To the Birman-Krein-Visic Theory. Solution to the Birman problem.

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Let A be a closed non-negative symmetric densely defined operator in a Hilbert space $\mathfrak{H}$ and let $\mathfrak{H}_1 := \operatorname{ran}(A+I)$. Stone and Friedrichs proved that the set $\operatorname{Ext}_A(0,\infty)$ of all nonnegative selfadjoint extensions $\widetilde{A} = \widetilde{A}^*$ of A is nonempty. Complete theory of extensions of $A \geq 0$ was built by M. Krein. In particular, he proved that $\operatorname{Ext}_A(0,\infty)$ contains the maximal (the Friedrichs) and the minimal (the Krein) extensions $\widehat{A}_F$ and $\widehat{A}_K$. They are uniquely characterized by means of the following inequalities: $\widehat{A}_K \leq \widetilde{A} \leq \widehat{A}_F$ for each $\widetilde{A} \in \operatorname{Ext}_A(0,\infty)$ which are understood in the sense of either quadratic forms or the resolvents.

Krein's theory has substantially been completed by M. Vicik and M. Birman.

If A is positive definite, then $\widehat{A}_K$ admits a representation $\widehat{A}_K = \widehat{A}'_K \oplus (\mathbb{O} \upharpoonright \mathfrak{N}_0)$ where $\mathfrak{N}_0 := \ker A^*$. The operator $\widehat{A}'_K$ is called the reduced Krein extension.

Krein proved the implication $(I_{\mathfrak{H}} + \widehat{A}_F)^{-1} \in \mathfrak{S}_\infty \implies (I_{\mathfrak{M}_0} + \widehat{A}'_K)^{-1} \in \mathfrak{S}_\infty$. We improve and complete Krein's result by showing that replacing $\widehat{A}_F$ by A turns this implication into the equivalence:

$$P_1(I_{\mathfrak{H}} + A)^{-1} \in \mathfrak{S}_\infty(\mathfrak{H}_1) \quad \Longleftrightarrow \quad (I_{\mathfrak{M}_0} + \widehat{A}'_K)^{-1} \in \mathfrak{S}_\infty(\mathfrak{M}_0), \tag{1}$$

where P_1 is the orthoprojection in $\mathfrak{H}$ onto $\mathfrak{H}_0$ and $\mathfrak{M}_0 = \mathfrak{N}_0^\perp$. It happens that this equivalence remains valid with the ideal $\mathfrak{S}_\infty$ of compact operators replaced by any symmetrically normed ideal $\mathfrak{S}$ (including ideals $\mathfrak{S}_p$, Σ_p, Σ_p^0, etc.)

Moreover, it turns out that under certain additional assumption on A the power asymptotic behaviour of the eigenvalues of these operators coincide, i.e. the following equivalence holds as $n \to \infty$:

$$\lambda_n\big(\widehat{A}_F)\big) = a^{-1} n^{1/p}\,(1 + o(1)) \iff \lambda_n\big(\widehat{A}'_K\big) = a^{-1} n^{1/p}\,(1 + o(1))\,.$$

In accordance with the Grubb result an extension $A_B = A_B^*$ is semibounded below only simultaneously with its boundary operator B (LSB-property of A) whenever $(I_{\mathfrak{H}} + A)^{-1} \in \mathfrak{S}_\infty$. An improvement of this result will also be discussed.

In early 2000s M.S. Birman posed the following problem.

Problem. *Assume that the operator $(I + A)^{-1} : \mathfrak{H}_1 \to \mathfrak{H}$ is compact. Is it true that the resolvent of the Friedrichs' extension $\widehat{A}_F$ of A is also compact?*

An answer to this question is negative and abstract counterexamples easy to built. In particular, they show that the Krein implication is not reversible.

Birman asked also to present examples of a non-negative symmetric differential operator A with compact inverse $(I + A)^{-1}$ and such that $(I + \widehat{A}_F)^{-1}$ is not. We will discuss a solution to this Birman problem for certain restrictions of Schrödinger operators $H(q) = -\Delta + q \geq 0$ in $\mathbb{R}^n$ with $\operatorname{dom}(H(q)) = W^{2,2}(\mathbb{R}^n)$.

Moreover, it will be shown that for certain $q \geq 0$ the spectrum of $\widehat{A}_F$ is purely absolutely continuous while $(I + A)^{-1}$ is compact.

The main results of the talk were announced in [1] and [2].

1. M. M.Malamud, To Birman–Krein–Vishik Theory, Doklady Mathematics, Vol. 107, No. 1 (2023), pp. 44–48.

2. M.M. Malamud, On the Birman problem on positive symmetric operators with compact inverse, Func. Anal. Appl., V.57, No 2 (2023), p. 111–116.