pairs of the form $\begin{pmatrix} p(z) \\ q(z) \end{pmatrix}$, where $p(z)$ and $q(z)$ are polynomials of degree at most $n-1$, when n is the maximal length of a Jordan chain of $\mathcal{A}$ at infinity. Note that the length of such chains cannot exceed $2\kappa+1$.

By Lemma 9.6 the subspace $\ker \pi_-$ of $\mathfrak{K}(Q)$ is contained in S_∞, hence $\dim \ker \pi_- \leq \dim S_\infty < \infty$ and $\pi_+ \ker \pi_-$ consists of polynomials of degree at most 2κ. An application of Lemma 8.9 yields the assertion.

$\square$

Corollary 9.8. *Let $M_1(z) \in \mathcal{M}^1_{\kappa_1}$ and $M_2(z) \in \mathcal{M}^1_{\kappa_2}$ satisfy (9.1) and (9.2), and assume that*

$$(0,1)M_1(z) = (0,1)M_2(z).$$

Then

$$M_1(z) = \begin{pmatrix} 1 & p(z) \\ 0 & 1 \end{pmatrix} M_2(z), \tag{9.18}$$

with a real polynomial $p(z)$, $\deg p \leq 2\max(\kappa_1,\kappa_2)+1$.

If the subspace $\ker \pi_-$ of $\mathfrak{K}(M_1)$ and the subspace $\ker \pi_-$ of $\mathfrak{K}(M_2)$ are both equal to $\{0\}$, then the polynomial $p(z)$ is a real constant.

Conversely, given $M_2(z) \in \mathcal{M}^1_{\kappa_2}$ and a real polynomial $p(z)$, $\deg p = n$, then $M_1(z)$ defined by (9.18) is contained in $\mathcal{M}^1_{\kappa_1}$ where $\kappa_1 \leq \kappa_2 + [\frac{n+1}{2}]$.

Proof : Let $Q_1(z) = \Psi(M_1)(z)$ and $Q_2(z) = \Psi(M_2)(z)$, then $(0,1)Q_1(z) = (0,1)Q_2(z)$. Since $Q_i(z)^* = Q_i(\overline{z})$, $i=1,2$, the functions $Q_1(z)$ and $Q_2(z)$ differ only in the left upper corner:

$$Q_1(z) = \begin{pmatrix} n_{11,1}(z) & n_{12}(z) \\ n_{21}(z) & n_{22}(z) \end{pmatrix},\ Q_2(z) = \begin{pmatrix} n_{11,2}(z) & n_{12}(z) \\ n_{21}(z) & n_{22}(z) \end{pmatrix}.$$

We also have $\mathrm{Ord}_w \det Q_i \geq \mathrm{Ord}_w n_{22}$, $i=1,2$, for $w \notin \rho(n_{22})$. Hence $\mathrm{Ord}_w(n_{11,1} - n_{11,2}) \geq 0$, i.e. $n_{11,1} - n_{11,2}$ is entire. By considering the integral representations of $n_{11,1}$ and $n_{11,2}$ as introduced in Section 2, we find $n_{11,1} = n_{11,2} + p(z)$, and the assertion follows.

If the subspace $\ker \pi_-$ of $\mathfrak{K}(M_1)$ and the subspace $\ker \pi_-$ of $\mathfrak{K}(M_2)$ both are equal to $\{0\}$, then, looking at the equivalent situation in $\mathfrak{K}(Q_1)$ and $\mathfrak{K}(Q_2)$ with a simultanuously chosen U as indicated at the beginning of the proof of Corollary 9.7, we see by (9.17) that the selfadjoint relations $\mathcal{A}_1$ in $\mathfrak{K}(Q_1)$ and $\mathcal{A}_2$ in $\mathfrak{K}(Q_2)$ are both operators. Hence the Q-functions satisfy

$$\lim_{y\to\infty} \frac{1}{y} \Im Q_1(iy) = 0,\ \lim_{y\to\infty} \frac{1}{y} \Im Q_2(iy) = 0.$$

This is only possible if $p(z)$ is equal to a real constant.

The converse follows again by considering Q_1 and Q_2, since a polynomial $p(z)$ is contained in $\mathcal{N}_\kappa$ with $\kappa \leq [\frac{\deg p+1}{2}]$, and since the sum of two Nevanlinna functions is again a Nevanlinna function.

$\square$

Remark 9.9. In the assumptions and conclusions of this section the roles played by the upper and lower row of a matrix $M(z) \in \mathcal{M}_\kappa^1$ are different. The corresponding results where the upper and lower row change their roles, can be obtained by considering the matrix $JM(z)J$.

10 A characterization of associated functions

In this and the following sections we will use the notation and results of [KW]. Let $\mathfrak{P} = \mathfrak{P}(E)$ be a fixed dB-Pontryagin space.

We identify the set of associated function of a dB-Pontryagin space as the set $\mathfrak{P}_-$ (as introduced in [KW]. First we have to relate elements of $\mathfrak{P}_-$ to entire functions. If $U \in \mathfrak{P}_-$, define an entire function $U(w)$ by

$$U(w) := [U, \begin{pmatrix} K(w,z) \\ \overline{w}K(w,z) \end{pmatrix}]_\pm . \tag{10.1}$$

Since $\mathcal{S}$ is minimal, Lemma 3.5 of [KW] implies that the correspondence between $U \in \mathfrak{P}_-$ and the entire function $U(w)$ is one-to-one. Note that, if $F \in \mathfrak{P}$, we have $(\iota F)(w) = F(w)$.

Lemma 10.1. *Let $\alpha \in [0, \pi)$ be given, and consider the selfadjoint relation $\mathcal{A}_\alpha$. We have for $U \in \mathfrak{P}_-$ and $w \in \rho(\mathcal{A}_\alpha)$*

$$(R_w^- U)(z) = \frac{U(z) - \frac{S_\alpha(z)}{S_\alpha(w)} U(w)}{z - w} . \tag{10.2}$$

Here R_w^- is the extension of the resolvent $(\mathcal{A}_\alpha - z)^{-1}$ to $\mathfrak{P}_-$ (compare Section 3 of [KW]).

Proof : A straightforward computation using Lemma 6.4 shows that

$$R_{\overline{w}}^+ K(z,t) = \frac{S_\alpha(\overline{z})}{S_\alpha(\overline{w})} \frac{1}{\overline{w} - \overline{z}} \begin{pmatrix} K(w,t) \\ \overline{w}K(w,t) \end{pmatrix} - \frac{1}{\overline{w} - \overline{z}} \begin{pmatrix} K(z,t) \\ \overline{z}K(z,t) \end{pmatrix} .$$

Hence

$$(R_w^- U)(z) = [R_w^- U, K(z,t)] = [U, R_{\overline{w}}^+ K(z,t)]_\pm = \frac{U(z) - \frac{S_\alpha(z)}{S_\alpha(w)} U(w)}{z - w} .$$

$\square$

Proposition 10.2. *If $U \in \mathfrak{P}_-$, the function $U(w)$ defined by (10.1) is associated to $\mathfrak{P}$. Conversely, any associated function can be represented in this way.*

Proof : Let $U \in \mathfrak{P}_-$ be given, then $R_w^- U \in \mathfrak{P}$ and by (10.2) and Lemma 4.5 we find $U \in \operatorname{Ass}\mathfrak{P}$.

To prove the converse part it suffices by Lemma 4.5 to show that for some $z_0 \in \mathbb{C}$ we can represent every function of the form $(z-z_0)F$, $F \in \mathfrak{P}$, as an element of $\mathfrak{P}_-$: Let $F \in \mathfrak{P}$ be given. Then there exists an element $U \in \mathfrak{P}_-$, such that $R_{z_0}^- U = F$. Since, by [KW] the kernel of R_z^- is not trivial and does not depend on z we may assume, for an appropriate choice of z_0, that $U(z_0) = 0$. The assertion now follows from Lemma 10.1.

$\square$

Proposition 10.3. *Let $\mathfrak{P} = \mathfrak{P}(E)$, $E = A - iB$, be a dB-Pontryagin space. We have $U \in \operatorname{Ass}\mathfrak{P}$ if and only if there exist real entire functions C and D, such that*

$$M(z) := \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix} \in \mathcal{M}_{\kappa'}^U$$

for some $\kappa' \in \mathbb{N} \cup \{0\}$, and $\operatorname{Ind}_0 \mathfrak{K}_+(M) = 0$.

Proof : If there exists a matrix M with the stated properties, we have $U \in \operatorname{Ass}\mathfrak{P}$ by Corollary 8.8.

Assume now that $U \in \operatorname{Ass}\mathfrak{P}$ is given. We have to construct a matrix $M(z)$ with the asserted properties.

Note that by Lemma 5.4 and Corollary 5.5 we can assume that E has no real zeros, i.e. that A and B have no common zeros.

By Proposition 10.2, we can consider U as an element of $\mathfrak{P}_-$. As such there exists a generalized U-resolvent matrix $W(z)$ of $\mathcal{S}$ (compare Theorem 4.9 of [KW]).

Since, by the results of [KW], we have $W \in \mathcal{M}_\kappa^1$, $\kappa = \operatorname{Ind}_- \mathfrak{P}$, Lemma 8.2 implies that $M(z) := U(z)(-J)W(z)J \in \mathcal{M}_\kappa^U$. By the construction of W and Lemma 6.4 we find that the first row of $M(z)$ equals $(A(z), B(z))$. Since $\mathcal{S}$ is minimal, it follows from [KW] that $\mathfrak{K}_+(M) = \mathfrak{K}(M)$, in particular $\operatorname{Ind}_0 \mathfrak{K}_+(M) = 0$. It remains to show that the entries of $M(z)$, or equivalently the entries of $U(z)W(z)$, are entire functions.

By [KW] the matrix kernel $H_W(w, z)$ can be written as

$$H_W(w, z) = \begin{pmatrix} -\mathcal{Q}(z) \\ \mathcal{P}(z) \end{pmatrix} (-\mathcal{Q}(w)^*, \mathcal{P}(w)^*),$$

with certain functionals $\mathcal{P}(z)$ and $\mathcal{Q}(z)$. It follows from Lemma 6.4 and the definition of $\mathcal{P}(z)$ that $U(z)\mathcal{P}(z)F = F(z)$ for any $F \in \mathfrak{P}$. In particular $U(z)\mathcal{P}(z)F$ is an entire function. Since the set of regular points of $\mathcal{S}$ equals $\mathbb{C}$, we find (using Lemma 5.1 of [KW]), that also $U(z)\mathcal{Q}(z)F$ is entire. Since the matrix $W(w)$ is invertible whenever $U(w) \neq 0$ and $U(\overline{w}) \neq 0$, we find that $U(z)W(z)$ is entire.

$\square$

Corollary 10.4. *The matrix M constructed in the proof of Proposition 10.3 satisfies $\mathfrak{K}_+(M) = \mathfrak{K}(M)$ and $\kappa' = \operatorname{Ind}_- \mathfrak{P}$. Moreover, $\pi_+ \mathfrak{K}(M) = \mathfrak{P}$ isometrically. Also, for the* *constructed matrix M we have*

$$\min(\mathrm{Ord}_w C, \mathrm{Ord}_w D) \geq \min(\mathrm{Ord}_w A, \mathrm{Ord}_w B), \ w \in \mathbb{C}, \tag{10.3}$$

i.e. the relation (9.2) holds for $JM(z)J$. The relation (9.1) holds for the matrix $\frac{1}{U(z)}JM(z)J$.

If we demand the matrix M in Proposition 10.3 to satisfy $\mathfrak{K}_+(M) = \mathfrak{K}(M)$, then C (D) are unique up to real multiples of A (B).

Proof : If A and B have no common zeros, the relation (10.3) holds since C and D are entire. The general case follows by applying Corollary 5.5.

To prove the remaining assertion apply Corollary 9.8.

$\square$

Remark 10.5. In particular we may apply Proposition 10.3 to the function $E(z)$. An elementary computation using Lemma 6.4 and Definition 4.8 of [KW] shows that for the matrix $M(z)$ we can choose

$$M_E(z) := \begin{pmatrix} A(z) & B(z) \\ -B(z) & A(z) \end{pmatrix}.$$

Then the mapping ψ as introduced in Corollary 8.8 is given by

$$\psi(F)(w) = iF(w).$$

11 Subspaces of dB-Pontryagin spaces

It is a main object of the theory developed in [dB7] to study the subspaces of a given dB-Hilbert space.

Definition 11.1. Let $\langle \mathfrak{P}, [.,.] \rangle$ be a dB-space. A closed subspace $\mathfrak{Q} \subseteq \mathfrak{P}$ is called a dB-subspace of $\mathfrak{P}$, if it is a dB-space.

Note that a closed subspace $\mathfrak{Q}$ of a dB-space $\mathfrak{P}$ is a dB-subspace if and only if it contains $F^\#$ and $\frac{z-\overline{w}}{z-w}F(z)$ whenever $F \in \mathfrak{Q}$ and $F \in \mathfrak{Q}$, $F(w) = 0$, respectively.

Unless explicitly stated the symbol $\mathfrak{Q} \subseteq \mathfrak{P}$ will mean in this and the remaining sections that $\mathfrak{Q} \subseteq \mathfrak{P}$ as a set of functions, and that $[.,.]_{\mathfrak{P}}|_{\mathfrak{Q}^2} = [.,.]_{\mathfrak{Q}}$.

Note that the property of $\mathfrak{Q}$ being a dB-subspace does not depend on the choice of an inner product:

Remark 11.2. Let $\langle \mathfrak{P}, (\mathcal{G}.,.) \rangle$ and $\langle \mathfrak{P}, (\mathcal{G}_1.,.) \rangle$ be dB-spaces on the same set $\mathfrak{P}$. A closed subspace $\mathfrak{Q}$ is a dB-subspace of $\langle \mathfrak{P}, (\mathcal{G}.,.) \rangle$ if and only if it is a dB-subspace of $\langle \mathfrak{P}, (\mathcal{G}_1.,.) \rangle$.

From Theorem 3.3 and Remark 11.2 we obtain an ordering theorem for subspaces of a dB-space.

Proposition 11.3. *Let $\langle \mathfrak{P}, [.,.] \rangle$ be a dB-space. If $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ are dB-subspaces of $\mathfrak{P}$ with $\mathfrak{d}(\mathfrak{Q}_1) = \mathfrak{d}(\mathfrak{Q}_2)$, then either $\mathfrak{Q}_1 \subseteq \mathfrak{Q}_2$ or $\mathfrak{Q}_2 \subseteq \mathfrak{Q}_1$.*

Proof : Choose an inner product on $\mathfrak{P}$ which turns $\mathfrak{P}$ into a dB-Hilbert space. Then $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ are dB-subspaces of $\mathfrak{P}$ with the same divisor. The assertion now follows from [dB7].

$\square$

Let us recall a result of [dB7] which shows how the subspaces of a given dB-Hilbert space are related.

Proposition 11.4. *Let $\mathfrak{P}_0$ be a dB-Hilbert space, and let $\mathfrak{P}_0 = \mathfrak{P}(E_0)$ with some $E_0 \in \mathcal{HB}_0$. Then there exists a number $s_- \in [-\infty, 0)$, and for each $t \leq 0$ there exist real entire functions A_t, B_t, such that ($E_t := A_t - iB_t$)*

(i) $E_t \in \mathcal{HB}_0$ *if* $t > s_-$, *and* A_t *and* B_t *are linearly dependent if* $t \leq s_-$.

(ii) $\operatorname{Ord}_x E_t = \operatorname{Ord}_x E_0$ *for* $x \in \mathbb{R}$.

(iii) $\lim_{t \searrow s_-} K_t(w,w) = 0$ *for* $w \in \mathbb{C}$.

(iv) *If* $s_- < t_1 \leq t_2 \leq 0$, *then* $\mathfrak{P}(E_{t_1}) \subseteq \mathfrak{P}(E_{t_2})$ *as a set of functions. If* $t_1 < t_2$, *then* $\mathfrak{P}(E_{t_1}) \neq \mathfrak{P}(E_{t_2})$ *as Hilbert spaces.*

Moreover, there exists a subset M_{sing} of $(s_-, 0)$ which is a union of open intervals, such that ($M_{reg} := (s_-, 0] \setminus M_{sing}$)

(v) $\mathfrak{P}(E_{t_r})$ *is contained isometrically in* $\mathfrak{P}(E_0)$ *if* $t_r \in M_{reg}$. *Conversely, if* $\mathfrak{Q}$ *is a dB-subspace of* $\mathfrak{P}(E_0)$, $\mathfrak{d}(\mathfrak{Q}) = \mathfrak{d}(\mathfrak{P})$, *then* $\mathfrak{Q} = \mathfrak{P}(E_{t_r})$ *for some* $t_r \in M_{reg}$.

If $t_s \in M_{sing}$, *then* $\mathfrak{P}(E_{t_s})$ *is not contained isometrically in* $\mathfrak{P}(E_0)$, *and if* $t_r^+ := \min\{t \in M_{reg} | t > t_s\}$, *then* $\mathfrak{P}(E_{t_s}) = \mathfrak{P}(E_{t_r})$ *as a set of functions.*

(vi) *If* $t_r \in M_{reg}$ *is not the left endpoint of an interval contained in* M_{sing}, *then*

$$\bigcap_{\substack{t > t_r \\ t \in M_{reg}}} \mathfrak{P}(E_t) = \mathfrak{P}(E_{t_r}).$$

Otherwise

$$\bigcap_{\substack{t > t_r \\ t \in M_{reg}}} \mathfrak{P}(E_t) = \mathfrak{P}(E_{t_r^+})$$

with $t_r^+ := \min\{t \in M_{reg} | t > t_r\}$, *and*

$$\dim \left[\Big(\bigcap_{\substack{t > t_r \\ t \in M_{reg}}} \mathfrak{P}(E_t) \Big) / \mathfrak{P}(E_{t_r}) \right] = 1.$$

(vii) *If* $t_r \in M_{reg}$ *and is not the right endpoint of an interval contained in* M_{sing}, *then*

$$\operatorname{cls} \bigcup_{\substack{t < t_r \\ t \in M_{reg}}} \mathfrak{P}(E_t) = \mathfrak{P}(E_{t_r}).$$

Otherwise

$$\operatorname{cls} \bigcup_{\substack{t<t_r \\ t\in M_{reg}}} \mathfrak{P}(E_t) = \mathfrak{P}(E_{t_r^-})$$

with $t_r^- := \max\{t \in M_{reg} | t < t_r\}$, *and*

$$\dim \left[\mathfrak{P}(E_{t_r}) / (\operatorname{cls} \bigcup_{\substack{t>t_r \\ t\in M_{reg}}} \mathfrak{P}(E_t)) \right] = 1.$$

By Remark 11.2 and Theorem 3.3 a similar result holds for the dB-subspaces of an arbitrary dB-space $\mathfrak{P}_0$. In the following we denote by $\mathfrak{P}_t$ the dB-subspace of $\mathfrak{P}_0$ which equals as a set of functions the space $\mathfrak{P}(E_t)$ introduced in Proposition 11.4, and which is endowed with the inner product induced by $\mathfrak{P}_0$. Note that, if $t_s \in M_{sing}$ and $t > t_s$ such that $(t_s, t) \subseteq M_{sing}$, then $\mathfrak{P}_{t_s} = \mathfrak{P}_t$ as linear spaces.

Some of the spaces $\mathfrak{P}_t$ may be degenerated, hence there need not exist functions, say $\tilde{E}_t$, which generate the space $\mathfrak{P}_t$ in the sense of Theorem 5.3. However, we will show in the following that this can happen only for finitely many values of $t \in M_{reg}$.

Clearly the function

$$t \mapsto \operatorname{Ind}_- \mathfrak{P}_t,\ t \in I,$$

is a nondecreasing function taking values in $\mathbb{N} \cup \{0\}$. If $t \in (s_-, 0]$ and $\mathfrak{P}_t$ is nondegenerated then, for each $s \in (s_-, 0]$, $s < t$ we have

$$\operatorname{Ind}_- \mathfrak{P}_s + \operatorname{Ind}_0 \mathfrak{P}_s \leq \operatorname{Ind}_- \mathfrak{P}_t.$$

Corollary 11.5. *Let* $\langle \mathfrak{P}, [.,.] \rangle$ *be a dB-space and consider the chain* $(\mathfrak{P}_t)_{t\in(s_-,0]}$. *If* $t, s \in (s_-, 0]$, $s < t$, *and* $\operatorname{Ind}_- \mathfrak{P}_s = \operatorname{Ind}_- \mathfrak{P}_t$, *then* $\mathfrak{P}_s^\circ \subseteq \mathfrak{P}_t^\circ$. *The set* $(s_-, 0]$ *is the union of at most* $\operatorname{Ind}_- \mathfrak{P} + \operatorname{Ind}_0 \mathfrak{P}$ *intervalls (possibly consisting of only one point), such that* $\operatorname{Ind}_- \mathfrak{P}_t$ *and* $\mathfrak{P}_t^\circ$ *are constant on each intervall.*

Proof : Let $\mathfrak{L}$ be a maximal negative subspace of $\mathfrak{P}_s$. Then $\dim \mathfrak{L} = \operatorname{Ind}_-(\mathfrak{P}_s) = \operatorname{Ind}_-(\mathfrak{P}_t)$, hence $\mathfrak{L}$ is maximal negative in $\mathfrak{P}_t$. It follows that the orthogonal complement $\mathfrak{L}^\perp$ in $\mathfrak{P}_t$ is positive semidefinite. Clearly, $\mathfrak{P}_s^\circ$ is a neutral subspace of $\mathfrak{L}^\perp$ and we conclude that $\mathfrak{P}_s^\circ \subseteq (\mathfrak{L}^\perp)^\circ$. Therefore $\mathfrak{P}_s^\circ \subseteq \mathfrak{P}_t^\circ$.

The remaining assertions follow immediately from the already proved.

□

The major part of this section is devoted to the proof of the following

Theorem 11.6. *Let* $\mathfrak{P}_0$ *be a dB-space. The set*

$$M_0 := \{t \in M_{reg} | \operatorname{Ind}_0 \mathfrak{P}_t \neq 0\}$$

is finite.

Similar as in Section 9 we make use of some results on 2×2-matrix valued Nevanlinna functions and of the Potapov-Ginzburg transformation.

Lemma 11.7. *Let*

$$M(z) = \begin{pmatrix} m_{11}(z) & m_{12}(z) \\ m_{21}(z) & m_{22}(z) \end{pmatrix} \in \mathcal{M}_0^1$$

be an entire matrix function, with real entries and assume that $\mathfrak{K}(M)$ is at least two dimensional. Then the Potapov-Ginzburg transformation $\Psi(M)(z)$ exists, is analytic on $\mathbb{C} \setminus \sigma(\Psi(M))$, where $\sigma(\Psi(M))$ is an isolated subset of $\mathbb{R}$, and has a representation

$$\Psi(M)(z) = C + zD + \sum_{t \in \sigma(\Psi(M))} \left(\frac{1}{t-z} - \frac{t}{t^2+1}\right) G_t. \tag{11.1}$$

Here C is a real selfadjoint 2×2-matrix, D is a real nonnegative 2×2-matrix, and G_t is a nonnegative 2×2-matrix of the form

$$G_t = \beta_t \begin{pmatrix} 1 \\ b_t \end{pmatrix} \begin{pmatrix} 1 & b_t \end{pmatrix} \tag{11.2}$$

where $0 < \beta_t \in \mathbb{R}$, $b_t \in \mathbb{R} \setminus \{0\}$, and where b_{t_1} and b_{t_2} have different sign for consecutive t_1 and t_2.

Proof : First of all note that $\det M(z) = 1$ and that therefore the right upper and the left lower entry of $\Psi(M)(z)$ coincide. It follows from [dB7] and from the assumption that $\dim \mathfrak{K}(M) \geq 2$, that no entry of $M(z)$ vanish identically, in fact no entry has a nonreal zero. Hence the Potapov-Ginzburg transformation exists. Since m_{21} and m_{22} have no common zeros and M is entire, M satisfies (9.1) and (9.2).

Clearly $\sigma(\Psi(M)) = \{z \in \mathbb{C} | m_{21}(z) = 0\}$ is an isolated subset of $\mathbb{R}$. Since $\Psi(M)(z) \in \mathcal{N}_0^{2\times 2}$, it can be written in the form (11.1) for selfadjoint C, and nonnegative D and G_t, $t \in \sigma(\Psi(M))$. Hence for $t \in \sigma(\Psi(M))$ the function $\Psi(M)(z)$ has a pole of order one at t with the residue $-G_t$. It follows from Lemma 9.3 that G_t has a nontrivial kernel.

Since m_{21} and m_{22} (m_{11} and m_{21}) have no common zeros, all entries of $\Psi(M)(z)$ have a pole at t of order one.

Clearly, the entries of $\Psi(M)(z)$ are real functions, hence C, D and G_t, $t \in \sigma(\Psi(M))$ are real. Thus we can write G_t in the form (11.2) with $b_t, \beta_t \neq 0$.

It remains to show that b_{t_1} and b_{t_2} have different sign for consecutive t_1 and t_2. As the lower left entry of $\Psi(M)(z)$ equals $\frac{1}{m_{21}(z)}$, it cannot have zeros on $\mathbb{R}$. Hence the residues of consecutive poles of $\frac{1}{m_{21}(z)}$ must have different sign.

$\square$

Proposition 11.8. *Let $M(z) \in \mathcal{M}_0^1$ be an entire matrix function, with real entries. Assume that there exist vectors $x_1, \ldots x_n \in \mathbb{R}^2 \setminus \{0\}$ and pairwise different numbers $t_1, \ldots, t_n \in \mathbb{R}$, such that*

$$\sum_{j=1}^{n} H_M(z, t_j) x_j = 0. \tag{11.3}$$

Then $\dim \mathfrak{K}(M) \leq 4n$.

Proof : If $\mathfrak{K}(M)$ is of dimension one, there is nothing to proof.

As the space $\mathfrak{K}(M)$ remains unchanged if we replace M by

$$M(z)\begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}$$

with $u_1, u_2, v_1, v_2 \in \mathbb{R}$, $u_1v_2 - u_2v_1 = 1$, we can assume without loss of generality that no point $t_1, \dots, t_n$ is contained in $\sigma(\Psi(M))$.

If we put $Q := \Psi(M)$, it follows from Lemma 8.9 that the condition (11.3) is equivalent to

$$\sum_{j=1}^{n} N_Q(z, t_j) y_j = 0$$

for certain vectors $y_1, \dots y_n \in \mathbb{R}^2 \setminus \{0\}$.

Applying the resolvent $f(z) \in \mathfrak{K}(\Psi(M)) \mapsto \frac{f(z)-f(\tau)}{z-\tau}$ for $\tau \in \mathbb{R} \setminus (\sigma(\Psi(M)) \cup \{t_1, \dots, t_n\})$ we get

$$\sum_{j=1}^{n} N_Q(z, t_j) \frac{y_j}{t_j - \tau} = N_Q(z, \tau) \sum_{j=1}^{n} \frac{y_j}{t_j - \tau}. \tag{11.4}$$

Let $\lambda_1, \dots, \lambda_m$ be the real zeros of $P(\tau) := \sum_{j=1}^{n} \frac{y_j}{t_j - \tau}$. Note that $m \leq n - 1$.

Now let $\tau \in \mathbb{R}$ tend to some $t \in \sigma(\Psi(M)) \setminus \{\lambda_1, \dots, \lambda_m\}$. Then the left hand side of (11.4) has a finite limit. But the norm of the right hand side is by Lemma 11.7 greater or equal than

$$\beta_t \frac{1}{(t-\tau)^2} P(\tau)^* \begin{pmatrix} 1 \\ b_t \end{pmatrix} \begin{pmatrix} 1 & b_t \end{pmatrix} P(\tau).$$

Hence $0 \neq P(t) \perp \begin{pmatrix} 1 \\ b_t \end{pmatrix}$. Since $b_t \neq 0$, neither the first component $p_1(\tau)$ nor the second component $p_2(\tau)$ of $P(\tau)$ vanishes at t.

Hence the real polynomial in τ

$$p(z) := p_1(\tau) p_2(\tau) \prod_{j=1,\dots,n} (t_j - \tau)^2$$

is nonzero for $\tau \in \sigma(\Psi(M)) \setminus \{\lambda_1, \dots, \lambda_m\}$. Moreover, for elements $\tau_1, \tau_2 \in \sigma(\Psi(M)) \setminus \{\lambda_1, \dots, \lambda_m\}$ which are consecutive in $\sigma(\Psi(M))$, the values $p(\tau_1)$ and $p(\tau_2)$ have different sign. Since the above polynomial has degree at most $2n-2$, an elementary consideration shows that $|\sigma(\Psi(M))| \leq 4n - 3$.

Since $\dim \mathfrak{K}(\Psi(M)) = \operatorname{rank}(D) + \sum_{t \in \sigma(\Psi(M))} \operatorname{rank}(G_t)$ we obtain the desired result.

$\square$

Proposition 11.9. *Let $\langle \mathfrak{P}_1, [.,.] \rangle$ and $\langle \mathfrak{P}_2, [.,.] \rangle$ be dB-spaces such that $\mathfrak{d}(\mathfrak{P}_1) = \mathfrak{d}(\mathfrak{P}_2)$, $\mathfrak{P}_1 \subseteq \mathfrak{P}_2$, $\mathfrak{P}_1^\circ \neq 0$, and*

$$\operatorname{Ind}_-(\mathfrak{P}_1) = \operatorname{Ind}_-(\mathfrak{P}_2).$$

If a dB-Hilbert space $\langle\mathfrak{P}_2,(.,.)_1\rangle$ can be constructed (compare Theorem 3.3) with a perturbation of rank n:

$$(F,G)_1=[F,G]+C\sum_{k=1}^n F(t_k)\overline{G(t_k)},\ F,G\in\mathfrak{P}_2,$$

for some $t_1,\dots,t_n\in\mathbb{R}$, then $\operatorname{codim}_{\mathfrak{P}_2}(\mathfrak{P}_1)\leq 4n$.

Proof : Clearly, with $\langle\mathfrak{P}_2,(.,.)_1\rangle$ also $\langle\mathfrak{P}_1,(.,.)_1\rangle$ is a dB-Hilbert space. Denote by $K_1(z,w)$, $K_2(z,w)$ the reproducing kernel of $\langle\mathfrak{P}_1,(.,.)_1\rangle$ and $\langle\mathfrak{P}_2,(.,.)_1\rangle$, respectively. Moreover, let $\mathcal{P}$ be the orthogonal projection of $\langle\mathfrak{P}_2,(.,.)_1\rangle$ onto $\langle\mathfrak{P}_1,(.,.)_1\rangle$.

As in the proof of Theorem 3.3 we can write $[F,G]=(\mathcal{G}_jF,G)_1,\ F,G\in\mathfrak{P}_j$ $(j=1,2)$, where

$$\mathcal{G}_j=I-C\sum_{k=1}^n(.,K_j(t_k,z))_1K_j(t_k,z),$$

hence $\mathfrak{P}_j^\circ=\ker\mathcal{G}_j$, and any function $F\in\mathfrak{P}_1^\circ\subseteq\mathfrak{P}_2^\circ$ is of the form

$$F=\sum_{k=1}^n\alpha_kK_2(t_k,z).$$

Since $\mathfrak{P}_1^\circ$ is invariant under $F\mapsto F^\#$, we may assume that $\alpha_k\in\mathbb{R},\ k=1,\dots n$. Since $F\in\mathfrak{P}_1^\circ$, and since $\mathcal{P}K_2(z,w)=K_1(z,w)$, we obtain

$$F=\sum_{k=1}^n\alpha_kK_1(t_k,z).$$

By $\mathfrak{d}(\mathfrak{P}_1)=\mathfrak{d}(\mathfrak{P}_2)$ it follows from [dB7] that there exists an entire matrix function $M(z)\in\mathcal{M}_0^1$ on $\mathbb{C}$ with real entries such that

$$(A_2(z),B_2(z))=(A_1(z),B_1(z))M(z).$$

We calculate (compare (12.3))

$$(\sum_{k=1}^n\alpha_kK_2(t_k,z),\sum_{k=1}^n\alpha_kK_2(t_k,z))_1=(\sum_{k=1}^n\alpha_kK_1(t_k,z),\sum_{k=1}^n\alpha_kK_1(t_k,z))_1+$$

$$+\sum_{k,l=1}^n(\alpha_kA_1(t_k),\alpha_kB_1(t_k))\frac{M(t_l)JM(t_k)^*-J}{t_l-t_k}\begin{pmatrix}\alpha_lA_1(t_l)\\ \alpha_lB_1(t_l)\end{pmatrix}.$$

Since the first and the second term in this equation both are equal to the norm of F, the last term vanishes. Then, by Proposition 11.8, the dimension of $\mathfrak{K}(M)$ is at most $4n$ dimensional. Since $\langle\mathfrak{P}_2,(.,.)_1\rangle\ominus\langle\mathfrak{P}_1,(.,.)_1\rangle$ is isomorphic to $\mathfrak{K}(M)$, we are finished.

$\square$

Proof : (of Theorem 11.6) Assume on the contrary that $|M_0|=\infty$. Then there exists some $\kappa\geq 0$, such that

$$M_{0,\kappa}:=\{t\in M_0|\mathrm{Ind}\_\mathfrak{P}_t=\kappa\}$$

is infinite. By Corollary 11.5 we may apply Proposition 11.9 to any two indices $t_1, t_2 \in M_{0,\kappa}$. However, since $M_{0,\kappa}$ is infinite, there exist indices t_1, t_2, such that the codimension of $\mathfrak{P}_{t_1}$ within $\mathfrak{P}_{t_2}$ exceeds the uniform bound given in Proposition 11.9, and we arrive at a contradiction.

$\square$

In the remaining part of this section we show that a point t_0 where the function $\mathrm{Ind}\_\mathfrak{P}(E_t)$ jumps is either the endpoint of an intervall contained in M_{sing} or $\mathfrak{P}(E_{t_0})^\circ \neq \{0\}$.

Lemma 11.10. *Let $\mathfrak{P}_n$, $n \in \mathbb{N}$, be a sequence of Pontryagin spaces, such that $\mathfrak{P}_n \supseteq \mathfrak{P}_{n+1}$ and $\bigcap_{n\in\mathbb{N}} \mathfrak{P}_n = \{0\}$. Then $\mathfrak{P}_n$ is a Hilbert space for some (and hence for all larger) n.*
Proof : Let $\|.\|$ be a positive definite norm on $\langle \mathfrak{P}_1, [.,.]\rangle$ induced by a fundamental symmetry. Since $\mathfrak{P}_n$ is nondegenerated, it is a closed subspace of $\mathfrak{P}_1$ with respect to the weak topology. Recall that, since $\mathrm{Ind}\_\mathfrak{P}_n < \infty$, the set $\{x \in \mathfrak{P}_n | [x,x] \leq 0\}$ is weakly closed. Consider the decreasing sequence of compact sets

$$M_n := \{x \in \mathfrak{P}_n | [x,x] \leq 0, \|x\| = 1\}.$$

By our assumptions $\bigcap_{n\in\mathbb{N}} M_n = \emptyset$, hence there exists some n such that $M_n = \emptyset$, and the assertion follows.

$\square$

Proposition 11.11. *Let $\mathfrak{P}_0 = \mathfrak{P}(E_0)$ be a dB-Pontryagin space and consider the chain $(\mathfrak{P}_t)_{t\in(s_-,0]}$ of subspaces with $\mathfrak{d}(\mathfrak{P}_t) = \mathfrak{d}(\mathfrak{P}_0)$. If $\mathfrak{P}_{t_0}$ is nondegenerated, there exists a number $\varepsilon > 0$, such that* $\mathrm{Ind}\_\mathfrak{P}_t$ *is constant on $(t_0 - \varepsilon, t_0 + \varepsilon) \cap M_{reg}$.*
Proof : We first consider $t < t_0$. If t_0 is the right endpoint of an intervall contained in M_{sing}, there is nothing to prove. Otherwise we have

$$\mathfrak{P}_{t_0} = \mathrm{cls} \bigcup_{\substack{t<t_r \\ t\in M_{reg}}} \mathfrak{P}_t.$$

Then there exists a maximal negative subspace which is contained in $\bigcup_{\substack{t<t_r \\ t\in M_{reg}}} \mathfrak{P}_t$, hence $\mathrm{Ind}\_\mathfrak{P}_t = \mathrm{Ind}\_\mathfrak{P}_{t_0}$ if $t < t_0$ and $t_0 - t$ is sufficiently small.

Now consider $t > t_0$. If t_0 is the left endpoint of an intervall contained in M_{sing}, we are done. If this is not the case, we have

$$\mathfrak{P}_{t_0} = \bigcap_{\substack{t>t_r \\ t\in M_{reg}}} \mathfrak{P}_t.$$

Applying Lemma 11.10 to $\mathfrak{P}_t \ominus \mathfrak{P}_{t_0}$, we find that $\mathrm{Ind}\_\mathfrak{P}_t = \mathrm{Ind}\_\mathfrak{P}_{t_0}$ if $t > t_0$ and $t - t_0$ is sufficiently small.

$\square$

12 Transfer matrices of subspaces

The aim of this section is to show that the dB-subspaces of a given dB-Pontryagin space are connected with entire matrix functions of the class $\mathcal{M}_\kappa^1$. First note that Lemma 4.5 implies the following

Lemma 12.1. *Let $\mathfrak{Q}$ and $\mathfrak{P}$ be dB-Pontryagin spaces. Then $\mathfrak{Q} \subseteq \mathfrak{P}$ as a set of functions, if and only if* $\operatorname{Ass}\mathfrak{Q} \subseteq \operatorname{Ass}\mathfrak{P}$

The remaining part of this section is devoted to the proof of the following result:

Theorem 12.2. *Let $\mathfrak{P}_a = \mathfrak{P}(E_a)$ and $\mathfrak{P}_b = \mathfrak{P}(E_b)$ be dB-Pontryagin spaces, $\mathfrak{d}(\mathfrak{P}_a) = \mathfrak{d}(\mathfrak{P}_b)$, let $\kappa_a = \operatorname{Ind}_-\mathfrak{P}_a$, $\kappa_b = \operatorname{Ind}_-\mathfrak{P}_b$, and write $E_a = A_a - iB_a$, $E_b = A_b - iB_b$. Then $\mathfrak{P}_a \subseteq \mathfrak{P}_b$ isometrically if and only if there exists a matrix function*

$$M(z) = \begin{pmatrix} m_{11}(z) & m_{12}(z) \\ m_{21}(z) & m_{22}(z) \end{pmatrix} \in \mathcal{M}^1_{\kappa_b-\kappa_a},$$

with the following properties:

(i) $M(z)$ is an entire function.

(ii) $(A_b, B_b) = (A_a, B_a)M$.

(iii) There exists no constant $\begin{pmatrix} u \\ v \end{pmatrix} \in \mathfrak{K}(M)$, such that $uA_a + vB_a \in \mathfrak{P}_a$.

If $\mathfrak{P}_a \subseteq \mathfrak{P}_b$ isometrically, then the matrix $M \in \mathcal{M}^1_{\kappa_b-\kappa_a}$ is uniquely determined by the properties (i) and (ii).

Corollary 12.3. *In the situation of Theorem 12.2, the mapping*

$$\begin{pmatrix} F_+ \\ F_- \end{pmatrix} \mapsto F_+A_a + F_-B_a, \ \begin{pmatrix} F_+ \\ F_- \end{pmatrix} \in \mathfrak{K}(M),$$

is an isometry of $\mathfrak{K}(M)$ onto $\mathfrak{P}_b \ominus \mathfrak{P}_a$.

Before be can give the proof of Theorem 12.2, we need some lemmata.

Lemma 12.4. *Let $\mathfrak{Q} \subseteq \mathfrak{P}$, $\mathfrak{d}(\mathfrak{Q}) = \mathfrak{d}(\mathfrak{P})$ be dB-Pontryagin spaces. If $\mathcal{S}$ is the operator of multiplication by the independent variable in the space $\mathfrak{P}$, then*

$$\mathfrak{Q} \not\subseteq \operatorname{ran}(\mathcal{S} - w), \ w \in \mathbb{C}.$$

Proof : Assume on the contrary that $\mathfrak{Q} \subseteq \operatorname{ran}(\mathcal{S} - w)$. Then, by Proposition 4.2, $\operatorname{Ord}_w F > \mathfrak{d}(\mathfrak{P})$ for all $F \in \mathfrak{Q}$. This contradicts $\mathfrak{d}(\mathfrak{P}) = \mathfrak{d}(\mathfrak{Q})$ by Lemma 4.1.

□

Lemma 12.5. *Let $\mathfrak{P} = \mathfrak{P}(E)$, $E = A - iB$, be a dB-Pontryagin space, and denote by M_E the matrix*

$$M_E(z) := \begin{pmatrix} A(z) & B(z) \\ -B(z) & A(z) \end{pmatrix}.$$

The space $\mathfrak{K}(M_E)$ consists of the pairs $\begin{pmatrix} F \\ iF \end{pmatrix}$ for $F \in \mathfrak{P}$. If $M \in \mathcal{M}_\kappa^1$, then

$$\{\begin{pmatrix} F_+ \\ F_- \end{pmatrix} \in \mathfrak{K}(M) | M_E(z) \begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix} \in \mathfrak{K}(M_E)\} = \{0\}.$$

The space

$$\mathfrak{L}_+ := \{\begin{pmatrix} F_+ \\ F_- \end{pmatrix} \in \mathfrak{K}(M) | \pi_+ M_E(z) \begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix} \in \mathfrak{K}(H_{M_E}^+)\}$$

has finite dimension. In fact, the space $\mathfrak{L}_+$ is algebraically isomorphic to $\ker \pi_+$. *If $\mathfrak{L}_+ \neq \{0\}$, it contains a constant.*

Proof : The first assertion follows from (8.2), compare also Remark 10.5. Now let $M \in \mathcal{M}_\kappa^1$ be given and consider the linear space

$$\mathfrak{L} := \{\begin{pmatrix} F_+ \\ F_- \end{pmatrix} \in \mathfrak{K}(M) | M_E(z) \begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix} \in \mathfrak{K}(M_E)\},$$

endowed with the inner product

$$[\begin{pmatrix} F_+ \\ F_- \end{pmatrix}, \begin{pmatrix} G_+ \\ G_- \end{pmatrix}] := [\begin{pmatrix} F_+ \\ F_- \end{pmatrix}, \begin{pmatrix} G_+ \\ G_- \end{pmatrix}]_{\mathfrak{K}(M)} +$$

$$+ [M_E \begin{pmatrix} F_+ \\ F_- \end{pmatrix}, M_E \begin{pmatrix} G_+ \\ G_- \end{pmatrix}]_{\mathfrak{K}(M_E)}.$$

Then $\mathfrak{L}$ can be identified with a closed subspace of $\mathfrak{K}(M) \times \mathfrak{K}(M_E)$. The maximal dimension of nonpositive subspace of $\mathfrak{L}$ cannot exceed $\mathrm{Ind}_- \mathfrak{K}(M) + \mathrm{Ind}_- \mathfrak{K}(M_E)$. Hence the space $\mathfrak{L}/\mathfrak{L}^\circ$ is a Pontryagin space.

We first show that $\mathfrak{L}$ does not contain a constant function: Assume that $\begin{pmatrix} u \\ v \end{pmatrix} \in \mathfrak{L}$, then

$$\begin{pmatrix} u \\ v \end{pmatrix} \in \mathfrak{K}(M), \;\; M_E(z) \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} uA(z) + vB(z) \\ -uB(z) + vA(z) \end{pmatrix} \in \mathfrak{K}(M_E).$$

By Corollary 8.4 we have $u\overline{v} \in \mathbb{R}$. Since the elements $\begin{pmatrix} F_+ \\ F_- \end{pmatrix}$ of $\mathfrak{K}(M_E)$ satisfy $F_- = iF_+$, we find

$$i(uA(z) + vB(z)) = -uB(z) + vA(z),$$

i.e. $iu = v$ and $iv = -u$. Hence $u = v = 0$.

The next step is to prove that $\mathfrak{L}$ is invariant under the difference quotient operator $\mathcal{R}_1$, and that $\mathcal{R}_1$ satisfies

$$[\mathcal{R}_1(a) \begin{pmatrix} F_+ \\ F_- \end{pmatrix}, \begin{pmatrix} G_+ \\ G_- \end{pmatrix}] - [\begin{pmatrix} F_+ \\ F_- \end{pmatrix}, \mathcal{R}_1(b) \begin{pmatrix} G_+ \\ G_- \end{pmatrix}] +$$

$$+(a-\overline{b})[\mathcal{R}_1(a)\begin{pmatrix} F_+ \\ F_- \end{pmatrix}, \mathcal{R}_1(b)\begin{pmatrix} F_+ \\ F_- \end{pmatrix}] = 0, \tag{12.1}$$

for $\begin{pmatrix} F_+ \\ F_- \end{pmatrix}, \begin{pmatrix} G_+ \\ G_- \end{pmatrix} \in \mathfrak{L}$ and $a, b, \in \mathbb{C}$.

Let $\begin{pmatrix} F_+ \\ F_- \end{pmatrix} \in \mathfrak{L}$. By Proposition 8.3, we have $\mathcal{R}_1(w)\begin{pmatrix} F_+ \\ F_- \end{pmatrix} \in \mathfrak{K}(M)$ and $\mathcal{R}_E(w)M_E\begin{pmatrix} F_+ \\ F_- \end{pmatrix} \in \mathfrak{K}(M_E)$. Since M_E satisfies (8.1) we obtain for $w \in \mathbb{C}$ with $E(w) \neq 0$

$$M_E\mathcal{R}_1(w)\begin{pmatrix} F_+ \\ F_- \end{pmatrix} = \mathcal{R}_E(w)M_E\begin{pmatrix} F_+ \\ F_- \end{pmatrix} -$$

$$-H_{M_E}(\overline{w}, z)(JM_E(\overline{w})^*)^{-1}\begin{pmatrix} F_+(w) \\ F_-(w) \end{pmatrix} \in \mathfrak{K}(M_E), \tag{12.2}$$

hence $\mathcal{R}_1(w)\begin{pmatrix} F_+ \\ F_- \end{pmatrix} \in \mathfrak{L}$ if $E(w) \neq 0$. Consider a number w_0 such that $E(w_0) = 0$. Then $\mathcal{R}_1(w)$ has a Laurent expansion at w_0. If a coefficient of some negative power of $(w - w_0)$ does not vanish, we can choose F_+, F_- and $z \in \mathbb{C}$ such that the function

$$\mathcal{R}_1(w)\begin{pmatrix} F_+ \\ F_- \end{pmatrix}(z) = \frac{\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix} - \begin{pmatrix} F_+(w) \\ F_-(w) \end{pmatrix}}{z - w}$$

has a singularity at w_0. This contradicts the fact that F_+ and F_- are entire. Hence $\mathcal{R}_1(w)$ is analytic at w_0, and the relation $\mathcal{R}_1(w_0)\begin{pmatrix} F_+ \\ F_- \end{pmatrix} \in \mathfrak{L}$ follows.

To prove (12.1) it is sufficient to consider the case $a = \overline{b}$. Then a computation using (8.1), (8.4) and (12.2) shows that (12.1) holds.

The isotropic part of $\mathfrak{L}$ is finite dimensional. Assume that $\mathfrak{L}^\circ \neq \{0\}$, then one easily shows that $\mathfrak{L}^\circ$ is a nontrivial finite dimensional invariant subspace of $\mathcal{R}_1(w)$. Hence there exists an eigenvector:

$$\mathcal{R}_1(w)\begin{pmatrix} F_+ \\ F_- \end{pmatrix} = \lambda\begin{pmatrix} F_+ \\ F_- \end{pmatrix}.$$

Since F_+ and F_- are entire, an elementary consideration shows that $\lambda = 0$. Hence $\begin{pmatrix} F_+ \\ F_- \end{pmatrix}$ is constant. This contradicts the above proved fact that $\mathfrak{L}$ does not contain any nonzero constant.

The previous consideration shows that $\mathfrak{L}^\circ = \{0\}$. Since for each $w \in \mathbb{C}$ the mappings

$$\begin{pmatrix} F_+ \\ F_- \end{pmatrix} \mapsto F_+(w), \quad \begin{pmatrix} F_+ \\ F_- \end{pmatrix} \mapsto F_-(w),$$

are continuous on $\mathfrak{L}$, there exists a matrix $L(w, z)$ of entire functions, such that

$$F_+(w) = [\begin{pmatrix} F_+ \\ F_- \end{pmatrix}, L(w, z)\begin{pmatrix} 1 \\ 0 \end{pmatrix}], \; F_-(w) = [\begin{pmatrix} F_+ \\ F_- \end{pmatrix}, L(w, z)\begin{pmatrix} 0 \\ 1 \end{pmatrix}].$$

By (12.1) we have for $a,b\in\mathbb{C}$, $a\neq\overline{b}$,

$$\frac{L(a,z)-L(a,b)}{z-b}=\frac{L(a,z)-L(\overline{b},z)}{\overline{a}-b}.$$

Hence $L(w,z)$ can be represented as

$$L(w,z)=\frac{Q(z)-Q(w)^*}{z-\overline{w}},$$

with the entire matrix function $Q(z):=zL(0,z)\in\mathcal{N}_\kappa^{2\times 2}$. For $x\in\mathbb{R}$ we have $\operatorname{Im}Q(x)=0$, and Corollary 2.3 implies that $Q(z)$ is a polynomial of degree at most $2(\mathrm{Ind}\_\mathfrak{K}(M)+\mathrm{Ind}\_\mathfrak{K}(M_E))+1$. From this we find that $\mathfrak{L}$ is finite dimensional. Unless $\mathfrak{L}=\{0\}$, the same consideration as in the previous step of the proof yields a contradiction.

In the next step consider the space

$$\mathfrak{K}(M_{E_a})\oplus\{M_{E_a}\begin{pmatrix}F_+\\F_-\end{pmatrix}\mid\begin{pmatrix}F_+\\F_-\end{pmatrix}\in\mathfrak{K}(M)\},$$

where the inner product in the second summand is given by

$$[M_{E_a}\begin{pmatrix}F_+\\F_-\end{pmatrix},M_{E_a}\begin{pmatrix}G_+\\G_-\end{pmatrix}]:=[\begin{pmatrix}F_+\\F_-\end{pmatrix},\begin{pmatrix}G_+\\G_-\end{pmatrix}]_{\mathfrak{K}(M)}.$$

The relation

$$\frac{M_b(z)JM_b(w)^*-E_a(z)J\overline{E_a(w)}}{z-\overline{w}}=\frac{M_{E_a}(z)JM_{E_a}(w)^*-E_a(z)J\overline{E_a(w)}}{z-\overline{w}}+$$

$$+M_{E_a}(z)\frac{M(z)JM(w)^*-J}{z-\overline{w}}M_{E_a}(w)^* \tag{12.3}$$

implies that this space is the space $\mathfrak{K}(M_b)$, where $M_b:=M_{E_a}M$ (compare with the concept of complemention of reproducing kernel spaces as in [ADSR1]). We have $\begin{pmatrix}F_+\\F_-\end{pmatrix}\in\mathfrak{L}_+$ if and only if

$$M_E\begin{pmatrix}F_+\\F_-\end{pmatrix}-\begin{pmatrix}F_+A+F_-B\\i(F_+A+F_-B)\end{pmatrix}=\begin{pmatrix}0\\(F_--iF_+)E\end{pmatrix}\in\mathfrak{K}(M_b).$$

Since $(F_--iF_+)E\neq 0$ by the previous part of the proof, $\mathfrak{L}_+$ is algebraically isomorphic to $\ker\pi_+$. Corollary 9.7 implies that $\dim\mathfrak{L}_+<\infty$. Note in this place that the matrix M_b satisfies (9.2), in fact with equality. The relation (12.2) shows that $\mathfrak{L}_+$ is invariant under $\mathcal{R}_1(w)$. Hence $R_1(w)$ has an eigenvector, which must be a constant.

$\square$

Proof : (of Theorem 12.2) In the first step of this proof we assume that $\mathfrak{P}_a\subseteq\mathfrak{P}_b$ is given. The function E_a is associated to $\mathfrak{P}_a$ and hence associated to $\mathfrak{P}_b$. Let the matrix $M_b\in\mathcal{M}_{\kappa_b}^{E_a}$ be as in Proposition 10.3, and assume that it is chosen such that $\mathfrak{K}_+(M_b)=\mathfrak{K}(M_b)$. The mapping $\psi=\pi_-(\pi_+)^{-1}$ assigns to each function $F\in\mathfrak{P}_b$ an entire function, such that (8.5) holds. In particular, if ψ is restricted to $\mathfrak{P}_a$, (8.5) holds. By Corollary 8.8 and Remark 9.9 there exists a number $\lambda \in \mathbb{R}$, such that for $F \in \mathfrak{P}_a$ we have $\psi F(z) = iF(z) + \lambda F(z)$. Since a change of ψ by an additve real multiple of the identity corresponds to a change of M_b by adding to the second row a real multiple of the first row, we can assume that M_b is chosen such that $\psi F(z) = iF(z)$ for $F \in \mathfrak{P}_a$. This guarantees that

$$\mathfrak{K}(M_{E_a}) \subseteq \mathfrak{K}(M_b). \tag{12.4}$$

Let $V_a \in \mathfrak{P}_{a,-}$ be such that, with the identification (10.1), we have $V_a(z) = E_a(z)$. By Remark 10.5 we know that $\frac{1}{E_a} J M_{E_a} J$ is the generalized V_a-resolvent matrix of $\mathcal{S} \subseteq \mathfrak{P}_a^2$ as introduced in Definition 4.8 of [KW]. If $V_b \in \mathfrak{P}_{b,-}$ is such that $V_b(z) = E_a(z)$, by the construction of M_b given in the proof of Proposition 10.3, the matrix $\frac{1}{E_a} J M_b J$ is the generalized V_b-resolvent matrix of $\mathcal{S} \subseteq \mathfrak{P}_b^2$ With the notation P_1' taken from Section 7 of [KW], an elementary computation using (10.1) and the definition of P_1' shows that the relation $V_b = P_1' V_a$ holds. Moreover, since (12.4) holds, the conventions made in the assumption of Theorem 7.4 of [KW] are satisfied.

Now we are in the situation to apply Theorem 7.4 of [KW], which shows together with Lemma 12.4 that the matrix $M := M_{E_a}^{-1} M_b$ is entire. Moreover, $M \in \mathcal{M}^1_{\kappa_b - \kappa_a}$. The conditions *(i)* and *(ii)* are therefore satisfied. The relation (12.3) shows that the mapping

$$\begin{pmatrix} F_+ \\ F_- \end{pmatrix} \mapsto M_{E_a} \begin{pmatrix} F_+ \\ F_- \end{pmatrix}$$

is an isometry of $\mathfrak{K}(M)$ onto $\mathfrak{K}(M_b) \ominus \mathfrak{K}(M_{E_a})$, in particular the condition *(iii)* is satisfied.

In the second step of the proof assume that a space $\mathfrak{P}_a = \mathfrak{P}(E_a)$ and a matrix $M \in \mathcal{M}^1_{\kappa_b - \kappa_a}$, which satisfies the listed properties, is given. Consider the matrix $M_b := M_{E_a} M$. Together with Lemma 12.5, the assumption *(iii)* implies that $\mathfrak{K}_+(M_b) = \mathfrak{K}(M_b)$. Hence, by the construction of $\mathfrak{K}(M_b)$ given in Lemma 12.5, we have with $(A_b, B_b) := (1, 0) M_b$, $E_b := A_b - iB_b$,

$$\mathfrak{P}_a \cong \mathfrak{K}(M_{E_a}) \subseteq \mathfrak{K}(M_b) = \mathfrak{K}_+(M_b) \cong \mathfrak{P}_b := \mathfrak{P}(E_b)$$

where the inclusion is isometric. The fact that $E_b \in \mathcal{HB}_{\kappa_b}$ is derived as follows: The conditions that $\frac{E_b^\#}{E_b} \in \mathcal{S}_{\kappa_b}$, and that E_b and $E_b^\#$ have no common nonreal zeros are obvious. Assume that A_b and B_b are linearly dependent. Then

$$\pi_+ H_{M_b}(w, z) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0.$$

Hence $\ker \pi_+ = \mathfrak{K}(M_b)$, a contradiction since $\mathfrak{K}(M_b) \supseteq \mathfrak{K}(M_{E_a})$.

In the third step we prove the uniqueness statement of Theorem 12.2. Consider the following situation: Let $\mathfrak{P}_a \subseteq \mathfrak{P}_b$ and $M \in \mathcal{M}^1_{\kappa_b - \kappa_a}$ be given, such that

$$(A_b, B_b) = (A_a, B_a) M.$$

Define a matrix $M_b' := M_{E_a} M$, and let M_b be as in the first step of the proof. By Lemma 12.5 and (12.3) we have $M_b' \in \mathcal{M}^{E_a}_{\kappa_b}$. By Corollary 9.8 there exists a polynomial $s(z)$, such that

$$M_b' = \begin{pmatrix} 1 & 0 \\ -s(z) & 1 \end{pmatrix} M_b.$$

The kernel relation

$$H_{M_b'}(w,z) = \begin{pmatrix} 1 & 0 \\ -s(z) & 1 \end{pmatrix} H_{M_b}(w,z) \begin{pmatrix} 1 & \overline{s(w)} \\ 0 & 1 \end{pmatrix} +$$

$$+E_a(z) \begin{pmatrix} 0 & 0 \\ 0 & \frac{s(z)-\overline{s(w)}}{z-\overline{w}} \end{pmatrix} \overline{E_a(w)}$$

and the fact that $\ker \pi_+ = \{0\}$ in $\mathfrak{K}(M_b)$ show that $\mathfrak{K}(M_b')$ is the direct and orthogonal sum of the space

$$\{ \begin{pmatrix} 1 & 0 \\ -s(z) & 1 \end{pmatrix} \begin{pmatrix} F_+ \\ F_- \end{pmatrix} | \begin{pmatrix} F_+ \\ F_- \end{pmatrix} \in \mathfrak{K}(M_b) \},$$

endowed with the inner product

$$[\begin{pmatrix} 1 & 0 \\ -s(z) & 1 \end{pmatrix} \begin{pmatrix} F_+ \\ F_- \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -s(z) & 1 \end{pmatrix} \begin{pmatrix} G_+ \\ G_- \end{pmatrix}] := [\begin{pmatrix} F_+ \\ F_- \end{pmatrix}, \begin{pmatrix} G_+ \\ G_- \end{pmatrix}]_{\mathfrak{K}(M_b)},$$

and the space $E_a(z)\mathfrak{K}(\frac{s(z)-\overline{s(w)}}{z-\overline{w}})$ with the corresponding inner product. Since

$$\mathrm{Ind}_- \mathfrak{K}(M_b') = \kappa_b = \mathrm{Ind}_- \mathfrak{K}(M_b),$$

we find $\mathrm{Ind}_- \mathfrak{K}(\frac{s(z)-\overline{s(w)}}{z-\overline{w}}) = 0$.

Choose $F \in \mathfrak{P}_a$. The definition of M_b' and the choice of M_b yields together with the above description of $\mathfrak{K}(M_b')$ that

$$\begin{pmatrix} F \\ iF \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -s(z) & 1 \end{pmatrix} \begin{pmatrix} F \\ iF \end{pmatrix} + E_a(z) \begin{pmatrix} 0 \\ s_1(z) \end{pmatrix} \tag{12.5}$$

for some polynomial $s_1(z)$. Hence

$$[\begin{pmatrix} F \\ iF \end{pmatrix}, \begin{pmatrix} F \\ iF \end{pmatrix}]_{\mathfrak{K}(M_b')} = [\begin{pmatrix} F \\ iF \end{pmatrix}, \begin{pmatrix} F \\ iF \end{pmatrix}]_{\mathfrak{K}(M_b)} + [s_1, s_1]_{\mathfrak{K}(\frac{s(z)-\overline{s(w)}}{z-\overline{w}})}.$$

Since $\mathfrak{K}(M_{E_a})$ is contained isometrically in $\mathfrak{K}(M_b')$ as well as in $\mathfrak{K}(M_b)$, we find

$$[s_1, s_1]_{\mathfrak{K}(\frac{s(z)-\overline{s(w)}}{z-\overline{w}})} = 0,$$

hence $s_1 = 0$. Comparing the left and right hand sides of the second row of (12.5), we find $s = 0$. Thus $M_b' = M_b$, and the matrix M equals the matrix constructed in the first step of the proof.

$\square$

In the following we denote for an entire matrix function $M(z)$ by $\mathfrak{t}(M)$ the trace of $M'(0)J$. It is proved in [dB7] that an entire matrix function $M \in \mathcal{M}_0^1$, $M(0) = 1$, satisfies $\mathfrak{t}(M) \geq 0$, and that $\mathfrak{t}(M) = 0$ if and only if $M = 1$. Together with Proposition 11.3 we obtain:

Corollary 12.6. *Let $\mathfrak{P}_1 = \mathfrak{P}(E_1)$, $\mathfrak{P}_2 = \mathfrak{P}(E_2)$ and $\mathfrak{P} = \mathfrak{P}(E)$ be dB-Pontryagin spaces. Assume that $\mathfrak{d}(\mathfrak{P}_1) = \mathfrak{d}(\mathfrak{P}_2) = \mathfrak{d}(\mathfrak{P})$, and that $\mathfrak{P}_1 \subseteq \mathfrak{P}$, $\mathfrak{P}_2 \subseteq \mathfrak{P}$. Denote by M_1 and M_2 the transfer matrices*

$$(A,B) = (A_1,B_1)M_1,\ \ (A,B) = (A_2,B_2)M_2.$$

(i) If $\mathrm{Ind}\_\mathfrak{P}_1 < \mathrm{Ind}\_\mathfrak{P}_2$*, then* $\mathfrak{P}_1 \subsetneq \mathfrak{P}_2$*.*

(ii) If $\mathrm{Ind}\_\mathfrak{P}_1 = \mathrm{Ind}\_\mathfrak{P}_2$*, then* $\mathfrak{P}_1 \subsetneq \mathfrak{P}_2$ *if and only if* $\mathfrak{t}(M_1) > \mathfrak{t}(M_2)$*.*

13 Factorization of transfer matrices

In this section we give an ordering theorem for transfer matrices of the class $\mathcal{M}_\kappa^1$.

Theorem 13.1. *Let $M_a \in \mathcal{M}_{\kappa_a}^1$, $M_b \in \mathcal{M}_{\kappa_b}^1$ and $M_c \in \mathcal{M}_{\kappa_c}^1$, $\kappa_c \geq \kappa_a, \kappa_b$, be given and assume that for some $M_{ac} \in \mathcal{M}_{\kappa_c-\kappa_a}^1$ and $M_{bc} \in \mathcal{M}_{\kappa_c-\kappa_b}^1$ the relations*

$$M_c = M_a M_{ac},\ M_c = M_b M_{bc}$$

hold. Then there exists either a matrix $M_{ab} \in \mathcal{M}_{\kappa_b-\kappa_a}^1$, with

$$M_b = M_a M_{ab},$$

or a matrix $M_{ba} \in \mathcal{M}_{\kappa_a-\kappa_b}^1$, with

$$M_a = M_b M_{ba}.$$

If we additionally assume that $M_a(0) = M_b(0) = M_c(0) = 1$, which is in fact no loss of generality, then we can give a condition which case occurs:

(i) If $\kappa_b > \kappa_a$ or $\kappa_b = \kappa_a$, $\mathfrak{t}(M_b) > \mathfrak{t}(M_a)$, there exists $M_{ab} \in \mathcal{M}_{\kappa_b-\kappa_a}^1$, such that $M_b = M_a M_{ab}$.

(ii) If $\kappa_b = \kappa_a$ and $\mathfrak{t}(M_b) = \mathfrak{t}(M_a)$, then $M_a = M_b$.

(iii) If $\kappa_b < \kappa_a$ or $\kappa_b = \kappa_a$, $\mathfrak{t}(M_b) < \mathfrak{t}(M_a)$, there exists $M_{ba} \in \mathcal{M}_{\kappa_a-\kappa_b}^1$, such that $M_a = M_b M_{ba}$.

Before we can give the proof of Theorem 13.1, we need some more information about nonisometric inclusions of spaces $\mathfrak{P}(E)$. We use in the following the theory of reproducing kernel Pontryagin spaces as given in [ADSR1] (compare also [ADSR2], [ADSR3]). For the theory of complementation we refer also to [dB8].

In the following let $\mathfrak{P}(E)$ be a dB-Pontryagin space such that $uA + vB \in \mathfrak{P}(E)$ and let $M \in \mathcal{M}_\kappa^1$ be such that $\begin{pmatrix} u \\ v \end{pmatrix} \in \mathfrak{K}(M)$. Put $E_c = (A_c, B_c)$, where

$$(A_c, B_c) = (A, B)M,$$

and assume that $\mathrm{Ind}\_\mathfrak{P}(E_c) = \mathrm{Ind}\_\mathfrak{P}(E) + \kappa$. In Lemma 12.5 we have introduced the space

$$\mathfrak{L}_+ = \{\begin{pmatrix} F_+ \\ F_- \end{pmatrix} \in \mathfrak{K}(M) | \pi_+ M_E(z) \begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix} \in \mathfrak{P}(E)\},$$

and shown that $\mathfrak{L}_+$ is finite dimensional, invariant under the difference quotient operator $\mathcal{R}_1(w)$ and contains only polynomials.

The size of the space $\mathfrak{L}_+$ mesures how far $\mathfrak{P}(E)$ is away from being contained isometrically in $\mathfrak{P}(E_c)$.

Lemma 13.2. *The orthogonal companion in $\mathfrak{P}(E)$ of the subspace*

$$\mathfrak{L}_1 := \{(1,0)M_E \begin{pmatrix} F_+ \\ F_- \end{pmatrix} \mid \begin{pmatrix} F_+ \\ F_- \end{pmatrix} \in \mathfrak{L}_+\}$$

is the largest dB-subspace of $\mathfrak{P}(E)$ which is contained isometrically in $\mathfrak{P}(E_c)$.

Proof : First we show that $\mathfrak{L}_1^\perp$ is a dB-space. Since $\mathfrak{L}_1$ is closed with respect to #, so is $\mathfrak{L}_1^\perp$. Assume that $F \perp \mathfrak{L}_1$, $F(w) = 0$. We have to show that

$$\frac{F(z)}{z-w} = \mathcal{R}_E(w)F(z) \perp \mathfrak{L}_1.$$

Using the fact that $\psi(F)(w) = iF(w) = 0$ (compare Remark 10.5), (8.4) and the relation (12.2), we find for $G = (1,0)M_E \begin{pmatrix} G_+ \\ G_- \end{pmatrix} \in \mathfrak{L}_1$

$$[\mathcal{R}_E(w)F, G] = [F, \mathcal{R}_E(\overline{w})G] = [F, \pi_+ M_E \mathcal{R}_1(\overline{w}) \begin{pmatrix} G_+ \\ G_- \end{pmatrix}] +$$

$$+[F, \pi_+ H_{M_E}(w,z)(JM_E(w)^*)^{-1} \begin{pmatrix} G_+(\overline{w}) \\ G_-(\overline{w}) \end{pmatrix}] = 0.$$

If $K(w,z)$ ($K_c(w,z)$) denotes the reproducing kernel of $\mathfrak{P}(E)$ ($\mathfrak{P}(E_c)$) and H_1 is the kernel

$$H_1(w,z) := (1,0)M_E(z)H_M(w,z)M_E(\overline{w}) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{13.1}$$

then the kernel relation

$$K_c(w,z) = K(w,z) + H_1(w,z)$$

holds. Since $\mathfrak{L}_1 = \mathfrak{P}(E) \cap \mathfrak{K}(H_1)$, the remaining assertions follow from [ADSR1].

□

Now consider the inner product on $\mathfrak{L}_+$ defined by

$$[\begin{pmatrix} F_+ \\ F_- \end{pmatrix}, \begin{pmatrix} G_+ \\ G_- \end{pmatrix}]_{\mathfrak{L}_+} := [\begin{pmatrix} F_+ \\ F_- \end{pmatrix}, \begin{pmatrix} G_+ \\ G_- \end{pmatrix}]_{\mathfrak{K}(M)} +$$

$$+[\pi_+ M_E \begin{pmatrix} F_+ \\ F_- \end{pmatrix}, \pi_+ M_E \begin{pmatrix} G_+ \\ G_- \end{pmatrix}]_{\mathfrak{P}(E)}. \tag{13.2}$$

Lemma 13.3. *The difference quotient operator $\mathcal{R}_1(w)$ satisfies*

$$[\mathcal{R}_1(w) \begin{pmatrix} F_+ \\ F_- \end{pmatrix}, \begin{pmatrix} G_+ \\ G_- \end{pmatrix}]_{\mathfrak{L}_+} - [\begin{pmatrix} F_+ \\ F_- \end{pmatrix}, \mathcal{R}_1(\overline{w}) \begin{pmatrix} G_+ \\ G_- \end{pmatrix}]_{\mathfrak{L}_+} = 0$$

for all $\begin{pmatrix} F_+ \\ F_- \end{pmatrix}, \begin{pmatrix} G_+ \\ G_- \end{pmatrix} \in \mathfrak{L}_+$ *and* $w \in \mathbb{C}$.

Proof : If we substitute the definition of the inner product $[.,.]_{\mathfrak{L}_+}$, we obtain differences of inner products $[.,.]_{\mathfrak{K}(M)}$ and $[.,.]_{\mathfrak{P}(E)}$. By Proposition 8.3 the difference of the terms $[.,.]_{\mathfrak{K}(M)}$ gives

$$-\begin{pmatrix} G_+(\overline{w}) \\ G_-(\overline{w}) \end{pmatrix} J \begin{pmatrix} F_+(w) \\ F_-(w) \end{pmatrix}.$$

Hence we have to show that

$$[\pi_+ M_E \mathcal{R}_1(w) \begin{pmatrix} F_+ \\ F_- \end{pmatrix}, \pi_+ M_E \begin{pmatrix} G_+ \\ G_- \end{pmatrix}]_{\mathfrak{P}(E)} -$$

$$-[\pi_+ M_E \begin{pmatrix} F_+ \\ F_- \end{pmatrix}, \pi_+ M_E \mathcal{R}_1(\overline{w}) \begin{pmatrix} G_+ \\ G_- \end{pmatrix}]_{\mathfrak{P}(E)} =$$

$$= \begin{pmatrix} G_+(\overline{w}) \\ G_-(\overline{w}) \end{pmatrix} J \begin{pmatrix} F_+(w) \\ F_-(w) \end{pmatrix}.$$

By (12.2) we have

$$[\pi_+ M_E \mathcal{R}_1(w) \begin{pmatrix} F_+ \\ F_- \end{pmatrix}, \pi_+ M_E \begin{pmatrix} G_+ \\ G_- \end{pmatrix}]_{\mathfrak{P}(E)} =$$

$$= [\mathcal{R}_E(w)(1,0) M_E \begin{pmatrix} F_+ \\ F_- \end{pmatrix}, (1,0) M_E \begin{pmatrix} G_+ \\ G_- \end{pmatrix}]_{\mathfrak{P}(E)} -$$

$$-[K(\overline{w}, z)(1,-i)(J M_E(\overline{w})^*)^{-1} \begin{pmatrix} F_+(w) \\ F_-(w) \end{pmatrix}, (1,0) M_E \begin{pmatrix} G_+ \\ G_- \end{pmatrix}],$$

and a similar expression for the other inner product term $[.,.]_{\mathfrak{P}(E)}$. By Corollary 8.8 we have

$$[\mathcal{R}_E(w)(1,0) M_E \begin{pmatrix} F_+ \\ F_- \end{pmatrix}, (1,0) M_E \begin{pmatrix} G_+ \\ G_- \end{pmatrix}] -$$

$$-[(1,0) M_E \begin{pmatrix} F_+ \\ F_- \end{pmatrix}, \mathcal{R}_E(\overline{w})(1,0) M_E \begin{pmatrix} G_+ \\ G_- \end{pmatrix}] =$$

$$= \frac{i}{E(w)E^\#(w)}((1,0) M_E(w) \begin{pmatrix} F_+(w) \\ F_-(w) \end{pmatrix} \overline{(1,0) M_E(\overline{w}) \begin{pmatrix} G_+(\overline{w}) \\ G_-(\overline{w}) \end{pmatrix}} -$$

$$-(1,0) M_E(w) \begin{pmatrix} F_+(w) \\ F_-(w) \end{pmatrix} \overline{i(1,0) M_E(\overline{w}) \begin{pmatrix} G_+(\overline{w}) \\ G_-(\overline{w}) \end{pmatrix}}).$$

The terms involving $K(w,z)$ $(K(\overline{w},z))$ can be computed by the reproducing kernel property of K and using the fact that $(J M_E(\overline{w})^*)^{-1} = \frac{-J M_E(w)}{E(w)E^\#(w)}$:

$$-(1,-i)\frac{-J M_E(w)}{E(w)E^\#(w)} \begin{pmatrix} F_+(w) \\ F_-(w) \end{pmatrix} \overline{(1,0) M_E(\overline{w}) \begin{pmatrix} G_+(\overline{w}) \\ G_-(\overline{w}) \end{pmatrix}} +$$

$$+(1,0) M_E(w) \begin{pmatrix} F_+(w) \\ F_-(w) \end{pmatrix} \overline{(1,-i)\frac{-J M_E(\overline{w})}{E(\overline{w})E^\#(\overline{w})} \begin{pmatrix} G_+(\overline{w}) \\ G_-(\overline{w}) \end{pmatrix}} =$$

$$= \begin{pmatrix} G_+(\overline{w}) \\ G_-(\overline{w}) \end{pmatrix}^* M_E(\overline{w})^* \frac{\begin{pmatrix} 1 \\ 0 \end{pmatrix}(1,-i)J + J\begin{pmatrix} 1 \\ i \end{pmatrix}(1,0)}{E(w)E^\#(w)} M_E(w) \begin{pmatrix} F_+(w) \\ F_-(w) \end{pmatrix}.$$

All together we obtain

$$[\pi_+ M_E \mathcal{R}_1(w) \begin{pmatrix} F_+ \\ F_- \end{pmatrix}, \pi_+ M_E \begin{pmatrix} G_+ \\ G_- \end{pmatrix}]_{\mathfrak{P}(E)} -$$

$$-[\pi_+ M_E \begin{pmatrix} F_+ \\ F_- \end{pmatrix}, \pi_+ M_E \mathcal{R}_1(\overline{w}) \begin{pmatrix} G_+ \\ G_- \end{pmatrix}]_{\mathfrak{P}(E)} = \begin{pmatrix} G_+(\overline{w}) \\ G_-(\overline{w}) \end{pmatrix}^* M_E(\overline{w})^* \cdot$$

$$\cdot \underbrace{\left(\frac{\begin{pmatrix} 1 \\ 0 \end{pmatrix}(i,0) + \begin{pmatrix} i \\ 0 \end{pmatrix}(1,0) + \begin{pmatrix} 1 \\ 0 \end{pmatrix}(1,-i)J + J\begin{pmatrix} 1 \\ i \end{pmatrix}(1,0)}{E(w)E^\#(w)} \right)}_{=\frac{J}{E(w)E^\#(w)}} \cdot$$

$$\cdot M_E(w) \begin{pmatrix} F_+(w) \\ F_-(w) \end{pmatrix} = \begin{pmatrix} G_+(\overline{w}) \\ G_-(\overline{w}) \end{pmatrix} \underbrace{\frac{M_E(\overline{w})^* J M_E(w)}{E(w)E^\#(w)}}_{=J} \begin{pmatrix} F_+(w) \\ F_-(w) \end{pmatrix},$$

which yields the assertion of the Lemma.

□

Since the element $\begin{pmatrix} u \\ v \end{pmatrix}$ is the only eigenvector of $R_1(w)$ in $\mathfrak{L}_+$, i.e. there is a Jordan chain of length $\dim \mathfrak{L}_+$ to the eigenvalue 0, we have:

Corollary 13.4. *If $\begin{pmatrix} u \\ v \end{pmatrix}$ is not neutral with respect to the inner product (13.2), then* $\dim \mathfrak{L}_+ = 1$.

Proof : Assume that $\mathcal{M}$ is an invariant subspace of $\mathfrak{L}_+$ for $\mathcal{R}_1(w)$, then $\mathcal{M}^\perp$ is invariant under $\mathcal{R}_1(\overline{w})$. If $\begin{pmatrix} u \\ v \end{pmatrix}$ is not neutral, it spans a one dimensional invariant subspace of $\mathcal{R}_1(w)$, which is orthocomplemented. By the above argument its orthogonal complement contains an eigenvector of $\mathcal{R}_1(\overline{w})$. This contradicts the fact that $\begin{pmatrix} u \\ v \end{pmatrix}$ is the only eigenvector of $\mathcal{R}_1(\overline{w})$ (compare Corollary 8.4), and we conclude that $\dim \mathfrak{L}_+ = 1$.

□

Proposition 13.5. *Let $\mathfrak{P}(E)$ and $\mathfrak{P}(E_c)$ be dB-Pontryagin spaces with* $\operatorname{Ind}_-\mathfrak{P}(E) \leq \operatorname{Ind}_-\mathfrak{P}(E_c)$*, and assume that there exists a matrix $M \in \mathcal{M}_\kappa^1$, $\kappa = \operatorname{Ind}_-\mathfrak{P}(E_c) - \operatorname{Ind}_-\mathfrak{P}(E)$, such that $(A_c, B_c) = (A, B)M$. Moreover assume that there exists a nonzero constant $\begin{pmatrix} u \\ v \end{pmatrix} \in \mathfrak{K}(M)$, such that $uA + vB \in \mathfrak{P}(E)$. Then $\mathfrak{P}(E)$ is contained in $\mathfrak{P}(E_c)$ as a* *set of functions, and the inclusion map is contractive but not isometric. The set $\mathfrak{P}(E)$ endowed with the inner product induced by $\mathfrak{P}(E_c)$ is a dB-subspace of $\mathfrak{P}(E_c)$ (and will be denoted by $\mathfrak{P}(E)^u$). The subspace $\mathfrak{P}(E)^l := \overline{\operatorname{dom} \mathcal{S}}$ of $\mathfrak{P}(E)$ is a dB-space and is contained isometrically in $\mathfrak{P}(E_c)$. At least one of the spaces $\mathfrak{P}(E)^u$ and $\mathfrak{P}(E)^l$ is nondegenerated.*

Proof : As in the proof of Lemma 13.2 we decompose the kernel $K_c(w,z)$ as

$$K_c(w,z) = K(w,z) + H_1(w,z),$$

where H_1 is defined as in (13.1). It follows from [ADSR1] that then

$$\operatorname{Ind}_-\mathfrak{P}(E_c) \leq \operatorname{Ind}_-\mathfrak{P}(E) + \operatorname{Ind}_-\mathfrak{K}(H_1) \leq \operatorname{Ind}_-\mathfrak{P}(E) + \operatorname{Ind}_-\mathfrak{K}(M). \tag{13.3}$$

By our assumption in (13.3) equality holds, i.e. $\mathfrak{P}(E)$ and $\mathfrak{K}(H_1)$ are complementary in $\mathfrak{P}(E_c)$. Moreover, $\operatorname{Ind}_-\mathfrak{K}(H_1) = \operatorname{Ind}_-\mathfrak{K}(M)$, hence $\mathfrak{L}_1$ is a Hilbert space with respect to the sum inner product.

In particular $\mathfrak{P}(E)$ is contained in $\mathfrak{P}(E_c)$ as a set of functions and the inclusion map is a contraction. The inclusion cannot be isometric, since the uniqueness part of Theorem 12.2 would then imply that $\begin{pmatrix} u \\ v \end{pmatrix} \notin \mathfrak{K}(M)$.

Consider the mapping

$$\varphi : \begin{pmatrix} F_+ \\ F_- \end{pmatrix} \mapsto (1,0)M_E \begin{pmatrix} F_+ \\ F_- \end{pmatrix} = AF_+ + BF_-.$$

Since $\operatorname{Ind}_-\mathfrak{K}(H_1) = \operatorname{Ind}_-\mathfrak{K}(M)$, the mapping φ is a partial isometry of $\mathfrak{K}(M)$ onto $\mathfrak{K}(H_1)$ and $\ker \varphi$ is a Hilbert space. Clearly $\mathfrak{P}(E) \cap \mathfrak{K}(H_1) = \varphi(\mathfrak{L}_+)$, and $\ker\varphi \subseteq \mathfrak{L}_+$. Since by the previous steps of this proof $\mathfrak{L}_1$ is a Hilbert space in the sum inner product, we conclude that $\mathfrak{L}_+$ endowed with the inner product (13.2) is a Hilbert space. Corollary 13.4 shows that $\dim \mathfrak{L}_+ = 1$, and Lemma 13.2 implies that $\overline{\operatorname{dom}\mathcal{S}}$ is contained isometrically in $\mathfrak{P}(E_c)$.

Since the relation $[\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}]_{\mathfrak{L}_+} > 0$ holds, we have

$$[uA + vB, uA + vB]_{\mathfrak{P}(E)} > 0 \text{ or } [\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}]_{\mathfrak{K}(M)} > 0.$$

In the first case $\mathfrak{P}(E)^l$ is nondegenerated. We show that in the second case $\mathfrak{P}(E)^u$ is nondegenerated. Assume that $[\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}]_{\mathfrak{K}(M)} = 1$ and put $(A^u, B^u) := (A,B)M_{\binom{u}{v}}$ with

$$M_{\binom{u}{v}} := \begin{pmatrix} 1 - u\overline{v}z & u\overline{u}z \\ -v\overline{v} & 1 + u\overline{v}z \end{pmatrix}.$$

Then $\mathfrak{P}(E^u) = \mathfrak{P}(E)$ as a set of functions. Consider the factorization

$$M = M_{\binom{u}{v}} \cdot \underbrace{(M_{\binom{u}{v}}^{-1} M)}_{:=M_1}.$$

Since $\mathfrak{K}(M_{\binom{u}{v}})$ is contained isometrically in $\mathfrak{K}(M)$ and since

$$M_{\binom{u}{v}} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}, \tag{13.4}$$

we find from [ADSR1] that $\begin{pmatrix} u \\ v \end{pmatrix} \notin \mathfrak{K}(M_1)$. By Theorem 12.2 the space $\mathfrak{P}(E^u)$ is contained isometrically in $\mathfrak{P}(E_c)$ and hence equals $\mathfrak{P}(E)^u$.

$\square$

Proof : (of Theorem 13.1) Let $E_0 \in \mathcal{HB}_0$ be such that the multiplication operator in the space $\mathfrak{P}(E)$ is densely defined, e.g. choose $E_0(z) = e^{-iz}$. Define functions

$$E_a := E_0 M_a,\ E_b := E_0 M_b,\ E_c := E_0 M_c,$$

then the space $\mathfrak{P}(E_0)$ is isometrically contained in $\mathfrak{P}(E_a)$, $\mathfrak{P}(E_b)$ and $\mathfrak{P}(E_c)$. We have

$$E_c = E_a M_{ac},\ E_c = E_b M_{bc},$$

hence we may consider the spaces $\mathfrak{P}(E_a)^l, \mathfrak{P}(E_a)^u, \mathfrak{P}(E_b)^l, \mathfrak{P}(E_b)^u$ with the convention that if, say, $\mathfrak{P}(E_a)$ is contained isometrically in $\mathfrak{P}(E_c)$ that then $\mathfrak{P}(E_a)^l = \mathfrak{P}(E_a)^u = \mathfrak{P}(E_a)$.

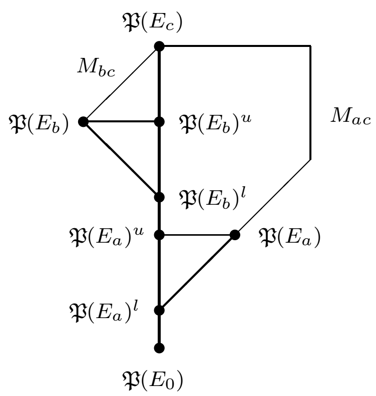
We assume first that both, $\mathfrak{P}(E_a)$ and $\mathfrak{P}(E_b)$, are not contained isometrically in $\mathfrak{P}(E_c)$. By Corollary 13.4 the codimension of $\mathfrak{P}(E_a)^l$ ($\mathfrak{P}(E_b)^l$) in $\mathfrak{P}(E_a)$ ($\mathfrak{P}(E_b)$) is one. The ordering theorem Proposition 11.3 shows that there occurs one of the following three situation:

(i) $\mathfrak{P}(E_a)^u \subseteq \mathfrak{P}(E_b)^l$,

(ii) $\mathfrak{P}(E_a)^u = \mathfrak{P}(E_b)^u$ and $\mathfrak{P}(E_a)^l = \mathfrak{P}(E_b)^l$,

(iii) $\mathfrak{P}(E_b)^u \subseteq \mathfrak{P}(E_a)^l$.

Consider the case *(i)*. Then we have the following picture:



Assume first that $[\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}]_{\mathfrak{K}(M_{ac})} = 1$, then, by Proposition 13.5, we have $\mathfrak{P}(E_a)^u = \mathfrak{P}(E_a^u)$ with $(A_a^u, B_a^u) = (A_a, B_a)M_{\binom{u}{v}}$. Since $\mathfrak{K}(M_{\binom{u}{v}})$ is a Hilbert space, $\mathrm{Ind}\_\mathfrak{P}(E_a^u) = \mathrm{Ind}\_\mathfrak{P}(E_a)$. Moreover, $\mathfrak{P}(E_a^u)$ is contained isometrically in $\mathfrak{P}(E_b)$, hence there exists a matrix $M_{a^u b} \in \mathcal{M}_\kappa^1$, $\kappa = \mathrm{Ind}\_\mathfrak{P}(E_b) - \mathrm{Ind}\_\mathfrak{P}(E_a^u)$, such that $(A_b, B_b) = (A_a^u, B_a^u)M_{a^u b}$. Since $\begin{pmatrix} u \\ v \end{pmatrix} \notin \mathfrak{K}(M_{a^u b})$, the matrix

$$M_{ab} := M_{\binom{u}{v}} M_{a^u b}$$

is contained in $\mathcal{M}_\kappa^1$, $\kappa = \mathrm{Ind}\_\mathfrak{P}(E_b) - \mathrm{Ind}\_\mathfrak{P}(E_a)$. Since $\mathfrak{P}(E_0)$ is contained isometrically in $\mathfrak{P}(E_a)$, there exists a matrix $M_{0,a} \in \mathcal{M}_\kappa^1$, $\kappa = \mathrm{Ind}\_\mathfrak{P}(E_a)$, such that $(A_a, B_a) = (A_0, B_0)M_{0a}$. Now we can write (A_c, B_c) in two different ways:

$$(A_c, B_c) = (A_0, B_0)M_{0a}M_{ac} = (A_0, B_0)M_{0a}M_{ab}M_{bc}.$$

By the uniqueness part of Theorem 12.2 we have

$$M_{0a}M_{ac} = M_{0a}M_{ab}M_{bc},$$

hence we obtain the desired factorization $M_{ac} = M_{ab}M_{bc}$.

Now assume that $[uA_a + vB_a, uA_a + vB_a]_{\mathfrak{P}(E_a)} = 1$. Then $\mathfrak{P}(E_a)^l$ is nondegerated, $\mathfrak{P}(E_a)^l = \mathfrak{P}(E_a^l)$ where

$$(A_a, B_a) = (A_a^l, B_a^l)M_{\binom{u}{v}}.$$

Consider the transfer matrix $M_{a^l b}$. Since the space $\mathfrak{P}(E_b)$ contains isometrically $\mathfrak{P}(E_a^u)$, it contains in particular the element $uA_a + vB_a$ which is equal to $uA_a^l + vB_a^l$ by (13.4). Hence $\begin{pmatrix} u \\ v \end{pmatrix} \in \mathfrak{K}(M_{a^l b})$. Moreover,

$$[\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}]_{\mathfrak{K}(M_{a^l b})} = [uA_a^l + vB_a^l, uA_a^l + vB_a^l]_{\mathfrak{P}(E_b)} \leq$$

$$\leq [uA_a^l + vB_a^l, uA_a^l + vB_a^l]_{\mathfrak{P}(E_a)} = 1, \tag{13.5}$$

Since $\begin{pmatrix} u \\ v \end{pmatrix}$ is an element of norm 1 in $\mathfrak{K}(M_{\binom{u}{v}})$, and hence of norm -1 in $\mathfrak{K}(M_{\binom{u}{v}}^{-1})$, it follows that the space $\mathfrak{K}(M_{\binom{u}{v}}^{-1}) \cap M_{\binom{u}{v}}^{-1}\mathfrak{K}(M_{a^l b})$ endowed with the sum inner product is not a Hilbert space. Hence, with the matrix

$$M_{ab} := M_{\binom{u}{v}}^{-1} M_{a^l b},$$

we find from [ADSR1] that

$$\mathrm{Ind}\_\mathfrak{K}(M_{ab}) < \mathrm{Ind}\_\mathfrak{K}(M_{a^l b}) + 1.$$

Since $(A_b, B_b) = (A_a, B_a)M_{ab}$ we must have in fact $\mathrm{Ind}\_\mathfrak{K}(M_{ab}) = \mathrm{Ind}\_\mathfrak{K}(M_{a^l b})$. Now we again write (A_c, B_c) in two different ways:

$$(A_c, B_c) = (A_{a^l}, B_{a^l})M_{\binom{u}{v}}M_{ab}M_{bc} = (A_{a^l}, B_{a^l})M_{\binom{u}{v}}M_{ac},$$

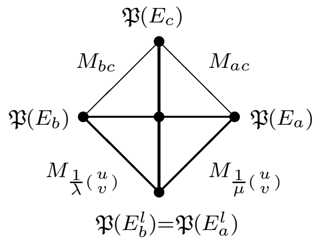
which implies as above that $M_{ac} = M_{ab}M_{bc}$.

The case *(iii)* is treated analogously, hence we may confine our attention to the situation *(ii)*. We consider for demonstration the case that $\mathfrak{P}(E_a)^l = \mathfrak{P}(E_b)^l$ is nondegenerated, and that

$$[uA_a + vb_a, uA_a + vB_a]_{\mathfrak{P}(E_a)} = \mu > 0,\ \ [uA_a + vb_a, uA_a + vB_a]_{\mathfrak{P}(E_b)} = \lambda > 0.$$

The other cases, i.e. $\mu > 0$ and $\lambda < 0$, or $\mu < 0$ and $\lambda < 0$, or if $\mathfrak{P}(E_a)^u = \mathfrak{P}(E_b)^u$ is nondegenerated, are treated similar.

In the considered case we have the picture:



Assume that $\mu \geq \lambda$, then

$$M_{ab} := M^{-1}_{\frac{1}{\mu}\binom{u}{v}} M_{\frac{1}{\lambda}\binom{u}{v}} = M_{(\frac{1}{\lambda}-\frac{1}{\mu})\binom{u}{v}} \in \mathcal{M}_0^1,$$

and we obtain the desired factorization $M_{ac} = M_{ab}M_{bc}$ by

$$(A_c, B_c) = (A_a^l, B_a^l) M_{\frac{1}{\mu}\binom{u}{v}} M_{ac} = (A_a^l, b_a^l) M_{\frac{1}{\mu}\binom{u}{v}} M_{ab} M_{ac},$$

and appealing to Theorem 12.2.

If one of $\mathfrak{P}(E_a)$ and $\mathfrak{P}(E_b)$ is contained isometrically in $\mathfrak{P}(E_c)$, the above proof works also, in fact with some simplifications.

□

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