The Szegő minimum problem.

Mikhail Sodin

Given a non-negative measure ρ on the unit circle $\mathbb{T}$, the Szegő minimum problem is to find the quantity

$$e_n(\rho)^2 = \min_{q_1,\dots,q_n} \int_{\mathbb{T}} \bigl|1 + q_1 t + \ldots + q_n t^n\bigr|^2 \, \mathrm{d}\rho(t).$$

A celebrated Szegő's theorem states that

$$\lim_{n\to\infty} e_n(\rho) = \exp\Bigl(\frac{1}{2}\int_{\mathbb{T}} \log \rho' \, \mathrm{d}m\Bigr),$$

where m is the Lebesgue measure, and $\rho' = \mathrm{d}\rho/\mathrm{d}m$ is the Radon-Nikodym derivative. Thus, $\lim_{n\to\infty} e_n(\rho) = 0$ if and only if the measure ρ has a divergent logarithmic integral. In spite of the classical nature and omnipresence of this result, little is known how properties of a measure ρ with divergent logarithmic integral affect the rate of decay of the sequence $e_n(\rho)$.

Several quantitative results in that direction were proven recently in joint works with Alexander Borichev, Anna Kononova, and Fedor Nazarov. Among them are a necessary and sufficient condition on ρ that guarantees a subexponential decay of $e_n(\rho)$, i.e., a solution to the Erdős-Turán problem, upper and lower bounds for $\log_- e_n(\rho)$ that match each other up to positive numerical factors, and the refutation of the possibility of a relative Szegő asymptotics.