NEVANLINNA-PICK INTERPOLATION: THE DEGENERATED CASE

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1 Introduction

Let f be a complex valued function which is meromorphic in the open upper half plane $\mathbf{C}^+$. We consider the kernel

$$N_f(z,z') = \frac{f(z)-\overline{f(z')}}{z-\overline{z'}} \quad (z,z' \in D_f). \tag{1}$$

Here and in the following D_f always denotes the set of all points of $\mathbf{C}^+$ where f is holomorphic. Recall that by definition the kernel N_f has κ ($\in \mathbb{N}_0$) positive (negative, respectively) squares on D_f if the following two conditions are satisfied:

1. For each $k \in \mathbb{N}$ and points $\zeta_1,\dots,\zeta_k \in D_f$ the hermitian form
$$\sum_{i,j=1}^{k} N_f(\zeta_i,\zeta_j)\xi_i\xi_j \tag{2}$$
has at most κ positive (negative, respectively) squares.

2. For some $k \in \mathbb{N}$ and points $\zeta_1,\dots,\zeta_k \in D_f$ the hermitian form *(2)* has exactly κ positive (negative, respectively) squares.

Definition 1 *Let $\pi,\nu \in \mathbb{N}_0$. Denote by N_ν^π the set of all functions f which are meromorphic in $\mathbf{C}^+$ and such that the kernel N_f has π positive and ν negative squares on D_f. Further let N^π and N_ν be the sets of those functions where only the number of positive (negative, respectively) squares is prescribed and equal to π (ν, respectively).*

The sets N_ν were introduced e.g. in [15]. In particular, N_0 is the Nevanlinna class consisting of all functions f holomorphic in $\mathbf{C}^+$, such that $\Im f(z) \geq 0$ for $z \in \mathbf{C}^+$.

It was shown essentially by G.Pick (see [16] and [17]) that for given distinct points $z_1, \ldots, z_n \in \mathbf{C}^+$ and points $w_1, \ldots, w_n \in \mathbf{C}$ a function $f \in \mathrm{N}_0$ satisfying

$$f(z_i) = w_i \text{ for } i = 1, \ldots, n \tag{3}$$

exists if and only if the so called Pick matrix

$$\mathbb{P} = \left(\frac{w_i - \overline{w_j}}{z_i - \overline{z_j}} \right)_{j,i=1}^{n} \tag{4}$$

is nonnegative definite, in this case f is uniquely determined if and only if $\mathbb{P}$ is singular. We will refer to a function f satisfying *(3)* as a solution of the interpolation problem with data $(z_1, w_1), \ldots, (z_n, w_n) \in \mathbf{C}^+ \times \mathbf{C}$. In this context the points $z_1, \ldots, z_n$ will always be distinct.

In this note we are interested in the case that $\mathbb{P}$ is singular. If, e.g., $\mathbb{P}$ has rank m $(< n)$ and is nonnegative definite it is shown that besides the unique solution of the interpolation problem in the set N_0, which is a rational function, there is no solution in any set N_ν with $0 < \nu < n - m$.

In fact this result is proved in greater generality: Let π_0 (ν_0, respectively) be the number of positive (negative, respectively) squares of the Pick matrix $\mathbb{P}$ and let $\operatorname{rank} \mathbb{P} = m$ with $m = \pi_0 + \nu_0 < n$. Then the interpolation problem *(3)* has a unique solution in the set $\mathrm{N}_{\nu_0}^{\pi_0}$, if and only if the data points do not belong to some exactly described 'small' exceptional set. This solution is given explicitly as a rational function. If either $\pi_0 < \pi < \pi_0 + (n - m)$ or $\nu_0 < \nu < \nu_0 + (m - n)$ then *(3)* has no solution in the set $\mathrm{N}^\pi \cup \mathrm{N}_\nu$. In a following note it will be shown that solutions of the problem *(3)* in the set N_ν^π with $\pi \geq \pi_0 + (m - n)$ and $\nu \geq \nu_0 + (m - n)$ always exist.

These results are proved in Section 3, using in fact ideas of G.Pick (see [16]), and in Section 4, using some geometric properties of certain inner product spaces and their relation to interpolation problems of the considered type.

In Sections 2 and 5 some results on rational functions and the sets N_ν^π are proved. Some of them are known, however, they are either used later or are immediate consequences of our results.

There are a lot of different approaches to questions of the considered type. In particular we refer to [2] and [3] where rational interpolation problems are studied. There the solution (solutions, respectively) with minimal McMillan degree also are of particular interest. Within the framework of rational interpolation not only values of the function itself, but also values for its derivatives can be prescribed (see e.g. [1] or [2]. Some generalizations to matrix valued interpolation problems can be found e.g. in [4], [6], [7] or [12]. Another approach to rational interpolation problems using the so called Iohvidov laws can be found e.g. in [10]. This method is not applicable in our case. We refer to [5] for further references.

Interpolation problems of Nevanlinna-Pick type have various applications. A whole part of the book [5] is devoted to connections with control and system theory (see also [9]).

2 A result on rational functions

Before stating the first theorem we recall the notion of degree and signature.

Let f be a rational function. The degree of f is defined as the number $\deg f$ of poles of f in $\overline{\mathbf{C}}$ counted according to their multiplicities. If $f(z) = \frac{p(z)}{q(z)}$ with relatively prime polynomials p and q the degree of f equals $\max(\deg p, \deg q)$, where the degree of a polynomial is defined as usual.

Let P be a hermitian $n \times n$-matrix. The signature $\operatorname{sign} P$ of P is the pair (π, ν) where π (ν, respectively) is the number of positive (negative, respectively) eigenvalues of P, counted according to their multiplicities. Obviously, the defect δ of P is $n - \pi - \nu$.

Further let f be a meromorphic function in $\mathbf{C}$ and denote by $D(f) \subseteq \mathbf{C}$ the domain of holomorphy of f. Then f is called real if for each $z \in D(f) \cap \overline{D(f)}$ the relation $f(\overline{z}) = \overline{f(z)}$ holds. If f is a rational function, say $f(z) = \frac{p(z)}{q(z)}$ (p, q relatively prime, q monic), then f is real if and only if p and q have real coefficients.

In analogy to the Pick matrix *(4)* we introduce the following notation.

Definition 2 *Let f be a complex valued function defined on some subset $D(f)$ of $\mathbf{C}^+$, let $k \in \mathbb{N}$ and $\zeta_1, \ldots, \zeta_k \in D(f)$. Denote by $\mathbb{P}_{\zeta_1,\ldots,\zeta_k}(f)$ the matrix*

$$\mathbb{P}_{\zeta_1,\ldots,\zeta_k}(f) = \left(\frac{f(\zeta_i) - \overline{f(\zeta_j)}}{\zeta_i - \overline{\zeta_j}} \right)_{j,i=1}^{k}.$$

It follows immediately that $f \in N_\nu^\pi$ if and only if for any $k \in \mathbb{N}$ and points $\zeta_1, \ldots, \zeta_k \in D_f$ the Pick matrix $\mathbb{P}_{\zeta_1,\ldots,\zeta_k}(f)$ has signature (π', ν') with $\pi' \le \pi$ and $\nu' \le \nu$, and if for some k and some points $\zeta_1, \ldots, \zeta_k \in D_f$ in both relations equality holds.

Theorem 1 *Let f be a real rational function. For $k \in \mathbb{N}, k \ge \deg f$ and distinct points $\zeta_1, \ldots, \zeta_k \in D_f$ the signature of the Pick matrix $\mathbb{P}_{\zeta_1,\ldots,\zeta_k}(f)$ does not depend on k and $\zeta_1, \ldots, \zeta_k$. If*

$$\operatorname{sign} \mathbb{P}_{\zeta_1,\ldots,\zeta_k}(f) = (\pi_f, \nu_f),$$

then $f \in N_{\nu_f}^{\pi_f}$.

Proof : Let $m = \deg f, k \geq m$ and let $K'(z, z')$ for $z, z' \in D_f$ be the kernel

$$K'(z, z') = \frac{p(z)q(\overline{z'}) - p(\overline{z'})q(z)}{z - \overline{z'}} = \sum_{r,s=0}^{m-1} a_{rs} z^r \overline{z'}^s .$$

Further let $\zeta_1, \ldots, \zeta_k$ be distinct points of D_f. The number of positive (negative, respectively) eigenvalues of the Pick matrix $\mathbb{P}_{\zeta_1, \ldots, \zeta_k}(f)$ equals the number of positive (negative, respectively) squares of the hermitian form

$$H_{\zeta_1, \ldots, \zeta_k}(y) = \sum_{r,s=1}^{k} K'(\zeta_r, \zeta_s) y_r \overline{y}_s$$

defined for $y = (y_1, \ldots, y_k) \in \mathbf{C}^k$. We consider the hermitian form

$$H(x) = \sum_{r,s=0}^{m-1} a_{rs} x_r \overline{x}_s$$

defined for $x = (x_0, \ldots, x_{m-1}) \in \mathbf{C}^m$, and the linear mapping $U_{\zeta_1, \ldots, \zeta_k} : \mathbf{C}^k \to \mathbf{C}^m$ represented with respect to the canonical bases by the matrix

$$U_{\zeta_1,\ldots,\zeta_k} = \begin{pmatrix} 1 & \cdots & 1 \\ \zeta_1 & \cdots & \zeta_k \\ \vdots & \ddots & \vdots \\ \zeta_1^{m-1} & \cdots & \zeta_k^{m-1} \end{pmatrix} .$$

For $y \in \mathbf{C}^k$ the following relation can be checked immediately:

$$H_{\zeta_1,\ldots,\zeta_k}(y) = H(U_{\zeta_1,\ldots,\zeta_k} y) .$$

As $U_{\zeta_1,\ldots,\zeta_k}$ has rank m and maps a definite subspace (of $\mathbf{C}^k$ considered with the form $H_{\zeta_1,\ldots,\zeta_k}$) injectively onto a definite subspace (of $\mathbf{C}^m$ considered with the form H) we find that the signature of $\mathbb{P}_{\zeta_1,\ldots,\zeta_k}(f)$ equals the signature of H. Therefore it does not depend on k and $\zeta_1, \ldots, \zeta_k$.

If points $\zeta_1, \ldots, \zeta_k$ with $k < m$ are given, choose arbitrary points $\zeta_{k+1}, \ldots, \zeta_m$. Then $\operatorname{sign} \mathbb{P}_{\zeta_1,\ldots,\zeta_m}(f) = (\pi_f, \nu_f)$ and therefore π_f (ν_f, respectively) are upper bounds for the signature numbers of $\mathbb{P}_{\zeta_1,\ldots,\zeta_k}(f)$.

$\square$

Corollary 1 *If $k > \deg f$, for any points $\zeta_1, \ldots, \zeta_k \in D_f$ the Pick matrix $\mathbb{P}_{\zeta_1,\ldots,\zeta_k}(f)$ is singular.*

Corollary 2 *Any real rational function f is contained in some set N_ν^π. Here the pair (π,ν) is the signature of $\mathbb{P}_{\zeta_1,\dots,\zeta_k}(f)$ where $k = \deg f$ and $\zeta_1,\dots,\zeta_k$ are $\deg f$ distinct points of D_f. Further $\pi+\nu \le \deg f$ holds.*

In Section 5 it will follow that in the last relation the equality sign holds.

3 The unique solution

Consider now the interpolation problem *(3)* with data $(z_1,w_1),\dots,(z_n,w_n) \in \mathbf{C}^+ \times \mathbf{C}$ and the corresponding Pick matrix *(4)* .

If sign $\mathbb{P} = (\pi_0,\nu_0)$ and $\nu < \nu_0$ or $\pi < \pi_0$ obviously no solution of the problem *(3)* exists in N_ν^π.

Let us introduce some notation (following e.g. [11]): If $P = (p_{ij})_{\substack{i=1,\dots n\\ j=1\dots m}}$ is a $(n\times m)$-matrix, $n' \le n$, $m' \le m$ and $\{i_1,\dots,i_{n'}\} \subseteq \{1,\dots,n\}$, $\{j_1,\dots,j_{m'}\} \subseteq \{1,\dots,m\}$ then

$$P\begin{pmatrix} i_1,\dots,i_{n'}\\ j_1,\dots,j_{m'}\end{pmatrix} = (p_{i_l j_k})_{\substack{l=1,\dots n'\\ k=1\dots m'}}.$$

Let $P_r = P\binom{1,\dots,r}{1,\dots,r}$ for $r \le n,m$ and define $P_0 = (1)$.

For a quadratic matrix $P = (p_{ij})_{i,j=1}^n$ we consider the matrix $P' = (p'_{ij})_{i,j=1}^n$ of the algebraic complements of P:

$$p'_{ij} = (-1)^{i+j}\left|\mathbb{P}\begin{pmatrix}1,\dots,i-1,i+1,\dots,n\\ 1,\dots,j-1,j+1\dots n\end{pmatrix}\right|.$$

In the sequel P' will be called the reciprocal matrix of P.

Lemma 1 *Consider the interpolation problem (3) with given data $(z_1,w_1),\dots,$ $(z_n,w_n) \in \mathbf{C}^+\times\mathbf{C}$ and let $l = \operatorname{rank}\mathbb{P}+1$. The data $(z_1,w_1),\dots,(z_n,w_n)$ can be ordered such that the Pick matrix $\mathbb{P}$ has the following properties:*

1. $|\mathbb{P}_{l-1}| \ne 0$,
2. *if $|\mathbb{P}_i| = 0$ for some i with $0 < i < l-1$, then $|\mathbb{P}_{i-1}|\cdot|\mathbb{P}_{i+1}| \ne 0$.*

For real symmetric matrices this lemma is proved e.g. in [14], §98 and remains valid in the complex hermitian case.

In the following we suppose that the data points are ordered according to Lemma 1 . Further, let $l = \operatorname{rank}\mathbb{P}+1$ and define numbers λ_i for $i = 1,\dots,l$ as

$$\lambda_i = (-1)^{i+l}\mathbb{P}\begin{pmatrix}1,\dots,i-1,i+1,\dots,l\\ 1,\dots,l-1\end{pmatrix}. \tag{5}$$

We associate with the given data the rational function

$$f_{min}(z) = \frac{\sum\limits_{i=1}^{l} \lambda_i \overline{w_i} \prod\limits_{\substack{j=1\\ j\neq i}}^{l} (z-\overline{z_j})}{\sum\limits_{i=1}^{l} \lambda_i \prod\limits_{\substack{j=1\\ j\neq i}}^{l} (z-\overline{z_j})}, \tag{6}$$

and the set

$$Z_{min} = \{z | \sum_{i=1}^{l} \lambda_i \prod_{\substack{j=1\\ j\neq i}}^{l} (z-\overline{z_j}) = 0\}.$$

Lemma 2 *The function f_{min} is real.*

Proof : We show that for each $z \notin \{z_1, \ldots, z_l, \overline{z_1}, \ldots, \overline{z_l}\} \cup Z_{min}$ the relation $f_{min}(\overline{z}) = \overline{f_{min}(z)}$ holds. Indeed

$$f_{min}(\overline{z}) - \overline{f_{min}(z)} = \frac{\left(\sum\limits_{i=1}^{l} \lambda_i \overline{w_i} \frac{1}{\overline{z}-\overline{z_i}}\right)\left(\sum\limits_{j=1}^{l} \overline{\lambda_j} \frac{1}{\overline{z}-z_j}\right) - \left(\sum\limits_{j=1}^{l} \overline{\lambda_j} w_j \frac{1}{\overline{z}-z_j}\right)\left(\sum\limits_{i=1}^{l} \lambda_i \frac{1}{\overline{z}-\overline{z_i}}\right)}{\left(\sum\limits_{i=1}^{l} \lambda_i \frac{1}{\overline{z}-\overline{z_i}}\right)\left(\sum\limits_{j=1}^{l} \lambda_j \frac{1}{\overline{z}-z_j}\right)}.$$

The numerator equals

$$\sum_{i,j=1}^{l} \lambda_i \overline{\lambda_j} \frac{w_j - \overline{w_i}}{z_j - \overline{z_i}} \left(\frac{1}{\overline{z}-\overline{z_i}} - \frac{1}{\overline{z}-z_j}\right) =$$

$$\sum_{i=1}^{l} \lambda_i \frac{1}{\overline{z}-\overline{z_i}} \sum_{j=1}^{l} \overline{\lambda_j} \frac{w_j - \overline{w_i}}{z_j - \overline{z_i}} - \sum_{j=1}^{l} \overline{\lambda_j} \frac{1}{\overline{z}-z_j} \sum_{i=1}^{l} \lambda_i \frac{w_j - \overline{w_i}}{z_j - \overline{z_i}}.$$

Both terms on the right hand side vanish as can be seen by expanding the determinant of the singular matrix $\mathbb{P}_l$:

$$\sum_{j=1}^{l} \overline{\lambda_j} \frac{w_j - \overline{w_i}}{z_j - \overline{z_i}} = 0 \text{ and } \sum_{i=1}^{l} \lambda_i \frac{w_j - \overline{w_i}}{z_j - \overline{z_i}} = 0$$

for $i = 1, \ldots, l$ ($j = 1, \ldots, l$, respectively).

□

Theorem 2 *Let $(z_1, w_1), \ldots, (z_n, w_n) \in \mathbf{C}^+ \times \mathbf{C}$, $sign\, \mathbb{P} = (\pi_0, \nu_0)$ and $rank\, \mathbb{P} = l - 1 < n$ for the corresponding Pick matrix (4) . Then the problem (3) has a solution $f \in \mathrm{N}_{\nu_0}^{\pi_0}$ if and only if one of the following three (equivalent) conditions is satisfied:*

(i) $deg\, f_{min} = l - 1$.

(ii) $\lambda_j \neq 0$ *for* $j = 1, \ldots, l-1$.

(iii) $z_j \notin Z_{min}$ *for* $j = 1, \ldots, l-1$ *or, written more explicitly,*

$$\sum_{i=1}^{l} \frac{\lambda_i}{z_j - \overline{z_i}} \neq 0 \text{ for } j = 1, \ldots, l-1.$$

In this case $f = f_{min}$ *which is given by (6) .*

Proof : First we prove the uniqueness statement. Suppose that the function $f \in \mathrm{N}_{\nu_0}^{\pi_0}$ satisfies *(3)* :

$$f(z_i) = w_i \text{ for } i = 1, \ldots, n.$$

Arrange the data according to Lemma 1 and consider for $z \in D_f$ and $w = f(z)$ the determinant

$$|\mathbb{P}_{l,z}| = \left| \begin{array}{ccc|c|c} \frac{w_1 - \overline{w_1}}{z_1 - \overline{z_1}} & \cdots & \frac{w_{l-1} - \overline{w_1}}{z_{l-1} - \overline{z_1}} & \frac{w_l - \overline{w_1}}{z_l - \overline{z_1}} & \frac{w - \overline{w_1}}{z - \overline{z_1}} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{w_1 - \overline{w_{l-1}}}{z_1 - \overline{z_{l-1}}} & \cdots & \frac{w_{l-1} - \overline{w_{l-1}}}{z_{l-1} - \overline{z_{l-1}}} & \frac{w_l - \overline{w_{l-1}}}{z_l - \overline{z_{l-1}}} & \frac{w - \overline{w_{l-1}}}{z - \overline{z_{l-1}}} \\ \hline \frac{w_1 - \overline{w_l}}{z_1 - \overline{z_l}} & \cdots & \frac{w_{l-1} - \overline{w_l}}{z_{l-1} - \overline{z_l}} & \frac{w_l - \overline{w_l}}{z_l - \overline{z_l}} & \frac{w - \overline{w_l}}{z - \overline{z_l}} \\ \hline \frac{w_1 - \overline{w}}{z_1 - \overline{z}} & \cdots & \frac{w_{l-1} - \overline{w}}{z_{l-1} - \overline{z}} & \frac{w_l - \overline{w}}{z_l - \overline{z}} & \frac{w - \overline{w}}{z - \overline{z}} \end{array} \right| . \tag{7}$$

Applying Jacobi's sign rule to the sequence

$$1 = |\mathbb{P}_0|, |\mathbb{P}_1|, \ldots, |\mathbb{P}_{l-1}| \neq 0, |\mathbb{P}_l| = 0, |\mathbb{P}_{l,z}|$$

it follows that $|\mathbb{P}_{l,z}|$ has the same sign as $|\mathbb{P}_{l-1}|$. Sylvester's identity implies

$$\begin{aligned} 0 \leq |\mathbb{P}_{l,z}| \cdot |\mathbb{P}_{l-1}| &= - \left| \mathbb{P}_{l,z} \begin{pmatrix} 1,\ldots,l-1,l \\ 1,\ldots,l-1,l+1 \end{pmatrix} \right|^2 = \\ &= - \left| \sum_{i=1}^{l} \lambda_i \frac{w - \overline{w_i}}{z - \overline{z_i}} \right|^2 , \end{aligned} \tag{8}$$

thus $\sum\limits_{i=1}^{l} \lambda_i \frac{w - \overline{w_i}}{z - \overline{z_i}} = 0$ which yields $w = f(z) = f_{min}(z)$.

In the following we show that f_{min} satisfies the interpolation conditions *(3)* if and only if (i), (ii) or (iii) is valid.

To establish the necessity of these conditions we note that (i) is an immediate consequence of Theorem 1 , (ii) is implied by (i) in any case and (i) also implies (iii) because f_{min} takes finite values at points z_j for $j = 1, \ldots, l-1$.

Suppose now that (ii) is satisfied. Then for $j = 1,\dots,l-1$ we have $f_{min}(\overline{z_j}) = \overline{w_j}$. If we assume (iii) we find

$$f_{min}(z_j) = \frac{\sum\limits_{i=1}^{l} \lambda_i \frac{\overline{w_i} - w_j}{z_j - \overline{z_i}}}{\sum\limits_{i=1}^{l} \frac{\lambda_i}{z_j - \overline{z_i}}} + w_j = w_j \text{ for } j = 1,\dots,l-1.$$

Thus (i),(ii) or (iii) imply $f_{min}(z_j) = w_j$ for $j = 1,\dots,l-1$. As $\lambda_l \neq 0$ we also have $f_{min}(z_l) = w_l$.

Now let $j \in \{l+1,\dots,n\}$ and consider the determinant *(7)* for $f = f_{min}$ and $z = z_j$. Then *(8)* implies that $|\mathbb{P}_{l,z_j}| = 0$ if and only if w is a solution of the linear equation

$$\sum_{i=1}^{l} \lambda_i \prod_{\substack{j=1 \\ j\neq i}}^{l} (z - \overline{z_j}) \cdot w = \sum_{i=1}^{l} \lambda_i \overline{w_i} \prod_{\substack{j=1 \\ j\neq i}}^{l} (z - \overline{z_j}).$$

This is the case for exactly one value of w, which is then equal to $f_{min}(z_j)$, or for all w or for no w. As $|\mathbb{P}_{l,z_j}| = 0$ for $w = w_j$ the last possibility cannot occur. The second possibility is also excluded because Theorem 1 already applies to f_{min} and we find $\deg f_{min} = l - 1$. The only case left is the first one and here we have $f_{min}(z_j) = w_j$.

Finally we again apply Theorem 1 to obtain $f \in \mathrm{N}_{\nu_0}^{\pi_0}$.

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4 Interpolation and inner product spaces

Now we associate with a complex function f defined on $D(f) \subseteq \mathbf{C}^+$ an inner product space.

Definition 3 *Let $f : D(f) \to \mathbf{C}$, $D(f) \subseteq \mathbf{C}^+$. Denote by H_f the following linear space of all formal sums*

$$\mathrm{H}_f = \{ \sum_{z\in D(f)} x_z e_z \,|\, x_z \in \mathbf{C}, x_z = 0 \textit{ for all but finitely many } z \in D(f)\},$$

equipped with the inner product defined by the relation

$$[e_z, e_w]_{\mathrm{H}_f} = \frac{f(z) - \overline{f(w)}}{z - \overline{w}} \;\; (z, w \in D(f)).$$

If no confusion can occur the index H_f at the inner product will be dropped. Note that by definition the elements e_z for $z \in D(f)$ are linearly independent.

If the function f is an extension of some function g (that is $D(g) \subseteq D(f)$ and $f|_{D(g)} = g$) then H_g can be considered in a canonical way as subspace of H_f.

Recall that for an inner product space L the isotropic subspace is the set $\mathrm{L}^\circ = \mathrm{L} \cap \mathrm{L}^\perp$. If $\mathrm{L}^\circ \neq \{0\}$ the space L is called degenerated.

Consider now the interpolation problem *(3)* with data $(z_1, w_1), \ldots, (z_n, w_n) \in \mathbf{C}^+ \times \mathbf{C}$. Let $\mathbb{P}$ be the corresponding Pick matrix, $\operatorname{sign} \mathbb{P} = (\pi_0, \nu_0)$ and $\delta = n - \pi_0 - \nu_0$.

With the given data we associate the inner product space $\mathrm{H}_{\hat{f}}$ for the function $\hat{f}$ defined as follows:

$$D(\hat{f}) = \{z_1, \ldots, z_n\},\ \hat{f}(z_i) = w_i \text{ for } i = 1, \ldots, n. \tag{9}$$

In the following the space $\mathrm{H}_{\hat{f}}$ is denoted by H. The dimension of each maximal positive (negative, respectively) subspace of H is π_0 (ν_0, respectively) and the dimension of H° is δ (> 0). If f is a solution of the interpolation problem *(3)* we evidently have $\mathrm{H} \subseteq \mathrm{H}_f$.

With each nonzero neutral vector h° of H we shall associate a real rational function f_{h° which will coincide with f_{min} from *(6)* .

Definition 4 *If $h^\circ \in \mathrm{H}^\circ \setminus \{0\}, h^\circ = \sum_{i=1}^{n} h_i^\circ e_{z_i}$, denote with f_{h° the rational function*

$$f_{h^\circ}(z) = \frac{\sum_{i=1}^{n} \overline{h_i^\circ}\, \overline{w_i} \prod_{\substack{j=1 \\ j\neq i}}^{n} (z - \overline{z_j})}{\sum_{i=1}^{n} \overline{h_i^\circ} \prod_{\substack{j=1 \\ j\neq i}}^{n} (z - \overline{z_j})}, \tag{10}$$

defined on its domain of holomorphy $D_{f_{h^\circ}}$ in $\mathbf{C}^+$.

Lemma 3 *Suppose that the interpolation problem (3) has a solution f in $\mathrm{N}_{\nu_0}^{\pi_0}$. Then there are δ linearly independent vectors $h_1^\circ, \ldots, h_\delta^\circ \in \mathrm{H}^\circ$, such that $f = f_{h_i^\circ}$ for $i = 1, \ldots, \delta$.*

Proof : The solution in $\mathrm{N}_{\nu_0}^{\pi_0}$ is given by formula *(6)* , which can be written as

$$f_{min}(z) = \frac{\sum_{i=1}^{n} \overline{\mu_i^{(1)}}\, \overline{w_i} \prod_{\substack{j=1 \\ j\neq i}}^{n} (z - \overline{z_j})}{\sum_{i=1}^{n} \overline{\mu_i^{(1)}} \prod_{\substack{j=1 \\ j\neq i}}^{n} (z - \overline{z_j})}$$

with $\mu_i^{(1)} = \overline{\lambda_i}$ for $i = 1, \ldots, l-1$, $\mu_l^{(1)} = \overline{\lambda_l} \neq 0$ and $\mu_i^{(1)} = 0$ for $i = l+1, \ldots, n$. From the definition *(5)* of the numbers λ_i we see that $(\mu_1^{(1)}, \ldots, \mu_{l-1}^{(1)}, \mu_l^{(1)})^T$ is a column of the reciprocal matrix of $\mathbb{P}\binom{1,\ldots,l-1,l}{1,\ldots,l-1,l}$. Together with $|\mathbb{P}_{l-1}| \neq 0$ and $\operatorname{rank} \mathbb{P} = l-1$ this implies that

$$\mathbb{P} \cdot (\mu_1^{(1)}, \ldots, \mu_{l-1}^{(1)}, \mu_l^{(1)}, 0, \ldots, 0)^T = 0$$

holds. In other words the vector $h_1^\circ = \sum_{i=1}^{n} \mu_i^{(1)} e_{z_i}$ is isotropic: $h_1^\circ \in \mathrm{H}^\circ$.

In the proof of Theorem 2 we have ordered the points $z_1, \ldots, z_n$ according to Lemma 1 . The conditions required in Lemma 1 concern only the properties of the first $l-1$ points. Thus the order of the remaining points is arbitrary, which means we could make the same calculations with the point $z_{(l-1)+r}$ $(r = 2, \ldots, \delta)$ instead of z_l. The function f_{min} constructed in this way can be written as

$$f_{min}(z) = \frac{\sum_{i=1}^{n} \overline{\mu_i^{(r)}}\, \overline{w_i} \prod_{\substack{j=1 \\ j \neq i}}^{n} (z - \overline{z_j})}{\sum_{i=1}^{n} \overline{\mu_i^{(r)}} \prod_{\substack{j=1 \\ j \neq i}}^{n} (z - \overline{z_j})}$$

where $(\mu_1^{(r)}, \ldots, \mu_{l-1}^{(r)}, \mu_{l-1+r}^{(r)})^T$ is again a column of the reciprocal matrix of $\mathbb{P}\binom{1,\ldots,l-1,l-1+r}{1,\ldots,l-1,l-1+r}$, $\mu_{l-1+r}^{(r)} \neq 0$ and $\mu_l^{(r)} = \ldots = \mu_{l-1+r-1}^{(r)} = \mu_{l-1+r+1}^{(r)} = \ldots = \mu_n^{(r)} = 0$. Again we get

$$\mathbb{P} \cdot (\mu_1^{(r)}, \ldots, \mu_{l-1}^{(r)}, 0, \ldots, 0, \mu_{l-1+r}^{(r)}, 0, \ldots, 0)^T = 0,$$

and therefore $h_r^\circ = \sum_{i=1}^{n} \mu_i^{(r)} e_{z_i} \in \mathrm{H}^\circ$.

Obviously the vectors $h_1^\circ, \ldots, h_\delta^\circ$ are linearly independent.

$\square$

Theorem 3 *Suppose that the interpolation problem (3) with data* $(z_1, w_1), \ldots, (z_n, w_n) \in \mathbf{C}^+ \times \mathbf{C}$ *has a solution* $f \in \mathrm{N}_{\nu_0}^{\pi_0}$ *where* (π_0, ν_0) *is the signature of the Pick matrix* $\mathbb{P}$, $n > \pi_0 + \nu_0$. *Then for arbitrary* $h^\circ \in \mathrm{H}^\circ, h^\circ \neq 0$ *we have* $f = f_{h^\circ}$.

Proof : Lemma 3 implies that the set L $(\subseteq \mathrm{H}^\circ)$ of all h° with $f(z) = f_{h^\circ}(z)$ contains $\delta = n - \pi_0 - \nu_0 = \dim \mathrm{H}^\circ$ linearly independent vectors. The theorem will be proved if we show that L (together with 0) is a subspace.

If $\alpha \in \mathbf{C}, \alpha \neq 0$ and $h^\circ \in \mathrm{L}$ then $f_{\alpha h^\circ}(z) = f_{h^\circ}(z)$. If $h_1^\circ, h_2^\circ \in \mathrm{L}$, $h_1^\circ \neq -h_2^\circ$ consider the representation *(10)* of $f_{h_1^\circ}$ and $f_{h_2^\circ}$:

$$f_{h_1^\circ}(z) = \frac{p_1(z)}{q_1(z)} \text{ and } f_{h_2^\circ}(z) = \frac{p_2(z)}{q_2(z)}.$$

Observe that the polynomials p_1, q_1 and p_2, q_2 need not be relatively prime. As $h_1^\circ \neq -h_2^\circ$ the polynomial $q_1 + q_2$ is not identically zero. Outside a circle containing all zeros of q_1, q_2 and $q_1 + q_2$ we find from $h_1^\circ, h_2^\circ \in \mathrm{L}$ that $p_1(z) = f(z)q_1(z)$ and $p_2(z) = f(z)q_2(z)$. Thus

$$\frac{p_1(z) + p_2(z)}{q_1(z) + q_2(z)} = f(z)$$

holds which shows that $h_1^\circ + h_2^\circ \in \mathrm{L}$.

$\square$

In the following we use the geometry of the inner product spaces H and H_f to show that there are no solutions of the problem *(3)* in the sets N^π for $\pi_0 < \pi < \pi + \delta$ and N_ν for $\nu_0 < \nu < \nu + \delta$ where again δ is the defect of $\mathbf{IP}$.

Lemma 4 *Let f be a complex valued continuous function defined on $D(f) \subseteq \mathbf{C}^+$ such that H_f is degenerated and let $h^\circ \in {\mathrm{H}_f}^\circ \setminus \{0\}$, say $h^\circ = \sum_{i=1}^m h_i^\circ e_{\zeta_i} \in {\mathrm{H}_f}^\circ$. Finally suppose that no point $z \in D(f)$ with $\sum_{i=1}^m \overline{h_i^\circ} \prod_{\substack{j=1 \\ j\neq i}}^m (z - \overline{\zeta_i}) = 0$ lies isolated in $D(f)$. Then for $z \in D(f)$*

$$f(z) = \frac{\sum_{i=1}^m \overline{h_i^\circ}\, \overline{f(z_i)} \prod_{\substack{j=1 \\ j\neq i}}^m (z - \overline{z_i})}{\sum_{i=1}^m \overline{h_i^\circ} \prod_{\substack{j=1 \\ j\neq i}}^m (z - \overline{z_i})}. \tag{11}$$

That is, f is the restriction of the rational function on the right hand side of (11) to $D(f)$.

Proof : For $z \in D(f)$ we find

$$0 = [e_z, h^\circ] = \sum_{i=1}^m \overline{h_i^\circ} \frac{f(z) - \overline{f(\zeta_i)}}{z - \overline{\zeta_i}}.$$

This gives

$$\sum_{i=1}^m \overline{h_i^\circ}(f(z) - \overline{f(\zeta_i)}) \prod_{\substack{j=1 \\ j\neq i}}^m (z - \overline{\zeta_i}) = 0$$

which yields *(11)*, at least if $\sum\limits_{i=1}^{m} \overline{h_i^\circ} \prod\limits_{\substack{j=1\\ j\neq i}}^{m} (z-\overline{\zeta_j}) \neq 0$.

□

The following lemma is proved by a straightforward calculation.

Lemma 5 *Let $h^\circ \in \mathrm{H}^\circ \setminus \{0\}, h^\circ = \sum\limits_{i=1}^{n} h_i^\circ e_{z_i}$ and define $u(z)$ as*

$$u(z) = \frac{1}{\sum\limits_{i=1}^{n} h_i^\circ \prod\limits_{\substack{j=1\\ j\neq i}}^{n} (\overline{z}-z_j)} \sum_{i=1}^{n} \left[h_i^\circ \prod_{\substack{j=1\\ j\neq i}}^{n} (\overline{z}-z_j) e_{z_i} \right].$$

Then for $z, w \in D_{f_{h^\circ}}$ the relation

$$[e_z, e_w] = \frac{f_{h^\circ}(z) - \overline{f_{h^\circ}(w)}}{z-\overline{w}} = [u(w), u(z)]_{\mathrm{H}} \tag{12}$$

holds.

Using this lemma we prove

Lemma 6 *Let $h^\circ \in \mathrm{H}^\circ \setminus \{0\}, h^\circ = \sum\limits_{i=1}^{n} h_i^\circ e_{z_i}$. Then $f_{h^\circ} \in \mathrm{N}_\nu^\pi$ for some numbers π and ν with $\pi \leq \pi_0$ and $\nu \leq \nu_0$.*

Proof : Let $m \in \mathbb{N}, \zeta_1, \ldots, \zeta_m \in D_{f_{h^\circ}}$ with

$$\sum_{i=1}^{n} h_i^\circ \prod_{\substack{j=1\\ j\neq i}}^{n} (\overline{z}-z_j) \neq 0 \text{ for } z = \zeta_1, \ldots, \zeta_m. \tag{13}$$

Consider the space $\mathbf{C}^m$ provided with the inner product

$$[e_i, e_j]_{\mathbf{C}^m} = \frac{f(\zeta_j) - \overline{f(\zeta_i)}}{\zeta_j - \overline{\zeta_i}} \text{ for } i, j = 1, \ldots, m,$$

where e_i denotes the i-th canonical basis vector of $\mathbf{C}^m$. Then Lemma 5 implies that the operator U defined by the relations $U e_i = u(\zeta_i)$ is an isometry from $\langle \mathbf{C}^m, [.,.]_{\mathbf{C}^m} \rangle$ into H. Thus π_0 (ν_0, respectively) is an upper bound for the dimension of positive (negative, respectively) subspaces of $\langle \mathbf{C}^m, [.,.]_{\mathbf{C}^m} \rangle$.

Because the set of points $z \in D_{f_{h^\circ}}$ satisfying *(13)* lies dense in $D_{f_{h^\circ}}$ the assertion follows.

□

Remark 1 Note that Lemma 4 cannot always be applied to the function $\hat{f}$ defined as in *(9)* because $D(\hat{f})$ is discrete. So we need not necessarily have $f_{h^\circ}(z_i) = w_i$ for $i = 1, \ldots, n$. But if no point z_i is a zero of $z \mapsto \sum_{i=1}^{n} \overline{h_i^\circ} \prod_{\substack{j=1 \\ j \neq i}}^{n} (z - \overline{z_j})$, we find that f_{h° extends $\hat{f}$ which means that $f_{h^\circ}(z_i) = w_i$ for all $i = 1, \ldots, n$. Then the above lemma shows that f_{h° is a solution in $\mathrm{N}_{\nu_0}^{\pi_0}$ of the interpolation problem *(3)* . Thus we can add to the conditions (i), (ii) and (iii) of Theorem 2 the following also equivalent condition.

(iv) *There exists a vector* $h^\circ \in \mathrm{H}^\circ \setminus \{0\}, h^\circ = \sum_{i=1}^{n} h_i^\circ e_{z_i}$, *such that no point* z_i $(i = 1, \ldots, n)$ *is a zero of* $z \mapsto \sum_{i=1}^{n} \overline{h_i^\circ} \prod_{\substack{j=1 \\ j \neq i}}^{n} (z - \overline{z_i})$.

Using some results about the geometry of inner product spaces (which can be found e.g. in [8] or [13]) we obtain the following theorem.

Theorem 4 *Consider the interpolation problem (3) with data* $(z_1, w_1), \ldots, (z_n, w_n) \in \mathbf{C}^+ \times \mathbf{C}$. *If* δ *denotes the defect of the Pick matrix* $\mathbb{P}$ *and sign* $\mathbb{P} = (\pi_0, \nu_0)$ *there is no solution* f *in any set* N^π *or* N_ν *with* $\pi_0 < \pi < \pi_0 + \delta$ *or* $\nu_0 < \nu < \nu_0 + \delta$.

Proof : Let f be a solution of the problem *(3)* with $f \in \mathrm{N}_\nu$ and $\nu_0 < \nu < \nu_0 + \delta$. Then we have $\mathrm{H} \subseteq \mathrm{H}_f$. Consider the nondegenerated inner product space $\mathrm{L} = \mathrm{H}_f / \mathrm{H}_f{}^\circ$. The largest dimension of a negative subspace of L equals ν.

In case $\mathrm{H} \cap \mathrm{H}_f{}^\circ = \{0\}$ we can consider H in a natural way as a subspace of L. If H^- denotes some ν_0-dimensional negative subspace of H, we can decompose L as $\mathrm{L} = \mathrm{H}^- [\dot{+}] (\mathrm{H}^-)^\perp$. Obviously $\mathrm{H}^\circ \subseteq (\mathrm{H}^-)^\perp$ and the largest dimension of a negative subspace of $(\mathrm{H}^-)^\perp$ equals $\nu - \nu_0$. Then the dimension of neutral subspaces of $(\mathrm{H}^-)^\perp$ is also bounded by $\nu - \nu_0$, which is impossible as $\dim \mathrm{H}^\circ = \delta > \nu - \nu_0$.

Thus we have $\mathrm{H} \cap \mathrm{H}_f{}^\circ \neq \{0\}$. For any vector $h^\circ \in \mathrm{H} \cap \mathrm{H}_f{}^\circ, h^\circ \neq 0$ Lemma 4 implies $f = f_{h^\circ}$.

Lemma 6 yields $f_{h^\circ} \in \mathrm{N}_{\nu'}$ with $\nu' \leq \nu_0$. This is a contradiction to $\nu > \nu_0$ which proves the assertion.

The remaining part of the theorem is proved analogously.

□

Remark 2 If a solution of *(3)* in $\mathrm{N}_{\nu_0}^{\pi_0}$ exists it is not necessary to use Lemma 6 in the proof of Theorem 4 . Because in this case Theorem 3 already implies

$f_{h^\circ} \in \mathrm{N}^{\pi_0}_{\nu_0}$.

5 Real rational functions

Using some ideas of the previous sections we will now complete the results of Section 2.

To start with, let f be any real rational function of degree k. Then for $n = k+1$ and any points $z_1, \ldots, z_n \in D_f$ the Pick matrix $\mathbb{P}_{z_1,\ldots,z_n}$ has signature (π_f, ν_f) and is singular. Furthermore, $f \in \mathrm{N}^{\pi_f}_{\nu_f}$. Thus we can consider f as the unique solution in the set $\mathrm{N}^{\pi_f}_{\nu_f}$ of the interpolation problem with data $(z_1, f(z_1)), \ldots, (z_n, f(z_n))$.

Consider the inner product space H assigned to this interpolation problem. By reversing the argument of Lemma 4 we get:

Lemma 7 *Let f be a real rational function. Then we have $\mathrm{H}^\circ \subseteq {\mathrm{H}_f}^\circ$.*

Proof : Let $h^\circ \in \mathrm{H}^\circ, h^\circ \neq 0$. By Theorem 3 we have $f(z) = f_{h^\circ}(z)$. An easy computation yields $[e_z, h^\circ] = 0$ for each $z \in D_f$, that is $h^\circ \in {\mathrm{H}_f}^\circ$.

□

Theorem 5 *Let f be a real meromorphic function in $\mathbf{C}$ and consider f restricted to the domain D_f. Then H_f degenerates if and only if f is a real rational function. In this case $dim\, \mathrm{H}_f/{\mathrm{H}_f}^\circ = deg\, f\ (< \infty)$.*

Proof : The first assertion follows from Lemma 4 together with Lemma 7 . To prove the second part of the theorem let $l = \dim \mathrm{H}_f/{\mathrm{H}_f}^\circ$ and $k = \deg f$. If we consider f as the unique solution of the above constructed interpolation problem we find $k = \operatorname{rank} \mathbb{P}$. Thus we find a k-dimensional nondegenerated subspace of H_f which implies $l \geq k$.

Assume that $l > k$. Linearly independent vectors of $\mathrm{H}_f/{\mathrm{H}_f}^\circ$ yield linearly independent vectors in H_f. The subspace they generate in H_f is nondegenerated. So we can find points $\zeta_1, \ldots, \zeta_m \in D_f$, such that $\operatorname{rank} \mathbb{P}_m \geq l > k$. But this is impossible as $\deg f = k$.

□

Because $\operatorname{rank} \mathbb{P} = \pi_f + \nu_f$ the above theorem shows that the inequality $\pi_f + \nu_f \leq \deg f$ given in Corollary 2 is in fact an equality. This immediately implies the following corollary.

Corollary 3 *Let* R_k *for* $k \in \mathbb{N}_0$ *be the set of real rational functions of degree* k*. Then we have*

$$\mathrm{R}_k = \bigcup_{\pi+\nu=k} \mathrm{N}_\nu^\pi \textit{ for } k \in \mathbb{N}.$$

Clearly the above union is a disjoint union.

Remark 3 Due to Corollary 3 a rational function f is an element of N_ν^π for some π and ν with $\pi+\nu = \deg f$. It is well known (see e.g. [15]) how the numbers π and ν are determined by f.

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