Automorphic Carathéodory - Julia Theorem and Related Boundary Interpolation

Alexander Kheifets

Let w be an analytic function on the unit disk, $|w(\zeta)| \leq 1$. Let t_0 be a point on the unit circle, $|t_0| = 1$. The classical Carathéodory - Julia Theorem states in particular that if w and w' have nontangential boundary values w_0, $|w_0| = 1$ and w_0', respectively, at this point t_0, then

$$t_0 \frac{w_0'}{w_0} \geq 0. \tag{1}$$

Moreover, the theorem states that $\dfrac{w(\zeta) - w_0}{\zeta - t_0}$ belongs to the Hardy class H^2.

Conversely, for every numbers w_0, $|w_0| = 1$ and w_0' such that (1) holds there exists an analytic function w on the unit disk, $|w(\zeta)| \leq 1$ with nontangential boundary values of w and w' at t_0 equal w_0 and w_0', respectively.

Let Γ be a Fuchsian group acting on the unit disk. Let β be a unitary character of this group. Let w be an analytic function on the unit disk, $|w(\zeta)| \leq 1$ which is β-automorphic, that is

$$w(\gamma(\zeta)) = \beta(\gamma) w(\zeta)$$

for every $\gamma \in \Gamma$. The goal of the work is to establish an analogue of (1) in this case: 0 in the righthand side of (1) will be replaced with a positive quantity that depends on β.

To have a meaningful construction one needs a condition on group Γ that guarantees existence for every character α of an α-automorfic function h such that $\dfrac{h(\zeta) - 1}{\zeta - t_0} \in H^2$. Recall that existence for every character α of an α-automorfic function $h \in H^2$ is equivalent to the famous Widom condition (given in terms of the Green function of group Γ) and necessary and sufficient condition of existence for every character α of an α-automorfic function h such that $\dfrac{h(\zeta)}{\zeta - t_0} \in H^2$ was established in a recent joint work with Peter Yuditskii (given in terms of the Martin function of Γ with singularity at t_0).