AN OPERATOR THEORETIC APPROACH TO DEGENERATED NEVANLINNA-PICK INTERPOLATION

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1 Introduction

In this paper we consider an interpolation problem of Nevanlinna-Pick type: If finitely many points $z_1, \dots, z_n$ of the open upper half plane $\mathbb{C}^+$ are given, we study the existence of functions $f \in \mathrm{N}_\nu^\pi$ assuming prescribed values $w_1, \dots, w_n \in \mathbb{C}$ in these points:

$$f(z_i) = w_i \text{ for } i = 1, \dots, n. \tag{1}$$

Furthermore a description of all solutions of (1) through selfadjoint extensions of a certain symmetric operator is given.

We recall the definition of the classes N_ν^π. If f is a complex function denote by $\rho(f)$ the domain of holomorphy of f.

Definition 1 *Let π and ν be nonnegative integers or ∞. Denote by N_ν^π the set of all functions f which are meromorphic in $\mathbb{C}^+$, such that the kernel*

$$\frac{f(z) - \overline{f(w)}}{z - \overline{w}} \text{ for } z, w \in \rho(f)$$

has π positive and ν negative squares, i.e., it satisfies

1. *For each $k \in \mathbb{N}$ and points $\zeta_1, \dots, \zeta_k \in \rho(f)$ the hermitian form*

$$\sum_{i,j=1}^{n} \frac{f(\zeta_i) - \overline{f(\zeta_j)}}{\zeta_i - \overline{\zeta_j}} \xi_i \overline{\xi_j} \tag{2}$$

   has at most π (ν, respectively) positive (negative, respectively) squares.

2. *If π (ν, respectively) is finite, then for some $k \in \mathbb{N}$ and points $\zeta_1, \dots, \zeta_k \in \rho(f)$ the hermitian form (2) has exactly π (ν, respectively) positive (negative, respectively) squares. If π (ν, respectively) equals ∞, then it is not*

possible to give a uniform bound for the number of positive (negative, respectively) squares of the hermitian forms (2).

We assign to the prescribed data the so-called Pick matrix

$$\mathbf{P} = \left(\frac{w_j - \overline{w_i}}{z_j - \overline{z_i}} \right)_{i,j=1}^{n} .$$

Denote by π_d (ν_d, respectively) the number of positive (negative, respectively) eigenvalues of $\mathbf{P}$, and let $\delta_d = n - (\pi_d + \nu_d)$ be its defect. Obviously it is impossible to find solutions f of (1) in some set N_ν^π with $\pi < \pi_d$ or $\nu < \nu_d$. In [12] it is shown that there are also no solutions in sets N_ν^π with $\pi_d < \pi < \pi_d + \delta_d$ or $\nu_d < \nu < \nu_d + \delta_d$. In case $\delta_d > 0$ there exists a unique solution in the set $\mathrm{N}_{\nu_d}^{\pi_d}$ if and only if the data points do not belong to a certain "small" exceptional set. In this note we prove that in each set N_ν^π with $\pi \geq \pi_d + \delta_d$ and $\nu \geq \nu_d + \delta_d$ there exist infinitely many solutions, and establish a one-to-one correspondence between solutions of (1) in N_ν^π and selfadjoint relations in certain Pontrjagin spaces. We always suppose, that at least one of the numbers π, ν is finite.

These results are well known in the case $\mathbf{P}$ does not degenerate (see e.g. [6], [10]). Therefore we can suppose that $\delta_d > 0$ holds, i.e. consider only the degenerated case $|\mathbf{P}| = 0$, although the developed method does not require any restrictions on the defect of $\mathbf{P}$.

In §2 we examine a connection between the sets N_ν^π, interpolation problems and inner product spaces. We assign to a complex function and especially to given interpolation data an inner product space and a symmetric operator. §3 proceeds with the definition of induced functions of an operator, and establishes the above mentioned correspondence (Theorem 1). In the remaining part of the paper we examine some more interpolation properties of the induced functions. §4 considers the case of semisimple eigenvalues in data points (Theorem 2), whereas §5 gives sufficient conditions for interpolation in the most general case (Theorem 3).

Besides the method used in this note there are many different approaches to interpolation problems of Nevanlinna-Pick type. Also there are various other generalizations of the classical Nevanlinna-Pick problem or of related questions. For an extensive bibliography on these subjects we refer to [2].

We use the notation and some results of the theory of Pontrjagin spaces and linear relations, which can be found in [1], [3], [4], [5] and [7].

2 Inner product spaces and interpolation

In this section we associate certain inner product spaces and symmetric operators with an interpolation problem. Denote in the following by $\mathbb{C}^0$ the set $\mathbb{C}^+ \cup \mathbb{C}^-$ of all nonreal complex numbers.

Definition 2 *Let $f : \mathcal{D}(f) \to \mathbb{C}$ with $\mathcal{D}(f) \subseteq \mathbb{C}^0$ be a complex function. Here $\mathcal{D}(f)$ is an arbitrary subset of $\mathbb{C}^0$. Denote by H_f the linear space of all formal sums*

$$\mathrm{H}_f = \{ \sum_{z \in \mathcal{D}(f)} x_z e_z | x_z \in \mathbb{C}, x_z = 0 \textit{ for all but finitely many } z \in \mathcal{D}(f)\},$$

equipped with the inner product defined by the relation

$$[e_z, e_w]_{\mathrm{H}_f} = \frac{f(z) - \overline{f(w)}}{z - \overline{w}} \textit{ for } z, w \in \mathcal{D}(f), z \neq \overline{w}.$$

If for some $z \in \mathcal{D}(f)$ also $\overline{z} \in \mathcal{D}(f)$, we assume that f is holomorphic at z and satisfies $f(\overline{z}) = \overline{f(z)}$ in a neighborhood of z. In this case we define

$$[e_z, e_{\overline{z}}] = f'(z).$$

If no confusion can occur the index H_f at the inner product will be dropped. Note that by definition the elements e_z for $z \in \mathcal{D}(f)$ are linearly independent.

If f is an extension of g, i.e. $\mathcal{D}(g) \subseteq \mathcal{D}(f)$ and $f|_{\mathcal{D}(g)} = g$, then H_g can be considered in a canonical way as subspace of H_f.

The index $\textsc{ind}(\mathrm{H})$ of an inner product space H is the pair (π, ν), where π (ν, respectively) equals the maximal dimension of a positive (negative, respectively) subspace of H. We agree that π (ν, respectively) is either a nonnegative integer or the symbol ∞.

Let f be a function which is meromorphic in $\mathbb{C}^+$. We extend f to a function $\hat{f}$, meromorphic in $\mathbb{C}^0$, by $\hat{f}(z) = \overline{f(\overline{z})}$ for $z \in \mathbb{C}^-$. Denote by $\rho(\hat{f})$ the domain of holomorphy of $\hat{f}$ in $\mathbb{C}^0$, then we will consider $\hat{f}$ as function with domain $\mathcal{D}\left(\hat{f}\right) = \rho(\hat{f})$. A function g with $\mathcal{D}(g) = \overline{\mathcal{D}(g)}$ which satisfies $g(\overline{z}) = \overline{g(z)}$ is called real.

Lemma 1 *Let f be a real meromorphic function in $\mathbb{C}^0$. Then $f \in \mathrm{N}_\nu^\pi$ if and only if $\textsc{ind}\left(\mathrm{H}_{\hat{f}}\right) = (\pi, \nu)$.*

The proof of Lemma 1 can be found e.g. in [10]. Note that if $\hat{f}$ is considered as function on $\rho(\hat{f}) \cap \mathbb{C}^+$, the above lemma just restates the definition of N_ν^π.

Thus the nontrivial assertion is, that, if f is extended to $\mathbb{C}^0$ (in the canonical way) the number of positive (negative, respectively) squares does not change. In the following we will always extend meromorphic functions on $\mathbb{C}^+$ to real meromorphic functions on $\mathbb{C}^0$.

Using a well known procedure we can produce a Pontrjagin space from H_f (see e.g. [8]): Let $f \in \mathrm{N}_\nu^\pi$, denote by $\mathrm{H}_f{}^\circ$ the isotropic part of H_f and consider the inner product space $\mathrm{H}_f/\mathrm{H}_f{}^\circ$. This space is nondegenerated and has index (π, ν). Thus completion gives a Pontrjagin space P_f, also with index (π, ν).

Remark 1 The above construction works in two steps:

$$\mathrm{H}_f \xrightarrow{p} \mathrm{H}_f/\mathrm{H}_f{}^\circ \xrightarrow{i} \mathrm{P}_f$$

with the canonical projection p and embedding i. Both mappings preserve the inner product and the notion of convergence (introduced as in [8]). Note that $\mathrm{H}_f/\mathrm{H}_f{}^\circ$ is topologically exactly the T_1-space associated to H_f.

Actually in the above construction either p or i is the identity mapping. This follows from the fact that $\mathrm{H}_f{}^\circ \neq \{0\}$ if and only if f is rational, and in this case $\pi + \nu = \mathrm{DEG}\,(f) < \infty$ (these results can be found e.g. in [12]).

With the given data we associate the function d defined on $\{z_1, \ldots, z_n\}$ by $d(z_i) = w_i$ for $i = 1, \ldots, n$, and consider the inner product space H_d. For further reference we state another lemma; for the proof we refer to [12].

Lemma 2 *Let f be a solution of (1) in N_ν^π with $\mathrm{H}_f{}^\circ \cap \mathrm{H}_d \neq \{0\}$. Then $\pi = \pi_d$ and $\nu = \nu_d$.*

With a function f we associate not only the inner product space H_f, but also a linear operator in this space.

Definition 3 *Let $f : \mathcal{D}(f) \to \mathbb{C}$ with $\mathcal{D}(f) \subseteq \mathbb{C}^0$ be a complex function. Denote by D_f the hyperplane*

$$D_f = \{ \sum_{z \in \mathcal{D}(f)} x_z e_z \in \mathrm{H}_f | \sum_{z \in \mathcal{D}(f)} x_z = 0 \},$$

and define an operator $\mathbf{S}_f$ with domain D_f as

$$\mathbf{S}_f(\sum_{z \in \mathcal{D}(f)} x_z e_z) = \sum_{z \in \mathcal{D}(f)} z x_z e_z.$$

Proposition 1 *The operator $\mathbf{S}_f$ is symmetric.*

Proof : Let $x, y \in D_f$, $x = \sum_{z\in\mathcal{D}(f)} x_z e_z$, $y = \sum_{w\in\mathcal{D}(f)} y_w e_w$. Then

$$[\mathbf{S}_f x, y] = \left[\sum_{z\in\mathcal{D}(f)} z x_z e_z, \sum_{w\in\mathcal{D}(f)} y_w e_w\right] = \sum_{z,w\in\mathcal{D}(f)} z x_z \overline{y_w}[e_z, e_w] =$$

$$= \sum_{z,w\in\mathcal{D}(f)} z x_z \overline{y_w} \frac{f(z) - \overline{f(w)}}{z - \overline{w}}.$$

In the same way we obtain

$$[x, \mathbf{S}_f y] = \sum_{z,w\in\mathcal{D}(f)} \overline{w} x_z \overline{y_w} \frac{f(z) - \overline{f(w)}}{z - \overline{w}}.$$

Subtraction gives

$$[\mathbf{S}_f x, y] - [x, \mathbf{S}_f y] = \sum_{z,w\in\mathcal{D}(f)} (z x_z \overline{y_w} - \overline{w} x_z \overline{y_w}) \frac{f(z) - \overline{f(w)}}{z - \overline{w}} =$$

$$= \sum_{z,w\in\mathcal{D}(f)} x_z \overline{y_w} (f(z) - \overline{f(w)}) = \sum_{z,w\in\mathcal{D}(f)} x_z \overline{y_w} f(z) - \sum_{z,w\in\mathcal{D}(f)} x_z \overline{y_w}\,\overline{f(w)} =$$

$$= \sum_{z\in\mathcal{D}(f)} \underbrace{\left(\sum_{w\in\mathcal{D}(f)} \overline{y_w}\right)}_{0} x_z f(z) - \sum_{w\in\mathcal{D}(f)} \underbrace{\left(\sum_{z\in\mathcal{D}(f)} x_z\right)}_{0} \overline{y_w}\,\overline{f(w)} = 0.$$

$\square$

We can restate the interpolation problem (1) as follows: which functions of N_ν^π extend d ? We will show that in some sense these extensions correspond to extensions of the operator $\mathbf{S}_d$. æ

3 Interpolation properties of induced functions

Definition 4 *Let* P *be a Pontrjagin space with* $\mathrm{H}_d \subseteq \mathrm{P}$. *Further let* $\mathbf{S}$ *be a selfadjoint relation in* P *with nonempty resolvent set extending* $\mathbf{S}_d$. *Denote by* $f_{\mathbf{S}}$ *the complex function*

$$f_{\mathbf{S}}(z) = \overline{w_1} + \frac{\Im w_1}{\Im z_1}(z - \overline{z_1}) + (z - \overline{z_1})(z - z_1)\left[(\mathbf{S} - z)^{-1} e_{z_1}, e_{z_1}\right]. \qquad (3)$$

We refer to $f_{\mathbf{S}}$ *as the induced function of* $\mathbf{S}$ *and consider* $f_{\mathbf{S}}$ *as function on its domain of holomorphy* $\rho(f_{\mathbf{S}})$ *in* $\mathbb{C}^0$.

By an elementary computation we obtain the representation

$$f_{\mathbf{S}}(z) = \Re w_1 + (z - \Re z_1)\frac{\Im w_1}{\Im z_1} + (z - z_1)(z - \overline{z_1})\left[(\mathbf{S} - z)^{-1} e_{z_1}, e_{z_1}\right]$$

for $f_{\mathbf{S}}$. From this it follows immediately that $f_{\mathbf{S}}$ is a real function.

Note that the resolvent set $\rho(\mathbf{S}) \cap \mathbb{C}^0$ is a dense subset of $\rho(f_{\mathbf{S}})$. Furthermore we have the following lemma.

Lemma 3 *Consider the inner product space* $\mathrm{H}_{f_{\mathbf{S}}}$ *and the subset*

$$D = \left\{ \sum_{z \in \rho(f_{\mathbf{S}})} x_z e_z \in \mathrm{H}_{f_{\mathbf{S}}} | x_z = 0 \textit{ for } z \notin \rho(\mathbf{S}) \right\}.$$

The linear manifold D *is dense in* $\mathrm{H}_{f_{\mathbf{S}}}$.

Proof : Let $z \in \rho(f_{\mathbf{S}}) \setminus \rho(\mathbf{S})$. Choose a sequence $(z_i)_{i \in \mathbb{N}}$, such that $z_i \in \rho(\mathbf{S})$ and $z_i \to z$. Then $e_{z_i} \in D$,

$$[e_{z_i}, e_w] = \frac{f(z_i) - \overline{f(w)}}{z_i - \overline{w}} \to \frac{f(z) - \overline{f(w)}}{z - \overline{w}} = [e_z, e_w] \text{ for } w \neq \overline{z},$$

$$[e_{z_i}, e_w] = \frac{f(z_i) - \overline{f(w)}}{z_i - \overline{w}} \to f'(z) = [e_z, e_w] \text{ for } w = \overline{z} \text{ and}$$

$$[e_{z_i}, e_{z_i}] = \frac{f(z_i) - \overline{f(z_i)}}{z_i - \overline{z_i}} \to \frac{f(z) - \overline{f(z)}}{z - \overline{z}} = [e_z, e_z].$$

Thus $e_{z_i} \to e_z$ which shows that $e_z \in \overline{D}$.

□

We define an operator $\mathbf{U}$ as

$$\mathbf{U} e_z = e_{z_1} + (z - z_1)\left(\mathbf{S} - z\right)^{-1} e_{z_1} \text{ for } z \in \rho(\mathbf{S}) \cap \mathbb{C}^0.$$

Proposition 2 *The operator* $\mathbf{U}$ *can be extended to an isometry from* $\mathrm{P}_{f_{\mathbf{S}}}$ *into* P.

Proof : We are going to check the relation

$$\frac{f_{\mathbf{S}}(z) - \overline{f_{\mathbf{S}}(w)}}{z - \overline{w}} = [\mathbf{U} e_z, \mathbf{U} e_w]_{\mathrm{P}}. \tag{4}$$

Once (4) is established the result follows, as $\mathbf{U}$ in first place is an isometric operator from D to P. As D is dense in $\mathrm{H}_{f_{\mathbf{S}}}$, we find $\mathbf{U}(\mathrm{H}_{f_{\mathbf{S}}}{}^{\circ}) \subseteq \mathrm{H}_{f_{\mathbf{S}}}{}^{\circ}$. Thus $\mathbf{U}$

yields an isometry from $\mathrm{P}_{f_{\mathbf{S}}}$ into P with domain $D/\mathrm{H}_{f_{\mathbf{S}}}{}^{\circ}$. Again as D is dense in $\mathrm{H}_{f_{\mathbf{S}}}$, $D/\mathrm{H}_{f_{\mathbf{S}}}{}^{\circ}$ is dense in $\mathrm{H}_{f_{\mathbf{S}}}/\mathrm{H}_{f_{\mathbf{S}}}{}^{\circ}$ and thus also in $\mathrm{P}_{f_{\mathbf{S}}}$. Now we can extend $\mathbf{U}$ by continuity to the whole of $\mathrm{P}_{f_{\mathbf{S}}}$ (see e.g. [11]).

It remains to verify (4). Note that we have

$$[\mathbf{U}e_z, \mathbf{U}e_w] = \left[\left(I + (z - z_1)(\mathbf{S} - z)^{-1}\right) e_{z_1}, \left(I + (w - z_1)(\mathbf{S} - w)^{-1}\right) e_{z_1}\right] =$$

$$= \left[\left(I + (\overline{w} - z_1)(\mathbf{S} - \overline{w})^{-1}\right)\left(I + (z - z_1)(\mathbf{S} - z)^{-1}\right) e_{z_1}, e_{z_1}\right].$$

The following formulas can be checked by a straightforward computation:

$$\left(I + (\overline{w} - z_1)(\mathbf{S} - \overline{w})^{-1}\right)\left(I + (z - z_1)(\mathbf{S} - z)^{-1}\right) =$$

$$= I + (z + \overline{w} - 2\Re z_1)(\mathbf{S} - z)^{-1} + (\overline{w} - \overline{z_1})(\overline{w} - z_1)(\mathbf{S} - z)^{-1}(\mathbf{S} - \overline{w})^{-1},$$

and

$$\frac{f_{\mathbf{S}}(z) - \overline{f_{\mathbf{S}}(w)}}{z - \overline{w}} = [e_{z_1} + (z + \overline{w} - 2\Re z_1)(\mathbf{S} - z)^{-1} e_{z_1} +$$
$$+ (\overline{w} - \overline{z_1})(\overline{w} - z_1)(\mathbf{S} - z)^{-1}(\mathbf{S} - \overline{w})^{-1} e_{z_1}, e_{z_1}].$$

From this (4) follows.

□

In the following definition we recall the notion of generating elements.

Definition 5 *Let* $\mathbf{A}$ *be a relation in a Pontrjagin space* P*, and let* $x \in \mathrm{P}$*. We call* x *a generating element for* $\mathbf{A}$ *if the relation*

$$\mathrm{P} = \overline{\langle x, (\mathbf{A} - z)^{-1} x | z \in \rho(\mathbf{A})\rangle}$$

holds ($\langle \ldots \rangle$ *denotes the linear span of a set of vectors).*

Proposition 3 *Let* $\mathbf{S}$ *be a selfadjoint relation in* P *extending* $\mathbf{S}_d$*. Suppose that* $z_1 \in \rho(\mathbf{S})$ *and that* e_{z_1} *is a generating element for* $\mathbf{S}$*. Then* $f_{\mathbf{S}} \in \mathrm{N}_{\nu}^{\pi}$*, where* (π, ν) *is the index of* P*.*

Proof : As $\mathbf{U} : \mathrm{P}_{f_{\mathbf{S}}} \to \mathrm{P}$ is an isometry, the index (π', ν') of $\mathrm{P}_{f_{\mathbf{S}}}$ is bounded by the index of P, i.e. $f_{\mathbf{S}} \in \mathrm{N}_{\nu'}^{\pi'}$ with $\pi' \le \pi$ and $\nu' \le \nu$.

From our assumptions we have

$$e_{z_1} = \mathbf{U}e_{z_1} \in \mathcal{R}(\mathbf{U}),$$

and for $z \in \rho(\mathbf{S}) \cap \mathbb{C}^0$:

$$(\mathbf{S} - z)^{-1} e_{z_1} = \frac{1}{z - z_1}(\mathbf{U}e_z - e_{z_1}) \in \mathcal{R}(\mathbf{U}).$$

Thus

$$\mathrm{P} = \overline{\langle e_{z_1}, (\mathbf{S}-z)^{-1} e_{z_1} | z \in \rho(\mathbf{S}) \cap \mathbb{C}^0 \rangle} \subseteq \overline{\mathcal{R}(\mathbf{U})} \subseteq \mathrm{P}$$

holds, which shows that $\mathcal{R}(\mathbf{U})$ is dense in P. Suppose e.g. π is finite. Then we find a π-dimensional positive subspace in $\mathcal{R}(\mathbf{U})$, and therefore $\pi' \geq \pi$. If ν is finite a similar argument shows $\nu' \geq \nu$. If, on the other hand, ν is infinite, then ν' must also be infinite, as no finite dimensional space can be dense in some infinite dimensional space.

□

Proposition 4 *Let* $\mathbf{S}$ *be a selfadjoint relation in* P *extending* $\mathbf{S}_d$*, and let* $i \in \{1, \ldots, n\}$*. If* $z_i \in \rho(\mathbf{S})$*, we have* $f_{\mathbf{S}}(z_i) = w_i$*.*

Proof : First consider the case $i = 1$. Then

$$f_{\mathbf{S}}(z_1) = \overline{w_1} + \frac{\Im w_1}{\Im z_1}(z_1 - \overline{z_1}) = w_1.$$

Let $i \neq 1$, then $e_{z_i} - e_{z_1} \in \mathcal{D}(\mathbf{S}_d)$ and

$$(\mathbf{S}_d - z_i)(e_{z_i} - e_{z_1}) = (z_i e_{z_i} - z_1 e_{z_1}) - (z_i e_{z_i} - z_i e_{z_1}) =$$

$$= (z_i - z_1) e_{z_1}.$$

Thus

$$(e_{z_i} - e_{z_1}, (z_i - z_1) e_{z_1}) \in \mathbf{S}_d - z_i \subseteq \mathbf{S} - z_i, \text{ and}$$
$$((z_i - z_1) e_{z_1}, e_{z_i} - e_{z_1}) \in (\mathbf{S} - z_i)^{-1}.$$

We compute

$$f_{\mathbf{S}}(z_i) = \overline{w_1} + \frac{\Im w_1}{\Im z_1}(z_i - \overline{z_1}) + (z_i - \overline{z_1})[e_{z_i} - e_{z_1}, e_{z_1}] =$$

$$= \overline{w_1} + \frac{\Im w_1}{\Im z_1}(z_i - \overline{z_1}) + (z_i - \overline{z_1}) \left(\frac{w_i - \overline{w_1}}{z_i - \overline{z_1}} - \frac{\Im w_1}{\Im z_1} \right) = w_i.$$

□

In the following we consider the eigenvalues of the selfadjoint extensions of $\mathbf{S}_d$. Denote by $\mathcal{K}(A)$ the kernel of the operator A.

Lemma 4 *Let* $\mathbf{S}$ *be a selfadjoint relation in* P *extending* $\mathbf{S}_d$*, and let* $\lambda \in \sigma_p(\mathbf{S}), \lambda \neq 0$*. Then for* $x \in \mathcal{K}(\mathbf{S}-\lambda)$ *and* $i, j = 1, \ldots, n$ *the relation*

$$(\lambda - \overline{z_i})[x, e_{z_i}] = (\lambda - \overline{z_j})[x, e_{z_j}] \tag{5}$$

holds. If in particular $\lambda = \overline{z_{i_0}}$ *for some* $i_0 \in \{1, \ldots, n\}$*, then*

$$[x, e_{z_j}] = 0 \text{ for } j \neq i_0.$$

Proof : As $x \in \mathcal{K}(\mathbf{S}-\lambda)$ we have $(x, \lambda x) \in \mathbf{S}$. Consider the element $e_{z_j} - e_{z_i}$ $(j \neq i)$. Then

$$(e_{z_j} - e_{z_i}, z_j e_{z_j} - z_i e_{z_i}) \in \mathbf{S}_d \subseteq \mathbf{S}.$$

As $\mathbf{S}$ is selfadjoint we find

$$\lambda[x, e_{z_j}] - \lambda[x, e_{z_i}] = [\lambda x, e_{z_j} - e_{z_i}] =$$

$$= [x, z_j e_{z_j} - z_i e_{z_i}] = \overline{z_j}[x, e_{z_j}] - \overline{z_i}[x, e_{z_i}],$$

which implies (5). The remaining assertion is obvious.

□

Lemma 5 *Let* $\mathbf{S}$ *be a selfadjoint relation in* P *with nonempty resolvent set extending* $\mathbf{S}_d$*, and suppose that* e_{z_1} *is a generating element for* $\mathbf{S}$*. Then* $z_j \in \rho(\mathbf{S})$ *for* $j \neq 1$*.*

Proof : It suffices to show that $\mathcal{K}(\mathbf{S} - \overline{z_j}) = \{0\}$ for $j = 2, \ldots, n$. If $x \in \mathcal{K}(\mathbf{S} - \overline{z_j})$ and $z \in \rho(\mathbf{S})$, then $(\mathbf{S} - \overline{z})^{-1} x \in \mathcal{K}(\mathbf{S} - \overline{z_j})$. The preceeding lemma yields

$$[(\mathbf{S} - \overline{z})^{-1} x, e_{z_1}] = 0.$$

As x is also orthogonal to e_{z_1} and e_{z_1} is a generating element for $\mathbf{S}$, we find $x \perp \mathrm{P}$, which implies $x = 0$.

□

Denote with $\mathcal{S}_d(\pi, \nu)$ the class of all selfadjoint relations which operate in a Pontrjagin space with index (π, ν), have e_{z_1} as generating element, extend $\mathbf{S}_d$and satisfy $z_1 \in \rho(\mathbf{S})$. Combining the above results we obtain the following statement.

Proposition 5 *The assignment* $\mathbf{S} \to f_{\mathbf{S}}$ *defines a mapping* Q *from* $\mathcal{S}_d(\pi, \nu)$ *into the set of all solutions of (1) in* N_ν^π*.*

Remark 2 Let (π_d, ν_d) be the signature of the Pick matrix $\mathbf{P}$ and let δ_d be its defect. Then $\textsc{ind}\,(\mathrm{H}_d) = (\pi_d, \nu_d)$ and $\textsc{dim}\,(\mathrm{H}_d{}^\circ) = \delta_d$. As P is a Pontrjagin space we conclude $\pi \geq \pi_d + \delta_d$ and $\nu \geq \nu_d + \delta_d$.

To show that every solution can be obtained in this way we proceed from a solution f of (1) in N_ν^π with $\pi \geq \pi_d + \delta_d$ and $\nu \geq \nu_d + \delta_d$, and construct a relation $\mathbf{S} \in \mathcal{S}_d(\pi, \nu)$ with $f = f_{\mathbf{S}}$. To do so consider the operator $\mathbf{S}_f$, which extends $\mathbf{S}_d$, as f extends d. As $\mathbf{S}_f$ is symmetric, it yields a symmetric relation in the Pontrjagin space P_f. Denote by $\mathbf{S}$ its closure.

Lemma 2 shows that H_d is embedded in P_f and therefore $\mathbf{S}$ is an extension of $\mathbf{S}_d$.

Lemma 6 *The relation $\mathbf{S}$ is selfadjoint. Furthermore we have $\rho(f) \subseteq \rho(\mathbf{S})$.*

Proof : We consider the range of $(\mathbf{S}_f - z)$ for $z \in \rho(f)$. Since

$$(\mathbf{S}_f - z)\left(\frac{e_w - e_z}{w - z}\right) = e_w \text{ for } w \neq z$$

we find that $\mathcal{R}\,(\mathbf{S}_f - z)$ is dense in H_f, and therefore that $\mathcal{R}\,(\mathbf{S} - z)$ is dense in P_f. Since $\mathbf{S}$ is a symmetric closed relation, the range of $\mathbf{S} - z$ is closed for all $z \in \mathbb{C}^0$. Thus $\mathcal{R}\,(\mathbf{S} - z) = \mathrm{P}_f$. From this we find

$$\mathcal{K}\,(\mathbf{S} - z) \subseteq \mathcal{R}\,(\mathbf{S} - \overline{z})^\perp = \{0\} \text{ for } z \in \mathbb{C}^0,$$

Thus $\mathbf{S}$ is selfadjoint and $\rho(\mathbf{S}) \cap \mathbb{C}^0 = \mathbb{C}^0 \setminus \sigma_p(\mathbf{S})$. As $\mathcal{K}\,(\mathbf{S} - z) = \{0\}$ for $z \in \rho(f)$ we find $\rho(f) \subseteq \rho(\mathbf{S})$.

□

Proposition 6 *Let $f \in \mathrm{N}_\nu^\pi$ be a solution of the interpolation problem (1). Then we have $f = f_{\mathbf{S}}$ for the selfadjoint relation $\mathbf{S}$ constructed above. Furthermore $\mathbf{S} \in \mathcal{S}_d(\pi, \nu)$.*

Proof : We first show that $\mathbf{S} \in \mathcal{S}_d(\pi, \nu)$. As $z_1 \in \rho(f)$ we find $z_1 \in \rho(\mathbf{S})$. Let $z \neq z_1$, then

$$(\mathbf{S} - z)^{-1} e_{z_1} = \frac{e_{z_1} - e_z}{z_1 - z},$$

which implies

$$e_z \in \left\langle e_{z_1}, (\mathbf{S} - z)^{-1} e_{z_1} | z \in \rho(\mathbf{S}) \right\rangle .$$

Thus

$$\mathrm{P}_f = \overline{\left\langle e_{z_1}, (\mathbf{S}-z)^{-1} e_{z_1} | z \in \rho(\mathbf{S}) \right\rangle},$$

which shows that e_{z_1} is an generating element for $\mathbf{S}$.

Let $z \in \rho(f), z \neq z_1$, then

$$f_{\mathbf{S}}(z) = \overline{w_1} + \frac{\Im w_1}{\Im z_1}(z - \overline{z_1}) + (z - \overline{z_1})[\frac{e_{z_1} - e_z}{z_1 - z}, e_{z_1}] =$$

$$= \overline{w_1} + \frac{\Im w_1}{\Im z_1}(z - \overline{z_1}) + (z - \overline{z_1})(\frac{f(z) - \overline{f(z_1)}}{z - \overline{z_1}} - \frac{\Im w_1}{\Im z_1}) = f(z).$$

□

Remark 3 As $\rho(f)$ is the domain of holomorphy of f, f cannot be extended. Thus $f_{\mathbf{S}}(z) = f(z)$ for $z \in \rho(f)$ implies $\rho(\mathbf{S}) \cap \mathbb{C}^0 \subseteq \rho(f)$. Together with Lemma 6 this shows $\rho(\mathbf{S}) \cap \mathbb{C}^0 = \rho(f) = \rho(f_{\mathbf{S}})$. Furthermore $|\mathbb{C}^{\pm} \setminus \rho(f)| \leq \min(\pi, \nu)$.

Due to Proposition 6 the mapping Q is surjective. We proceed determining the kernel of Q.

Definition 6 *Let* P_1 *and* P_2 *be Pontrjagin spaces extending* H_d*, and let* $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *be selfadjoint relations with nonempty resolvent set in* P_1 *and* P_2 *respectively. We call* $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *unitary equivalent with respect to* H_d*, if there exists a unitary operator* $\mathbf{U} : \mathrm{P}_1 \to \mathrm{P}_2$ *which leaves* H_d *pointwise invariant and satisfies*

$$\mathbf{U} \circ (\mathbf{S}_1 - z)^{-1} = (\mathbf{S}_2 - z)^{-1} \circ \mathbf{U} \text{ for } z \in \rho(\mathbf{S}_1) \cap \rho(\mathbf{S}_2). \tag{6}$$

Lemma 7 *Let* $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *be relations in* P_1 *and* P_2 *respectively, which are unitary equivalent with respect to* H_d*. Then the functions* $f_{\mathbf{S}_1}$ *and* $f_{\mathbf{S}_2}$ *coincide.*

Proof : Let $z \in \rho(\mathbf{S}_1) \cap \rho(\mathbf{S}_2)$, then we have (6), which shows

$$[(\mathbf{S}_1 - z)^{-1} e_{z_1}, e_{z_1}] = [\mathbf{U}^{-1} \circ (\mathbf{S}_2 - z)^{-1} \circ \mathbf{U} e_{z_1}, e_{z_1}] =$$

$$= [(\mathbf{S}_2 - z)^{-1} \mathbf{U} e_{z_1}, \mathbf{U} e_{z_1}] = [(\mathbf{S}_2 - z)^{-1} e_{z_1}, e_{z_1}],$$

and therefore $f_{\mathbf{S}_1}(z) = f_{\mathbf{S}_2}(z)$. Since $\rho(\mathbf{S}_1) \cap \rho(\mathbf{S}_2)$ is dense in $\rho(f_{\mathbf{S}_1})$ and $\rho(f_{\mathbf{S}_2})$ we find $f_{\mathbf{S}_1} = f_{\mathbf{S}_2}$.

□

A certain converse to the above lemma is given by

Lemma 8 *Let* $\mathbf{S}$ *be a selfadjoint relation of class* $\mathcal{S}_d(\pi,\nu)$. *Denote by* $\mathbf{S}_{f_\mathbf{S}}$ *the selfadjoint relation in the Pontrjagin space* $\mathrm{P}_{f_\mathbf{S}}$, *which has been constructed for the proof of Proposition 6, starting from the function* $f_\mathbf{S}$. *Then* $\mathbf{S}$ *and* $\mathbf{S}_{f_\mathbf{S}}$ *are unitary equivalent with respect to* H_d.

Proof : Consider the isometric operator $\mathbf{U}$ defined in Proposition 2. We have shown in Proposition 3 that under the assumptions of the lemma $\mathcal{R}(\mathbf{U})$ is dense in P. Thus $\mathbf{U}$ is actually unitary. It remains to prove (6).

Consider the vector $e_z \in \mathrm{P}_{f_\mathbf{S}}$ for some $z \in \rho(\mathbf{S}) \cap \rho(\mathbf{S}_{f_\mathbf{S}})$, then

$$\mathbf{U}e_z = e_{z_1} + (z - z_1)(\mathbf{S} - z)^{-1} e_{z_1},$$

where the right hand side is an element of P. We compute for $w \in \rho(\mathbf{S}) \cap \rho(\mathbf{S}_{f_\mathbf{S}})$:

$$(\mathbf{S} - w)^{-1} \circ \mathbf{U}e_z = (\mathbf{S} - w)^{-1} e_{z_1} + (z - z_1)(\mathbf{S} - w)^{-1}(\mathbf{S} - z)^{-1} e_{z_1} =$$
$$= (\mathbf{S} - w)^{-1} e_{z_1} + \frac{z - z_1}{z - w}\left[(\mathbf{S} - z)^{-1} e_{z_1} - (\mathbf{S} - w)^{-1} e_{z_1}\right] =$$
$$= \frac{z - z_1}{z - w}(\mathbf{S} - z)^{-1} e_{z_1} - \frac{w - z_1}{z - w}(\mathbf{S} - w)^{-1} e_{z_1}.$$

On the other hand we have

$$\left(\frac{e_z - e_w}{z - w}, e_z\right) \in \mathbf{S}_{f_\mathbf{S}} - w,$$

which implies

$$(\mathbf{S}_{f_\mathbf{S}} - w)^{-1} e_z = \frac{e_z - e_w}{z - w}.$$

Thus

$$\mathbf{U} \circ (\mathbf{S}_{f_\mathbf{S}} - w)^{-1} e_z = \mathbf{U}\left(\frac{e_z - e_w}{z - w}\right) =$$
$$= \frac{1}{z - w}\left[(z - z_1)(\mathbf{S} - z)^{-1} e_{z_1} - (w - z_1)(\mathbf{S} - w)^{-1} e_{z_1}\right]$$

and (6) holds.

□

Let $\pi_d + \delta_d \leq \pi$ and $\nu_d + \delta_d \leq \nu$. Then we obtain by combining Proposition 5 and Proposition 6 with Lemma 7 and Lemma 8 the following result.

Theorem 1 *The mapping* Q *which assigns to each selfadjoint relation* $\mathbf{S}$ *its induced function* $f_\mathbf{S}$ *establishes a bijective correspondence between the set of all solutions of (1) in* N_ν^π *and the class* $\mathcal{S}_d(\pi,\nu)$ *modulo unitary equivalence with respect to* H_d.

Remark 4 Since unitary equivalent relations have the same spectrum and $\rho(\mathbf{S}_{f_\mathbf{S}}) \cap \mathbb{C}^0 = \rho(f_\mathbf{S})$, we have

$$\rho(\mathbf{S}) \cap \mathbb{C}^0 = \rho(f_\mathbf{S}) \text{ for each relation } \mathbf{S} \in \mathcal{S}_d(\pi, \nu).$$

4 Semisimple eigenvalues in data points

We consider in this paragraph the case that for a selfadjoint extension $\mathbf{S}$ of $\mathbf{S}_d$ (in an Pontrjagin space P) with nonempty resolvent set the point z_j for some $j \in \{1, \dots, n\}$ is a semisimple eigenvalue, i.e., that

$$\mathcal{K}\left(\mathbf{S} - z_j\right)^2 = \mathcal{K}\left(\mathbf{S} - z_j\right) \neq \{0\}.$$

Denote the kernel $\mathcal{K}\left(\mathbf{S} - z_j\right)$ with K^+ and $\mathcal{K}\left(\mathbf{S} - \overline{z_j}\right)$ with K^-. The subspaces K^+ and K^- are neutral, skewly linked and finite dimensional. Thus P decomposes as

$$\mathrm{P} = K^\perp [\dot{+}] \left(K^+ \dot{+} K^-\right) \tag{7}$$

where $K^\perp$ denotes the orthogonal complement of $K = K^+ \dot{+} K^-$. According to this decomposition we write vectors $x \in \mathrm{P}$ as $x = x^\perp + x^+ + x^-$.

The selfadjoint relation $\mathbf{S}$ also decomposes with respect to (7).

Proposition 7 *Denote with* $\mathbf{S}'$ *the relation* $\mathbf{S}' = \mathbf{S} \cap \left(K^\perp\right)^2 = \mathbf{S}|_{K^\perp}$ *of* $K^\perp$. *Then we have*

$$\mathbf{S} = \mathbf{S}' \, (\dot{+}) \, z_j I|_{K^+} \, (\dot{+}) \, \overline{z_j} I|_{K^-}. \tag{8}$$

The relation $\mathbf{S}'$ *is selfadjoint in the Pontrjagin space* $K^\perp$.

Proof : Obviously $\mathbf{S}', z_j I|_{K^+}, \overline{z_j} I|_{K^-} \subseteq \mathbf{S}$. Thus the inclusion "$\supseteq$" of (8) holds. To establish the converse inclusion let $(x, y) \in \mathbf{S}$, $x = x^\perp + x^+ + x^-$. Then $(x^+, z_j x^+)$ and $(x^-, \overline{z_j} x^-)$ are elements of $\mathbf{S}$. Consider $u = y - z_j x^+ - \overline{z_j} x^-$, then $(x^\perp, u) \in \mathbf{S}$. As $\mathbf{S}$ is selfadjoint we have for $v \in \mathcal{K}\left(\mathbf{S} - z_j\right)$

$$[u, v] = [x^\perp, z_j v] = 0,$$

and for $v \in \mathcal{K}\left(\mathbf{S} - \overline{z_j}\right)$

$$[u, v] = [x^\perp, \overline{z_j} v] = 0.$$

Thus $u \in K^{\perp}$, and

$$(x,y) = (x^{\perp}, u) + (x^{+}, z_j x^{+}) + (x^{-}, \overline{z_j} x^{-}).$$

As the domains (ranges, respectively) of the relations on the right hand side of (8) are disjoint subspaces the sum in (8) is direct.

Obviously the restriction of $\mathbf{S}$ to $K^{\perp}$ is closed, and the inclusion $\mathbf{S}' \subseteq \mathbf{S}'^{*}$ holds, as $\mathbf{S}$ is selfadjoint. To prove the converse inclusion let $(u,v) \in \mathbf{S}'^{*}$. Then we have $[u,y] = [v,x]$ for $(x,y) \in \mathbf{S}'$ by definition, and for $(x, z_j x) \in I|_{K^+}$ we find $[u, z_j x] = 0 = [v,x]$, as $u, v \in K^{\perp}$. Similar $[u, \overline{z_j} x] = 0 = [v,x]$ holds for $(x, \overline{z_j} x) \in I|_{K^-}$. As (8) holds, we conclude $(u,v) \in \mathbf{S}^* = \mathbf{S}$. Furthermore $u, v \in K^{\perp}$ implies $(u,v) \in \mathbf{S}'$.

□

To prove the main result of this paragraph we have to do some computations, which are given in the following lemmata.

Lemma 9 *Using the above notation we have for $i \in \{1, \ldots, n\}$:*

$$e_{z_i}^{-} = \frac{z_j - \overline{z_j}}{z_i - \overline{z_j}} e_{z_j}^{-},$$

and for $i \neq j$

$$e_{z_i}^{+} = 0.$$

Proof : For $i = j$ the assertion is trivial, so assume $i \neq j$. Then

$$(e_{z_i} - e_{z_j}, (z_i - \overline{z_j}) e_{z_i} - (z_j - \overline{z_j}) e_{z_j}) \in \mathbf{S}_d - \overline{z_j} \subseteq \mathbf{S} - \overline{z_j}.$$

Comparing components with respect to (7) gives

$$({e_{z_i}}^{\perp} - {e_{z_j}}^{\perp}, (z_i - \overline{z_j}) {e_{z_i}}^{\perp} - (z_j - \overline{z_j}) {e_{z_j}}^{\perp}) \in \mathbf{S}' - \overline{z_j},$$

$$(z_j - \overline{z_j}) e_{z_i}^{+} - (z_j - \overline{z_j}) e_{z_j}^{+} = (z_i - \overline{z_j}) e_{z_i}^{+} - (z_j - \overline{z_j}) e_{z_j}^{+}, \tag{9}$$

$$0 = (z_i - \overline{z_j}) e_{z_i}^{-} - (z_j - \overline{z_j}) e_{z_j}^{-}. \tag{10}$$

From (9) and (10) the assertion of the lemma follows.

□

Lemma 10 *Let $z \in \rho(\mathbf{S})$. Then*

$$[(\mathbf{S} - z)^{-1} e_{z_1}, e_{z_1}] = [(\mathbf{S}' - z)^{-1} {e_{z_1}}^{\perp}, {e_{z_1}}^{\perp}] +$$

$$+ \frac{1}{z_j - z} [e_{z_1}^{+}, e_{z_1}^{-}] + \frac{1}{\overline{z_j} - z} [e_{z_1}^{-}, e_{z_1}^{+}]. \tag{11}$$

Proof : As K^+ and K^- are invariant under $(\mathbf{S}-z)^{-1}$, so is $K^\perp$. Thus $(\mathbf{S}'-z)^{-1}$ is an operator defined on the whole of $K^\perp$. Furthermore

$$(\mathbf{S}-z)^{-1}\, {e_{z_1}}^\perp = (\mathbf{S}'-z)^{-1}\, {e_{z_1}}^\perp.$$

We compute

$$[(\mathbf{S}-z)^{-1}\,({e_{z_1}}^\perp + e_{z_1}^+ + e_{z_1}^-), ({e_{z_1}}^\perp + e_{z_1}^+ + e_{z_1}^-)] =$$

$$= [(\mathbf{S}-z)^{-1}\, {e_{z_1}}^\perp, {e_{z_1}}^\perp] + [(\mathbf{S}-z)^{-1}\, e_{z_1}^+, e_{z_1}^-] + [(\mathbf{S}-z)^{-1}\, e_{z_1}^-, e_{z_1}^+].$$

As $(e_{z_1}^+, (z_j - z)e_{z_1}^+) \in \mathbf{S}-z$ and $(e_{z_1}^-, (\overline{z_j} - z)e_{z_1}^-) \in \mathbf{S}-z$, we find

$$(\mathbf{S}-z)^{-1}\, e_{z_1}^+ = \frac{1}{z_j - z} e_{z_1}^+ \text{ and } (\mathbf{S}-z)^{-1}\, e_{z_1}^- = \frac{1}{\overline{z_j} - z} e_{z_1}^-$$

which proves (11).

□

Theorem 2 *Let $j \in \{1, \ldots, n\}$ and let $\mathbf{S}$ be a selfadjoint relation extending $\mathbf{S}_d$ and satisfying*

$$\mathcal{K}\,(\mathbf{S}-z_j)^2 = \mathcal{K}\,(\mathbf{S}-z_j) \neq \{0\}.$$

Then $f_{\mathbf{S}}(z_j) = w_j$ if and only if $[e_{z_j}^+, e_{z_j}^-]$ vanishes.

Proof : Consider first the case $j \neq 1$. Then

$$(e_{z_j} - e_{z_1}, (z_j - z_1)e_{z_1}) \in \mathbf{S} - z_j,$$

which implies

$$\left(\frac{{e_{z_j}}^\perp - {e_{z_1}}^\perp}{z_j - z_1}, {e_{z_1}}^\perp\right) \in \mathbf{S}' - z_j.$$

As $\mathcal{K}\,(\mathbf{S}'-z_j) = \{0\}$ we have $z_j \in \rho(\mathbf{S}')$. Thus

$$(\mathbf{S}'-z)^{-1}\, {e_{z_1}}^\perp = \frac{{e_{z_j}}^\perp - {e_{z_1}}^\perp}{z_j - z_1},$$

and, with regard to $e_{z_1}^+ = 0$ and (11):

$$\lim_{z\to z_j}(z-z_1)[(\mathbf{S}-z)^{-1}\, e_{z_1}, e_{z_1}] = \lim_{z\to z_j}(z-z_1)[(\mathbf{S}'-z)^{-1}\, {e_{z_1}}^\perp, {e_{z_1}}^\perp] =$$

$$= (z_j - z_1)[(\mathbf{S}'-z_j)^{-1}\, {e_{z_1}}^\perp, {e_{z_1}}^\perp] = [{e_{z_j}}^\perp - {e_{z_1}}^\perp, {e_{z_1}}^\perp] =$$

$$= [e_{z_j} - e_{z_1}, e_{z_1}] - [e^+_{z_j}, e^-_{z_1}]. \tag{12}$$

In the proof of Proposition 4 we have seen that $f_{\mathbf{S}}(z_j) = w_j$ is equivalent to

$$\lim_{z \to z_j} (z - z_1)[(\mathbf{S} - z)^{-1} e_{z_1}, e_{z_1}] = [e_{z_j} - e_{z_1}, e_{z_1}].$$

Due to (12) this is the case if and only if $[e^+_{z_j}, e^-_{z_1}] = 0$, which is equivalent to $[e^+_{z_j}, e^-_{z_j}] = 0$, as $e^-_{z_1}$ and $e^-_{z_j}$ are linearly dependend.

In the case $j = 1$ the interpolation condition $f_{\mathbf{S}}(z_1) = w_1$ is satisfied if and only if the expression $[(\mathbf{S} - z)^{-1} e_{z_1}, e_{z_1}]$ is bounded at z_1. As $z_1 \in \rho(\mathbf{S}')$ in this case this is, in view of (11), equivalent to $[e^+_{z_1}, e^-_{z_1}] = 0$.

$\square$

5 Interpolation through arbitrary selfadjoint extensions

Let $\mathbf{S}$ be a selfadjoint extension of $\mathbf{S}_d$ in a Pontrjagin space with index (π, ν). Then the first part of the proof of Proposition 3 implies that $f_{\mathbf{S}} \in \mathrm{N}^{\pi'}_{\nu'}$ for some $\pi' \leq \pi$ and $\nu' \leq \nu$. From Proposition 4 it follows that, if a data point z_j is in the resolvent set of $\mathbf{S}$, the function $f_{\mathbf{S}}$ satisfies $f_{\mathbf{S}}(z_j) = w_j$.

In this section we consider an arbitrary selfadjoint extension $\mathbf{S}$ of $\mathbf{S}_d$ and give a weaker condition, which is also sufficient for $f_{\mathbf{S}}(z_j) = w_j$.

We start with two lemmata about restrictions and factorizations of relations.

Lemma 11 *Let $\mathbf{A}$ be a relation in a Pontrjagin space P, let L be a (closed) subspace of P and consider the relation $\mathbf{A}' = \mathbf{A}|_{\mathrm{L}}$. If $z \in \rho(\mathbf{A})$ and L is invariant under $(A - z)^{-1}$, then $z \in \rho(\mathbf{A}')$ and*

$$(\mathbf{A} - z)^{-1}|_{\mathrm{L}} = (\mathbf{A}' - z)^{-1}. \tag{13}$$

Proof : Obviously $\mathcal{K}(\mathbf{A}' - z) = \{0\}$. Let $x \in \mathrm{L}$ and consider $y = (\mathbf{A} - z)^{-1} x$, then $y \in \mathrm{L}$. Thus $(y, x) \in (\mathbf{A} - z) \cap \mathrm{L}^2 = (\mathbf{A}' - z)$ which implies $(\mathbf{A}' - z)^{-1} x = y$. This proves (13) and shows that $(\mathbf{A}' - z)^{-1}$ is defined on the whole of L. As $(\mathbf{A} - z)^{-1}$ is bounded, so is $(\mathbf{A}' - z)^{-1}$ and thus $z \in \rho(\mathbf{A}')$.

$\square$

Note that the subspace L may be degenerated. We define a relation $\mathbf{A}''$ in $\mathrm{L}/\mathrm{L}^\circ$ as

$$(x + \mathrm{L}^\circ, y + \mathrm{L}^\circ) \in \mathbf{A}'' \iff (x, y) \in \mathbf{A}'.$$

Lemma 12 *In the situation of Lemma 11 suppose that* L° *is invariant under* $(\mathbf{A}' - z)^{-1}$. *Then* $z \in \rho(\mathbf{A}'')$ *and*

$$(\mathbf{A}'' - z)^{-1}(x + \mathrm{L}^\circ) = (\mathbf{A}' - z)^{-1} x + \mathrm{L}^\circ.$$

Proof : Let $x + \mathrm{L}^\circ \in \mathcal{K}(\mathbf{A}'' - z)$, i.e. $(x + \mathrm{L}^\circ, \mathrm{L}^\circ) \in (\mathbf{A}'' - z)$. Then there exist $x', y' \in \mathrm{L}$ such that $x' - x \in \mathrm{L}^\circ, y' \in \mathrm{L}^\circ$ and $(x', y') \in (\mathbf{A}' - z)$. Thus $x' = (\mathbf{A}' - z)^{-1} y'$ which implies $x' \in \mathrm{L}^\circ$ and consequently $x + \mathrm{L}^\circ = \mathrm{L}^\circ$, hence $\mathcal{K}(\mathbf{A}'' - z) = \{0\}$.

Let $x \in \mathrm{L}$ and consider $y \in (\mathbf{A}' - z)^{-1} x$. Then $(y, x) \in (\mathbf{A}' - z)$ and therefore $(y + \mathrm{L}^\circ, x + \mathrm{L}^\circ) \in (\mathbf{A}'' - z)$. This implies $y + \mathrm{L}^\circ = (\mathbf{A}'' - z)^{-1}(x + \mathrm{L}^\circ)$ and shows that $(\mathbf{A}'' - z)^{-1}$ is everywhere defined.

It remains to show that $(A'' - z)^{-1}$ is continous. Consider the decomposition

$$\mathrm{P} = \mathrm{L}_1[\dot{+}](\mathrm{L}^\circ \dot{+} \hat{\mathrm{L}^\circ})[\dot{+}]\mathrm{L}_2$$

with $\mathrm{L} = \mathrm{L}_1[\dot{+}]\mathrm{L}^\circ$. Let $\hat{x}_n \in \mathrm{L}/\mathrm{L}^\circ$ be a sequence which converges to $\hat{x}$. We may assume that the representators x_n and x of $\hat{x}_n$ and $\hat{x}$ are elements of L_1. As L_1 is orthocomplemented the sequence x_n converges to x in the topology of P. Thus $(A - z)^{-1} x_n \to (A - z)^{-1} x$, which shows that also $(A'' - z)^{-1} \hat{x}_n \to (A'' - z)^{-1} \hat{x}$.

□

We now apply the above lemmata to the following situation. Let $\mathbf{S}$ be a selfadjoint extension of $\mathbf{S}_d$ in a Pontrjagin space P with nonempty resolvent set. Choose the subspace L as

$$\mathrm{L} = \overline{\langle e_{z_i}, (\mathbf{S} - w)^{-1} e_{z_i} | w \in \rho(\mathbf{S}), i = 1, \ldots, n \rangle}. \tag{14}$$

Lemma 13 *The subspaces* L *and* L° *are invariant under the resolvents* $(\mathbf{S} - z)^{-1}$, $z \in \rho(\mathbf{S})$.

Proof : Let $i \in \{1, \ldots, n\}$. Then for $z, w \in \rho(\mathbf{S}), z \neq w$ we have $(\mathbf{S} - z)^{-1} e_{z_i} \in \mathrm{L}$ and

$$(\mathbf{S} - z)^{-1}(\mathbf{S} - w)^{-1} e_{z_i} = (z - w)\left((\mathbf{S} - z)^{-1} e_{z_i} - (\mathbf{S} - w)^{-1} e_{z_i}\right) \in \mathrm{L}.$$

As $(\mathbf{S} - z)^{-1}$ depends continously on z also

$$(\mathbf{S} - w)^{-1}(\mathbf{S} - w)^{-1} e_{z_i} \in \mathrm{L}.$$

As $\mathbf{S}$ is selfadjoint this also implies that L° is invariant under each resolvent.

□

In the sequel we use the notation developed above. The next lemma provides the main tool for proving the below theorem.

Lemma 14 *Let* $e_{z_j} \in \mathcal{R}(\mathbf{S}' - z_j)$. *Then* $\mathcal{R}(\mathbf{S}' - z_j) = \mathrm{L}$.

Proof : As $\mathbf{S}'$ is closed it suffices to show that $\mathcal{R}(\mathbf{S}' - z_j)$ is dense in L. As $\mathcal{R}(\mathbf{S}' - z_j)$ is invariant under each resolvent $(\mathbf{S}' - z)^{-1}$ it suffices to show $e_{z_i} \in \mathcal{R}(\mathbf{S}' - z_j)$ for $i = 1, \ldots, n$. Due to the assumption of the lemma we may assume $i \neq j$. Then $e_{z_i} - e_{z_j} \in \mathrm{L}$ and

$$(e_{z_i} - e_{z_j}, (z_i - z_j)e_{z_i}) \in (\mathbf{S}' - z_j).$$

$\square$

Theorem 3 *Let* $\mathbf{S}$ *be a selfadjoint relation with nonempty resolvent set extending* $\mathbf{S}_d$ *and operating in a Pontrjagin space* P *with index* (π, ν). *Suppose*

$$e_{z_j} \in \mathcal{R}(\mathbf{S} - z_j)\,|_{\mathrm{L}} \text{ for } j = 1, \ldots, n,$$

where the subspace L *is defined as in (14). Then the induced function* $f_{\mathbf{S}}$ *is a solution of the interpolation problem (1) in some set* $\mathrm{N}_{\nu'}^{\pi'}$ *with* $\pi' \leq \pi$ *and* $\nu' \leq \nu$.

Proof : The relation $\mathbf{S}''$ is symmetric and closed. Furthermore $\rho(\mathbf{S}) \subseteq \rho(\mathbf{S}'')$ which implies $\rho(\mathbf{S}'') \cap \mathbb{C}^{\pm} \neq \emptyset$. Thus $\mathrm{DIM}\,(\mathcal{K}(\mathbf{S}'' - z_j)) - \mathrm{DIM}\left(\mathcal{R}(\mathbf{S}'' - z_j)^{\perp}\right) = 0$ on $\mathbb{C}^0$. As $\mathcal{R}(\mathbf{S}'' - z_j) = \mathrm{L}/\mathrm{L}^{\circ}$ we must also have $\mathcal{K}(\mathbf{S}'' - z_j) = \{0\}$ and therefore $z_j \in \rho(\mathbf{S}'')$.

Let $j \neq 1$, then we have

$$(e_{z_j} - e_{z_1} + \mathrm{L}^{\circ}, (z_j - z_1)e_{z_1} + \mathrm{L}^{\circ}) \in (\mathbf{S}'' - z_j),$$

and thus

$$\lim_{z \to z_j} (z - z_1)[(\mathbf{S} - z)^{-1} e_{z_1}, e_{z_1}] = \lim_{z \to z_j} (z - z_1)[(\mathbf{S}'' - z)^{-1} e_{z_1} + \mathrm{L}^{\circ}, e_{z_1} + \mathrm{L}^{\circ}] =$$

$$= (z_j - z_1)[(\mathbf{S}'' - z_j)^{-1} e_{z_1} + \mathrm{L}^{\circ}, e_{z_1} + \mathrm{L}^{\circ}] =$$
$$= (z_j - z_1)[e_{z_j} - e_{z_1} + \mathrm{L}^{\circ}, e_{z_1} + \mathrm{L}^{\circ}],$$

which implies $\lim_{z \to z_j} f_{\mathbf{S}}(z) = w_j$.

Consider now the case $j = 1$. Then $z_1 \in \rho(\mathbf{S}'')$ implies that $[(\mathbf{S}'' - z)^{-1} e_{z_1} + \mathrm{L}^{\circ}, e_{z_1} + \mathrm{L}^{\circ}]$ is bounded at z_1. Thus $\lim_{z \to z_1} (z - z_1)[(\mathbf{S} - z)^{-1} e_{z_1}, e_{z_1}] = 0$ and $f_{\mathbf{S}}(z_1) = w_1$.

$\square$

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References

[1] R.Arens: *Operational calculus of linear relations,*
Pac.J.Math. 11 (1961),9-23.

[2] J.A.Ball & I.Gohberg & L.Rodman: *Interpolation of rational matrix functions,*
Birkhäuser Verlag, Basel, 1990.

[3] J.Bognar: *Indefinite inner product spaces,*
Springer Verlag, Berlin, 1974.

[4] A.Dijksma & H.S.V.de Snoo: *Symmetric and selfadjoint relations in Krein spaces I,*
Operator Theory: Adv. and Appl. 24 (1987),145-166.

[5] A.Dijksma & H.S.V.de Snoo: *Symmetric and selfadjoint relations in Krein spaces II,*
Ann.Acad.Sci.Fenn. (AI) 12 (1987),199-216.

[6] A.Dijksma & H.Langer: *preprint,*
.

[7] I.S.Iohvidov & M.G.Krein & H.Langer: *Introduction to the spectral theory of operators in spaces with an indefinite metric,*
Akademie Verlag, Berlin, 1982.

[8] M.G.Krein & H.Langer: *Defect subspaces and generalized resolvents of an hermitian operator in the space* Π_κ*,*
Functional Anal.Appl. 5 (1971/72) 136-146, 217-220.

[9] M.G.Krein & H.Langer: *Über die Q-Funktion eines* π*-hermiteschen Operators im Raume* Π_κ*,*
Acta Sci.Math. (Szeged) 34 (1973),191.

[10] M.G.Krein & H.Langer: *Über einige Fortsetzungsprobleme, die eng mit der Theorie hermitescher Operatoren im Raume* Π_κ *zusammenhängen. (I.) Einige Funktionenklassen und ihre Darstellungen,*
Math.Nachr. 77 (1977), 187-236.

[11] J.v/d Ploeg: *Operator functions and associated reproducing kernel Pontrjagin spaces,*
University of Groningen, preprint.

[12] H.Woracek: *Nevanlinna-Pick interpolation: The degenerated case,*
preprint.

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