Trace formulas for pairs of nonselfadjoint operators. Langer's contribution to the theory and its development.

Mark Malamud

The talk is devoted to perturbation determinants and trace formulas for a pair of operators with the trace class resolvent difference.

For a pair of selfadjoint operators $\{A_0, A_1\}$ with the trace class resolvent difference this formula as well as a concept of the spectral shift function were introduced by I.M. Lifshitz (the case of finite dimensional perturbations) and M. G. Krein (general case). The first work treated a pair of non-selfadjoint operators is due to H. Langer [1]. More precisely, he generalized Krein's formula for a pair $\{T_1, T_2\}$ of bounded operators with the spectra lying within the unit disc and computed the spectral shift function via the perturbation determinant of the pair $\{T_1, T_2\}$.

We will discuss the present state of this topic including certain recent results as well as Langer's influence on the theory.

Following [2]-[5] we will discuss the existence of complex valued spectral shift function for a pair of maximal dissipative operators $\{L_1, L_2\}$ with trace class resolvent difference. For such pairs of operators the Krein-Langer type trace formulas are established for a class of operator Lipschitz functions. The proof is substantially relied on boundary triplet technique and the method of double operator integrals.

For instance, we plan to discuss a formula for the spectral shift function of a pair of m-dissipative extensions $\{L_1, L_2\}$ of a symmetric operator as well as the formula for the corresponding perturbation determinant of the pair $\{L_1, L_2\}$. Both objects are expressed via the Weyl function and boundary operators.

The problem of existence **a real valued spectral shift function** for such pairs of operators will also be discussed. Applications to boundary value problems for differential operators will be discussed too.

The talk is based on our joint works with H. Neidhardt and V. Peller [2]–[5]. Some recent results in this direction will be discussed too.

References

[1] Heinz Langer, *Eine Erweiterung der Spurformel der Storungstheorie.*
Math. Nachr., **30**, (1965), 123–135.
[2] M.M. Malamud, H. Neidhardt, *Perturbation determinants for singular perturbations.*
Russ. J. Math. Phys. **21**, (2014), 55–98.
[3] M. Malamud, H. Neidhardt, *Trace formulas for additive and non-additive perturbations.*
Adv. Math. **274**, (2015), 736–832.
[4] M.M. Malamud, H. Neidhardt, V.V. Peller, *Absolute continuity of spectral shift.*
J. Funct. Anal. **276**, (2019), 1575–1621.
[5] M. M. Malamud, H. Neidhardt, and V. V. Peller,
Real-Valued Spectral Shift Functions for Contractions and Dissipative Operators,
Doklady Mathematics, **110, No. 2**, (2024), 399–403.